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**Existence of front solutions in
degenerate reaction diffusion systems**

by

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Existence of front solutions in degenerate reaction diffusion systems

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Abstract: We investigate the dynamics of a system of two reaction diffusion equations in one space dimension, and study the effect of a small and of a vanishing diffusion coefficient in one equation. The analysis is restricted to competing species with two stable equilibria. We show that the system has traveling front solutions and analyze their wave-speed. It turns out that the diffusive species can propagate at a finite rate, while the non-diffusive species is blocked. We characterize the two cases with the help of an Lyapunov function.

1 Introduction

Reaction diffusion equations are a common model in the description of nonlinear systems such as chemical processes, ecological systems, or nerve-pulse propagation. Recently, Luckhaus and Triolo [6] studied a discrete stochastic model for tumor growth and derived a limiting continuous model for the densities u and v of malignant and healthy tissue. In one space dimension the system can be written as

$$\begin{aligned}\partial_t u &= \partial_{xx} u + f(u, v), \\ \partial_t v &= g(u, v).\end{aligned}\tag{1.1}$$

In the discrete system the malignant cells are mobile which results in a positive diffusivity in the first equation, but the healthy cells are immobile and there is no diffusion in the v -equation. A feature of the nonlinearities in (1.1) is competition, which results from the fact that in the discrete model the two species are competing for space. In the continuous model competition is expressed by

$$\partial_v f(u, v) \leq 0, \quad \partial_u g(u, v) \leq 0.\tag{1.2}$$

Due to competition, (1.1) defines a monotone dynamical system. The monotonicity has many analytical consequences, for instance, a comparison principle for solutions holds. In the case of a positive diffusion in the second equation the existence and stability of traveling fronts is well-understood [8]. We analyze traveling waves for (1.1). We will see that the missing diffusivity changes the qualitative picture.

Our setting will be such that at $x = -\infty$ only healthy cells are present, $(u, v) = (0, 1) = S_-$, while at $x = +\infty$ only tumor cells are present, $(u, v) = (1, 0) = S_+$. These are supposed to be the only stable uniform states. We will construct a Lyapunov function for the system and thus associate an energy $H(u, v)$ to each state. Under the condition

$$H(S_-) > H(S_+), \tag{1.3}$$

we expect that the system prefers state $S_+ = (1, 0)$ over $S_- = (0, 1)$, and therefore that fronts travel to the left. Our analysis verifies this picture. Instead, for the opposite inequality in (1.3), there are no waves traveling to the right. A blocking occurs and only stationary states exist. The phenomenon has some similarity to the blocking of propagation in an inhomogeneous medium with a highly varying diffusivity. Also in this case the existence of steady states prevents propagation. In [4], we analyze the stability of the traveling waves. While waves traveling to the left are stable, the stationary states are unstable, and an invasion of v at a logarithmic rate is possible.

While the above describes the picture for many nonlinearities g , for some applications we have to study a more general situation. The energy H is then no longer the decisive quantity, and in (1.3) we must replace $H(f, g)$ by a function $A(f, g)$, which coincides with H for nonlinearities g for which the set of nontrivial zeroes is the graph of a monotone function.

A similar competitive system has been studied by Aronson, Tesei and Weinberger in [1] as a model for pattern formation. They considered (1.1) on a bounded domain and with a diffusion coefficient in the first equation that depends on the second species. They show the existence of many discontinuous steady states and verify their stability. In their model, the assumptions on the reaction term differ from ours, in particular, the bistable case is not considered.

A bistable system with diffusion for both species was considered by Hosono and Mimura [5]. They obtained traveling wave solutions for a small diffusivity in the second equation via a singular perturbation method from stationary waves of the limit system. Our characterization of wave pinning enlarges the range of applicability of their approach (cp. section 4).

1.1 Main assumptions

In the proposed model the nonlinearities f and g are continuously differentiable and satisfy assumptions 1.-4. below. In this article we always assume that 1.-4. are satisfied by f and g . For an illustration see Figure 1.

1. Preserving positivity: $f(0, v) = 0 = g(u, 0)$ for all $u, v \in [0, 1]$.
2. Bistability: There are exactly two linearly stable equilibria $(u, v) = (1, 0)$ and $(u, v) = (0, 1)$, and two linearly unstable equilibria $(0, 0)$ and (u_s, v_s) .
3. Strict Competition: $\partial_v f, \partial_u g < 0$ for $u, v \in (0, 1)$.
4. The non-trivial solutions of $g(u, v) = 0$ have the form $u = \Gamma(v)$, $v \in (0, 1)$, for a continuous function $\Gamma : [0, 1] \rightarrow [0, \infty)$ with $\Gamma(1) = 0$ and $\Gamma(0) \in (0, 1)$. We assume that Γ has no local minimum in $(0, 1)$, but possibly a maximum which is then unique. Observe also that the range of Γ may exceed $[0, 1]$.

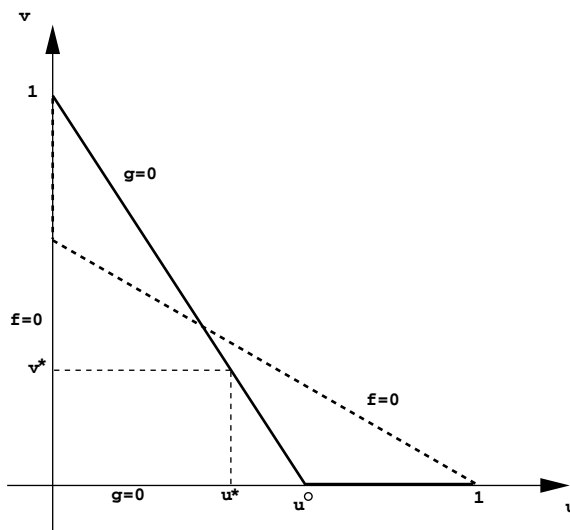


Figure 1: The zero-sets in the phase diagram

By linearly stable (unstable) we mean that the two eigenvalues of the Jacobian have nonzero real part, stable refers to both eigenvalues having negative real part and unstable refers to the case where at least one has positive real part. Note that our assumption implies that $(0, 0)$ is totally unstable and (u_s, v_s) is a saddle.

1.2 Main results

We show that for all pairs of nonlinearities (f, g) satisfying assumptions 1.-4. there exists a monotone traveling wave solution to system (1.1) connecting S_- with S_+ , i.e., there exists a triple (u, v, c) representing a solution $(u, v)(x + ct)$ of (1.1). The profiles $(u, v)(\xi)$ solve the system of ordinary differential equations

$$\begin{aligned} cu' &= u'' + f(u, v), \\ cv' &= g(u, v), \end{aligned} \tag{TW}$$

$$(u, v)(-\infty) = S_- = (0, 1), \quad (u, v)(+\infty) = S_+ = (1, 0).$$

Here $'$ denotes differentiation by the wave coordinate ξ . Furthermore, we have a complete characterization of the possible wave speeds c according to a quantity $A = A(f, g)$, which is related to the Lyapunov function and is defined in (3.2) below.

Theorem 1.1. *If $A < 0$, there exists a monotone solution of (TW) with $c > 0$. If $A \geq 0$, there is a monotone solution of (TW) with $c = 0$.*

Moreover, the existence of a monotone traveling wave with $c > 0$ implies $A < 0$. The existence of a monotone standing wave implies $A \geq 0$.

Our approach to prove the existence of traveling waves is independent of the value of A . The characterization of the speed is given a-posteriori.

In the case $c = 0$ we will show that the monotone standing wave is unique (see Theorem 3.1). However, there are families of non-monotone standing waves (see Lemma 5.1). In [4] we show a uniqueness and stability result for traveling waves in the case $c > 0$, and an instability result for the monotone stationary solutions.

