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**Spectral Estimates and Non-Selfadjoint
Perturbations of Spheroidal Wave
Operators**

by

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Abstract

We derive a spectral representation for the oblate spheroidal wave operator, which is holomorphic in the aspherical parameter Ω in a neighborhood of the real line. For real Ω , estimates are derived for all eigenvalue gaps uniformly in Ω .

The proof of the gap estimates is based on detailed estimates for complex solutions of the Riccati equation. The spectral representation for complex Ω is derived using the theory of slightly non-selfadjoint perturbations.

1 Introduction

Recently an integral representation was derived for solutions of the scalar wave equation in the Kerr black hole geometry [3]. This result relies crucially on a spectral representation for the oblate spheroidal wave operator for complex values of the aspherical parameter Ω (also referred to as “ellipticity parameter” or “semifocal distance”). In the present paper we rigorously derive this spectral representation. The reason why this problem deserves to be worked out in a separate paper is that most of our methods apply in a much more general context. Namely, the core of the paper is to derive estimates for the eigenvalue gaps $\lambda_{n+1} - \lambda_n$ for real Ω , which are uniform in Ω and n . To this end, we need to control the eigenvalues and the behavior of the wave functions in detail. Our method is based on invariant region estimates for the complex Riccati equation and applies to general one-dimensional Schrödinger problems. In particular, it gives refined error estimates for WKB approximations. We regard the spheroidal wave equation as a model problem for working out these estimates.

Despite the vast literature on spectral estimates for the Schrödinger equation (see e.g. [10] and the references therein), gap estimates are rarely found in the standard literature. Most papers are concerned with the two lowest eigenvalues [7, 12], or they apply in special situations like a nearly constant potential [8]. Probably, this is because gap estimates depend sensitively on the detailed form of the potential (as one sees in the example of a double-well potential), making it difficult to get general results. Our method requires that the potential is piecewise monotone and that we have explicit control of its derivatives.

We now introduce our problem and state our results. The spheroidal wave equation can be written as the eigenvalue equation

$$\mathcal{A}\Theta = \lambda\Theta \tag{1.1}$$

where \mathcal{A} is the elliptic differential operator on the unit sphere

$$\mathcal{A} = -\frac{d}{d \cos \vartheta} \sin^2 \vartheta \frac{d}{d \cos \vartheta} + \frac{1}{\sin^2 \vartheta} (\Omega \sin^2 \vartheta + k)^2 \quad (1.2)$$

which involves two parameters $k \in \mathbb{Z}$ and $\Omega \in \mathbb{C}$. We consider \mathcal{A} as an operator in the Hilbert space $\mathcal{H} = L^2(S^2)$ with domain of definition $D(\mathcal{A}) = C^\infty(S^2)$. We will fix k throughout, whereas Ω is a variable parameter. Note that the potential of this operator is in general complex,

$$\operatorname{Im} \left(\frac{(\Omega \sin^2 \vartheta + k)^2}{\sin^2 \vartheta} \right) = 2 (\operatorname{Re} \Omega \sin^2 \vartheta + k) \operatorname{Im} \Omega, \quad (1.3)$$

and therefore \mathcal{A} is symmetric only if Ω is real. In previous works asymptotic expansions for individual eigenvalues are derived [4, 9], and it is shown numerically that eigenvalues can degenerate for non-real Ω [5], but rigorous estimates or completeness statements are not given. Our main result is the following spectral representation for Ω in a neighborhood of the real line.

Theorem 1.1 *For any given $k \in \mathbb{Z}$ and $c > 0$, we define the open set $U \subset \mathbb{C}$ by the condition*

$$|\operatorname{Im} \Omega| < \frac{c}{1 + |\operatorname{Re} \Omega|}. \quad (1.4)$$

Then there is a positive integer N and a family of operators $Q_k(\Omega)$ on \mathcal{H} defined for $k \in \mathbb{N} \cup \{0\}$ and $\Omega \in U$ with the following properties:

- (i) *The Q_k are holomorphic in Ω .*
- (ii) *Q_0 is a projector on an N -dimensional invariant subspace of \mathcal{A} . For $k > 0$, the Q_n are projectors on one-dimensional eigenspaces of \mathcal{A} with corresponding eigenvalues $\mu_k(\Omega)$. These eigenvalues satisfy a bound of the form*

$$|\mu_k(\Omega)| \leq C(k) (1 + |\Omega|) \quad (1.5)$$

for suitable constants $C(k)$. Furthermore, there is a parameter $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ and $\Omega \in U$,

$$|\mu_k(\Omega)| \geq k \varepsilon. \quad (1.6)$$

- (iii) *The Q_k are complete, i.e.*

$$\sum_{k=0}^{\infty} Q_k = \mathbf{1}$$

with strong convergence of the series.

- (iv) *The Q_k are uniformly bounded, i.e. for all $k \in \mathbb{N}_0$,*

$$\|Q_k\| \leq c_1 \quad (1.7)$$

with c_1 independent of Ω and k .

If c is sufficiently small, $c < \varepsilon$, or the real part of Ω is sufficiently large, $|\operatorname{Re} \Omega| > C(c)$, one can choose $N = 1$, i.e. \mathcal{A} is diagonalizable with non-degenerate eigenvalues.

To avoid misunderstandings, we point out that by “projector on an invariant subspace of \mathcal{A} ” we mean an operator Q which is idempotent and commutes with \mathcal{A} . But Q will in general not be symmetric.

In our proof we shall treat the imaginary part of the potential (1.3) as a slightly non-selfadjoint perturbation in the spirit of [6, Chap. V, §4, Sec. 5], see also [2, Chapter 12]. For this method to be applicable, we need good control of the eigenvalues of the corresponding selfadjoint problem. Our starting point is the following standard spectral decomposition of \mathcal{A} in the case of *real* Ω .

Theorem 1.2 *For any given $k \in \mathbb{Z}$ and $\Omega \in \mathbb{R}$, the operator \mathcal{A} is essentially selfadjoint, positive and compact resolvent. It is invariant on the even and odd parity subspaces \mathcal{H}^\pm defined by*

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \quad \text{with} \quad \mathcal{H}^\pm = \{\phi \in \mathcal{H} \text{ with } \phi(\pi - \vartheta) = \pm\phi(\vartheta)\}.$$

We denote the eigenvalues of \mathcal{A} restricted to \mathcal{H}^\pm by λ_n^\pm and count them with multiplicities,

$$0 \leq \lambda_1^\pm < \lambda_2^\pm < \lambda_3^\pm \leq \dots$$

The crucial and most delicate part is the following gap estimate uniform in n and Ω .

Theorem 1.3 *For any given $k \in \mathbb{Z}$ and $\gamma > 0$, there is a positive integer N such that*

$$\lambda_{n+1}^\pm - \lambda_n^\pm > \gamma \quad \text{for all } n \geq N \text{ and } \Omega \in \mathbb{R}.$$

If γ is sufficiently small or $|\Omega|$ is sufficiently large, one can choose $N = 1$.

The paper is organized as follows. In Section 2 we prove Theorem 1.2 and reduce Theorem 1.3 to gap estimates for a real Schrödinger equation on the interval $\vartheta \in [0, \frac{\pi}{2}]$ with suitable boundary conditions. In Sections 2–6 we introduce the complex Riccati equation and develop general techniques for analyzing its solutions. In Section 7 and 8 we apply these techniques to the spheroidal wave operator and prove Theorem 1.3. Finally, in Section 8 we use perturbative methods to prove Theorem 1.1.

2 Basic Considerations, the Schrödinger Equation

Until the end of Section 7 we will consider the spheroidal wave equation (1.1) for real Ω . Using that (1.2) is symmetric under the transformations $\Omega \rightarrow -\Omega$ and $k \rightarrow -k$, we can clearly assume that

$$\Omega > 0.$$

Obviously, the operator \mathcal{A} is symmetric, elliptic and positive. Standard elliptic theory [11] yields that \mathcal{A} is essentially selfadjoint and has compact resolvent. Furthermore, (1.2) is symmetric under the transformation $\vartheta \rightarrow \pi - \vartheta$, and thus the parity subspaces \mathcal{H}^\pm are invariant under \mathcal{A} . This gives Theorem 1.2.

In order to better understand Theorem 1.3, it is useful to consider the limits $n \rightarrow \infty$ and $\Omega \rightarrow \infty$. For fixed Ω and large n , Weyl’s asymptotics applies and yields that the eigenvalues of \mathcal{A} behave for large n like the eigenvalues of the Laplacian on the sphere,

$$\lambda_n \sim n^2 \quad \text{and} \quad \lambda_{n+1} - \lambda_n \sim n.$$

Therefore, it is obvious that the statement of Theorem 1.3 holds for any fixed Ω and sufficiently large $N = N(\Omega)$. The estimate

$$|\lambda_n(\Omega) - \lambda_n(\Omega')| \leq \|\mathcal{A}(\Omega) - \mathcal{A}(\Omega')\|_\infty \leq |\Omega - \Omega'| (\Omega + \Omega' + 2k)$$

yields that eigenvalues of \mathcal{A} are locally Lipschitz in Ω , uniformly in n . This shows that the constant $N(\Omega)$ can be chosen locally uniformly in Ω . If conversely we fix n , the asymptotic expansion of the spheroidal eigenvalue λ_n for large Ω (see [4] or [9])

$$\lambda_n(\Omega) = \begin{cases} 2(n+1)\Omega + O(1) & \text{if } n-k \text{ is even,} \\ 2n\Omega + O(1) & \text{if } n-k \text{ is odd.} \end{cases} \quad (2.1)$$

shows us that for each n we can make the eigenvalue gap arbitrarily large by choosing Ω sufficiently large. We conclude that it remains to show that the eigenvalue gaps are bounded uniformly as both N and $|\Omega|$ become large. This is the hard part of Theorem 1.3, and we state it as a separate Lemma.

Lemma 2.1 *For any given $k \in \mathbb{Z}$ and $c > 0$, there are constants $N \in \mathbb{N}$ and $\Omega_0 > 0$ such that*

$$\lambda_{n+1}^\pm - \lambda_n^\pm > c \quad \text{for all } n \geq N \text{ and } \Omega > \Omega_0.$$

The proof of this lemma requires detailed eigenvalue estimates. We will complete it in Section 7, and this will also finish the proof of Theorem 1.3.

Let us collect some basic facts about the spheroidal wave functions Θ_n^\pm . First of all, standard elliptic regularity theory applied to the yields that they are smooth functions on S^2 . The spheroidal wave operator (1.2) is singular at the poles $\vartheta = 0, \pi$. Introducing the variable $u = \vartheta$ (or analogously $u = \pi - \vartheta$), the spheroidal wave equation has near $u = 0$ the asymptotic form

$$\left(-\frac{d^2}{du^2} - \frac{1}{u} \frac{d}{du} + \frac{k^2}{u^2} \right) \Theta = 0.$$

Computing the fundamental solutions of this equation and using Sturm-Liouville theory, one sees that the spheroidal wave functions have near $u = 0$ the asymptotic form $\Theta \sim u^0, \log u$ (if $k = 0$) and $\Theta \sim u^{\pm k}$ (if $k \neq 0$). The cases $\Theta \sim \log u$ and $\Theta \sim u^{-|k|}$ cannot occur because they would lead to a singularity of Θ at the poles. We conclude that near the poles the spheroidal wave functions have the following asymptotic form,

$$\Theta_n^\pm(u) = c u^{|k|} + \mathcal{O}(u^{|k|+1}) \quad \text{with } c \neq 0. \quad (2.2)$$

Furthermore, the node theorem [13, Theorem 14.10] tells us about the number of zeros of the spheroidal wave functions. In our setting, the statement of the node theorem can easily be derived as follows. We know from Sturm-Liouville theory that the spheroidal wave functions depend smoothly on the parameter Ω . If the number of zeros of the function Θ_n^\pm changed at some Ω , there would be a $u \in (0, \pi)$ with $\Theta_n^\pm(u) = 0 = (\Theta_n^\pm)'(u)$, in contradiction to the fact that Θ_n^\pm does not vanish identically. We conclude that the number of zeros of Θ_n^\pm is independent of Ω , and therefore it suffices to consider the case $\Omega = 0$, when the spheroidal wave functions reduce to the spherical harmonics. We thus obtain

$$\left. \begin{array}{l} \Theta_n^+(\vartheta) \\ \Theta_n^-(\vartheta) \end{array} \right\} \text{ has on the interval } (0, \pi) \left\{ \begin{array}{l} 2n \text{ zeros} \\ 2n + 1 \text{ zeros} \end{array} \right. \quad (2.3)$$

(note that we do not count the zeros at $\vartheta = 0, \pi$).

