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On hyperbolic vectorfields on stratified spaces

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Abstract The goal of this article is twofold. First, we want to generalize the stable/unstable manifold theorem for a normal hyperbolic invariant set to the situation where we have only Lipschitz conditions instead of C^1-smoothness assumptions. This was already claimed in the book of Hirsch/Shub/Pugh but not worked out. We will need the notion of a generalized Jacobian as well as the Implicit Function Theorem with Lipschitz conditions introduced by Clarke. Second, we apply this result to give a good generalization of the notion of a hyperbolic fixed point to a stratified vectorfield on a stratified space.

1 Introduction

The goal of this article is twofold. First using the notion of a generalized Jacobian as well as the Inverse Function Theorem with Lipschitz conditions - both tools have been developed in [Cla90] - we generalize the stable/unstable manifold theorem for a normal hyperbolic invariant submanifold (see [HPS77]) to the situation where we have only Lipschitz conditions instead of C^1-smoothness assumptions. That this should be possible has already been claimed in [HPS77]. Second, we apply this result to give a good generalization of the notion of a hyberbolic fixed point to stratified vectorfields on abstract stratified spaces. Hereby we restrict ourselves to such vectorfields which satisfy a so-called radiality assumption, i.e. they point inward a tubular neighborhood of a stratum. This condition yields similar vectorfields as introduced by M.H.Schwartz [Sch91] on Whitney-stratified spaces. Our definition of a stratified hyperbolic vector field will capture the essential properties of hyperbolic fixed points in the smooth case, namely persistence under perturbation as well as topological stability.
Several notions of index, as for example the GSV-index and the Schwartz-index, have been introduced for singularities of stratified vectorfields. For singularities of radial vectorfields [Sch91] shows that the Schwartz-index can be calculated on the stratum which contains the singular point. Moreover a generalization of the Poincaré-Hopf index theorem for radial vectorfields holds (see also [KT94] for further generalizations). With the tools developed here we can now study a different kind of index, namely the Conley index [Con78], for a stratified

hyperbolic vectorfield and show that the Conley index can equally be calculated on the stratum of the point. One can moreover study Morse type inequalities.

2 Dominated splitting for Lipschitz maps

In this section we give the definition for a bi-Lipschitz map to admit a dominated splitting on an invariant submanifold. Then we study the stable manifold for such an invariant submanifold. This is a generalization of the theory of normal hyperbolic dynamics [HPS77, Wig94] in the presence of Lipschitz conditions.
Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map. Denote by Ω_F the set of points where F fails to be differentiable. For a point $x \in \mathbb{R}^n - \Omega_F$ one denotes by $JF(x)$ the Jacobian in x. Let us recall the notion of the generalized Jacobian, denoted by $\partial F(x)$(see [Cla90]), at a point $x \in \Omega_F$. It is the convex hull (denoted by co) of all $n \times n$ matrices obtained as the limit of a sequence of the form $JF(x_i)$, where $x_i \to x$ and $x_i \notin \Omega_F$. Symbolically one can write:

$$\partial F(x) = \text{co}\{\lim JF(x_i), x_i \to x, x_i \notin \Omega_F\}.$$

$\partial F(x)$ is a nonempty compact convex set, and $\partial F(x)$ is upper semi-continuous (see [Cla90], Prop.2.6.2).
Let M be a manifold, let $F : M \to M$ be a Lipschitz map. Since the generalized Jacobian behaves well under coordinate transformation, the notion is also well-defined for Lipschitz maps between manifolds.
We now generalize the notion of dominated splitting, which plays an important role for the proof of the stability conjecture [Mañ88], from the C^1-case to the Lipschitz case.

Definition 2.1. *Let $F : M \to M$ be a bi-Lipschitz map with inverse G. Let L be an F-invariant C^1- (or Lipschitz) submanifold. We say that F admits a dominated splitting on L if there exists a ∂F invariant splitting of the tangential space $T_pM = E^s_p \oplus E^u_p, p \in L$, and a Riemannian metric on M such that*

$$\|B(p)_{|E^s_p}\| > \lambda^{-1} \text{ for all } B(p) \in \partial G(p)$$

and

$$\|A(p)_{|E^u_p}\| > \mu > 1 \text{ for all } A(p) \in \partial F(p)$$

where λ and μ are constants satisfying $\lambda\mu^{-1} < 1$ and $\mu^{-1} < 1$.
Moreover the splitting is assumed to depend continuously on the base point $p \in L$.

Remark 2.2. *(1) By ∂F-invariance of the splitting we mean that for every $A \in \partial F(p)$ there is $A(E^{s/u}_p) \subset E^{s/u}_{F(p)}$, i.e. A has diagonal form with respect to the decomposition $E^s_p \oplus E^u_p$.*

(2) One can deduce easily that the splitting is ∂G-invariant as well: All matrices $B \in \partial G(p)$ which are limits of matrices have diagonal form with respect to the splitting. Moreover the property of having diagonal form is preserved under the operation of convex closure.

(3) Note that from

$$\|A(p)_{|E^u}\| > \mu > 1 \text{ for all } A(p) \in \partial F(p)$$

we can deduce that $\|B(F(p))_{|E^u}\| < \mu^{-1} < 1$ for all $B \in \partial G(F(p))$. This is easily seen for matrices which occur as limits, and the statement for arbitrary matrices is derived therefrom by applying the triangle inequality. Note also that the converse is true in the C^1-case, but not in general.

(4) The contraction/expansion properties can be expressed independently of the Riemannian metric if one uses multiples F^n.

(5) One can show that the continuity of the splitting follows from the other conditions.

Denote by $N := \dim M$ and $k := \mathrm{rk} E^u$.