2 Existence of traveling wave solutions

To prove the existence of traveling wave solutions to (1.1) we use the method of vanishing viscosity. For $\varepsilon > 0$ we consider traveling waves for the system

$$\begin{aligned} u_t - \partial_{xx}u &= f(u, v), \\ v_t - \varepsilon^2 \partial_{xx}v &= g(u, v). \end{aligned} \tag{1.1_\varepsilon}$$

The advantage of this approach is that convergence of traveling waves of (1.1_\varepsilon) as $\varepsilon \rightarrow 0$ can be proved without appealing to the regularity of the limit. In fact, the limit will be discontinuous in the case $A > 0$. We do not divide our approach into the two possible cases of propagating and stationary waves.

A traveling wave solution of (1.1_ε) is a heteroclinic orbit $(u, v)(\xi)$, $\xi = x + ct$, of the system of ordinary differential equations

$$\begin{aligned} cu' &= u'' + f(u, v), \\ cv' &= \varepsilon^2 v'' + g(u, v), \\ (u, v)(-\infty) &= (0, 1), \quad (u, v)(+\infty) = (1, 0), \end{aligned} \tag{TW_ε}$$

connecting the desired equilibria asymptotically. Existence of a monotone wave $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ for any $\varepsilon > 0$ is assured by Theorem 3.2, p. 174 in [8]. To employ the result directly one has to transform (TW_ε) into a monotone system by considering the vector-valued variable $(u, 1 - v)$.

Theorem 2.1. *Let (f, g) satisfy the main assumptions 1. to 4. Let $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ be a family of monotone traveling wave solutions of (TW_ε), $\varepsilon \in (0, \varepsilon_0)$. After suitable translation, the waves $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ converge for $\varepsilon \rightarrow 0$ to (u, v, c) , being a monotone traveling wave solution of (TW) in the following sense:*

If $c \neq 0$ then (u, v, c) is a classical solution and the convergence is in $C^{2,\alpha} \times C^{1,\alpha}$ on compact intervals. There holds $u' > 0 > v'$.

If $c = 0$ then (u, v, c) is a weak solution and we have $C^{1,\alpha}$ -convergence for u , whereas convergence is only pointwise for v . Furthermore, $u' > 0$ and there exists a unique point, say 0 , with the property $v(\xi) > 0$ if and only if $\xi < 0$. Also, $v'(\xi) < 0$ for $\xi < 0$. Thus, we have better convergence for v away from 0 .

An important step in the proof of Theorem 2.1 will be to show uniform bounds for the speed c_ε .

Theorem 2.2. *Let $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ be any traveling wave solution of (TW_ε). Then there holds*

$$-2\sqrt{K}\varepsilon \leq c_\varepsilon \leq 2\sqrt{L}$$

with positive constants $K := \sup_{0 < v < 1} \frac{g(0,v)}{v}$ and $L := \sup_{0 < u < 1} \frac{f(u,0)}{u}$.

The lower bound implies an interesting consequence for the speed as ε tends to zero.

Corollary 2.3 (Exclusion of negative speeds). *Let (u, v, c) be a traveling wave arising as a limit point as in Theorem 2.1. Then there holds $c \geq 0$.*

All speed estimates are obtained by using test functions (\tilde{u}, \tilde{v}) which are sub- or super-solutions of (1.1_ε). We use the comparison principle for

diffusive competitive systems, see Theorem 5.5, p. 244 of [8]. We say that $(\tilde{u}, \tilde{v})(x, t)$ is a sub-solution in the moving frame $\xi = x + \tilde{c}t$ if it satisfies

$$\begin{aligned}\partial_t \tilde{u} + \tilde{c} \partial_\xi \tilde{u} - \partial_{\xi\xi} \tilde{u} &\leq f(\tilde{u}, \tilde{v}), \\ \partial_t \tilde{v} + \tilde{c} \partial_\xi \tilde{v} - \varepsilon^2 \partial_{\xi\xi} \tilde{v} &\geq g(\tilde{u}, \tilde{v}).\end{aligned}$$

We recall that the system is competitive which leads to the opposite inequality in the second line. The parabolic comparison principle in the weak form yields that for a sub-solution (\tilde{u}, \tilde{v}) in the frame \tilde{c} and an instationary solution (u, v) the inequalities

$$\begin{aligned}\tilde{u}(t, \xi) &\leq u(t, \xi), \\ \tilde{v}(t, \xi) &\geq v(t, \xi),\end{aligned}$$

remain satisfied for all times if they are satisfied initially. We recall that for weak sub-solutions jumps in the derivative are allowed if the sign condition $[\partial_\xi \tilde{u}] \geq 0 \geq [\partial_\xi \tilde{v}]$ is satisfied. Super-solutions are defined analogously.

Proof of Theorem 2.2. i) Lower bound. We claim that a sub-solution can be constructed in the form

$$\begin{aligned}\underline{c} &= -2\sqrt{K}\varepsilon, \\ \underline{u} &= u_m \max\{1 - e^{-2\sqrt{K}\varepsilon\xi}, 0\}, \\ \underline{v} &= \begin{cases} \min\{v_m e^{-\frac{\sqrt{K}}{\varepsilon}\xi}, 1\}, & \xi \leq M/\varepsilon, \\ \underline{v}(M/\varepsilon)(1 + e^{-2\frac{\sqrt{K}}{\varepsilon}(\xi - M/\varepsilon)})/2, & \xi > M/\varepsilon. \end{cases}\end{aligned}$$

Here K is chosen such that $\frac{g(u, v)}{v} \leq \frac{g(0, v)}{v} \leq K$, which is possible by $\partial_u g \leq 0$. We choose $u_m \in (\Gamma(0), 1)$ freely and then $v_m \in (0, 1)$ such that $u_m > \sup_{0 < v < v_m} \Gamma(v)$ and $f(u, v) \geq 0$ for all $u \leq u_m$, $v \leq v_m$. The latter is possible since $(0, 0)$ is an unstable equilibrium. We can now choose $M > 0$, independent of ε , such that

$$\underline{u}(M/\varepsilon) = u_m \left(1 - e^{-\sqrt{K}M}\right) \geq \sup_{0 < v < v_m} \Gamma(v).$$

Note that $[\partial_\xi \underline{u}] \geq 0 \geq [\partial_\xi \underline{v}]$, where the derivatives are discontinuous.

We consider the equation for v . In the case $\frac{\varepsilon}{\sqrt{K}} \ln v_m < \xi < M/\varepsilon$ the inequality $g(\underline{u}, \underline{v}) \leq K\underline{v}$ implies

$$-\underline{c} \partial_\xi \underline{v} + \varepsilon^2 \partial_{\xi\xi} \underline{v} + g(\underline{u}, \underline{v}) \leq -\underline{c} \partial_\xi \underline{v} + \varepsilon^2 \partial_{\xi\xi} \underline{v} + K\underline{v} = 0 = \partial_t \underline{v}.$$

For $\xi > M/\varepsilon$ the definition of M implies $g(\underline{u}, \underline{v}) \leq 0$ and thus

$$-\underline{c} \partial_\xi \underline{v} + \varepsilon^2 \partial_{\xi\xi} \underline{v} + g(\underline{u}, \underline{v}) \leq -\underline{c} \partial_\xi \underline{v} + \varepsilon^2 \partial_{\xi\xi} \underline{v} = 0 = \partial_t \underline{v}.$$

We now consider the equation for u . There holds $f(\underline{u}, \underline{v}) \geq 0$. For $\xi < 0$ this is implied by $\underline{u}(\xi) = 0$, for $\xi > 0$ by the choice of u_m, v_m . Hence for all $\xi \neq 0$ we have

$$-\underline{c}\partial_\xi \underline{u} + \partial_{\xi\xi} \underline{u} + f(\underline{u}, \underline{v}) \geq -\underline{c}\partial_\xi \underline{u} + \partial_{\xi\xi} \underline{u} = 0 = \partial_t \underline{u}.$$