Next we reformulate the eigenvalue problem (1.1) as a boundary value problem for a Schrödinger-type equation on the interval $0 \leq \vartheta \leq \frac{\pi}{2}$. In the variable $u = \vartheta \in [0, \pi]$, the angular operator (1.2) can be written as

$$\mathcal{A} = -\frac{1}{\sin u} \frac{d}{du} \sin u \frac{d}{du} + \frac{1}{\sin^2 u} (\Omega \sin^2 u + k)^2.$$

In order to get rid of the first order term, we introduce the function Y by

$$Y = \sqrt{\sin u} \Theta. \quad (2.4)$$

Then

$$B Y = \lambda Y,$$

where

$$\begin{aligned} B &= -\frac{1}{\sqrt{\sin u}} \frac{d}{du} \sin u \frac{d}{du} \frac{1}{\sqrt{\sin u}} + \frac{1}{\sin^2 u} (\Omega \sin^2 u + k)^2 \\ &= -\frac{d^2}{du^2} + \frac{1}{2} \frac{\cos^2 u}{\sin^2 u} - \sqrt{\sin u} \left(\frac{1}{\sqrt{\sin u}} \right)'' + \frac{1}{\sin^2 u} (\Omega \sin^2 u + k)^2 \\ &= -\frac{d^2}{du^2} - \frac{1}{4} \frac{\cos^2 u}{\sin^2 u} - \frac{1}{2} + \frac{1}{\sin^2 u} (\Omega \sin^2 u + k)^2. \end{aligned}$$

Thus Y satisfies the so-called *Schrödinger equation*

$$\left(-\frac{d^2}{du^2} + V \right) Y = 0, \quad (2.5)$$

where V is the potential

$$V = \Omega^2 \sin^2 u + \left(k^2 - \frac{1}{4} \right) \frac{1}{\sin^2 u} - \mu \quad (2.6)$$

with

$$\mu = \lambda - 2\Omega k + \frac{1}{4}. \quad (2.7)$$

The behavior near the poles (2.2) and the normalization condition gives rise to the following boundary conditions at $u = 0$,

$$\begin{cases} Y_n^\pm(0) = 0 & \text{if } k \neq 0 \\ \lim_{u \searrow 0} u^{-\frac{1}{2}} |Y_n^\pm(u)| < \infty & \text{if } k = 0 \end{cases} \quad (2.8)$$

The boundary conditions at $u = \frac{\pi}{2}$ are obvious from the parity,

$$\begin{cases} (Y_n^+)' \left(\frac{\pi}{2} \right) = 0 \\ Y_n^- \left(\frac{\pi}{2} \right) = 0. \end{cases} \quad (2.9)$$

The eigenvalue equation (1.1) is equivalent to the Schrödinger equation (2.5) with boundary conditions (2.8, 2.9). Finally, the node theorem (2.3) yields that

$$Y_n^\pm \text{ has } n \text{ zeros on } (0, \frac{\pi}{2}). \quad (2.10)$$

3 The Complex Riccati Equation

Let Y_1 and Y_2 be two real fundamental solutions of the Schrödinger equation (2.5) for a general real and smooth potential V . Then their Wronskian

$$w := Y_1(u) Y_2'(u) - Y_1'(u) Y_2(u)$$

is a constant; we assume in what follows that $w > 0$. We combine the two real solutions to the complex function

$$z = Y_1 + iY_2,$$

and denote its polar decomposition by

$$z = \rho e^{i\varphi} \tag{3.1}$$

with real functions $\rho(u) \geq 0$ and $\varphi(u)$. By linearity, z is a solution of the complex Schrödinger equation

$$z'' = Vz. \tag{3.2}$$

Note that z has no zeros because at every u at least one of the fundamental solutions does not vanish. Thus the function y defined by

$$y = \frac{z'}{z} \tag{3.3}$$

is smooth. Moreover, y satisfies the complex Riccati equation

$$y' + y^2 = V. \tag{3.4}$$

The fact that the solutions of the complex Riccati equation are smooth will be helpful for getting estimates. Conversely, from a solution of the Riccati equation one obtains the corresponding solution of the Schrödinger equation by integration,

$$\log z|_u^v = \int_u^v y. \tag{3.5}$$

Using (3.1) in (3.3) gives separate equations for the amplitude and phase of z ,

$$\rho' = \rho \operatorname{Re} y, \quad \varphi' = \operatorname{Im} y,$$

and integration gives

$$\log \rho|_u^v = \int_u^v \operatorname{Re} y \tag{3.6}$$

$$\varphi|_u^v = \int_u^v \operatorname{Im} y. \tag{3.7}$$

Furthermore, the Wronskian gives a simple algebraic relation between ρ and y . Namely, w can be expressed by $w = \operatorname{Im}(\bar{z} z') = \rho^2 \operatorname{Im} y$ and thus

$$\rho^2 = \frac{w}{\operatorname{Im} y}. \tag{3.8}$$

Since ρ^2 and w are non-negative, we see that

$$\operatorname{Im} y(u) > 0 \quad \text{for all } u. \tag{3.9}$$

The boundary conditions for the Schrödinger equation can easily be translated into conditions for y . To this end we write the solutions Y_n^\pm of the Schrödinger equation as $Y_n^\pm = \text{Im}(e^{-i\alpha} z_n^\pm)$ with a suitable phase factor $e^{-i\alpha}$. Then a Dirichlet condition can be written as $\varphi = \alpha \bmod \pi$. A Neumann boundary condition gives

$$0 = \text{Im}(e^{-i\alpha} yz) = \text{Re}\left(e^{-i(\alpha + \frac{\pi}{2})} yz\right) = \left[\text{Re } y \cos(\varphi - \alpha - \frac{\pi}{2}) - \text{Im } y \sin(\varphi - \alpha - \frac{\pi}{2})\right] \rho$$

and thus

$$\varphi = \alpha + \frac{\pi}{2} + \arctan\left(\frac{\text{Re } y}{\text{Im } y}\right).$$

According to (3.7) and (3.9) the function $\varphi(u)$ is monotone increasing. Therefore the number of zeros of Y , (2.10) tells us how often φ crossed the points $\bmod \pi$. This allows us to completely determine the “phase shifts” on the interval $(0, \frac{\pi}{2})$,

$$\varphi_n^+ \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} + \arctan\left(\frac{\text{Re } y(\frac{\pi}{2})}{\text{Im } y(\frac{\pi}{2})}\right) + n\pi \quad (3.10)$$

$$\varphi_n^- \Big|_0^{\frac{\pi}{2}} = (n+1)\pi \quad (3.11)$$

(we use the usual convention that the arc tangent takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$). Using (3.7) these boundary conditions can be expressed purely in terms of y and the integral of the imaginary part of y .

For the gap estimates we need to control how y depends on λ . To this end, we differentiate through the complex Riccati equation (3.4) and use that $\partial_\lambda V = -1$ according to (2.6) and (2.7). This gives the linear ODE

$$y'_\lambda = -1 - 2yy_\lambda,$$

where the λ -derivative is denoted by a subscript. This equation can immediately be integrated using variation of constants. Applying (3.5), we obtain

$$z^2 y_\lambda \Big|_u^v = - \int_u^v z^2. \quad (3.12)$$

Substituting into this relation the integration-by-parts formula

$$\int_u^v z^2 = \int_u^v \frac{1}{2y} (z^2)' = \frac{z^2}{2y} \Big|_u^v + \int_u^v \frac{V - y^2}{2y^2} z^2$$

we obtain the identity

$$z^2 y_\lambda \Big|_u^v = - \frac{z^2}{2y} \Big|_u^v - \int_u^v \frac{V - y^2}{2y^2} z^2. \quad (3.13)$$

In our estimates we will work both with (3.12) and (3.13).

4 Invariant Region estimates

In this section we describe estimates for the complex Riccati equation (3.4) with initial conditions at $u = 0$,

$$y' = V - y^2, \quad y(0) = y_0 \quad (4.1)$$

on the interval $[0, u_{\max})$ with $u_{\max} \in \mathbb{R}^+ \cup \{\infty\}$. In what follows the potential $V \in C^\infty([0, u_{\max}))$ can be any real and smooth function. The next lemma is the key to all the estimates in this section.

Lemma 4.1 *Let α be a real function on $[0, u_{\max}]$ which is continuous and piecewise C^1 . For a constant $T_0 \geq 1$ we introduce the functions σ , U and T by*

$$\sigma(u) = \exp\left(2 \int_0^u \alpha\right) \quad (4.2)$$

$$U(u) = V - \alpha^2 - \alpha' \quad (4.3)$$

$$T(u) = T_0 \exp\left(\frac{1}{2} TV_{[0,u]} \log |\sigma^2 U|\right). \quad (4.4)$$

Furthermore, we define the functions β , R and m by

$$\beta = \frac{\sqrt{|U|}}{2} \left(T + \frac{1}{T}\right) \quad (4.5)$$

$$R = \frac{\sqrt{|U|}}{2} \left(T - \frac{1}{T}\right) \quad (4.6)$$

$$m = \alpha + i\beta. \quad (4.7)$$

Suppose that $U \leq 0$ on $[0, u_{\max}]$. If a solution y of the boundary value problem (4.1) satisfies at $u = 0$ the condition

$$|y - m| \leq R, \quad (4.8)$$

then this condition holds for all $u \in [0, u_{\max}]$.

Before coming to the proof we briefly discuss the statement of this lemma. If α is a real solution of the Riccati equation, the function U as given by (4.3) vanishes identically, and thus $\beta \equiv 0 \equiv R$. In this case, the above lemma reduces to the trivial statement that $y(0) = \alpha$ implies that $y = \alpha$ on $[0, u_{\max}]$. It is more interesting to consider the case that $\alpha = \operatorname{Re} y$ with y a complex solution of the Riccati equation. Then

$$U = \operatorname{Re}(V - \alpha^2 - \alpha') = \operatorname{Re}(V - z^2 - z') - (\operatorname{Im} y)^2 = -(\operatorname{Im} y)^2 < 0.$$

Moreover, from (3.6) we can immediately compute σ ,

$$\sigma(u) = \exp\left(2 \int_0^u \operatorname{Re} y\right) = \frac{\rho^2(u)}{\rho^2(0)}.$$

Substituting these relations into (3.8) yields

$$\sigma^2 U = -\frac{\rho^4(u)}{\rho^4(0)} (\operatorname{Im} y)^2 = -\frac{w^2}{\rho^4(0)}.$$

Hence the function $\log |\rho^2 U|$ is a constant, and its total variation in (4.4) vanishes. This means that T is a constant, and thus β and R are constant multiples of $\operatorname{Im} y$. Our Lemma states that the circles of radius $R(u)$ around the point $m(u) = \alpha(u) + i\beta(u)$ are invariant under the flow of the Riccati equation.