Theorem 2.3. *Let $F : M \to M$ be a bi-Lipschitz map admitting a dominated splitting like in Def. 2.1 on an F-invariant submanifold L. Then the local stable set*

$$W^s(L, F) = \{x \in U(L) \mid \lim_{n\to\infty} F^n(x) \in L\}$$

is a Lipschitz submanifold of M of dimension $N - k$.

Remark 2.4. *(1) With the assumption of the theorem, one can show the existence of an unstable Lipschitz foliation in a similar way (thus generalizing Theorem 5.6.1 in [Wig94]).*

(2) By assuming the dominated splitting condition for the inverse G one can show the existence of the unstable manifold and the stable foliation. The submanifold $\widetilde{L} \subset \widetilde{X}$ is called normally hyperbolic if both splitting conditions hold.

Proof of Theorem 2.1: We will construct the unstable manifold for the inverse map G. The proof follows the lines of [HPS77, Wig94] and uses the Hadamard graph transformation. One has to be careful only when showing the well-definedness of the graph transform: At this point we apply the Inverse Function Theorem for Lipschitz maps (see [Cla90], Theorem 7.1.1).
One can assume without loss of generality that the vector bundles E^u and E^s are smooth and that E^s is a trivial vector bundle over $\exp(E^s)$ (see [HPS77],

Proof of Theorem 4.1). Denote by Lip_γ the set of all Lipschitz sections $(id, \varphi) : \exp(E^s) \to E^u$ with Lipschitz constant γ. Denote by graph$\varphi = \{(x, \varphi(x)), x \in \exp(E^s)\}$. One has to construct a graph transform

$$\begin{array}{cccc} \mathcal{G}: & \mathrm{Lip}_\gamma & \to & \mathrm{Lip}_\gamma \\ & \varphi & \mapsto & \mathcal{G}\varphi \end{array},$$

where $\mathrm{graph}(\mathcal{G}\varphi) = G\mathrm{graph}(\varphi)$.

Let us show the well-definedness of the graph transform: We chose local charts $\{U(p) \simeq \mathbb{R}^{N-k} \oplus \mathbb{R}^k\}_{p \in L}$, such that $G : U(p) \simeq U(G(p))$. Because of the ∂G-invariance of the splitting all $B(p) \in \partial G(p)$ have the form

$$B(p) = \begin{pmatrix} B^{uu}(p) & B^{su}(p) \\ 0 & B^{ss}(p) \end{pmatrix},$$

where $B^{uu} \in M(N-k, N-k), B^{su}(N-k, k), B^{ss} \in M(k, k)$ are block matrices. Let $\varphi : \mathbb{R}^{N-k} \to \mathbb{R}^k$ be a Lipschitz map with $L(\varphi) < \gamma$ and $\varphi(0) = 0$. One has to show that $\{G(x, \varphi(x)), x \in \mathbb{R}^{N-k}\}$ is also the graph of a Lipschitz map with Lipschitz constant $< \gamma$. We show that

$$\pi_s \circ G \circ (id, \varphi) : \mathbb{R}^{N-k} \to \mathbb{R}^N \to \mathbb{R}^N \to \mathbb{R}^{N-k}$$

has a left inverse. (We denoted by π_s and π_u the projections $\pi_s : \mathbb{R}^N \to \mathbb{R}^{N-k}$ and $\pi_u : \mathbb{R}^N \to \mathbb{R}^k$.)

From the Jacobian chain rule (see [Cla90], Theorem 2.6.6) it follows that $\partial(\pi_s \circ G \circ (id, \varphi))(p) \in B^{uu}(p) + B^{su}(p)L(\varphi)$. From the dominated splitting condition one knows that $\|B^{uu}\| > \lambda^{-1}$ and thus that $B^{uu}(p)$ has maximal rank for all $B(p) \in \partial G(p)$. Thus for small enough γ also all matrices in $\partial(\pi_s \circ G \circ (id, \varphi))(p)$ have maximal rank. As a composition of Lipschitz maps, $\pi_s \circ G \circ (id, \varphi)$ is Lipschitz and we apply the Inverse Function Theorem for Lipschitz maps. Thus $\pi_s \circ G \circ (id, \varphi)$ is invertible with inverse h, and with Lemma 2.5 and 2.6 the Lipschitz constant of the inverse is $L(h) = \lambda + O(\epsilon)$. Thus we have shown that $\{G(x, \varphi(x)), x \in \mathbb{R}^{N-k}\}$ is also a graph. By using $\lambda\mu^{-1} < 1$, one can compute, like in the smooth case, that the Lipschitz constant of $\mathcal{G}\varphi$ is smaller than γ. Exploiting that $\mu^{-1} < 1$, one deduces that $\mathcal{G}$ is a contraction on Lip_γ (w.r.t the C^0-topology). The assertions of the theorem follow since the unstable manifold $W^u(L, G)$ is the unique fixed point of the graph transformation $\mathcal{G}$.

□

Let us now show the postponed lemmas:

Lemma 2.5. *With the notations of the theorem the map $\pi_s \circ G \circ (id, 0) : \mathbb{R}^{N-k} \to \mathbb{R}^{N-k}$ is bi-Lipschitz and the inverse has Lipschitz constant $< \lambda$.*

Proof: The generalized Jacobian $B^{uu} \in \partial(\pi_s \circ G \circ (id, 0))$ has maximal rank and thus by applying the Inverse Function Theorem for Lipschitz functions we can deduce that there is a Lipschitz inverse. The matrix $A^{uu}(p) \in \partial F(p), p \in L$ satisfies $\|A^{uu}(p)\| < \lambda$ and thus, because of the upper semi-continuity of the generalized Jacobian, we have $\|A^{uu}(x)\| < \lambda$ for x in a small neighborhood of p. □