We have thus proved that $(\underline{c}, \underline{u}, \underline{v})$ is a sub-solution. Note that for any $\varepsilon > 0$ the profile $(\underline{u}, \underline{v})$ remains in both components at a positive distance from $S_+ = (1, 0)$ for all $\xi > 0$. Furthermore, $(\underline{u}, \underline{v})(\xi) = (0, 1)$ for all $\xi < 0$ with $|\xi|$ large. Thus, any initial data (u_0, v_0) with $(u_0, v_0)(\xi) \rightarrow (1, 0)$ as $\xi \rightarrow \infty$ can be shifted to be comparable with $(\underline{u}, \underline{v})$. This implies that no traveling wave solution can travel at a slower speed than the comparison solution, $c_\varepsilon \geq \underline{c} = -2\sqrt{K}\varepsilon$.

ii) Upper bound. The definition of Γ , i.e., $g(\Gamma(t), t) = 0$ for all $0 < t < 1$, implies

$$\frac{d}{ds}\Gamma^{-1}(s) = -\frac{\partial_u g(s, \Gamma^{-1}(s))}{\partial_v g(s, \Gamma^{-1}(s))}.$$

For small s we know that $\partial_v g(s, \Gamma^{-1}(s))$ is close to $\partial_v g(0, 1) < 0$, i.e., $\frac{d}{ds}\Gamma^{-1}(s)$ is well-defined. Furthermore, there exists $\gamma > 0$ such that $\Gamma^{-1}(s) \geq 1 - \gamma s$ for small $s \geq 0$. By the continuity of $\partial_u f$ there exists a small $\delta > 0$ such that $f(s, t) \leq 0$ for $s \leq \delta$ and $t \geq 1 - 2\gamma\delta$ and $g(s, t) \geq 0$ for $s \leq \delta, t \leq 1 - \gamma\delta$.

For the upper bound we claim that

$$\begin{aligned} \bar{c} &= 2\sqrt{L}, \\ \bar{u} &= \begin{cases} \min\{\delta e^{\sqrt{L}\xi}, 1\}, & \xi \geq -M, \\ \bar{u}(-M)(1 + e^{2\sqrt{L}(\xi+M)})/2, & \xi < -M, \end{cases} \\ \bar{v} &= \max\{(1 - 2\gamma\delta)(1 - e^{\frac{\bar{c}}{2}\xi}), 0\}. \end{aligned}$$

is a super-solution for M sufficiently large. The condition $[\partial_\xi \bar{u}] \leq 0 \leq [\partial_\xi \bar{v}]$ is satisfied.

The inequality in the u -equation is trivially satisfied if $\bar{u} = 1$. If $\xi > -M$ and $\bar{u} < 1$ we deduce from $f(\bar{u}, \bar{v}) \leq f(\bar{u}, 0) \leq L\bar{u}$

$$-\bar{c}\partial_\xi \bar{u} + \partial_{\xi\xi} \bar{u} + f(\bar{u}, \bar{v}) \leq -\bar{c}\partial_\xi \bar{u} + \partial_{\xi\xi} \bar{u} + L\bar{u} = 0 = \partial_t \bar{u}.$$

For $\xi < -M$, M sufficiently large, the conditions on γ, δ ensure $f(\bar{u}, \bar{v}) \leq 0$. Thus,

$$-\bar{c}\partial_\xi \bar{u} + \partial_{\xi\xi} \bar{u} + f(\bar{u}, \bar{v}) \leq -\bar{c}\partial_\xi \bar{u} + \partial_{\xi\xi} \bar{u} = 0 = \partial_t \bar{u}.$$

Inequality in the v -equation: For $\xi < 0$ we have $g(\bar{u}, \bar{v}) \geq g(\delta, \bar{v}) \geq 0$, hence,

$$-\bar{c}\partial_\xi \bar{v} + \varepsilon^2 \partial_{\xi\xi} \bar{v} + g(\bar{u}, \bar{v}) \geq -\bar{c}\partial_\xi \bar{v} + \varepsilon^2 \partial_{\xi\xi} \bar{v} = 0 = \partial_t \bar{v}.$$

Thus, $(\bar{c}, \bar{u}, \bar{v})$ is indeed a super-solution. Since \bar{u} and \bar{v} are uniformly bounded away from 0 and 1, respectively, and $(\bar{u}, \bar{v})(\xi) = (1, 0)$ for $\xi > 0$, we can always shift initial data (u_0, v_0) with $(u_0, v_0)(\xi) \rightarrow (0, 1)$ for $\xi \rightarrow -\infty$ to be comparable with the super-solution. We conclude $c_\varepsilon \leq \bar{c}$. \square

Proof of Theorem 2.1. Due to the translational invariance of the the problem (TW_ε) , we have to impose a normalization condition. We consider those wave solutions $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ with $u_\varepsilon(0) = u_s$, where (u_s, v_s) is the unique saddle point of (f, g) in $(0, 1)^2$.

1) *Uniform bounds.* First we show that equation (TW_ε) implies uniform $C^{1,1}$ -bounds for u_ε and uniform C^0 -bounds for v_ε . Recall the wave monotonicity $u'_\varepsilon > 0 > v'_\varepsilon$. At a position where $|u'_\varepsilon|$ is maximal we have

$$|c_\varepsilon u'_\varepsilon| = |f(u_\varepsilon, v_\varepsilon)| \leq \sup_{0 < s, t < 1} |f(s, t)| =: C_f,$$

since u_ε and v_ε map to $[0, 1]$. Thus, $|u''_\varepsilon| \leq \sup_\xi |c_\varepsilon u'_\varepsilon| + \sup_\xi |f(u_\varepsilon, v_\varepsilon)| \leq 2C_f$. The bound on u''_ε implies a bound on u'_ε by interpolation. For all $\xi_0 \in \mathbb{R}$ we use

$$1 > |u_\varepsilon(\xi) - u_\varepsilon(\xi_0)| = \left| (\xi - \xi_0)u'_\varepsilon(\xi_0) + \int_{\xi_0}^\xi \int_{\xi_0}^s u''_\varepsilon(t) dt ds \right|$$

to deduce

$$u'_\varepsilon(\xi_0) \leq \inf_\xi (|\xi - \xi_0|^{-1} + |\xi - \xi_0|C_f) = 2\sqrt{C_f}.$$

2) *Convergence of a subsequence.* We recall that the speeds c_ε are bounded uniformly in ε by Theorem 2.2. From the monotonicity and the bounds for $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ we obtain for a subsequence the convergence to a monotone profile (u, v, c) (the convergence is in $C^{1,\alpha}$ on compact intervals for u and pointwise for v). The bounds for the speed imply $c \geq 0$. Since u is a weak solution for $\varepsilon = 0$, we obtain $u \in C^{1,1}$. Also $cv' = g(u, v)$ is satisfied in the weak sense. Thus, $c > 0$ implies $v \in C^{1,1}$ and hence $u \in C^{2,1}$. In the case $c = 0$ the profiles are confined to the zero level set of g consisting of the axis $\{(s, 0) : s \in [0, 1]\}$ and the curve $\{(\Gamma(t), t) : t \in [0, 1]\}$. The assumptions on $\Gamma(t)$ imply that the monotone profile v can jump at most once, from the curve down to the axis.

3) *Boundary values.* The limit (u, v, c) is a solution of (TW) ; it remains to check the limits for $|\xi| \rightarrow \infty$. The monotonicity of the profiles yields

$u(-\infty) \leq u_s \leq u(\infty)$ and $1 \geq v(-\infty) \geq v(\infty) \geq 0$. Since u is continuous we have $u(0) = u_s$. Furthermore, $(u, v)(\pm\infty)$ are equilibria of (f, g) and therefore contained in the set $\{(1, 0), (u_s, v_s), (0, 1), (0, 0)\}$. By the maximum principle, u is either strictly increasing or $u \equiv u_s$ identically. This implies, that we are left with three possibilities:

- i) (u, v, c) is a traveling wave solution of (TW) satisfying the boundary condition.
- ii) v vanishes identically and u is a strictly monotone traveling wave increasing from $u(-\infty) = 0$ to $u(\infty) = 1$,
- iii) $(u, v) \equiv (u_s, v_s)$.