If no solution of the Riccati equation is known (and this will of course be the usual situation), one can take for α the real part of an *approximate solution* of the complex Riccati equation. In this case the function $\log |\rho^2 U|$ will not be constant, but we can hope that its total variation is small. If this is the case, our lemma gives an “improved approximative solution” m together with a rigorous error estimate R . A good candidate for an approximate solution would be the usual wave function obtained by “gluing together”

suitable WKB wave functions and Airy functions. We remark that the above lemma might even be useful for getting rigorous error estimates for *numerical* solutions of the Schrödinger or Riccati equations. In this case, one would have to estimate the total variation of $\log |\rho^2 U|$ from above, and this might be doable numerically if one has some control of the accuracy of the numerical calculation.

Proof of Lemma 4.1. For $\varepsilon > 0$ we set

$$T_\varepsilon(u) = T_0 \exp\left(\frac{1}{2} \int_0^u \left| \frac{\sigma^2 U'}{\sigma^2 U} \right| + \varepsilon e^{-u}\right) \quad (4.9)$$

and let R_ε and m_ε be the functions obtained from (4.5)–(4.7) if one replaces T by T_ε . Since $T_\varepsilon(0) = T(0)$ and $\lim_{\varepsilon \searrow 0} T_\varepsilon = T$, it suffices to show that for all $\varepsilon > 0$ the following statement holds,

$$|y - m_\varepsilon|(0) \leq R_\varepsilon(0) \implies |y - m_\varepsilon|(u) \leq R_\varepsilon(u) \text{ for all } u \in [0, u_{\max}].$$

In order to prove this statement, we will show that the assumption

$$|y - m_\varepsilon|(u) = R_\varepsilon(u) \quad (4.10)$$

implies that

$$|y - m_\varepsilon|'(u) < R_\varepsilon'(u). \quad (4.11)$$

In what follows we will often omit the subscript ε .

Assume that (4.10) holds and that $U \leq 0$. Then we can represent y as

$$y = m + R e^{i\varphi} \quad (4.12)$$

with $\varphi \in [0, 2\pi)$. Furthermore, it follows immediately from (4.5), (4.6), and (4.3) that

$$R^2 = U + \beta^2 = V - \alpha^2 + \beta^2 - \alpha'. \quad (4.13)$$

Using the above relations together with (3.4), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{du} |y - m|^2 &= (\operatorname{Re} y - \alpha) (\operatorname{Re} y - \alpha)' + (\operatorname{Im} y - \beta) (\operatorname{Im} y - \beta)' \\ &\stackrel{(3.4)}{=} (\operatorname{Re} y - \alpha) (V - (\operatorname{Re} y)^2 + (\operatorname{Im} y)^2 - \alpha') + (\operatorname{Im} y - \beta) (-2\operatorname{Re} y \operatorname{Im} y - \beta') \\ &= \operatorname{Re} y [V - |y|^2 - \alpha' + 2\alpha \operatorname{Re} y + 2\beta \operatorname{Im} y] \\ &\quad - \beta' \operatorname{Im} y - \alpha |y|^2 - \alpha V + \alpha \alpha' + \beta \beta' \\ &\stackrel{(4.10)}{=} \operatorname{Re} y [V - R^2 + |m|^2 - \alpha'] - \beta' \operatorname{Im} y - \alpha |y|^2 - \alpha V + \alpha \alpha' + \beta \beta' \\ &\stackrel{(4.12)}{=} R \cos \varphi [V - R^2 + |m|^2 - \alpha' - 2\alpha^2] - R \sin \varphi [\beta' + 2\alpha\beta] - 2\alpha R^2 \\ &\stackrel{(4.13)}{=} -2\alpha R^2 - R (\beta' + 2\alpha\beta) \sin \varphi \leq -2\alpha R^2 + R |\beta' + 2\alpha\beta|. \end{aligned}$$

Using that $\frac{d}{du} |y - m|^2 = 2R|y - m|'$, we obtain the simple inequality

$$|y - m|' \leq -2\alpha R + |\beta' + 2\alpha\beta|.$$

Hence in order to prove (4.11), it suffices to show that

$$R' > -2\alpha R + |\beta' + 2\alpha\beta|.$$

Using (4.2), we write the last inequality in the equivalent form

$$(\sigma R)' > |(\sigma\beta)'|. \quad (4.14)$$

In order to prove this inequality, we first use (4.5) and (4.6) to write the functions $\sigma\beta$ and σR as

$$\left. \begin{aligned} \sigma\beta &= \frac{1}{2} \left(\sqrt{|\sigma^2 U|} T + \sqrt{|\sigma^2 U|} T^{-1} \right) \\ \sigma R &= \frac{1}{2} \left(\sqrt{|\sigma^2 U|} T - \sqrt{|\sigma^2 U|} T^{-1} \right). \end{aligned} \right\} \quad (4.15)$$

By definition of T_ε (4.9),

$$\frac{T'}{T} = \frac{1}{2} \left| \frac{|\sigma^2 U'|}{|\sigma^2 U|} \right| + \varepsilon e^{-u}.$$

It follows that

$$\begin{cases} (\sqrt{|\sigma^2 U|} T^{-1})' = -\varepsilon e^{-u} (\sqrt{|\sigma^2 U|} T^{-1}) & \text{if } |\sigma^2 U'| \geq 0 \\ (\sqrt{|\sigma^2 U|} T)' = \varepsilon e^{-u} (\sqrt{|\sigma^2 U|} T) & \text{if } |\sigma^2 U'| < 0. \end{cases}$$

Hence when we differentiate through (4.15) and set $\varepsilon = 0$, either the first or the second summand drop out in each equation, and we obtain $(\sigma R)' = |\sigma\beta'|$. If $\varepsilon > 0$, an inspection of the signs of the additional terms gives (4.14). \blacksquare

The question arises how the function α in the above lemma is to be chosen. In order to explain our method, we first consider the WKB wave functions

$$\phi(u) = |V|^{-\frac{1}{4}} \exp\left(\pm i \int^u \sqrt{|V|}\right),$$

which should be good approximations to fundamental solutions in the “semiclassical” regime $V \ll 0$. The corresponding function y is

$$y(u) = \frac{\phi'}{\phi} = \pm i \sqrt{|V|} - \frac{V'}{4V}.$$

Lemma 4.1 should give a good estimate only if m is close to the exact solution y . This leads us to choose for the function $\alpha = \operatorname{Re} m$ the corresponding expression in the WKB approximation,

$$\alpha = -\frac{V'}{4V} \quad \text{in the “semiclassical regime”}.$$

This gives rise to the following estimate.

Theorem 4.2 *Assume that the potential V is negative and monotone increasing on $[0, u_{max})$, and that the following condition holds,*

$$K := \frac{\sup |V''| + \operatorname{TV} V''}{V_{max}^2} + \sup \frac{V'^2}{|V|^3} \leq 1, \quad (4.16)$$

where $V_{max} := \sup V \leq 0$ (and the supremum as well as the total variation are taken on the interval $[0, u_{max})$). Then the solution y of the boundary value problem (4.1) with initial condition

$$y_0 = i\sqrt{|V(0)|} - \frac{V'(0)}{4V(0)} \quad (4.17)$$

satisfies on $[0, u_{\max})$ the inequalities

$$\left| y - i\sqrt{|V|} + \frac{V'}{4V} \right| \leq 20 \sqrt{|V|} K \quad (4.18)$$

$$\operatorname{Im} y \geq \frac{\sqrt{|V|}}{10}. \quad (4.19)$$

Proof. We introduce on $[0, u_{\max}]$ the function α by

$$\alpha(u) = -\frac{V'}{4V}.$$

Then from (4.3),

$$\alpha' = -\frac{V''}{4V} + \frac{V'^2}{4V^2} \quad (4.20)$$

$$U = V \left(1 + \frac{V''}{4V^2} + \frac{5V'^2}{16|V|^3} \right). \quad (4.21)$$

Using the inequality (4.16) we get

$$2V \leq U \leq \frac{V}{2}. \quad (4.22)$$

In particular, U is negative.

The inequalities (4.21) and (4.16) allow us to estimate $\sqrt{|U|} - \sqrt{|V|}$,

$$\left| \sqrt{|U|} - \sqrt{|V|} \right| = \frac{|U - V|}{\sqrt{|U|} + \sqrt{|V|}} \leq \sqrt{|V|} \left| \frac{U - V}{V} \right| \leq \frac{\sqrt{|V|}}{2} K. \quad (4.23)$$

Dividing by $\sqrt{|V|}$ and $\sqrt{|U|}$, we obtain furthermore

$$\frac{1}{1+K} \leq \sqrt{\frac{|V|}{|U|}} \leq 1+K. \quad (4.24)$$

Choosing $T_0 = 1 + K$, we have the following estimates at $u = 0$,

$$\begin{aligned} |y - m| &= |\sqrt{|V|} - \beta| = \left| \sqrt{|V|} - \frac{\sqrt{|U|}}{2} \left((1+K) + \frac{1}{1+K} \right) \right| \\ &= \frac{\sqrt{|U|}}{2} \left| (1+K) + \frac{1}{1+K} - 2\sqrt{\frac{|V|}{|U|}} \right|. \end{aligned}$$

Applying (4.24) we obtain

$$|y - m| \leq \frac{\sqrt{|U|}}{2} \left((1+K) - \frac{1}{1+K} \right) = R.$$

We conclude that the inequality (4.8) holds at $u = 0$.

Hence we can apply Lemma 4.1 and obtain that (4.8) holds for all $u \in [0, u_{\max})$. Combining this with the inequalities (4.22) and (4.24) we obtain

$$\begin{aligned} \left| y - i\sqrt{|V|} + \frac{V'}{4V} \right| &\leq |y - m| + |\beta - \sqrt{|V|}| \leq R + |\beta - \sqrt{|V|}| \\ &= R + \frac{\sqrt{|U|}}{2} \left(T + \frac{1}{T} - 2\sqrt{\frac{|V|}{|U|}} \right) = \sqrt{|U|} \left(T - \sqrt{\frac{|V|}{|U|}} \right) \\ &\leq 2\sqrt{|V|} \left(T - \frac{1}{1+K} \right) \leq 2\sqrt{|V|} (T - 1 + K) \end{aligned} \quad (4.25)$$

$$\operatorname{Im} y \geq \beta - R = \frac{\sqrt{|U|}}{T} \geq \frac{\sqrt{|V|}}{\sqrt{2}T}. \quad (4.26)$$

It remains to estimate the function T , (4.4). We first compute σ and $\sigma^2 U$,

$$\sigma = \sqrt{\frac{V_0}{V(u)}} \quad (4.27)$$

$$\frac{\sigma^2 U}{|V_0|} = -1 - \frac{V''}{4V^2} - \frac{5V'^2}{16|V|^3}, \quad (4.28)$$

where we set $V_0 = V(0)$. Applying (4.16) we immediately obtain the inequalities

$$\frac{1}{2} \leq \left| \frac{\sigma^2 U}{V_0} \right| \leq 2.$$

The lower bound allows us to leave out the logarithm in the total variation in the definition of T ; namely,

$$\operatorname{TV}_{[0,u]} \log |\sigma^2 U| = \int_0^u \left| \frac{(\sigma^2 U)'}{\sigma^2 U} \right| \leq 2 \int_0^u \left| \frac{(\sigma^2 U)'}{|V_0|} \right| = 2 \operatorname{TV}_{[0,u]} \frac{\sigma^2 U}{|V_0|}.$$

We substitute (4.28) into this equation and estimate the total variation of the individual terms using (4.16) as well as the monotonicity of V ,

$$\begin{aligned} \operatorname{TV}_{[0,u]} \frac{V''}{V^2} &\leq \int_0^u \frac{|V''|}{V^2} + 2 \int_0^u \frac{|V''| V'}{(-V)^3} \leq \frac{\operatorname{TV} V'' + \sup |V''|}{V_{\max}^2} \\ \operatorname{TV}_{[0,u]} \frac{V'^2}{(|V|^3)} &\leq \int_0^u \frac{2|V''| V'}{(|V|^3)} + \int_0^u \frac{3|V'|^3}{V^4} \leq \frac{\sup |V''|}{V_{\max}^2} + \int_0^u \frac{3|V'|^3}{V^4}. \end{aligned}$$

In the last term we can integrate by parts,

$$\int_0^u \frac{3|V'|^3}{V^4} = \int_0^u V'^2 ((-V)^{-3})' = \frac{V'^2}{|V|^3} \Big|_0^u - \int_0^u \frac{2V'' V'}{|V|^3} \leq \sup \frac{V'^2}{|V|^3} + \frac{\sup |V''|}{V_{\max}^2}.$$

Collecting all the terms and using (4.16) we conclude that

$$\operatorname{TV}_{[0,u]} \log |\sigma^2 U| \leq 2K.$$

We substitute this bound into (4.4) and use that $T_0 = 1 + K$ to obtain the bound

$$T - 1 = (1 + K) e^K - 1 \leq |e^K - 1| + K e^K \leq 2e K.$$

Using this bound in (4.25) and (4.26) concludes the proof. \blacksquare

The condition (4.16) will clearly be violated when $|V|$ becomes very small. This is not astonishing because the WKB approximation fails near the zeros of the potential. In this “quantum regime”, there is no good candidate for α , and therefore we simply take

$$\alpha = \text{const} \quad \text{in the “quantum regime”}.$$

We state the corresponding estimate in such a way that it can easily be “pasted together” with the result of Lemma 4.2.