Lemma 2.6. *Let $\mathcal{B} : \mathbb{R}^{N-k} \to \mathbb{R}^{N-k}$ be a bi-Lipschitz map with inverse $\mathcal{A}$. Assume that all matrices in $\partial\mathcal{B}(0)$ have maximal rank and that the Lipschitz constant of $\mathcal{A}$ satisfies $L(\mathcal{A}) < \lambda$. Let β be a Lipschitz map with $L(\beta) < \delta$. Then the map $\mathcal{B} + \beta$ is bi-Lipschitz and its inverse has Lipschitz constant $\lambda + O(\delta)$.*

Proof: From the fact that the maximal rank condition is open and that the generalized Jacobian is a compact set, one deduces that all matrices in $\partial(\mathcal{B} + \beta)(0)$ have maximal rank. Then the invertibility of $\mathcal{B}+\beta$ follows from the Inverse Function Theorem for Lipschitz maps. Let us write $\mathcal{A} + \alpha$ for the inverse of $\mathcal{B} + \beta$. From $(\mathcal{B} + \beta) \circ (\mathcal{A} + \alpha) = id$ we deduce that $\mathcal{B} \circ \alpha + \beta \circ \alpha = -\beta \circ \mathcal{A}$. By multiplying $\mathcal{A}$ from the left we obtain $\alpha = -\mathcal{A} \circ \beta \circ \alpha - \mathcal{A} \circ \beta \circ \mathcal{A}$. For the Lipschitz constants we obtain the following inequality:

$$L(\alpha) \leq L(\mathcal{A})L(\beta)L(\alpha) + L^2(\mathcal{A})L(\beta).$$

Thus $L(\alpha) \leq \frac{\lambda^2\delta}{1-\lambda\delta}$.

□

3 Definition of a stratified hyperbolic vectorfield

3.1 Preliminaries on stratified spaces

In this section we recall the basic definitions from the theory of stratified spaces. These spaces have first been introduced by Mather [Mat70] and Thom [Tho69]. For stratified vectorfields there is no canonical notion of continuity, but Mather [Mat70] introduced controlled vectorfields which have the nice property that they still induce a continuous flow. Here we modify the notion of controlled vectorfield slightly. These control conditions arrive as natural conditions for the gradient vectorfield of a stratified Morse function as defined in [Lud03].

Definition 3.1. *(see [Ver84], paragraph 1.2.1)*
A topological space X with a family of locally closed subsets $\mathcal{S}$ and a tubular system $\{(T_S, \pi_S, \rho_S)\}_{S\in\mathcal{S}}$ is an abstract stratification if the following conditions hold:

(1) X is a "nice" topological space, i.e. Hausdorff, locally compact, paracompact and with countable basis of the topology.

(2) $\mathcal{S}$ is a locally finite family of pair-wise disjoint locally closed subsets $S \subset X$ called strata such that $X = \bigcup_{S\in\mathcal{S}} S$. Each stratum S is a smooth manifold without boundary in the induced topology.

(3) *The stratification fulfills the frontier condition, i.e.* $\mathcal{S}$ *is a partially ordered set with order relation* $S \le R \Leftrightarrow S \cap \overline{R} \neq \emptyset \Leftrightarrow S \subset \overline{R}$.

(4) *The tubular system for a stratified space is a family of tuples*

$$\{(T_S, \pi_S : T_S \to S, \rho_S : T_S \to \mathbb{R}_{\ge 0})\}_{S \subset \mathcal{S}},$$

where T_S *is an open neighborhood of* S, $\pi_S : T_S \to S$ *is a continuous retraction and* $\rho : T_S \to \mathbb{R}_{\ge 0}$ *is continuous and such that* $\rho_S^{-1}(0) = S$. *For each pair of strata* (S, R) *with* $S < R$ *the map*

$$(\pi_S, \rho_S) : R \cap T_S \to S \times \mathbb{R}_{>0}$$

is smooth and submersive.

(5) *The tubular system is controlled, i.e. for each pair of strata* (S, R) *with* $S \subset \overline{R}$ *and all* $x \in T_S \cap T_R$ *the following conditions are satisfied:*

$(C\pi)$ $\quad \pi_S \pi_R(x) = \pi_S(x),$

$(C\rho)$ $\quad \rho_S \pi_R(x) = \rho_S(x).$

Definition 3.2. *A stratified* (C^k-) *vectorfield on an abstract stratified space* X *is given by a family* $\xi = \{\xi_S : S \to TS\}_{S \in \mathcal{S}}$ *of* (C^k-) *vectorfields on each stratum.*

We will study the following control conditions which are a slight modification of the ones introduced by Mather:

Definition 3.3. *A stratified vectorfield on* X *is called controlled if for all pairs of strata* (S, R) *with* $S \subset \overline{R}$ *and all* $x \in T_S \cap R$ *the following conditions are satisfied:*

$(C\pi')$ $\quad d\pi_S \xi_R(x) = \xi_S(\pi_S(x)) + \rho_S^2(x)\chi(\pi_S(x))$, *where* χ *is a stratified bounded vectorfield.*

$(C\rho')$ $\quad | d\rho_S \xi_R(x) | \le A\rho_S(x), d\rho_S \xi_R(x) \le 0.$

Note that the second part of $(C\rho')$, namely $d\rho_S \xi_R(x) < 0$, means that the vectorfield points always inward a tubular neighborhood of each stratum. Thus we included a (inverse) radiality assumption in our definition of controlled vectorfields. (There will be a slight diefference here with respect to [Sch91], since there a radial vectorfield is pointing *outward* a tubular neighborhoof of a stratum.) As mentioned before, control conditions insure the (local) integrability of stratified vectorfields. A slight modification of the arguments in the control theory of Mather (Lemma 2.3 in [Ver84]) yields the existence of the flow. From the condition $(C\rho')$ one can deduce that the flow does not leave a stratum in finite time ([dPW95], Prop. 2.5.1).