First we want to exclude case ii). Here the position where v_ε equals v_s diverges to $-\infty$ as ε tends to 0. Since the resulting wave u solves

$$cu' = u'' + f(u, 0),$$

an integration over \mathbb{R} gives $c > 0$ (the nonlinearity is of KPP type). Now we consider other translates $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ of $(u_\varepsilon, v_\varepsilon)$, satisfying the new normalization $\tilde{v}_\varepsilon(0) = v_s$. We know $\tilde{u}_\varepsilon(0) \rightarrow 0$ as $\varepsilon \rightarrow 0$ since v_ε was tending to zero in the first normalization. We take a further subsequence $\varepsilon \rightarrow 0$ such that $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ converges on compact intervals to (\tilde{u}, \tilde{v}) , in $C^{1,\alpha}$ for \tilde{u} and in C^α for \tilde{v} , where now the profile \tilde{u} vanishes identically. Thus, recalling $c > 0$, the decreasing profile \tilde{v} is a classical solution of

$$c\tilde{v}' = g(0, \tilde{v}) > 0.$$

This contradicts $c > 0$ and the monotonicity of \tilde{v} .

To exclude case iii) we consider the translates $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ of $(u_\varepsilon, v_\varepsilon)$ satisfying $\tilde{u}_\varepsilon(0) = \frac{1}{2}u_s$. Arguing as before, we obtain a subsequence $\varepsilon \rightarrow 0$ and strictly monotone profiles (\tilde{u}, \tilde{v}) , now satisfying the asymptotic boundary conditions $(\tilde{u}, \tilde{v})(\infty) = (u_s, v_s)$, and $(\tilde{u}, \tilde{v})(-\infty) = (0, 1)$. The fact that $(\tilde{u}, \tilde{v}, c)$ solves the traveling wave system (TW) except for the boundary conditions will provide a contradiction.

Note that $c \geq 0$ and the monotonicity of \tilde{v} imply $0 \geq c\tilde{v}' = g(\tilde{u}, \tilde{v})$. Integrating the equation

$$c\tilde{u}' = \tilde{u}'' + f(\tilde{u}, \tilde{v})$$

over \mathbb{R} we obtain

$$0 \leq cu_s = \int_{\mathbb{R}} f(\tilde{u}, \tilde{v}) d\xi.$$

We claim that the integral on the right hand side is negative, giving the contradiction. The condition $g(\tilde{u}, \tilde{v}) \leq 0$ implies $u_s \geq \tilde{u} \geq \Gamma(\tilde{v})$. But $\partial_v f < 0$ implies $f(\tilde{u}, \tilde{v}) \leq 0$ for $\tilde{v} > v_s$ by $\partial_v f < 0$ and $f(\Gamma(t), t) < 0$ for $t \in (v_s, 1)$. Note that \tilde{u} is non-constant which implies the strict inequality.

4) *Strict monotonicity if $c > 0$.* We note here that $c > 0$ implies $u' > 0$ and $v' < 0$. For the u -component this followed already by excluding the constant limit profiles in 3). The weak monotonicity $v' \leq 0$ implies $g(u, v) \leq 0$ on \mathbb{R} . We claim $g(u, v) < 0$, which would imply the strict inequality $v' < 0$. Assuming there is a point ξ_0 with $g(u, v)(\xi_0) = 0$ we conclude $\frac{d}{d\xi} g(u, v) = \partial_u g u' < 0$. Since this leads to a sign change of the function $g(u, v)(\xi)$ we arrive at a contradiction. \square

3 Characterization of the speed

3.1 A Lyapunov function

The system (1.1) possesses a free energy function H and a Lyapunov function E which yield further conditions for the wave speed. In fact, the energy characterizes the two cases of standing waves ($c = 0$) and propagating waves ($c > 0$).

For functions $(u, v)(t, x)$ satisfying $u, v \in [0, 1]$, $(u, v)(t, -\infty) = (0, 1)$, and $(u, v)(t, \infty) = (1, 0)$ we define

$$E((u, v)) := \int_{\mathbb{R}} \left\{ \frac{1}{2} |\partial_x u|^2 + H(u, v) - H(1, 0) \chi_{\mathbb{R}_+} \right\} dx, \quad (3.1)$$

where $\chi_{\mathbb{R}_+}$ is the characteristic function of $\{x \in \mathbb{R} : x > 0\}$ and

$$H(u, v) = - \int_0^u f(\sigma, v) d\sigma - \int_v^1 \int_0^{\Gamma(\tau)} \partial_v f(\sigma, \tau) d\sigma d\tau.$$

We recall that $\sigma = \Gamma(\tau)$ is the unique nontrivial solution of $g(\sigma, \tau) = 0$. We normalized such that $H(0, 1) = H(S_-) = 0$. The constant term $H(1, 0) \chi_{\mathbb{R}_+}$ is inserted to ensure finiteness of the integral for solutions sharing the asymptotics of the traveling wave.

Next we verify that E is decreasing along solutions of (1.1). In the calculation we perform a partial integration without boundary terms using

$\partial_x u(t, \pm\infty) = 0$, which is justified for $\partial_x u, \partial_{xx} u, \partial_t u, \partial_t \partial_x u \in L^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$.

$$\begin{aligned} \partial_t E((u, v)) &= \int_{\mathbb{R}} (-\partial_{xx} u) \partial_t u + \partial_u H \partial_t u + \partial_v H \partial_t v \\ &= \int_{\mathbb{R}} (-\partial_{xx} u - f(u, v)) \partial_t u \\ &\quad + \int_{\mathbb{R}} \left(-\int_0^u \partial_v f(\sigma, v) d\sigma + \int_0^{\Gamma(v)} \partial_v f(\sigma, v) d\sigma \right) \partial_t v. \end{aligned}$$

We exploit that $\partial_v f$ is nonpositive and that

$$g(u, v) < 0 \text{ for } u < \Gamma(v), \quad g(u, v) > 0 \text{ for } u > \Gamma(v),$$

to conclude that E is decreasing.

The Lyapunov function allows to obtain a condition on the uniqueness and existence of standing waves. Let

$$u_1 := \min \left\{ 1, \sup_{t \in (0,1)} \Gamma(t) \right\}, \quad v_1 := \Gamma^{-1}(u_1).$$

Here we defined for s in the range of Γ the unique maximal inverse $\Gamma^{-1}(s) := \max\{t \in (0, 1] : \Gamma(t) = s\}$. From our assumptions on Γ we know that Γ decreases strictly monotone on the interval $(v_1, 1)$. Note that for monotone Γ we have $v_1 = 0$ and $u_1 = \Gamma(0)$.

We additionally define the following number $A = A(f, g) \in \mathbb{R}$,

$$\begin{aligned} A &:= H(1, 0) - H(u_1, 0) + H(u_1, v_1) \\ &= -\int_0^{u_1} f(s, \Gamma^{-1}(s)) ds - \int_{u_1}^1 f(s, 0) ds. \end{aligned} \tag{3.2}$$

Theorem 3.1. *If the system (TW) has a monotone standing wave solution $(u, v, c = 0)$, then $A \geq 0$.*

Vice versa, if $A \geq 0$, there is a monotone solution (u, v, c) of (TW) with $c = 0$. This wave is unique in the class of standing waves. For monotone Γ and $A = 0$ the solution is continuous.

There are two special cases where we can easily check the condition $A \geq 0$. For monotone Γ there holds $v_1 = \Gamma^{-1}(u_1) = 0$ and therefore

$$\begin{aligned} A = H(1, 0) &= -\int_0^1 f(s, v) ds - \int_0^1 \int_0^{\Gamma(t)} \partial_v f(s, t) ds dt \\ &= \int_0^1 f(s, \Gamma^{-1}(s)) ds. \end{aligned}$$

Secondly, if $\sup \Gamma$ exceeds 1, we have $u_1 = 1$ and therefore

$$\begin{aligned} A = H(1, v_1) &= - \int_0^1 f(s, v_1) ds - \int_{v_1}^1 \int_0^{\Gamma(t)} \partial_v f(s, t) ds dt \\ &= - \int_0^1 f(s, \Gamma^{-1}(s)) ds > 0, \end{aligned}$$

since $f(s, \Gamma^{-1}(s)) < 0$. The latter follows because otherwise we had more than one non-trivial equilibrium, contradicting our assumptions. Thus, in the case $u_1 = 1$ the theorem ensures the existence of a standing wave.