Theorem 4.3 *Assume that the potential V is negative and monotone (increasing or decreasing) on $[0, u_{\max})$, and that for some constant $\kappa > 0$ the following condition holds,*

$$\sqrt{|V_0|} u_{\max} \leq \kappa \tag{4.29}$$

(with $V_0 = V(0)$). Then any solution y of the boundary value problem (4.1) which is bounded by

$$|y_0| \leq c_1 \sqrt{|V_0|}, \quad \text{Im } y_0 \geq \frac{\sqrt{|V_0|}}{c_1}$$

for a suitable constant $c_1 \geq 1$ satisfies on $[0, u_{\max})$ the inequalities

$$|y| \leq c_2 \sqrt{\|V\|_\infty}, \quad \text{Im } y \geq \frac{1}{c_2} \frac{|V_0|}{\sqrt{\|V\|_\infty}}$$

where $\|V\|_\infty := \sup_{[0, u_{\max})} |V|$ and c_2 is a constant which depends only on κ and c_1 .

Proof. Let α be the constant function $\alpha = \sqrt{|V_0|}$. Then the function $U = V - \alpha^2$ is clearly negative. A simple calculation shows that by choosing $T_0 = 2c_1(1+c_1)^2$, we can arrange that $|y_0 - m(0)| \leq R(0)$. Lemma 4.1 yields that $|y - m| \leq R$ for all $u \in [0, u_{\max})$.

Since α is a constant, $\sigma(u) = e^{2\alpha u}$ and thus

$$|\sigma^2 U| = e^{4\alpha u} (\alpha^2 - V).$$

As a consequence,

$$\frac{|\sigma^2 U|'}{|\sigma^2 U|} \leq 4\alpha - \frac{V'}{\alpha^2 - V}.$$

If we integrate and use (4.29), we obtain the following bound for T ,

$$\begin{aligned} T &\leq T_0 e^{2\alpha u} \left(\frac{\sqrt{\alpha^2 - V_0}}{\sqrt{\alpha^2 - V}} + \frac{\sqrt{\alpha^2 - V}}{\sqrt{\alpha^2 - V_0}} \right) \\ &\leq T_0 e^{2\alpha u} \frac{4\sqrt{\|V\|_\infty}}{\alpha} \leq 4T_0 e^{2\kappa} \sqrt{\frac{\|V\|_\infty}{|V_0|}} =: T_{\max}. \end{aligned}$$

Finally we bound y by

$$\begin{aligned} |y| &\leq |y - m| + |m| \leq R + \alpha + \beta \\ &= \sqrt{|U|} T + \alpha \leq (2T + 1) \alpha \\ \text{Im } y &\geq R - \beta = \frac{\sqrt{|U|}}{T} \geq \frac{\alpha}{T}. \end{aligned}$$

These are the desired inequalities if we set $c_2 = 2T_{max} + 1 = 8c_1 (1 + c_1)^2 e^{2\kappa} + 1$. \blacksquare

As is obvious from (2.6, 2.7), the potential V has a singularity at $u = 0$. We will now explain how Lemma 4.1 can be used for estimates near such a singular point. We will restrict attention to the case $k = 0$, but our method applies similarly to general k . In order to find a good candidate for the function α , we consider on the interval $[0, u_{max})$ the Schrödinger equation with a potential which at $u = u_{max}$ has the same singular behavior as (2.6),

$$z''(u) = -\frac{1}{4(u_{max} - u)^2} z. \quad (4.30)$$

Setting $v = u_{max} - u$, this differential equation has the two fundamental solutions $\phi_1 = \sqrt{v}$ and $\phi_2 = \sqrt{v} \log v$, and therefore the function

$$z = \sqrt{v}(1 - i \log v)$$

is a complex solution. The corresponding solution of the complex Riccati equation is given by

$$y = \frac{z'}{z} = -\frac{1}{2v} + \frac{i}{v(1 - i \log v)} = \left(-\frac{1}{2v} - \frac{\log v}{v(1 + \log^2 v)} \right) + \frac{i}{v(1 + \log^2 v)}. \quad (4.31)$$

Choosing α equal to the real part of this function gives rise to the following estimate.

Lemma 4.4 *Suppose that the potential V is on $[0, u_{max})$ of the form*

$$V = -\frac{1}{4} \frac{1}{(u_{max} - u)^2} + B(u)$$

with a function B which is monotone (decreasing or increasing) and satisfies the inequality

$$u_{max}^2 (1 + \log^2 u_{max})^2 \|B\|_\infty \leq \frac{1}{8} \quad (4.32)$$

(with $\|B\|_\infty := \sup_{[0, u_{max})} |B|$). Then any solution y of the boundary value problem (4.1) with initial conditions bounded by

$$|y_0| \leq C \sqrt{|V_0|}, \quad \text{Im } y_0 \geq \frac{\sqrt{|V_0|}}{C}$$

for any constant $C \geq 1$ satisfies on $[0, u_{max})$ the inequalities

$$|y| \leq \frac{64 C^3}{u_{max} - u} \quad (4.33)$$

$$\text{Im } y \leq 64 C^3 (1 + \log^2 u_{max}) \frac{1}{(u_{max} - u) \log^2(u_{max} - u)} \quad (4.34)$$

$$\text{Im } y \geq \frac{1}{64 C^3 (1 + \log^2 u_{max})} \frac{1}{(u_{max} - u) \log^2(u_{max} - u)}. \quad (4.35)$$

Proof. We set $v = u_{max} - u$ and choose for α the real function

$$\alpha = -\frac{1}{2v} - \frac{\log v}{v(1 + \log^2 v)}.$$

Using that $\alpha = \operatorname{Re} y$ with y according to (4.31) and that y is a solution of the complex Riccati equation corresponding to the Schrödinger equation (4.30), we obtain

$$U = V - \alpha^2 - \alpha' = \operatorname{Re}(V - y^2 - y') - (\operatorname{Im} y)^2 = B - \frac{1}{v^2 (1 + \log^2 v)^2}. \quad (4.36)$$

Our assumption (4.32) yields that U is negative.

At $u = 0$, the potentials V and U can easily be bounded by above and below,

$$\begin{aligned} -1 &\leq 4v^2 V = -1 + 4v^2 B \leq -\frac{1}{2} \\ -1 &\leq 4v^2 (1 + \log^2 v)^2 U = -1 + v^2 (1 + \log^2 v)^2 B \leq -\frac{1}{2} \end{aligned}$$

and in particular

$$\frac{1}{2} \leq \frac{(1 + \log^2 v) \sqrt{|U|}}{|\sqrt{V}|} \leq 2.$$

A simple calculation shows that by choosing $T_0 = 2C(1 + C)^2 (1 + \log^2 u_{\max})$, we can arrange that $|y_0 - m(0)| \leq R(0)$. Lemma 4.1 yields that $|y - m| \leq R$ for all $u \in [0, u_{\max}]$.

Writing the function α in the form

$$\alpha = \frac{d}{du} \log \left(\sqrt{v (1 + \log^2 v)} \right)$$

we can immediately compute $\sigma^2 U$,

$$\begin{aligned} \sigma^2 &= v^2 (1 + \log^2 v)^2 \\ |\sigma^2 U| &= 1 - v^2 (1 + \log^2 v)^2 B. \end{aligned}$$

Using the bound (4.32) we obtain

$$\operatorname{TV}_{[0,u]} \log |\sigma^2 U| \leq 2 \operatorname{TV}_{[0,u]} |\sigma^2 U| \leq 4 u_{\max}^2 (1 + \log^2 u_{\max})^2 \|B\|_{\infty} \leq 2, \quad (4.37)$$

and thus T is bounded by $T \leq T_0 e^2 \leq 64C^3 (1 + \log^2 u_{\max})$. Finally we combine the above estimates with the inequalities

$$|y| \leq R + |\alpha| + \beta, \quad R - \beta \leq \operatorname{Im} y \leq R + \beta. \quad \blacksquare$$

The estimate (4.34) is very useful because it shows that the pole of $\operatorname{Im} y$ at $u = 0$ is integrable.

5 Convexity Estimates

The estimates of the previous section gave us good control of the solutions of the boundary value problem (4.1) provided that the potential is negative. In this section we now proceed with estimates in the case that V is *positive*, $V \geq 0$. Under this assumption it is a simple observation that ρ^2 is convex, because

$$(\rho^2)'' = (\bar{z}z)'' = 2(V + |y|^2) \rho^2 > 0. \quad (5.1)$$

This fact will be essential for the estimates in this section.

We begin with a lemma which bounds ρ from below.

Lemma 5.1 *Suppose that V is positive and monotone increasing on $[0, u_{\max})$. Then every solution of the boundary value problem (4.1) satisfies on $[0, u_{\max})$ the inequality*

$$\rho \geq \rho_0 \frac{\operatorname{Im} y_0}{|y_0|}$$

(with $\rho_0 = |z(0)|$ and z any solution of the corresponding complex Schrödinger equation (3.2)).