Lemma 3.4. *Let ξ be a stratified C^1-vectorfield on X satisfying the contol conditions $(C\pi')$ and $(C\rho')$. Then the following holds:*

(1) The flow Φ induced by

$$\frac{\partial \Phi}{\partial t}(x,t) = \xi(x)$$

is defined for all time t and is continuous.

(2) The restriction of Φ to each stratum is C^1.

(3) The flow can leave the large stratum only in positive infinite time.

3.2 Local structure of stratified spaces. Resolution

In this section we recall the local cone-like structure of stratified spaces (see e.g. [Ver84] for a proof). In [BHS92] there was defined a resolution of an abstract stratified space (see loc.cit. for the relation of this definition to the one formerly given by Verona) which we equally recall here. Moreover we will show how to lift the time-1-map of the flow induced by a controlled stratified vectorfield to the resolution.
As a consequence of Thom's first isotopy lemma (see e.g. [Ver84]), abstract stratified spaces have a cone-like structure: Let q be a point lying in a stratum S ($\dim S = s$) of the stratified space. There is a neighborhood U of q in X which is homeomorphic (by a strata preserving homeomorphism) to $(-\epsilon,\epsilon)^s \times \mathrm{cone}(L)$. Here L denotes the normal link in q. The normal link is independent of the choice of $q \in S$. It carries an (induced) structure of an abstract stratified space of smaller depth, where the depth of X is defined as follows:

$$\mathrm{depth}(X) := \max\{m \mid \text{ there exists a chain of strata } X_0 < X_1 < ... < X_m\}.$$

We will assume in the following that the stratified space X is compact and has a stratum X_{max} lying densely in X, i.e. $\overline{X_{max}} = X$.
A resolution $\Pi : \widetilde{X} \to X$ of an abstract stratified space can be defined [BHS92] by using an induction on the depth of the stratification . The resolution $\widetilde{X}$ is a smooth manifold and the restriction $\Pi_{|\Pi^{-1}(X_{max})} : \Pi^{-1}(X_{max}) \to X_{max}$ is a trivial covering. Using the local cone structure of the space, the resolution can be described locally as follows:

$$\begin{array}{rccc} \Pi : & (-\epsilon,\epsilon)^i \times \widetilde{L} \times (-\epsilon,\epsilon) & \to & (-\epsilon,\epsilon)^i \times \mathrm{cone}(L), \\ & (x,l,r) & \mapsto & (x,[l, \mid r \mid]). \end{array}$$

Here $[l, \mid r \mid]$ denotes the class of (l,r) in the cone, and $\widetilde{L}$ is a desingularisation of the link L, which has been defined in a former step of the induction, since $\mathrm{depth}(L) < \mathrm{depth}(X)$.

Let ξ be a stratified controlled vectorfield on X, which by Lemma 3.4 induces a flow Φ. Denote by $F : X \to X$ resp. $G : X \to X$ the time-1-map for the stratified flow Φ resp. its inverse. We would like to construct lifts $\widetilde{F} : \widetilde{X} \to \widetilde{X}$ resp. $\widetilde{G} : \widetilde{X} \to \widetilde{X}$ of F resp. G. (Note that the construction in [BHS92] to give these maps on the resolution does not apply here since the required control conditions on F and G do not hold.)
From now on we will assume that the vectorfield $\xi_{max} : X_{max} \to TX_{max}$ is C^1-bounded and thus uniformly Lipschitz. Thus also the restrictions of G and F to the stratum X_{max} are uniformly Lipschitz. For an $x \in \Pi^{-1}(X_{max})$, denote by X^x_{max} the connected component of x. Then one has a well-defined map $\widetilde{F} : \Pi^{-1}(X_{max}) \to \Pi^{-1}(X_{max})$ given by $\widetilde{F}(x) = \Pi^{-1}F(\Pi(x)) \cap X^x_{max}$. Similarly, one can define the inverse $\widetilde{G} : \Pi^{-1}(X_{max}) \to \Pi^{-1}(X_{max})$. The following lemma gives a resolution of the dynamics on X:

Lemma 3.5. *Let X_{max} be the (largest) dense stratum of a stratified space X. Let $F : X_{max} \to X_{max}$ be a C^1-diffeomorphism with inverse $G : X_{max} \to X_{max}$. Assume that F and G have bounded partial derivatives. Denote by $\widetilde{F}, \widetilde{G} : \Pi^{-1}(X_{max}) \to \Pi^{-1}(X_{max})$ the C^1 maps defined above. Then there exist continuations $\widetilde{F} : \widetilde{X} \to \widetilde{X}$ resp. $\widetilde{G} : \widetilde{X} \to \widetilde{X}$ which are uniformly Lipschitz. Moreover $\widetilde{G} \circ \widetilde{F} = \widetilde{F} \circ \widetilde{G} = id$ and thus $\widetilde{G}$ is the inverse of $\widetilde{F}$.*

Proof: The preimage of X_{max} in $\widetilde{X}$ is dense and one applies [Die85] Paragraph 3.15.6. □
Thus we obtain the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{F} & : & \widetilde{X} & \to & \widetilde{X} \\ & & \Pi \downarrow & & \downarrow \Pi \\ F & : & X & \to & X \end{array} \qquad (D)\ ,$$

where the projection Π is strata-wise smooth and $\widetilde{F}$ is the continuation given by Lemma 3.5.

3.3 Definition of a stratified hyperbolic fixed point

With these preparations we can give the definition of a stratified hyperbolic fixed point. We will call a fixed point $q \in X$ stratified hyperbolic if the $\widetilde{F}$-invariant set $\Pi^{-1}(q) \simeq \widetilde{L}$ satisfies the dominated splitting criterium presented in paragraph 2.

Definition 3.6.
Let ξ be a stratified $(C\pi' + C\rho')$-controlled C^1-vectorfield such that its restriction to X_{max} is uniformly Lipschitz. Denote by F the time-1-map for ξ. A singular point $q \in S$ of ξ is called stratified hyperbolic if the following conditions are satisfied:

(1) $q \in S$ is a hyperbolic fixed point for the restriction $F_S := F_{|S} : S \to S$.