For traveling wave solutions (u, v, c) we calculate

$$\begin{aligned} c \left(-\frac{1}{2}|u'|^2 + H(u, v) \right)' &= c \left((-u'' + \partial_u H)u' + \partial_v H v' \right) \\ &= - \left(c^2 |u'|^2 + \int_{\Gamma(v)}^u \partial_v f(s, v) ds \int_{\Gamma(v)}^u \partial_u g(s, v) ds \right). \end{aligned}$$

Since the traveling wave connects the equilibrium $S_- = (0, 1)$ with $S_+ = (1, 0)$, an integration over \mathbb{R} yields

$$c \cdot [H(1, 0) - H(0, 1)] \leq 0. \quad (3.3)$$

For $c > 0$ the inequality is strict. From $c > 0$ and $v_1 = 0$ we can conclude from this $A = H(1, 0) < 0$.

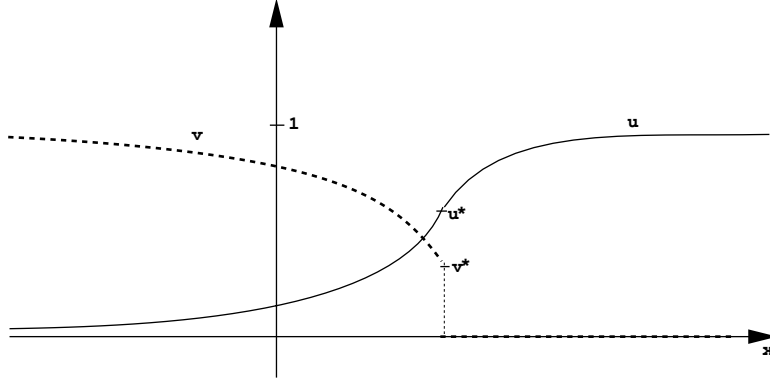


Figure 2: A standing wave solution

For $A \geq 0$ the qualitative behavior of a corresponding standing wave is depicted in Figure 2. We remark that the profile v is non-smooth; there is a unique point, where v or v' has a discontinuity.

Proof. i) Let us assume that there exists a monotone standing wave solution (u, v, c) with $c = 0$ of system (TW). We want to show $A \geq 0$. Recall that the elliptic equation provides uniform $C^{1,1}$ -estimates for u .

From (TW) and $c = 0$ we know that $g(u, v) = 0$ for all ξ . According to our assumption on Γ , the monotonicity of the profiles (u, v) implies that, after a suitable translation, $(u, v)(\xi)$ follows the curve $\{(\Gamma(t), t) : 0 < t \leq 1\}$ for $\xi < 0$ and the axis $\{(s, 0) : 0 < s \leq 1\}$ for $\xi > 0$. Thus, again by monotonicity of u and v , we have $u^* := u(0) \in (0, u_1]$, $v(\xi) = 0$ for $\xi > 0$, and $v(\xi) = \Gamma^{-1}(u(\xi))$ for $\xi < 0$.

We exploit that

$$-\frac{d}{du}H(u, \Gamma^{-1}(u)) = f(u, \Gamma^{-1}(u)) \quad \text{and} \quad -\frac{d}{du}H(u, 0) = f(u, 0).$$

imply that the profile u satisfies

$$\begin{aligned} -\frac{1}{2}|u'(\xi)|^2 + H(u(\xi), v(\xi)) &= H(0, 1) \quad \text{for } \xi < 0, \\ -\frac{1}{2}|u'(\xi)|^2 + H(u(\xi), 0) &= H(1, 0) \quad \text{for } \xi > 0. \end{aligned}$$

The continuity of u' implies that there must hold equality in $\xi = 0$, i.e.

$$H(1, 0) - H(u^*, 0) = -H(u^*, \Gamma^{-1}(u^*)). \quad (3.4)$$

In the following we show that the existence of a solution $u^* \in (0, u_1]$ of (3.4) is equivalent with $A \geq 0$. Furthermore, u^* will be unique.

Let us define for all $u \in [0, u_1]$

$$\begin{aligned} A(u) &:= H(1, 0) - H(u, 0) + H(u, \Gamma^{-1}(u)) \\ &= -\int_0^u f(s, \Gamma^{-1}(s)) ds - \int_u^1 f(s, 0) ds, \end{aligned}$$

such that $A = A(u_1)$ and a solution u^* of (3.4) is characterized by the property $A(u^*) = 0$. We evaluate $A(0) = H(1, 0) - H(0, 0) + H(0, 1) = -\int_0^1 f(s, 0) ds < 0$ and calculate $A'(u) = f(u, 0) - f(u, \Gamma^{-1}(u)) > 0$ on $(0, u_1)$. Hence, there is a solution $u^* \in (0, u_1]$ satisfying $A(u^*) = 0$ if and only if $A = A(u_1) \geq 0$. The solution is unique by the monotonicity of $A(\cdot)$.

ii) We now prove that for $A \geq 0$ there exists a standing wave solution. We will construct such a wave using the solution u^* of $A(u^*) = 0$, i.e. of (3.4). We define

$$\begin{aligned} F_-(u) &:= -\int_0^u f(s, \Gamma^{-1}(s)) ds, & \text{for } u \in [0, u_1], \\ F_+(u) &:= \int_u^1 f(s, 0) ds, & \text{for } u \in [0, 1]. \end{aligned}$$

For $u \in [0, u_1]$ we recover $A(u) = F_-(u) - F_+(u)$. We note the identities $F_-(u) = H(u, \Gamma^{-1}(u)) - H(0, 1)$ and $F_+(u) = H(u, 0) - H(1, 0)$.

We can now explicitly construct a wave with $c = 0$. On $\xi < 0$ we find $u(\xi)$ as the solution of the (backward in 'time' ξ) ordinary differential equation

$$u'(\xi) = \sqrt{2F_-(u(\xi))}, \quad u(0) = u^*.$$

From the boundedness of f on $[0, 1]^2$ we know that F_{\pm} is Lipschitz for all $s \in [0, u_1]$. Furthermore, bounded derivatives $\partial_u f$ give $F_-(s) \leq Cs^2$. Finally, there holds $F_- > 0$ on $s \in (0, u_1]$. For $F_-(u_1)$ this follows from $F_+(u_1) > 0$ and $A(u_1) \geq 0$. Also, if there were a nonpositive minimum at $\tilde{u} \in (0, u_1)$, it followed that $0 = F'_-(\tilde{u}) = f(\tilde{u}, \Gamma^{-1}(\tilde{u}))$, i.e., it is located at the unique saddle point $\tilde{u} = u_s$. In this case $f(s, \Gamma^{-1}(s)) < 0$ for all $s \in (0, \tilde{u})$ by our assumptions. Thus, $F_-(\tilde{u}) > 0$, contradicting the assumed nonpositive minimum.

Thus, the o.d.e. above has a Lipschitz nonlinearity implying existence and uniqueness of the trajectory $u(\xi)$. In particular, $u(\xi)$ exists for all $\xi < 0$ with $u(\xi) \rightarrow 0$ as $\xi \rightarrow -\infty$.

Now let us consider $\xi > 0$. Let $u(\xi)$ be a solution of the initial value problem

$$u'(\xi) = \sqrt{2F_+(u(\xi))}, \quad u(0) = u^*.$$

Observe that F_+ is uniformly Lipschitz in $u \in [0, 1]$. Furthermore, $F_+(s) = 0$ only for $s = 1$. We conclude that the solution $u(\xi)$ is unique and monotone for all $\xi > 0$, satisfying $u(\xi) \rightarrow 1$ as $\xi \rightarrow +\infty$. The uniqueness of the profile $u(\xi)$ follows from the uniqueness of u^* . Finally we define $v(\xi) = \Gamma^{-1}(u(\xi))$ for $\xi < 0$ and $v(\xi) = 0$ for $\xi > 0$.