Proof. Differentiating through the equation $\rho' = \rho \operatorname{Re}(y)$ and using the complex Riccati equation (3.4), we get

$$\rho'' = \rho (\operatorname{Re} y)^2 + \rho \operatorname{Re}(V - y^2) = (V + (\operatorname{Im} y)^2) \rho,$$

and substituting in (3.8) we obtain the following differential equation for ρ ,

$$\rho'' = V\rho + \frac{w^2}{\rho^3}. \quad (5.2)$$

We let $\underline{\rho}(u)$ be the solution of the boundary value problem

$$\underline{\rho}'' = \frac{w^2}{\underline{\rho}^3} \quad \text{with} \quad \underline{\rho}(0) = \rho_0, \quad \underline{\rho}'(0) = \rho'(0) \equiv \rho_0 \operatorname{Re} y_0. \quad (5.3)$$

Then the function $\delta = \rho - \underline{\rho}$ satisfies the conditions $\delta(0) = 0 = \delta'(0)$ and $\delta'' \geq 0$. Hence $\delta \geq 0$ on $[0, u_{\max})$, and we conclude that $\underline{\rho}$ is a lower barrier for ρ ,

$$\rho \geq \underline{\rho} \quad \text{on} \quad [0, u_{\max}).$$

The function $\underline{\rho}$ is a solution of (5.2) in the case $V \equiv 0$. Therefore, $\underline{\rho}$ can be written explicitly in the form $\underline{\rho} = |\underline{z}|$ with \underline{z} a solution of the complex Schrödinger equation without potential with Wronskian is equal to w , i.e.

$$\underline{z}'' = 0 \quad \text{and} \quad \operatorname{Im}(\overline{\underline{z}} \underline{z}') = w = \rho_0^2 \operatorname{Im} y_0.$$

A short calculation shows that the solution of the boundary value problem (5.3) has the simple form

$$\underline{\rho}(u) = \rho_0 |1 + y_0 u|.$$

Computing the minimum of this function gives the result. ■

This lemma has the following immediate consequence. Due to the convexity of ρ ,

$$\sup_{[0, u)} \rho \leq \rho_0 + \rho(u) = \rho(u) \left(1 + \frac{\rho_0}{\rho(u)}\right) \leq \rho(u) \left(1 + \frac{|y_0|}{\operatorname{Im} y_0}\right)$$

and hence

$$\sup_{[0, u)} \rho \leq \rho(u) \frac{2|y_0|}{\operatorname{Im} y_0}. \quad (5.4)$$

In regions where the potential V is large, we expect that ρ should increase exponentially. The next lemma quantifies this exponential increase of ρ by showing that in the “semiclassical regime” $V \gg 0$, the integral over ρ^2 is much smaller than the supremum of ρ^2 .

Lemma 5.2 *Suppose that V is positive and monotone increasing on $[0, u_{max})$. Then every solution of the boundary value problem (4.1) satisfies on $[0, u_{max})$ the inequality*

$$\int_0^u \rho^2 \leq L \sup_{[0,u)} \rho^2$$

with L given by

$$L = \sup \left(\frac{3}{\sqrt{V}} + \frac{V'}{V^2} \right) + \text{TV} \frac{V'}{V^2}. \quad (5.5)$$

Proof. We substitute the differential equation for ρ^2 , (5.1), into the integral,

$$\int_0^u \rho^2 = \frac{1}{2} \int_0^u \frac{1}{V + |y|^2} (\rho^2)''.$$

Integrating by parts gives

$$\int_0^u \rho^2 = \frac{(\rho^2)'}{2(V + |y|^2)} \Big|_0^u - \frac{1}{2} \int_0^u \left(\frac{1}{V + |y|^2} \right)' (\rho^2)'.$$

Using the estimates

$$\begin{aligned} \left| \frac{(\rho^2)'}{V + |y|^2} \right| &\leq \frac{2\rho^2 |y|}{V + |y|^2} \leq \frac{2\rho^2 |y|}{2\sqrt{V} |y|} = \frac{\rho^2}{\sqrt{V}} \\ \left| \left(\frac{1}{V + |y|^2} \right)' \right| &\leq \frac{V' + 2|y| |V - y^2|}{(V + |y|^2)^2} \leq \frac{V'}{V^2} + \frac{2|y|}{V + |y|^2} \leq \frac{V'}{V^2} + \frac{1}{\sqrt{V}} \end{aligned}$$

we obtain

$$\int_0^u \rho^2 \leq \sup_{[0,u)} \frac{\rho^2}{\sqrt{V}} + \frac{1}{2} \int_0^u \left(\frac{V'}{V^2} + \frac{1}{\sqrt{V}} \right) |(\rho^2)'|.$$

When integrating by parts once again we must be careful because the function $(\rho^2)'$ may change signs. However, since ρ^2 is convex, it changes signs at most once, and therefore we get positive boundary terms at most twice,

$$\int_0^u \left(\frac{V'}{V^2} + \frac{1}{\sqrt{V}} \right) \left| \frac{d}{du} \rho^2 \right| \leq 2 \sup_{[0,u)} \left(\rho^2 \frac{V'}{V^2} + \frac{\rho^2}{\sqrt{V}} \right) + \int_0^u \left| \left(\frac{V'}{V^2} + \frac{1}{\sqrt{V}} \right)' \right| \rho^2.$$

Finally, we can estimate the last integral by

$$\begin{aligned} \int_0^u \left| \left(\frac{V'}{V^2} + \frac{1}{\sqrt{V}} \right)' \right| \rho^2 &\leq \sup_{[0,u)} \rho^2 \text{TV}_{[0,u)} \left(\frac{V'}{V^2} + \frac{1}{\sqrt{V}} \right) \\ &= \sup_{[0,u)} \rho^2 \left(\text{TV}_{[0,u)} \frac{V'}{V^2} + \sup_{[0,u)} \frac{1}{\sqrt{V}} \right), \end{aligned}$$

where in the last step we used the monotonicity of V . ■

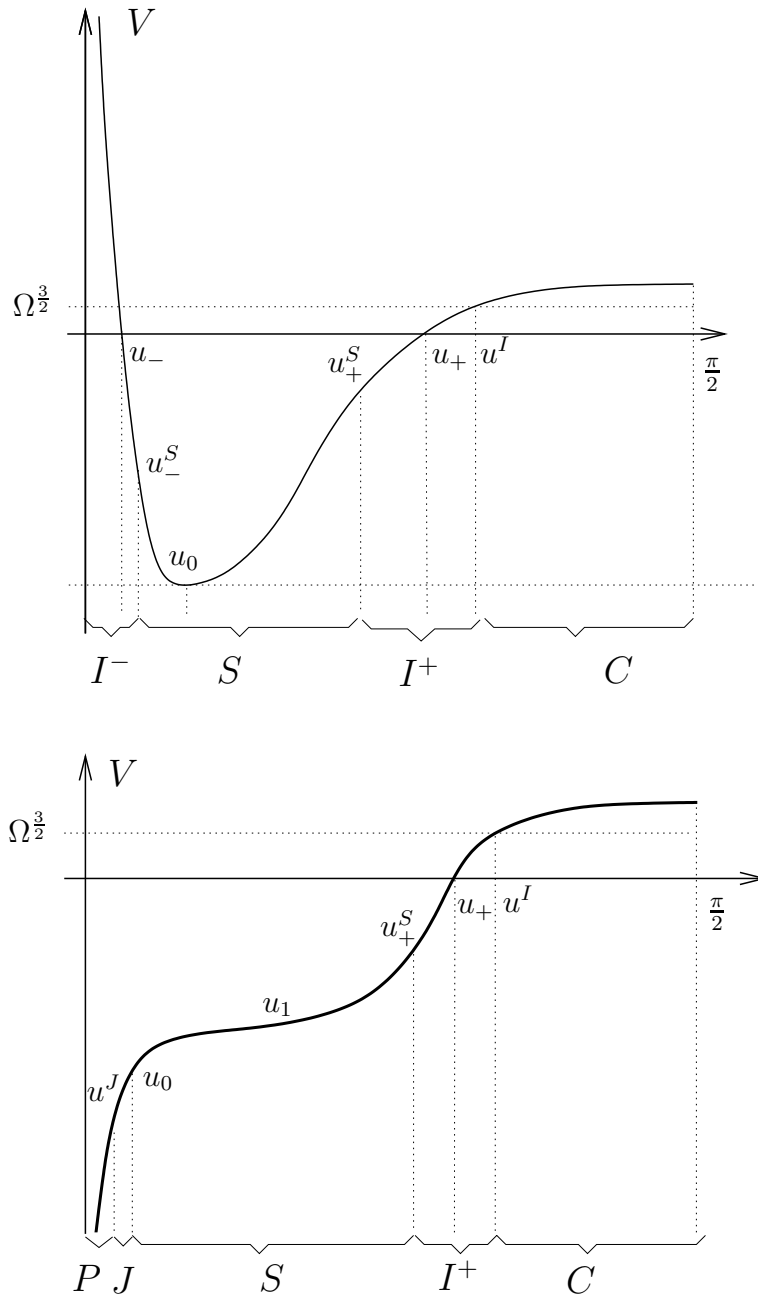


Figure 1: The potential V in the cases $k \neq 0$ (top) and $k = 0$ (bottom).

6 Elementary Properties of the Potential

In this section we shall analyze the potential V for large λ and Ω ; more precisely in the range

$$\Omega > \Omega_0 \quad \text{and} \quad \lambda > 2\Lambda\Omega \quad (6.1)$$

for parameters Ω_0 and Λ , which we can choose as large as we want. Then the potential looks qualitatively as in Figure 1. In the case $k \neq 0$, V has a unique minimum u_0 given by

$$\sin^2 u_0 = \frac{1}{\Omega} \sqrt{k^2 - \frac{1}{4}}, \quad (6.2)$$

where the potential is negative,

$$V(u_0) = \Omega \left(2 \sqrt{k^2 - \frac{1}{4}} + 2k \right) - \lambda - \frac{1}{4} < -\Lambda\Omega$$

(where in the last step we possibly increased Λ). V is strictly decreasing on the interval $[0, u_0]$ and tends to infinity as $u \searrow 0$. Thus there is a unique $u_- \in [0, u_0]$ with $V(u_-) = 0$. On the interval $[u_0, \frac{\pi}{2}]$, V is strictly increasing. Thus there is at most one $u_+ \in (u_0, \frac{\pi}{2}]$ with $V(u_+) = 0$. If no such u_+ exists, we set $u_+ = \frac{\pi}{2}$. Setting $\Delta u = \kappa/\sqrt{\Lambda\Omega}$ and choosing Λ sufficiently large (for any given parameter κ), it is easily verified that $V(u_0 \pm \Delta u) < 0$. As a consequence

$$|V(u_0)| |u_{\pm} - u_0|^2 > \kappa^2.$$

Using monotonicity, we can thus uniquely introduce points $u_+^S \in (u_0, u_+)$ and $u_-^S \in (u_-, u_0)$ by the condition that

$$|V(u_{\pm}^S)| |u_{\pm} - u_{\pm}^S|^2 = \kappa^2. \quad (6.3)$$

Finally, we introduce the point $u^I \in (u_+, \frac{\pi}{2}]$ by the condition $V(u^I) = \Omega^{\frac{3}{2}}$. If no such point exists, we set $u^I = \frac{\pi}{2}$.

In the case $k = 0$, V is monotone increasing on the whole interval $(0, \frac{\pi}{2}]$. We set

$$u^J = \frac{1}{8\sqrt{\lambda} \log^2 \lambda}, \quad u_0 = \frac{\kappa}{\sqrt{\lambda}} \quad \text{and} \quad u_1 = \frac{1}{\sqrt{\Omega}}.$$

The points u_+ , $u_+^S \in (u_1, u_+)$ and u^I are introduced as in the case $k \neq 0$.

We consider on $(0, \frac{\pi}{2}]$ the solution y of the complex Riccati equation (3.4) with initial condition

$$y(u_0) = i\sqrt{|V(u_0)|} - \frac{V'(u_0)}{4V(u_0)}. \quad (6.4)$$

The next lemmas make the following statements precise: The intervals S (as introduced in Figure 1) are “semiclassical” in the sense that Theorem 4.2 applies. On C we can use the convexity estimate of Lemma 5.2. The interval P near the pole can be treated by Theorem 4.4. Finally, the “intermediate regions” I_{\pm} and J can be controlled with Theorem 4.3 and Lemma 5.1.