(2) The continuation map $\widetilde{F} : \widetilde{X} \to \widetilde{X}$ of Lemma 3.5 admits a $\partial\widetilde{F}$-invariant dominated splitting

$$T_p\widetilde{X} = E^s_p \oplus E^u_p, p \in \Pi^{-1}(q)$$

along $\Pi^{-1}(q) \simeq \widetilde{L}$.

Moreover the splitting is such that $E^s_p = T_0\mathbb{R} \oplus T_l\widetilde{L} \oplus E^{ss}_q$, where $p = (q, l, 0)$ (in the local coordinates of paragraph 3.2). Here $T_0\mathbb{R}$ denotes the component in the radial direction and $T_qS = E^{ss} \oplus E^u$ is the DF_S-invariant splitting into stable and unstable directions for the hyperbolic fixed point q of $F_S : S \to S$.

3.4 Properties of hyperbolic fixed points

In this section we show the stable/unstable manifold theorem for stratified hyperbolic fixed points. In analogy to the smooth case one can moreover construct a stable and unstable foliation in a neighborhood of the fixed point and deduce the local stability of stratified hyperbolic fixed points.
Let $q \in S$ be a stratified hyperbolic fixed point lying in the stratum S. Let d be a metric on X. Then the local unstable $W^u(q, \xi)$ and stable set $W^s(q, \xi)$ are defined as:

$$W^u(q, \xi) = \{x \in U(q) \mid \lim_{t\to-\infty} d(\Phi(t, x), q) \to 0\},$$

$$W^s(q, \xi) = \{x \in U(q) \mid \lim_{t\to\infty} d(\Phi(t, x), q) \to 0\}.$$

Theorem 3.7. *Let $q \in S$ be a stratified hyperbolic fixed point for the stratified vectorfield like in Def. 3.6. Then:*
(1) The local unstable manifold $W^u(q, \xi)$ is an embedded submanifold of the stratum S of dimension $\dim E^u$.
(2) The local stable set $W^s(q, \xi)$ is an abstract stratified space. The restriction to each stratum is C^1.

Proof: (1) Because of the control condition $(C\rho')$, the flow does not leave the stratum S in positive time. Moreover a stratified hyperbolic fixed point is also a hyperbolic fixed point of the smooth vectorfield $\xi_S : S \to TS$. Thus we can deduce the unstable manifold theorem directly from smooth theory (cf. e.g. [PdM82], Theorem 6.2).
(2) We will prove the assertion for the local stable set $W^s(q, F) = \{x \in U(q) \mid \lim_{n\to\infty} d(F^n(x), q) \to 0\}$ of the time-1-map F. One can then argue as in [KH95] 17.4.3 to get $W^s(q, \xi) = W^s(q, F)$. From the definition of a stratified hyperbolic fixed point one knows that $\widetilde{F}$ satisfies the dominated splitting condition along $\Pi^{-1}(q) \simeq \widetilde{L}$. We apply Lemma 2.3 to construct the stable manifold $W^s(\Pi^{-1}(p), \widetilde{F}))$, which is a Lipschitz submanifold of $\widetilde{X}$. The

local stable manifold for the stratified hyperbolic fixed point q is given by $W^s(q, F) = \Pi(W^s(\widetilde{L}, \widetilde{F}))$. The fibers of Π are transversal to S and thus we can deduce that $W^s(p, F) = \Pi(W^s(L, \widetilde{F}))$ is an abstract stratified space (see [KTL89]).

The proof that the stable manifold is stratawise C^1 follows by exploring both the Lipschitz property and the control conditions. The proof is postponed to section 3.6. □

In the smooth case the hyperbolicity of a critical point is equivalent to the local structural stability. The next proposition shows that stratified hyperbolic fixed points are structurally stable.

We will say that two stratified vectorfields are C^1-close if their restriction to each stratum is C^1-close in the strong (or Whitney) topology.

Proposition 3.8. *Let X be an abstract stratified space. Let ξ and ξ' be two $(C\pi')$, $(C\rho')$ controlled stratified vectorfields which are C^1-close.*

(1) Let $q \in S$ be a stratified hyperbolic fixed point of ξ. Then ξ' has also a stratified hyperbolic fixed point $q' \in S$ lying in a neighborhood of q. The stable sets $W^s(q, \xi)$ and $W^s(q', \xi')$ are strata-wise C^1-close (in the strong topology). Equally the unstable manifolds $W^u(q, \xi)$ and $W^u(q', \xi')$ are C^1-close.

(2) There exist neighborhoods U resp. U' of q and q' resp. as well as a strata-preserving homeomorphism $h : U \to U'$ which yields a topological conjugacy of the induced time-1-maps $F : X \to X$ and $F' : X \to X$, i.e. the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{h} & U' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ U & \xrightarrow{h} & U'. \end{array}$$

Proof:

(1) Since $q \in S$ is a hyperbolic fixed point of ξ_S, every small perturbation ξ'_S has a hyperbolic fixed point $q' \in S$ in a neighborhood of q. One has to check that the dominated splitting condition is satisfied for $\Pi^{-1}(q')$ where the pair of constants (λ, μ) is replaced by $(\lambda + \epsilon, \mu - \epsilon)$. This is done by showing that the cone criterium (see Paragraph 3.6) is satisfied for the Lipschitz perturbation $\widetilde{F'}$ of $\widetilde{F}$. Since the arguments are as in Paragraph 3.6 we omit the details.