$A = 0$ implies $A(u_1) = A = 0$ and therefore $u^* = u_1$. For monotone Γ we have $\Gamma^{-1}(u_1) = 0$, so v is continuous. \square

Proof of Theorem 1.1. The existence of traveling waves is shown in Theorem 2.1. In Corollary 2.3 we have seen that $c \geq 0$ holds for these waves. Standing waves were considered in Theorem 3.1, we have seen that $A \geq 0$ is equivalent to the existence of a traveling wave $(u, v, c = 0)$.

In Theorem 3.2 we already verified that for the solutions constructed there, $c > 0$ is impossible in the case $A \geq 0$. To conclude the proof of Theorem 1.1 it remains to show that the existence of a monotone traveling wave (u, v, c) with $c > 0$ always implies $A < 0$.

Let (u, v, c) be a monotone traveling wave with $c > 0$. We want to show that there must hold $A < 0$. The argument from part 1) of the proof of Theorem 2.1 gives $u \in C^{2,1}$, $v \in C^{1,1}$ with u strictly increasing. The

equation $cv' = g(u, v)$ gives $g(u, v) \leq 0$. Hence $v \geq \phi(u)$, where $\phi(u) = \Gamma^{-1}(u)$ for $u \leq u_1$ and $\phi(u) = 0$ for $u > u_1$. In particular it follows that there must hold $u_1 < 1$.

Recalling that f is competitive we obtain from (TW)

$$cu' = u'' + f(u, v) \leq u'' + f(u, \phi(u)).$$

Thus, multiplying with u' and integrating over \mathbb{R} we get

$$0 < c \int_{\mathbb{R}} |u'|^2 = \int_0^1 f(s, \phi(s)) ds,$$

since the traveling wave connects the equilibria $(0, 1)$ and $(1, 0)$ at $-\infty$ and $+\infty$, respectively. Here we used that the $t \rightarrow \int_0^t f(s, \phi(s)) ds$ is a Lipschitz function on $[0, 1]$. Furthermore, the definition of A and our estimates for ϕ yield $\int_0^1 f(s, \phi(s)) ds < -A$. Thus, the existence of a monotone traveling wave with $c > 0$ implies the inequality $A < 0$. \square

The next theorem shows that if system (1.1) is in the standing wave regime $A \geq 0$, then the diffusive system (1.1 $_{\varepsilon}$) allows only slow waves.

Theorem 3.2. *Consider (1.1 $_{\varepsilon}$) under assumptions 1. to 4. In addition, we assume for a more readable proof that the nonlinearities f, g are C^2 in $[0, 1]^2$, $\Gamma'(t) \neq 0$ for all $t \in (v_1, 1]$. If $A > 0$ then there exist $K, M, \varepsilon_0 > 0$ such that any traveling wave solution $(u_{\varepsilon}, v_{\varepsilon}, c_{\varepsilon})$ connecting $S_- = (0, 1)$ with $S_+ = (1, 0)$ satisfies for all $0 < \varepsilon < \varepsilon_0$*

$$|c_{\varepsilon}| \leq M\varepsilon. \quad (3.5)$$

Proof. A lower bound $c_{\varepsilon} \geq -2\sqrt{K}\varepsilon$ is implied by Theorem 2.2. For the upper bound we construct a super-solution to the speed $\hat{c} = M\varepsilon$, where M is chosen later sufficiently large.

Analogously to the construction of the standing wave we may obtain profiles (\hat{u}, \hat{v}) serving as a super-solution for the speed \hat{c} . The super-solution will be close to the standing wave profile. To allow initial data with an arbitrary decay to the asymptotic states, the super-solution has to differ from the standing wave for large $|\xi|$. Recall that if $A > 0$ the profile v of the standing wave is discontinuous at $\xi = 0$. Thus, an admissible test-function \hat{v} satisfying $-\hat{c}\partial_{\xi}\hat{v} + \varepsilon^2\partial_{\xi\xi}\hat{v} + g(\hat{u}, \hat{v}) \geq 0$ has to be altered in the neighborhood of $\xi = 0$ in an appropriate way.

For $a = -\partial_u f(0, 1) > 0$ we can choose $0 < \nu < a$ such that $\frac{4\nu}{\sqrt{a(1-\frac{\nu}{a})}} = M\varepsilon$. We recall from the proof of Theorem 2.2 that $\partial_v g(0, 1) < 0$ and $a > 0$ implies that for $\nu > 0$ there exists $\delta > 0$ such that $|\partial_u f(s, \Gamma^{-1}(s)) + a| < \nu$ for all $s \in [0, \delta]$.

The unique solution of $A(u^*) = 0$ satisfies $u^* < u_1$ if $A = A(u_1) > 0$. We choose $u_0 \in (u^*, u_1)$. In close analogy to the proof of Theorem 3.1 we define for $u \in [0, 1]$

$$F_0(u) := - \int_0^u f(s, \phi_0(s)) ds, \quad \phi_0(s) := \begin{cases} \Gamma^{-1}(s), & 0 \leq s \leq u_0, \\ 0, & u_0 < s \leq 1. \end{cases}$$

There holds

$$\begin{aligned} F_0(1) &= - \int_0^1 f(s, \phi_0(s)) ds \\ &> - \int_0^{u_*} f(s, \Gamma^{-1}(s)) ds - \int_{u_*}^1 f(s, 0) ds = A(u^*) = 0. \end{aligned}$$

We recall that $F_0 = F_-$ on $[0, u_0]$. Since F_- and F_+ are positive on $(0, u_1]$ and $[0, 1)$, respectively, we have $F_0 = F_- > 0$ on $(0, u_0]$ and $F_0(s) = F_+(s) + F_0(1) > F_0(1) > 0$ for $s \in [u_0, 1)$. Furthermore, $\sqrt{2F_0}$ is uniformly Lipschitz on $[0, 1]$, vanishing only at 0.

For $\xi < 0$ we get the profile $\hat{u}(\xi)$ by solving

$$\hat{u}'(\xi) = \sqrt{2F_0(\hat{u}(\xi))}, \quad \hat{u}(0) = 1$$

in backward direction. The properties of F_0 ensure that $\hat{u}(\xi)$ exists for all $\xi < 0$, is strictly increasing in ξ , and satisfying $\lim_{\xi \rightarrow -\infty} \hat{u}(\xi) = 0$. Since $F_0 = F_-$ on $[0, u_0]$ we know that $\hat{u}(\xi)$ coincides with a translate of the unique standing wave profile for all $\xi \leq \xi^*$, with $\hat{u}(\xi^*) = u^* < u_0$.

There exist a unique $\xi_- < 0$ such that $\hat{u}(\xi_-) = \delta$. The definition of δ implies that the following inequalities hold

$$|f(s, \Gamma^{-1}(s)) + as| < \nu s, \quad |2F_0(s) - as^2| < \nu s^2 \quad \text{for all } s \in [0, \delta].$$

Thus, $\hat{u}'(\xi_-) = \sqrt{2F_0(\delta)} < \sqrt{a + \nu\delta}$. We re-define for $\xi < \xi_-$

$$\hat{u}(\xi) = \delta \left(\tilde{\nu} + (1 - \tilde{\nu}) \exp\left\{ \frac{\sqrt{a + \nu}}{1 - \tilde{\nu}} (\xi - \xi_-) \right\} \right),$$

where $\tilde{\nu} = \frac{\nu}{a}$. There holds $\lim_{\xi \rightarrow \xi_-} \hat{u}(\xi) = \delta = \hat{u}(\xi_-)$. Also,

$$\lim_{\xi \rightarrow \xi_-} \hat{u}'(\xi) = \sqrt{a + \nu\delta} > \sqrt{2F_0(\delta)} = \hat{u}'(\xi_-).$$

Thus, the new profile is continuous with a jump in the derivative of negative sign. We remark that for small ε we have that δ is strictly less than u^* . Note that the new asymptotic state $\hat{u}(-\infty)$ is strictly positive.