Lemma 6.1 *For every $\delta > 0$ and $k \in \mathbb{Z}$ there are parameters $\kappa, \Lambda, \Omega_0 > 0$ such that for all Ω, λ in the range (6.1), the quantity K as defined by (4.16) is on the interval S bounded by*

$$K \leq \delta.$$

Proof. The third derivative of V can be written in the form

$$V''' = \frac{\cos u}{\sin^5 u} (\text{polynomial in } \sin^2 u \text{ of degree } 3).$$

Hence V''' has on any interval $[u, v] \subset [0, \frac{\pi}{2}]$ at most 4 zeros. Thus after splitting up $[u, v]$ into at most four subintervals, V''' has on each subinterval a fixed sign. On any such subinterval $[\underline{u}, \underline{v}]$ we can apply the estimate

$$\int_{\underline{u}}^{\underline{v}} |V'''| du \leq |V''(\underline{u})| + |V''(\underline{v})|.$$

This makes it possible to control the total variation of V'' in (4.16) by $8 \sup |V'''|$. We conclude that it suffices to show that on the interval I the following two inequalities hold,

$$\frac{V'^2}{|V|^3} \leq \delta \tag{6.5}$$

$$\frac{|V''|}{V^2} \leq \delta. \tag{6.6}$$

We treat three cases separately.

First case: $k = 0$ and $u_+ \geq \frac{3\pi}{8}$.

On the interval $[\frac{5\pi}{16}, \frac{\pi}{2}]$, the potential V is concave; more precisely,

$$-\Omega^2 \leq V'' \leq -\frac{\Omega^2}{4}.$$

Integration yields for all $\tau \in (\frac{5\pi}{16}, u_+]$ the following bounds for V' and V ,

$$\begin{aligned} \frac{\Omega^2}{4} \left(\frac{\pi}{2} - \tau \right) &\leq V'(\tau) \leq \Omega^2 \left(\frac{\pi}{2} - \tau \right) \\ V(u_+) - V(\tau) &\geq \frac{\Omega^2}{8} (u_+ - \tau) (\pi - u_+ - \tau) \geq \frac{\Omega^2}{8} (u_+ - \tau)^2. \end{aligned}$$

$V(u_+)$ is either zero or else it is negative and bounded by $V(u_+) = V(\frac{\pi}{2}) \leq \Omega^2 - \lambda$. Hence

$$|V(\tau)| \geq \max(\lambda - \Omega^2, 0) + \frac{\Omega^2}{8} (u_+ - \tau)^2. \tag{6.7}$$

Using the last inequality in the defining equation for u_+^S , (6.3), we obtain that (possibly after increasing Ω_0), $u_+^S \in (\frac{5\pi}{16}, u_+]$ and

$$u_+ - u_+^S \leq 2 \kappa^{\frac{2}{3}} \Omega^{-\frac{2}{3}}.$$

Again using (6.3), we conclude that

$$-V(u_+^S) \geq \frac{\kappa^{\frac{2}{3}}}{4} \Omega^{\frac{4}{3}},$$

and since V is monotone increasing, this also gives a lower bound for $|V|$ on the whole interval I .

Next we bound V' and V'' on I . A short calculation shows that

$$|V'| + |V''| \leq 4\Omega^2 \quad \text{on } [u_1, \frac{\pi}{2}]. \quad (6.8)$$

Conversely, on the interval $[u_0, u_1]$ we can bound V' and V'' in terms of suitable powers of λ ,

$$|V'|^2 \leq \frac{\lambda^3}{\kappa^6} \quad \text{and} \quad |V''| \leq \frac{2\lambda^2}{\kappa^4} \quad \text{on } [u_0, u_1]. \quad (6.9)$$

Combining (6.8) and (6.9) with (6.7) gives the result.

Second case: $k = 0$ and $u_+ < \frac{3\pi}{8}$.

On the interval $[u_0, u_1]$, a simple estimate shows that

$$|V| \geq \frac{\lambda}{2}.$$

Combing this with (6.9) and choosing κ sufficiently large, we obtain the inequalities (6.5, 6.6) on $[u_0, u_1]$. Conversely, on $[u_1, \frac{3\pi}{8}]$, V'' can be bounded in terms of higher powers of the first derivatives; more precisely,

$$|V''| \leq 10 |V'|^{\frac{4}{3}}.$$

This inequality allows us to deduce (6.6) from (6.5). Hence it remains to prove (6.5) on the interval $[u_1, u_+^S]$.

On the interval $[u_1, u_+^S]$, the potential V is either convex or at least else the second derivative of V is small compared to $|V'|^{\frac{4}{3}}$. More precisely, by choosing Ω_0 sufficiently large we can arrange that

$$V'' \geq -\kappa^{-\frac{2}{3}} V'^{\frac{4}{3}}. \quad (6.10)$$

We shall derive an upper bound for $\Delta u := u_+ - u_+^S$; for ease in notation the subscript '+' will be omitted. We rewrite (6.10) as

$$\frac{d}{du} \left(V'^{\frac{1}{3}} \right) = -\frac{1}{3} \frac{V''}{V'^{\frac{4}{3}}} \leq \frac{\kappa^{-\frac{2}{3}}}{3}.$$

We integrate from u^S to $\tau \in [u^S, u]$ to obtain

$$V'(\tau) \geq \left(V'(u^S)^{-\frac{1}{3}} + \frac{\kappa^{-\frac{2}{3}}}{3} (\tau - u^S) \right)^{-3}.$$

Integrating τ over the interval $[u^S, u]$, we obtain for $\Delta V := V(u) - V(u^S)$ the estimate

$$\Delta V \geq \frac{3\kappa^{\frac{2}{3}}}{2} V'(u^S)^{\frac{2}{3}} \left(1 - \frac{1}{(1+\alpha)^2} \right) \quad \text{with} \quad \alpha := \frac{\kappa^{-\frac{2}{3}}}{3} V'(u^S)^{\frac{1}{3}} \Delta u.$$

The inequality

$$1 - \frac{1}{(1+\alpha)^2} \geq \frac{\alpha}{1+\alpha},$$

gives

$$\Delta V \geq \frac{V'(u^S)}{2} \frac{\Delta u}{1+\alpha}.$$

By definition of u^S , (6.3), we know that $\Delta V \cdot (\Delta u)^2 = \kappa^2$. Hence multiplying the last inequality by $(\Delta u)^2$ we obtain

$$\kappa^2 \geq \frac{V'(u^S)}{2} \frac{(\Delta u)^3}{1 + \alpha}.$$

Putting in the definition of α gives the inequality

$$(\Delta u)^3 - \frac{2}{3} \left(\kappa^{\frac{2}{3}} V'(u^S)^{-\frac{1}{3}} \right)^2 \Delta u - 2 \left(\kappa^{\frac{2}{3}} V'(u^S)^{-\frac{1}{3}} \right)^3 \leq 0.$$

Since the polynomial $x^3 - \frac{2x}{3} - 2$ is positive for $x \geq 2$, we conclude that

$$\Delta u \leq 2 \kappa^{\frac{2}{3}} V'(u^S)^{-\frac{1}{3}}.$$

Using again the relation $\Delta V (\Delta u)^2 = \kappa^2$, we get an upper bound for ΔV ,

$$\Delta V \geq \frac{\kappa^{\frac{2}{3}}}{4} V'(u^S)^{\frac{2}{3}}. \quad (6.11)$$

This proves the inequality (6.5) at $u = u^S$.

Next we will want to show that (6.5) holds on the whole interval $(u_1, u^S]$. To this end, we introduce on this interval the function f by

$$f = V'^2 + \frac{4^3}{\kappa^2} V^3.$$

We saw above that $f(u^S) < 0$; our goal is to show that $f \leq 0$ on $(u_1, u^S]$. Let $(v, u^S]$ with $u_1 \leq v < u^S$ be the maximal interval on which f is negative. We apply (6.10) to obtain

$$\begin{aligned} f'(v) &= V' \left(2V'' + \frac{3 \cdot 4^3}{\kappa^2} V^2 \right) \\ &\geq V' \left(-2\kappa^{-\frac{2}{3}} V'^{\frac{4}{3}} + \frac{3 \cdot 4^3}{\kappa^2} V^2 \right) \geq \frac{160}{\kappa^2} V'^2 > 0, \end{aligned}$$

where in the last line we used that $f(v) \leq 0$. The last inequality contradicts the maximality of the interval $(v, u^S]$ unless $v = u_1$. This concludes the proof in the second case.

Third case: $k \neq 0$.

On the interval $(u_-^S, u_0]$, the proof of the second case goes through without changes if we consider the integral backwards and set $u_1 = u_0 - 1/(4\sqrt{\Omega})$. On the remaining interval $[u_0, u_+^S]$, we can use the proof of the first case and the second case after setting $u_1 = u_0 + 1/(4\sqrt{\Omega})$. ■

Lemma 6.2 *For sufficiently large Λ and Ω_0 , the parameter L as defined by (5.5) is for all Ω, λ in the range (6.1) bounded by*

$$L \leq \frac{3}{\sqrt{\Omega_0}}.$$

Proof. We already saw in (6.7) that V' and V'' satisfy on C the bound $|V'| + |V''| \leq 4\Omega^2$. On the other hand, it is clear from the definition of u^I that $|V| \geq \Omega^{\frac{3}{2}}$ on C . This immediately gives the lemma. \blacksquare

Lemma 6.3 *For sufficiently large Λ and Ω_0 , the potential*

$$B(u) := V(u) + \frac{1}{4u^2}$$

satisfies on the interval P the inequality

$$(u_-^I)^2 (1 + \log^2 u_-^I)^2 \|B\|_\infty \leq \frac{1}{8}.$$

Proof. A short calculation shows that B is bounded from above by

$$|B(u)| \leq \Omega^2 u^2 + 2\lambda.$$

From the definition of u_-^I and the fact that λ is large, we can clearly assume that

$$\frac{1}{\lambda} < u_-^I < \frac{1}{\sqrt{\lambda}}.$$

These inequalities give rise to the following estimate,

$$\begin{aligned} (u_-^I)^2 (1 + \log^2 u_-^I)^2 \|B\|_\infty &\leq \frac{1}{64 \lambda \log^4 \lambda} (1 + \log^2 \lambda)^2 \|B\|_\infty \\ &\leq \frac{1}{32 \lambda} \|B\|_\infty \leq \frac{1}{32 \lambda} \left(\frac{\Omega^2}{\lambda} + 2\lambda \right) \leq \frac{1}{8}, \end{aligned}$$

where in the last step we used (6.1). \blacksquare

Lemma 6.4 *For every $\delta > 0$ and $k \in \mathbb{Z}$ there are parameters $\kappa, \Lambda, \Omega_0 > 0$ such that for all Ω, λ in the range (6.1),*

$$|I_\pm| \leq \delta.$$

Proof. We choose Ω_0 so large that $u_0 < \frac{\delta}{4}$. Then clearly $|I_-| \leq \delta$. Furthermore, it is readily verified that the potential is increasing on the interval $K := [\frac{\delta}{4}, \frac{\pi}{2} - \frac{\delta}{4}]$ at the following rate,

$$V(v) - V(u) \geq c(v - u)^2 \Omega^2 \quad \text{for all } u, v \in K, v > u,$$

where c is independent of λ and Ω . This implies that

$$|[u_+^S, u_+] \cap K| \leq c^{-\frac{1}{4}} \sqrt{\frac{\kappa}{\Omega}},$$

because otherwise (6.3) would be violated. Furthermore, the condition $V(u^I) \leq \Omega^{\frac{3}{2}}$ implies that

$$|[u_+, u^I] \cap K| \leq \sqrt{c} \Omega^{-\frac{1}{4}}.$$

We conclude that by increasing Ω_0 , we can arrange that $|I_+ \cap K| \leq \frac{\delta}{2}$ and thus $|I_+| \leq \delta$. \blacksquare

7 Spectral Estimates for the Selfadjoint Problem

In this section we shall prove Lemma 2.1. We begin by reducing the problem to an estimate for y_λ .