(2) By using the above methods one can construct a flow invariant stable/unstable foliation in the neighborhood of a stratified hyperbolic point. The unstable leaves are C^1 submanifolds of dimenison k completely contained in a stratum, whereas the stable leaves are strata-wise C^1-smooth abstract stratified spaces, transverse to the stratum S. The proof of local stability follows from

the existence of these foliations as in the geometric proof of local stability in the case of a (smooth) hyperbolic fixed point (see e.g. [PdM82], Chapter 1.7).

Remark 3.9. *(1) Note that we can not always construct a topological conjugacy on the resolution $\widetilde{X}$. A necessary condition for doing so is that the maps $\widetilde{F}_{|\widetilde{L}} : \widetilde{L} \to \widetilde{L}$ and $\widetilde{F'}_{|\widetilde{L'}} : \widetilde{L'} \to \widetilde{L'}$ on the invariant manifolds are conjugate.*
(2) Normal hyperbolicity is not a necessary condition for topological stability. Thus in the stratified case the hyperbolicity of a critical point is not equivalent to topological stability.

3.5 Conley Index

In this section we study the Conley index of a stratified hyperbolic fixed point and show that it can be calculated in the stratum S (which contains the critical point). Let us recall the notion of the homological Conley index as introduced by Conley [Con78].
Let A be a metric space, $\varphi : A \times \mathbb{R} \to A$ be a continuous flow on A.

Definition 3.10. *Let I be an isolated flow-invariant set. A pair of compact sets (N_1, N_0) with $N_0 \subset N_1$ is called index pair for I if the following conditions are satisfied:*
(i) I is the maximal invariant set in $\overline{N_1 - N_0}$ and $I \cap N_0 = \emptyset$.
(ii) N_0 is positively invariant, i.e., if $y \in N_0$ and $\varphi(y, [0,t]) \subset N_1$ then $\varphi(y, [0,t]) \subset N_0$.
(iii) N_0 is an exit set for N_1, i.e., if $y \in N_1$ and $\varphi(y,t) \notin N_1$ for a $t > 0$ then there exists $t_0, 0 \leq t_0 < t$ such that $\varphi(y, [0, t_0]) \in N_1$ and $\varphi(y, t_0) \in N_0$.

Two index pairs for an isolated invariant set are homotopic and thus the notion of the Conley index is well-defined:

Definition 3.11. *The homotopical Conley index of I is defined as the homotopy type of an arbitrary index pair (N_1, N_0).*

The existence of the stable foliation yields a continuous projection $\pi' : U(q) \to W^u(q)$ in a neighborhood $U(q)$ of the stratified hyperbolic fixed point q. This projection commutes with the flow. It is now an easy exercise to prove the following:

Proposition 3.12. *Let q be a stratified hyperbolic fixed point of a stratified space lying in the stratum S. Let (N_1, N_0) be an index pair for $I = \{q\}$ in S (for the restricted flow). Then $(N_1^X, N_0^X) := (\pi')^{-1}(N_1, N_0) \cap \{\rho_S(x) \leq \rho_0\}$ is an index pair for I in X. The two pairs (N_1, N_0) and (N_1^X, N_0^X) are homotopic.*

This means that the Conley index of q can be defined on the stratum S. This is an analogous result as the one shown for the Schwartz index of radial vectorfields (see [Sch91]).

Let ξ be a stratified controlled vectorfield having only stratified hyperbolic fixed points. For a stratified hyperbolic fixed point $q \in X$ denote by $k(q)$ the rank of the unstable bundle. The Conley index of q is the homotopy type of the pointed sphere $(S^{k(p)}, *)$. The Morse-Conley inequalities now can be written as:

$$\sum_{q,\xi(q)=0} t^{k(q)} = \sum_{i=0}^{N} b_i(X)t^i + (1+t)Q(t),$$

where $b_i(X)$ are the Betti numbers of X and $Q(t)$ is a polynomial with non-negative coefficients. Putting $t = -1$ yields the Hopf-index theorem for the stratified hyperbolic vectorfield ξ, and the result again compares with [Sch91].

3.6 Proof of the Smoothness Assumption in Theorem 3.7

As in the smooth case we will use a cone criterium to prove stratawise C^1-smoothness of the local stable set of a stratified hyperbolic fixed point. The cone criterium will follow from the control conditions on ξ.
Let $\widetilde{F} : \widetilde{X} \to \widetilde{X}$ be a bi-Lipschitz map and $\widetilde{L}$ an $\widetilde{F}$-invariant submanifold of $\widetilde{X}$. Let us denote by $\partial\widetilde{F}(x)$ the generalized Jacobian in x and by $\lim \widetilde{F}(x) \subset \partial\widetilde{F}(x) = \{\lim J\widetilde{F}(x_i), x_i \to x, x_i \notin \Omega_{\widetilde{F}}\}$. We will say that $\widetilde{F}$ satisfies the cone criterium along $\widetilde{L}$ if there exists a familly $\{\mathcal{H}_p\}_{p\in\widetilde{L}}, \mathcal{H}_p \subset T_p\tilde{X}$ of horizontal cones with constant core dimension as well as a familly of vertical cones $\{\mathcal{V}_p\}_{p\in\widetilde{L}}, \mathcal{V}_p \subset T_p\widetilde{X}$, such that the following conditions are satisfied:

(1) $B^{-1}\mathcal{V}_p \subset \mathrm{Int}\mathcal{V}_{\widetilde{F}(p)}$ for all $p \in \widetilde{L}$ and all $B \in \partial\widetilde{G}(\widetilde{F}(p))$,

(2) $A^{-1}\mathcal{H}_{\widetilde{F}(p)} \subset \mathrm{Int}\mathcal{H}_p$ for all $p \in \widetilde{L}$ and all $A \in \partial\widetilde{F}(p)$,

(3) $\|A(u,v)\| > \mu'\|(u,v)\|$ for all $(u,v) \in \mathcal{V}_p$ and all $A \in \lim \widetilde{F}(p)$,

(4) $\|B(u,v)\| > \lambda'^{-1}\|(u,v)\|$ for all $(u,v) \in A^{-1}\mathcal{H}_{\widetilde{F}(p)}$ and all $B \in \lim \widetilde{G}(\widetilde{F}(p))$.