We extend \hat{u} by setting $\hat{u}(\xi) = 1$ for all $\xi > 0$. Obviously, the jump in the derivative satisfies $[\partial_\xi \hat{u}](0) < 0$. There is a unique $\xi_0 \in (\xi^*, 0)$ such that

$\hat{u}(\xi_0) = u_0$. Let us define $\hat{v}(\xi) := \phi_0(\hat{u}(\xi))$ for $\xi \leq \xi_0$. Since $\phi_0(s) = \Gamma^{-1}(s)$ for $s \leq u_0$ we have $v(0) = \Gamma^{-1}(u_0) < v^*$ and $v'(0_-) = \frac{d}{du}\Gamma^{-1}(u_0)u'(\xi_0) < 0$. We extend \hat{v} by setting

$$\hat{v}(\xi) := \max\{\hat{v}(0) + \varepsilon(e^{(\xi-\xi_0)/\varepsilon} - 1)\hat{v}'(0_-), 0\} \text{ for all } \xi > \xi_0.$$

Hence, there is a unique $\xi_1 \in (\xi_0, 0)$ satisfying $\hat{v}(\xi_1) > 0$ for all $\xi < \xi_1$. We remark that $\xi_1 - \xi_0 \leq C\varepsilon \ln \varepsilon^{-1}$.

Both profiles $(\hat{u}(\xi), \hat{v}(\xi))$ are C^2 except at the points $E = \{\xi_-, \xi^*, \xi_0, 0\}$. Furthermore, the construction ensures that for each $\xi_e \in E$ we have $[\partial_\xi \hat{u}](\xi_e) \leq 0 \leq [\partial_\xi \hat{v}](\xi_e)$. We will now verify that (\hat{u}, \hat{v}) is a super-solution for the speed \hat{c} .

First, consider $\xi < \xi_-$. There holds

$$\begin{aligned} -\hat{c}\partial_\xi \hat{u} + \partial_{\xi\xi} \hat{u} + f(\hat{u}, \hat{v}) &\leq -\hat{c}\partial_\xi \hat{u} + \partial_{\xi\xi} \hat{u} + (\nu - a)\hat{u} \\ &\leq \delta \left(-\hat{c}\sqrt{a+\nu} + \frac{a+\nu}{1-\tilde{\nu}} - (a-\nu)(1-\tilde{\nu}) \right) \exp\left\{\frac{\sqrt{a+\nu}}{1-\tilde{\nu}}(\xi - \xi_-)\right\} \leq 0 \end{aligned}$$

since

$$-\hat{c}\sqrt{a+\nu} + \frac{a+\nu}{1-\tilde{\nu}} - (a-\nu)(1-\tilde{\nu}) < -\hat{c}\sqrt{a} + \frac{4\nu}{1-\tilde{\nu}} = 0$$

by the definition of \hat{c} .

We next consider $\xi_- < \xi < 0$, $\xi \notin E$. The profile \hat{u} satisfies

$$\partial_{\xi\xi} \hat{u} + f(\hat{u}, \hat{v}) = \partial_{\xi\xi} \hat{u} + f(\hat{u}, \phi_0(\hat{u})) = 0.$$

Thus, $\hat{c} > 0$, $\partial_\xi \hat{u} > 0$ implies

$$-\hat{c}\partial_\xi \hat{u} + \partial_{\xi\xi} \hat{u} + f(\hat{u}, \hat{v}) \leq -\hat{c}\partial_\xi \hat{u} < 0 = \partial_t \hat{u}.$$

Now consider the equation for the profile v . For all $\xi > \xi_1$ we have $\hat{v}(\xi) = 0$. Thus, the inequality $-\hat{c}\partial_\xi \hat{v} + \varepsilon^2 \partial_{\xi\xi} \hat{v} + g(\hat{u}, \hat{v}) \geq 0$ holds trivially for all $\xi > \xi_1$, $\xi \notin E$.

Recalling $\hat{c} = M\varepsilon$ we estimate for $\xi < \xi_0$, $\xi \notin E$

$$\begin{aligned} -\hat{c}\partial_\xi \hat{v} + \varepsilon^2 \partial_{\xi\xi} \hat{v} &= -\hat{c}\phi_0' \partial_\xi \hat{u} + \varepsilon^2 (\phi_0' \partial_{\xi\xi} \hat{u} + \phi_0'' (\partial_\xi \hat{u})^2) \\ &\geq \varepsilon (-\phi_0') \partial_\xi \hat{u} \left(M - \varepsilon \left(\frac{\partial_{\xi\xi} \hat{u}}{\partial_\xi \hat{u}} + \frac{\phi_0''}{\phi_0'} \partial_\xi \hat{u} \right) \right). \end{aligned}$$

The definition of ϕ_0 ensures that

$$\frac{\phi_0''}{\phi_0'}(\hat{u}) = \frac{-\Gamma''}{(\Gamma')^2}(\Gamma^{-1}(\hat{u}))$$

is bounded since $\hat{u} < u_0 < u_1$ for $\xi > \xi_0$. Furthermore, the definition of F_0 implies a uniform (in ε and ξ) bound on $\partial_\xi \hat{u}$ as well as $\partial_{\xi\xi} \hat{u}/\partial_\xi \hat{u}$. Choosing M sufficiently large, we therefore have

$$-\hat{c}\partial_\xi \hat{v} + \varepsilon^2 \partial_{\xi\xi} \hat{v} \geq \frac{1}{2}M\varepsilon(-\phi'_0(\hat{u}))\partial_\xi \hat{u} > 0.$$

Since the construction implies $g(\hat{u}(\xi), \hat{v}(\xi)) = 0$ for all $\xi \notin (\xi_0, \xi_1)$ we obtain

$$-\hat{c}\partial_\xi \hat{v} + \varepsilon^2 \partial_{\xi\xi} \hat{v} + g(\hat{u}, \hat{v}) \geq 0.$$

Finally we turn to the small interval $\xi \in (\xi_0, \xi_1)$. We compute

$$\begin{aligned} -\hat{c}\partial_\xi \hat{v} + \varepsilon^2 \partial_{\xi\xi} \hat{v} &= (-\hat{v}'(0))\varepsilon(M-1)e^{(\xi-\xi_0)/\varepsilon} \\ &= (M-1)\left(\hat{v}(0) + \varepsilon(-\hat{v}'(0)) - \hat{v}\right). \end{aligned}$$

Observe for $\xi \in (\xi_0, \xi_1)$ that $\hat{u}'' + f(\hat{u}, 0) = 0$ implies

$$\begin{aligned} g(\hat{u}, \hat{v}) &\geq g(\hat{u}(0) + \xi\hat{u}'(0), \hat{v}) = \int_{\Gamma^{-1}(\hat{u}(0) + \xi\hat{u}'(0))}^{\hat{v}} \partial_v g(\hat{u}(0) + \xi\hat{u}'(0), t) dt \\ &\geq -C_g |\Gamma^{-1}(\hat{u}(0) + \xi\hat{u}'(0)) - \hat{v}| \\ &= -C_g |\hat{v}(0) + \xi\hat{v}'(0) + o(\xi) - \hat{v}|, \end{aligned}$$

since Γ^{-1} is C^2 close to $u_0 = \hat{u}(0)$. Summarizing, we obtain

$$\begin{aligned} &-\hat{c}\partial_\xi \hat{v} + \varepsilon^2 \partial_{\xi\xi} \hat{v} + g(\hat{u}, \hat{v}) \\ &\geq (M-1)\left(\hat{v}(0) + \varepsilon(-\hat{v}'(0)) - \hat{v}\right) - C_g |\hat{v}(0) + \xi\hat{v}'(0) + o(\xi) - \hat{v}| \\ &\geq \frac{1}{2}M\left(\hat{v}(0) + \varepsilon(-\hat{v}'(0)) - \hat{v}\right) - C_g |\hat{v}'(0)(\xi + \varepsilon) + o(\xi)| \\ &\geq \frac{1}{2}M\varepsilon - C\varepsilon \geq 0, \end{aligned}$$

for M chosen sufficiently large and ε sufficiently small.