Lemma 7.1 *Assume that for any given $k \in \mathbb{Z}$ and $\varepsilon > 0$, there are constants $\Lambda, \Omega_0 > 0$ such that*

$$\int_0^{\frac{\pi}{2}} \operatorname{Im} y_\lambda \leq \varepsilon \quad (7.1)$$

for all Ω and λ in the range (6.1). Then Lemma 2.1 holds.

Proof. According to the asymptotics (2.1) it suffices to consider λ in the range $\lambda > \Lambda\Omega$ for sufficiently large Λ . Let us consider the quotient $\operatorname{Re}(y)/\operatorname{Im}(y)$ in (3.10). According to Theorem 4.2 and Theorem 4.3 there is $c > 0$ such that for all $\Omega > \Omega_0$ and $\lambda > \Lambda\Omega$,

$$\frac{\operatorname{Re} y}{\operatorname{Im} y} > -c \quad \text{on } [u_0, u_+]. \quad (7.2)$$

On the interval $[u_+, \frac{\pi}{2}]$, ρ^2 is convex, and using the identity

$$\frac{\operatorname{Re} y}{\operatorname{Im} y} = \frac{\rho^2}{w} \operatorname{Re} y = \frac{1}{2w} (\rho^2)'$$

one sees that $\operatorname{Re}(y)/\operatorname{Im}(y)$ is monotone increasing. We conclude that the inequality (7.2) also holds at $u = \frac{\pi}{2}$, and thus

$$-\frac{\pi}{2} < -\arctan c < \arctan\left(\frac{\operatorname{Re} y(\frac{\pi}{2})}{\operatorname{Im} y(\frac{\pi}{2})}\right) < \frac{\pi}{2}.$$

Using the last bounds in (3.10) one sees that for two neighboring eigenvalues, the phases must differ at least by $\delta := \frac{\pi}{2} - \arctan c$,

$$\varphi_{n+1} - \varphi_n \Big|_0^{\frac{\pi}{2}} \geq \delta. \quad (7.3)$$

From (3.11) one sees that this inequality is also true for the states of odd parity (with $\delta = \pi$). Applying (3.7) and the mean value theorem, we conclude that there is $\lambda \in [\lambda_n^\pm, \lambda_{n+1}^\pm]$ such that

$$(\lambda_{n+1}^\pm - \lambda_n^\pm) \int_0^{\frac{\pi}{2}} \operatorname{Im} y_\lambda \geq \delta.$$

Hence the upper bound (7.1) gives the desired gap estimate. ■

We establish (7.1) by deriving separate estimates in the regions S , I_\pm , C and near the pole.

Lemma 7.2 *For any given $k \in \mathbb{Z}$ and $\varepsilon > 0$, there are constants $\Lambda, \Omega_0 > 0$ such that on the interval S ,*

$$|y_\lambda| \leq \varepsilon.$$

Proof. Differentiating the initial condition (6.4) gives

$$y_\lambda(u_0) = \frac{i}{2\sqrt{|V(u_0)|}} + \frac{V'(u_0)}{4V^2(u_0)}.$$

This can be estimated using Lemma 6.1,

$$|y_\lambda(u_0)| \leq \frac{1}{2\sqrt{|V(u_0)|}} \left(1 + \frac{|V'(u_0)|}{|V(u_0)|^{\frac{3}{2}}} \right) \leq \frac{1}{\sqrt{|V(u_0)|}}. \quad (7.4)$$

For given $u \in S$, we compute $y_\lambda(u)$ via (3.13). This gives rise to the estimate

$$|y_\lambda(u)| \leq \frac{1}{2|y|} + \operatorname{Im} y(u) \int_{u_0}^u \frac{|V - y^2|}{2|y^2|} \frac{1}{\operatorname{Im} y}.$$

According to Lemma 6.1 we can apply the estimates of Theorem 4.2,

$$\begin{aligned} |y| &\leq 22\sqrt{|V|} \\ |y - i\sqrt{|V|}| &\leq \sqrt{|V|}(20K + \sqrt{K}) \leq 21\sqrt{K}\sqrt{|V|} \\ |V - y^2| &= |(y - i\sqrt{|V|})(y + i\sqrt{|V|})| \\ &\leq 21\sqrt{K}\sqrt{|V|} 23\sqrt{|V|} \leq 500|V|\sqrt{K}, \end{aligned}$$

and thus, using the monotonicity of V on the intervals $[u_0, u]$ or $[u, u_0]$,

$$\begin{aligned} |y_\lambda(u)| &\leq \frac{10}{\sqrt{|V(u)|}} + \frac{10}{\sqrt{|V(u)|}} \int_{u_0}^u \frac{|V - y^2|}{|y|^2} 20\sqrt{V} \\ &\leq \frac{10}{\sqrt{|V(u)|}} + 200 \int_{u_0}^u \frac{|V - y^2|}{|y|^2} \leq \frac{10}{\sqrt{|V(u)|}} + 5\pi 10^6 \sqrt{K}, \end{aligned}$$

According to Lemma 6.1, this can be made arbitrarily small. ■

Lemma 7.3 *For any given $k \in \mathbb{Z}$ and $\varepsilon > 0$, there are constants $\Lambda, \Omega_0 > 0$ such that on the intervals I_\pm ,*

$$|y_\lambda| \leq \varepsilon.$$

Proof. We only consider the interval I_+ ; the proof for I_- is analogous. For any $v \in (u_+^S, u_+]$, we compute y_λ via (3.12) with $u = u_+^S$,

$$y_\lambda(v) = -\frac{z^2(u_+^S) y_\lambda(u_+^S)}{z^2(v)} - \frac{1}{z^2(v)} \int_{u_+^S}^v z^2.$$

According to Lemma 6.4 we can apply Theorem 4.3. This gives the estimate

$$|y_\lambda(v)| \leq |y_\lambda(u_+^S)| \frac{\operatorname{Im} y(v)}{\operatorname{Im} y(u_+^S)} + \operatorname{Im} y(v) \int_{u_+^S}^v \frac{1}{\operatorname{Im} y} \leq c_2^2 |y_\lambda(u_+^S)| + c_2^2 (v - u_+^S).$$

This can be made arbitrarily small according to Lemma 7.2 and Lemma 6.4.

If $v \in (u_+, u^I]$, the change of y_λ on the interval (u_+, v) can be estimated similarly using Lemma 5.1. ■

Lemma 7.4 For any given $k \in \mathbb{Z}$ and $\varepsilon > 0$, there are constants $\Lambda, \Omega_0 > 0$ such that on the interval C ,

$$|y_\lambda| \leq \varepsilon.$$

Proof. We again compute y_λ via (3.12). This gives for any $v \in C$ the estimate

$$|y_\lambda(v)| \leq \frac{\rho^2(u^I)}{\rho^2(v)} |y_\lambda(u^I)| + \frac{1}{\rho^2(v)} \int_{u^I}^v \rho^2.$$

The first summand can be made arbitrarily small according to Lemma 7.3 and Lemma 5.1, whereas the second summand can be handled with Lemma 5.2 and Lemma 6.2. \blacksquare

Lemma 7.5 For any given $k \in \mathbb{Z}$ and $\varepsilon > 0$, there are constants $\Lambda, \Omega_0 > 0$ such that for all λ and Ω in the range (6.1),

$$\int_{P \cup J} |y_\lambda| \leq \varepsilon.$$

Proof. A short calculation shows that on the interval J ,

$$\frac{\lambda}{8\kappa^2} \leq |V(u_0)| \leq \frac{\lambda}{\kappa^2}, \quad 8\lambda \log^4 \lambda \leq |V(u^J)| \leq 64\lambda \log^4 \lambda, \quad |J| \leq \frac{\kappa}{\sqrt{\lambda}}.$$

In particular, $|V(u_0)| |J|^2 \leq \kappa^2$, and so we can apply Lemma 6.2 to obtain on J the estimates

$$|y| \leq 8c_2 \sqrt{\lambda} \log^2 \lambda, \quad \operatorname{Im} y \geq \frac{1}{8c_2 \kappa^2} \frac{\sqrt{\lambda}}{\log^2 \lambda}. \quad (7.5)$$

These estimates allow us to bound y_λ on J again using (3.12). Namely, for all $v \in J$,

$$|y_\lambda(v)| \leq |y_\lambda(u_0)| \frac{\operatorname{Im} y(v)}{\operatorname{Im} y(u_0)} + \operatorname{Im} y(v) \int_v^{u_0} \frac{1}{\operatorname{Im} y}.$$

Estimating the factor $|y_\lambda(u_0)|$ by (7.4) we obtain

$$|y_\lambda(v)| \leq c_3 \frac{\log^4 \lambda}{\sqrt{\lambda}} \quad \text{on } J. \quad (7.6)$$

with $c_3 = 256c_2\kappa^3$. By increasing Λ this can be made arbitrarily small.

On the interval P , we apply Lemma 4.4 with $C = 64c_2\kappa^2 \log^4 \lambda$. This gives the estimate

$$\frac{1}{c_4 \log^p \lambda} \frac{1}{v \log^2 v} \leq \operatorname{Im} y(v) \leq \frac{c_4 \log^p \lambda}{v \log^2 v} \quad (7.7)$$

with $p = 14$ and a constant c_4 which is independent of Λ and Ω . We again estimate y_λ using (3.12),

$$|y_\lambda(v)| \leq |y_\lambda(u^J)| \frac{\operatorname{Im} y(v)}{\operatorname{Im} y(u^J)} + \operatorname{Im} y(v) \int_v^{u^J} \frac{1}{\operatorname{Im} y}.$$

Estimating $y(u^J)$ and $y_\lambda(u^J)$ by (7.5, 7.6) and using (7.7) we get for all $v \in P$,

$$|y_\lambda(v)| \leq c_5 \frac{\log^{p+6} \lambda}{\lambda} \frac{1}{v \log^2 v} + c_5 \frac{\log^{2p} \lambda}{v \log^2 v} \int_v^{u^J} \tau \log^2 \tau \, d\tau$$

for a suitable constant c_5 . This expression is not bounded as $v \searrow 0$. But the pole is integrable, and the calculation

$$\begin{aligned} \int_0^{u^J} \frac{dv}{v \log^2 v} &= -\frac{1}{\log u^J} \\ \int_0^{u^J} \frac{dv}{v \log^2 v} \int_v^{u^J} \tau \log^2 \tau d\tau &= \int_0^{u^J} d\tau \tau \log^2 \tau \int_0^\tau \frac{dv}{v \log^2 v} \\ &= -\int_0^{u^J} \tau \log \tau d\tau = \frac{1}{2} (u^J)^2 (1 - \log u^J) \end{aligned}$$

shows that by increasing Λ , we can make the resulting integrals over P arbitrarily small. \blacksquare

This completes the proof of Theorem 1.3.

8 Slightly Non-Selfadjoint Perturbations

It remains to prove Theorem 1.1. In preparation, we split up the spheroidal wave operator as

$$\mathcal{A} = \mathcal{A}_0 + W$$

with

$$\begin{aligned} \mathcal{A}_0 &= -\frac{d}{d \cos \vartheta} \sin^2 \vartheta \frac{d}{d \cos \vartheta} + \frac{1}{\sin^2 \vartheta} (\operatorname{Re} \Omega \sin^2 \vartheta + k)^2 \\ W &= 2i \operatorname{Re} \Omega \operatorname{Im} \Omega - (\operatorname{Im} \Omega)^2 \sin^2 \vartheta. \end{aligned}$$

The symmetric operator \mathcal{A}_0 clearly satisfies the hypothesis of Theorems 1.2 and 1.3, whereas the complex potential W is uniformly bounded according to assumption (1.4),

$$|W| \leq 2c + c^2 =: \frac{\rho}{2}. \quad (8.1)$$

Our method is to treat W as a slightly non-selfadjoint perturbation as introduced by Kato [6, Chap. V, §4, Sec. 5]; see in particular [6, Theorem 4.15a]. Unfortunately, the latter theorem is not quite strong enough for our purpose. For clarity, we here repeat the basic ideas of Kato and give a detailed proof of our main theorem.