Note that the following weaker conditions follow from the cone criterium:

Lemma 3.13. *Let $\widetilde{F}$ satisfy the cone criterium, then the following conditions hold:*
(1') $A\mathcal{V}_p \subset \mathcal{V}_{\widetilde{F}(p)}$ for all $A \in \partial\widetilde{F}(p)$ and $p \in \widetilde{L}$,
(2') $B\mathcal{H}_{\widetilde{F}(p)} \subset \mathrm{Int}\mathcal{H}_p$ for all $B \in \partial\widetilde{G}(\widetilde{F}(p))$ and all $p \in \widetilde{L}$.

Proof: $A\mathcal{V} \subset \mathcal{V}$ is deduced from (1) for all $A \in \lim \widetilde{F}$ because in this case one can write $A = B^{-1}$ for some $B \in \partial\widetilde{G}$. The general case $A \in \partial\widetilde{F}$ follows from the special case since $\mathcal{V}$ is convex and $\partial\widetilde{F}$ is the convex closure of $\lim \widetilde{F}$. □

We will now show that the cone criterium is satisfied in a sufficiently small neighborhood of $\widetilde{L}$.
Let us first recall that the control condition $(C\pi')$ on the stratified vectorfield ξ implies the following control condition on the time-1-map $F : X \to X$:

$$(C\pi') \qquad \pi_S \circ F(z) = F \circ \pi_S(z) + O(\rho_S^2) \text{ for all } z \in T_S.$$

This induces the following condition for the induced map $\widetilde{F} : \widetilde{X} \to \widetilde{X}$:

$$\widetilde{F}(x, l, r) = \widetilde{F}(x, 0, 0) + O(r^2),$$

where (x, l, r) are the coordinates on the resolution like in Paragraph 3.2. The perturbation term $O(r^2)$ (which can depend on l and x) is Lipschitz and strata-wise smooth.

Lemma 3.14. *Let $q \in S$ be a stratified hyperbolic fixed point of the stratified vectorfield ξ satisfying the control conditions $(C\pi'), (C\rho')$ and being uniformly Lipschitz on X_{max}. Then the cone criterium is satisfied for the induced dynamics $\widetilde{F} : \widetilde{X} \to \widetilde{X}$ in a neighborhood of $\Pi^{-1}(q) \simeq \widetilde{L}$.*

Proof: We choose the following notation for $A \in \partial\widetilde{F}$ and $B \in \partial\widetilde{G}$:

$$A = \begin{pmatrix} A^{ss} & A^{su} \\ A^{us} & A^{uu} \end{pmatrix}, \quad B = \begin{pmatrix} B^{uu} & B^{us} \\ B^{su} & B^{ss} \end{pmatrix},$$

where $A^{ss}, B^{uu} \in M(N-k, N-k); A^{su}, B^{us} \in M(N-k, k); A^{us}, B^{su} \in M(k, N-k); A^{uu}, B^{ss} \in M(k, k)$. From the dominated splitting condition we know that $\|A^{uu}\| > \mu$ and therefor $\|(A^{uu})^{-1}\| < \mu^{-1}$ for all $A \in \partial\widetilde{F}(p), p \in \widetilde{L}$. By triangle inequality we can deduce $\|B^{ss}\| < \mu^{-1}$ for all $B \in \partial\widetilde{G}(\widetilde{F}(p))$. Because of the upper semicontinuity of $\partial\widetilde{G}$ we know that in a neighborhood of $\widetilde{L}$ there is $\|B^{ss}\| < \mu^{-1} + O(\epsilon)$ for all $B \in \partial\widetilde{G}(\widetilde{F}(x))$. Thus $\|A^{uu}(x)\| > \mu - \epsilon$ for all $A \in \lim \widetilde{F}$.
From the upper semicontinuity of $\partial\widetilde{F}$ we deduce that

$$\|A^{ss}\| < \lambda + \epsilon \text{ and } \|A^{us}\| < \epsilon, \|A^{su}\| < \epsilon,$$

for all $A \in \partial\widetilde{F}(x)$, x in a neighborhood of $\widetilde{L}$.
Let $(u, v) \in \mathcal{V}$, where $\mathcal{V} := \{(u, v) \mid \|u\| < \gamma\|v\|\}$ is a vertical γ-cone, i.e. $\|u\| < \gamma\|v\|$. We show that $A(u, v) =: (u', v') \in \mathcal{V}$:
For all $A \in \partial\widetilde{F}(x), x \in U(\widetilde{L})$, we have

$$\|u'\| = \|A^{ss}u + A^{su}v\| \leq \|A^{ss}u\| + \|A^{su}v\| \leq (\lambda + \epsilon)\|u\| + \epsilon\|v\|.$$

Since (u, v) lies in a vertical γ-cone we deduce that

$$\|u'\| \leq ((\lambda + \epsilon)\gamma + \epsilon)\|v\|.$$

For all $A \in \lim \widetilde{F}(x), x \in U(\widetilde{L})$ we have:

$$\|v'\| = \|A^{us}u + A^{uu}v\| \geq (\mu - \epsilon)\|v\| - \epsilon\|u\| \geq (\mu - \epsilon - \epsilon\gamma)\|v\|.$$

Putting together the estimates for $\|u'\|$ and $\|v'\|$ we deduce

$$\frac{\|u'\|}{\|v'\|} \leq \frac{\gamma(\lambda + \epsilon) + \epsilon}{\mu - \epsilon - \epsilon\gamma} < \gamma,$$

for γ and ϵ chosen small enough using the normal hyperbolicity condition $\lambda\mu^{-1} < 1$.

We have shown that $A\mathcal{V} \subset \mathcal{V}$ for all $A \in \lim \widetilde{F}$, but since $\mathcal{V}$ is convex we have $A\mathcal{V} \subset \mathcal{V}$ for all $A \in \partial\widetilde{F}$. Equivalently $A^{-1}\mathcal{H} \subset \mathcal{H}$ for the horizontal $1/\gamma$-cone. Thus we have verified the condition (2) for the cone criterium.

The proof of (1) is similar.

(3) and (4) can be deduced by a similar argument and following the estimates in the smooth case, see Lemma 6.2.11 of [KH95].

□

We now deduce:

Proposition 3.15. *Let $q \in S$ be a stratified hyperbolic fixed point of the stratified vectorfield ξ satisfying the control conditions $(C\pi'), (C\rho')$ and being uniformly Lipschitz on X_{max}. Then there exists a continuous $\partial\widetilde{F}$-invariant splitting in a neighborhood of $\Pi^{-1}(p)$.*

The proof relies on the following lemma of linear algebra, which we recall for the convenience of the reader (see [KH95], Prop. 6.2.12):

Lemma 3.16. *Let (λ', μ') be a pair with $\lambda' < \mu'$. Let $\mathcal{L}_m : \mathbb{R}^{N-k} \times \mathbb{R}^k \to \mathbb{R}^{N-k} \times \mathbb{R}^k$ be a sequence of linear maps satisfying the following conditions:*

(1) $\mathcal{L}_m \mathcal{V}^{\gamma_m} \subset \mathrm{Int}\mathcal{V}^{\gamma_{m+1}}$,

(2) $\mathcal{L}_m^{-1}\mathcal{H}^{\gamma'_{m+1}} \subset \mathrm{Int}\mathcal{H}^{\gamma'_m}$,

(3) $\|\mathcal{L}_m(u,v)\| > \mu'\|(u,v)\|$ *for all* $(u,v) \in \mathcal{V}^{\gamma_m}$,

(4) $\|\mathcal{L}_m(u,v)\| < \lambda'\|(u,v)\|$ *for all* $(u,v) \in \mathcal{L}_m^{-1}\mathcal{H}^{\gamma'_{m+1}}$.

Then

$$E^u := \bigcap_i \mathcal{L}_{m-1} \circ \ldots \circ \mathcal{L}_{m-i}\mathcal{V}^{\gamma_{m-i}}$$

is a k-dimensional vector space in $\mathcal{V}^{\gamma_m}$ and

$$E^s = \bigcap_i \mathcal{L}_m^{-1} \circ \ldots \circ \mathcal{L}_{m+i}^{-1}\mathcal{H}^{\gamma'_{m+i+1}}$$

is an $(N-k)$-dimensional vector space in $\mathcal{H}^{\gamma'_m}$.

Remark 3.17. *The subspace E^u is the uniquely determined $\mathcal{L}$-invariant vector space in the vertical cone, whereas E^s is the uniquely determined $\mathcal{L}$-invariant vector space in the horizontal cone.*

Proof of Proposition 3.15: One applies Lemma 3.16 for $\mathcal{L} = A \in \lim \widetilde{F}$ and $\mathcal{L}^{-1} = B \in \lim \widetilde{G}$. A priori, since we want a splitting which is invariant under all $A \in \partial\widetilde{F}$, we should rather take

$$E_p^s = \bigcap_{\mathcal{L}^{-1}\in\partial\widetilde{G}} \bigcap_i \mathcal{L}^{-1}_{\widetilde{G}(p)} \circ ... \circ \mathcal{L}^{-1}_{\widetilde{G}^i(p)} \mathcal{H}_{\widetilde{G}^i(p)}$$

and

$$E_p^u = \bigcap_{\mathcal{L}\in\partial\widetilde{F}} \bigcap_i \mathcal{L}_{\widetilde{F}(p)} \circ ... \circ \mathcal{L}_{\widetilde{F}^i(p)} \mathcal{V}_{\widetilde{F}^i(p)}.$$

thus considering the intersection for all possible choices of $A \in \partial\widetilde{F}$, resp. $B \in \partial\widetilde{G}$. Therefor Lemma 3.16 would not apply for a general Lipschitz map satisfying a cone criterium. But in our situation we have made more smoothness assumptions: First of all we have no problems for points $x \in \Pi^{-1}(X_{max})$ since there $\widetilde{F}$ is smooth and moreover $\widetilde{F}^n(x) \in \Pi^{-1}(X_{max})$ because of the control condition $(C\rho')$ for the flow. In points $x \in \widetilde{X} - \Pi^{-1}(X_{max})$ we know from the control condition $(C\pi')$ that $A(T\mathbb{R} \oplus TL) = T\mathbb{R} \oplus TL$ for all $A \in \partial\widetilde{F}(x)$ and moreover $\partial\widetilde{F}_{|TS}$ is a single point. This follows from the stratawise smoothness of the flow.

□

As a corollary of Proposition 3.15 we obtain the stratawise smoothness of the local stable set:

Corollary 3.18. *Let $q \in S$ be a stratified hyperbolic fixed point of the stratified vectorfield ξ satisfying the control conditions $(C\pi'), (C\rho')$ and being uniformly Lipschitz on X_{max}. Then the stable set $W^s(q,\xi)$ is stratawise C^1.*

Proof of Corollary 3.18: One can apply the proof in [KH95] Theorem 6.2.8 (step 4 of the proof) to see that the stable manifold $W^s(\Pi^{-1}(q), \widetilde{F})$ is C^1, the tangent space beeing given by the continuous family E^s obtained in Proposition 3.15. By using the strata-wise smoothness of the projection $\Pi : \widetilde{X} \to X$ one deduces the statement for $W^s(q, F) = \Pi(W^s(\Pi^{-1}(q), \widetilde{F}))$.

□

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