Proof of Theorem 1.1. Throughout the proof we restrict all operators either to \mathcal{H}_+ or \mathcal{H}_- . Applying Theorems 1.2 and 1.3 to the operator \mathcal{A}_0 and $\gamma = 8\pi\rho$, we obtain for the eigenvalues $0 \leq \lambda_1 < \lambda_2 < \dots$ of \mathcal{A}_0 the estimates

$$\lambda_{n+1} - \lambda_n > \gamma \quad \text{for all } n \geq N \text{ and } \Omega \in \mathbb{R}.$$

For all $\lambda \notin \sigma(\mathcal{A}_0)$, the resolvent $R_\lambda^0 := (\lambda - \mathcal{A}_0)^{-1}$ exists and satisfies the bound

$$\|R_\lambda^0\| \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(\mathcal{A}_0))}. \quad (8.2)$$

Since the spectrum of \mathcal{A}_0 is real, we have in particular

$$\|R_\lambda^0\| \leq \frac{1}{|\operatorname{Im} \lambda|}.$$

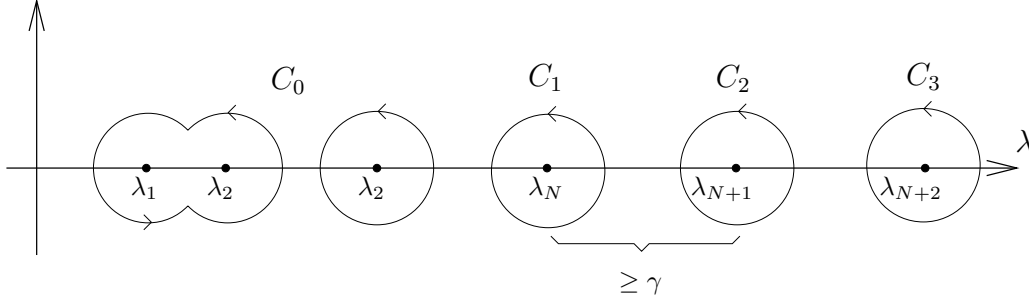


Figure 2: The contours C_k .

Around each λ_n we draw a circle of radius ρ . The first N circles may intersect, and we take the outermost lines to define the contour C_0 ,

$$C_0 = \partial(B_\rho(\lambda_1) \cup \dots \cup B_\rho(\lambda_N)).$$

All the following circles do not intersect and give rise to the contours

$$C_k = \partial B_\rho(\lambda_{N+k}), \quad k \geq 1$$

(see Figure 2). Since the distance of these contours to the spectral points of \mathcal{A}_0 is at least ρ , we have for λ on any of these contours,

$$\|R_\lambda^0 W\| \leq \|R_\lambda^0\| \|W\| \leq \frac{1}{2}.$$

Hence the operator $1 + R_\lambda^0 W$ is invertible with a Neumann series. We conclude that the resolvent $R_\lambda := (\lambda - \mathcal{A})^{-1} = ((\lambda - \mathcal{A}_0)(1 + R_\lambda^0 W))^{-1} = (1 + R_\lambda^0 W)^{-1} R_\lambda^0$ exists and

$$\|R_\lambda\| \leq 2 \|R_\lambda^0\|. \quad (8.3)$$

This allows us to introduce the operators Q_k as the following contour integrals

$$Q_k = \oint_{C_k} R_\lambda d\lambda.$$

The Cauchy integral formula together and the resolvent identity

$$R_\lambda R_{\lambda'} = -\frac{1}{\lambda - \lambda'} (R_\lambda - R_{\lambda'})$$

immediately yield that the operators Q_k are projectors onto invariant subspaces of \mathcal{A} , and that they are holomorphic in Ω . Furthermore, they are uniformly bounded because according to (8.3, 8.2) and the definition of the contours,

$$\|Q_k\| \leq \frac{1}{2\pi} \oint_{C_k} 2 \|R_\lambda^0\| \leq 2N.$$

We introduce the operators P_K as the finite sums

$$P_K = \sum_{k=1}^K Q_k.$$

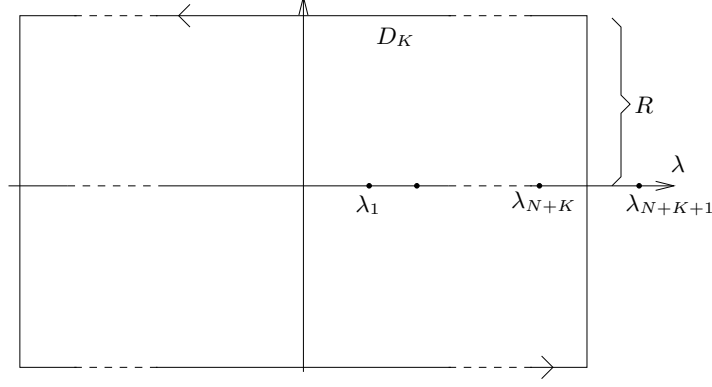


Figure 3: The contours D_K .

For the unperturbed operator \mathcal{A}_0 , we introduce similarly the projectors Q_k^0 and P_K^0 . Let us derive estimates for the difference $P_K - P_K^0$. We first write it as the contour integral

$$P_K - P_K^0 = \frac{1}{2\pi i} \oint_{D_K} (R_\lambda - R_\lambda^0) d\lambda,$$

where D_K is a rectangle with side lengths $\lambda_{N+K} + \lambda_{N+K+1}$ and $2R$ centered at the origin (see Figure 3). Since $\text{dist}(D_K, \sigma(\mathcal{A}_0)) > \rho$, the inequality (8.3) again holds. Using the resolvent identity

$$R_\lambda - R_\lambda^0 = R_\lambda W R_\lambda^0$$

together with (8.1, 8.2), we obtain for any λ on the contour D_K the estimate

$$\|R_\lambda - R_\lambda^0\| \leq \frac{4\rho}{\gamma^2 + 4(\text{Im } \lambda)^2}.$$

This inequality allows us to take the limit $R \rightarrow \infty$ to obtain the estimate

$$\|P_K - P_K^0\| \leq \frac{4\pi\rho}{\gamma} = \frac{1}{2}. \quad (8.4)$$

This estimate can be improved if the operator $P_K - P_K^0$ is restricted to the range of P_L^0 , $L < K$. Namely, applying the bound

$$\|R_\lambda^0 P_L^0\| \leq \max_{n=1, \dots, N+L} |\lambda - \lambda_n|^{-1}$$

to the resolvent identity gives for any λ on the contour C_K the bound

$$\|(P_K - P_K^0) P_L^0\| \leq \frac{4\rho}{|\gamma + 2i \text{Im } \lambda| |\lambda - \lambda_{N+L}|}.$$

Substituting this estimate into the contour integral, taking the limit $R \rightarrow \infty$ and estimating the resulting integral as follows,

$$\int_0^\infty \frac{dx}{\sqrt{a^2 + x^2} \sqrt{b^2 + x^2}} \leq \int_0^{\sqrt{ab}} \frac{dx}{ab} + \int_{\sqrt{ab}}^\infty \frac{dx}{\sqrt{ab} x^2} \leq \frac{2}{\sqrt{ab}},$$

we conclude that

$$\|(P_K - P_K^0) P_L^0\| \leq \frac{4\rho}{\sqrt{\gamma(\lambda_{N+K} - \lambda_{N+L})}}, \quad L < K. \quad (8.5)$$

The inequality (8.4) allows us to determine the rank of the operators P_K . Namely, for every Ψ in the range of P_K^0 ,

$$\|P_K \Psi\| \geq \|P_K^0 \Psi\| - \|(P_K - P_K^0) \Psi\| \geq \frac{1}{2} \|\Psi\|.$$

In particular, Ψ is not in the kernel of P_K . This shows that the rank of P_K is greater or equal to the rank of P_K^0 . Interchanging the roles of P_K and P_K^0 , we see that P_K and P_K^0 have the same rank. Since P_K^0 is the projector on the eigenspaces of \mathcal{A}_0 corresponding to the eigenvalues $\lambda_1, \dots, \lambda_{N+K}$, the dimension of its range is $N+k$. We conclude that Q_0 is a projector on an N -dimensional invariant subspace of \mathcal{A} and Q_1, Q_2, \dots are projectors on 1-dimensional eigenspaces.

The inequalities (8.4, 8.5) imply completeness: Let $\Psi \in \mathcal{H}$ and $\varepsilon > 0$. Since the spectral projectors of the unperturbed problem converge strongly (i.e. $s - \lim_{L \rightarrow \infty} P_L^0 = 1$), there is $L \in \mathbb{N}$ such that $\|\Psi - P_M^0 \Psi\| < \varepsilon$ for all $M \geq L$. According to (8.4), $\|P_K - 1\| \leq \|P_K - 1\| \leq \|P_K - P_K^0\| + \|P_K^0 - 1\| \leq 3$. Hence

$$\begin{aligned} \|(P_K - 1) \Psi\| &\leq \|(P_K - 1) (\Psi - P_L^0 \Psi)\| + \|(P_K - P_K^0) P_L^0 \Psi\| + \|(P_K^0 - 1) P_L^0 \Psi\| \\ &\leq 5\varepsilon + \|(P_K - P_K^0) P_L^0 \Psi\|, \end{aligned}$$

and the estimate (8.5) shows that the last term can be made arbitrarily small by choosing K sufficiently large.

It remains to prove the inequalities (1.5) and (1.6). Combining the gap estimates of Theorem 1.3 with the fact that $\mu_k \in B_\rho(\lambda_{N+k})$, one immediately obtains (1.6). The inequality (1.6) follows similarly from the asymptotics (2.1). \blacksquare

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References

- [1] M. S. ASHBAUGH, *The universal eigenvalue bounds of Payne-Pólya-Weinberger, Hile-Protter, and H. C. Yang*, Proc. Indian Acad. Sci. Math. Sci. **112** (2002), no. 1, 3–30.
- [2] E.A. CODDINGTON AND N. LEVINSON, *The Theory of Ordinary Differential Equations*, McGraw Hill (1955)
- [3] F. FINSTER, N. KAMRAN, J. SMOLLER, S. T. YAU, *An Integral Spectral Representation of the Propagator for the Wave Equation in the Kerr Geometry*, gr-qc/0310024 (2003).
- [4] FLAMMER, CARSON, *Spheroidal wave functions*, Stanford University Press, Stanford, California (1957).

- [5] C. HUNTER AND B. GUERRIERI, *The Eigenvalues of the Angular Spheroidal Wave Equation*, Stud. Appl. Math. **66** (1982), 217–240.
- [6] T. KATO, *Perturbation theory for linear operators*, Springer (1966).
- [7] R. LAVINE, *The eigenvalue gap for one-dimensional convex potentials*, Proc. Amer. Math. Soc. **121** (1994), no. 3, 815–821.
- [8] C. K. LAW AND YU-LING HUANG, *Eigenvalue ratios and eigenvalue gaps of Sturm-Liouville operators*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 337–347.
- [9] J. MEIXNER AND F. W. SCHÄFKE, *Mathieusche Funktionen und Sphäroidfunktionen*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band LXXI, Springer (1954).
- [10] M. REED AND B. SIMON, *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, Academic Press, New York (1978).
- [11] M.E. TAYLOR, *Partial Differential Equations II*, Springer (1996).
- [12] I.M. SINGER, B. WONG, S.-T. YAU, S. S.-T. YAU, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Sci. (4) **12** (1985).
- [13] J. WEIDMANN, *Spectral Theory of Ordinary Differential Operators*, Lect. Notes in Math. 1258, Springer (1980).

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