Hence we have shown that $(\hat{u}, \hat{v}, \hat{c})$ is a super-solution which provides the inequality $c \leq \hat{c} = +M\varepsilon$. □

4 Examples

4.1 Lotka-Volterra Nonlinearities

In this subsection we consider the for parameters $\alpha, \beta > 1$, and $\delta > 0$ the nonlinearities

$$\begin{aligned} f(u, v) &= u(1 - u - \alpha v), \\ g(u, v) &= \delta v(1 - v - \beta u). \end{aligned}$$

In our parameter range the system is bistable and competitive. We find $\Gamma(v) = \frac{1-v}{\beta}$, $u_1 = 1/\beta$, and

$$H(u, v) = -\frac{1}{2}u^2(1 - \alpha v - \frac{2}{3}u) - \frac{\alpha}{6\beta^2}(v - 1)^3.$$

The Lyapunov function E satisfies

$$\frac{d}{dt}E = -\int u_t^2 + \frac{\alpha\delta}{2\beta^2}v(1 - v + \beta u)(1 - v - \beta u)^2 \leq 0.$$

Furthermore, along the heteroclinic wave orbit we have

$$c\left(-\frac{1}{2}|u'|^2 + H(u, v)\right)' = -c^2|u'|^2 - \frac{\alpha\delta}{2\beta^2}v(1 - v + \beta u)(1 - v - \beta u)^2.$$

Since $A = A(u_1) = H(1, 0) = \frac{1}{6}(-1 + \frac{\alpha}{\beta^2})$, the speed criterion reads

$$\begin{aligned} c > 0 &\iff \beta^2 > \alpha, \\ c = 0 &\iff \beta^2 \leq \alpha. \end{aligned}$$

Hence, for a stationary wave, i.e. $c = 0$, we have to assume $\beta^2 \leq \alpha$. Using the Lyapunov function we can construct the monotone, but singular (in v) solution. Let us fix $v \equiv 0$ on $x > 0$ and $v = \Gamma^{-1}(u) = 1 - \beta u$ on $x < 0$. Equation (3.4) requires $v^* = 1 - \beta u^*$, $u^* = u(0)$ to be a solution of

$$1 = \alpha(u^*)^2(1 + 2v^*) = \frac{\alpha}{\beta^2}(1 - v^*)^2(1 + 2v^*).$$

Observe that $(1 - v^*)^2(1 + 2v^*)$ decreases strictly monotonically from 1 to 0 for v^* varying from 0 to 1. Hence, for $\beta^2 \leq \alpha$ there is a unique solution $v^* \in [0, 1)$, which in turn defines also u^* . We remark that $v^* \rightarrow 0$ for $\alpha/\beta^2 \rightarrow 1$ and $v^* \rightarrow 1$ for $\alpha/\beta^2 \rightarrow \infty$.

4.2 A system by Hosono and Mimura

We study the nonlinearities introduced in [5],

$$F = u(a - bu - kv), \quad G = v\left(a - bv - \frac{ku}{1 + ev}\right).$$

It is shown in [5] with singular perturbation methods that there are slow waves for $0 < \varepsilon < \varepsilon_0$ assuming (A.4) [5, p. 441], namely

$$k/b > 3 \quad \text{and} \quad e \gg 1. \tag{4.1}$$

The latter condition has a geometric interpretation. It ensures that the function $\Gamma(v)$ has a maximum which is larger than 1. Hence, the curve $(\Gamma(v), v)$ consists of two continuous branches in the region $\{(u, v) \in [0, 1]^2\}$. One branch is connected to $(0, 1)$ and monotonically increasing, the other decreases and connects to the axis $(u, 0)$. Thus, there is no continuous path in the zero level set of g connecting the states $(0, 1)$ and $(1, 0)$ and staying completely in $[0, 1]^2$. Therefore, a stationary profile (u, v) necessarily has a jump.

With the help of the Lyapunov function we can provide the precise parameter regime for stationary waves in the limit system with $\varepsilon = 0$. Thus, we considerably improve the range of applicability of the singular perturbation result which relies on the assumption $c_\varepsilon = O(\varepsilon)$.

Scaling t, x, u, v we arrive at the form

$$f = u(1 - u - \kappa v), \quad g = v \left(1 - v - \frac{\kappa u}{1 + \eta v} \right)$$

with $\kappa = \frac{k}{b} > 1$, $\eta = \frac{a}{b}e > 0$.

Now we consider the Lyapunov function. We have $\Gamma(t) = \frac{1}{\kappa}(1-t)(1+\eta t)$ showing that Γ is monotone for $\eta \leq 1$. In this case $A(u_1) = \bar{H}(1, 0)$ where

$$H(1, 0) = \frac{1}{6} \left(-1 + \frac{1 + \frac{1}{2}\eta + \frac{1}{10}\eta^2}{\kappa} \right).$$

Hence, for $\eta \leq 1$ there holds $A(u_1) \geq 0$ if and only if $k \leq \kappa_1 := 1 + \frac{1}{2}\eta + \frac{1}{10}\eta^2$.

Furthermore, there holds $u_1 = 1$ if and only if $\eta > 1$ and $\kappa \leq \kappa_3 := (\eta + 1)^2 / (4\eta)$.

If $\eta > 1$ and $\kappa > \kappa_3$ we have $u_1 = \kappa_3 / \kappa < 1$ and $v_1 = (1 - 1/\eta) / 2 > 0$. A direct calculation shows that $A(u_1) \geq 0$ if and only if

$$\kappa \leq \kappa_2 := \frac{23\eta^5 + 85\eta^4 + 110\eta^3 + 50\eta^2 - 5\eta - 7}{160\eta^3}.$$

We remark that $\kappa_1 = \kappa_2 = 8/5$ for $\eta = 1$.

Since $\kappa_3 < \kappa_2$ for all $\eta > 1$ we obtain that there holds $A(u_1) \geq 0$ if and only if

$$\kappa \leq \kappa_0 = \begin{cases} \kappa_1, & \eta \leq 1, \\ \kappa_2, & \eta > 1. \end{cases}$$

Thus, for any given $\eta > 0$, depending on the value of $\kappa > 1$, propagating and standing waves are possible for the limit system (TW).

Theorem 3.2 yields: *For all $\eta > 0$ there exist $\kappa_0 = \kappa_0(\eta) > 1$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \leq \kappa_0$ the unique traveling wave solutions $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ connecting $(0, 1)$ with $(1, 0)$ satisfies*

$$|c_\varepsilon| \leq 2\varepsilon. \quad (4.2)$$

Note, that $\kappa_0 \rightarrow 1$ as $\eta_0 \rightarrow 0$. Thus, standing waves still exist for small values of $e = \frac{b}{a}\eta$, showing that (4.1) is not optimal. Also for large values of η the critical κ_0 is not linear in η , as suggested by the geometric property of Γ leaving the unit square $[0, 1]$, cf. κ_3 and the condition $(ae+b)^2/(4bke) > a/b$ following lemma 3.1, [5, p. 437]. In fact, it is quadratic in η which is seen from the formula for κ_2 , given the weaker condition $e^2/k > \text{const.}$ We remark that condition (4.2) is precisely assumption (A.3) on [5, p. 437].

Having verified the assumptions for the singular perturbation approach we can use their result on the expansion of the speed. Thus, using (4.2), the speed actually satisfies

$$\lim_{\varepsilon \rightarrow 0} \frac{c_\varepsilon}{\varepsilon} = -c^* < 0.$$

5 Large sets of equilibria

Due to the absence of diffusion in the equation for v we can characterize stationary solutions of (1.1) by the scalar equation

$$\begin{aligned} 0 &= \partial_{xx}u + f(u, v), \\ v(x) &= \psi(x)\Gamma^{-1}(u(x)), \end{aligned}$$

where the function $\psi : \mathbb{R} \rightarrow \{0, 1\}$ decides whether g vanishes at the point $x \in \mathbb{R}$ due to $v = 0$ or due to $u = \Gamma(v)$. In order to match the asymptotic boundary data we impose $u(-\infty) = 0$, $u(+\infty) = 1$ and $\psi(-\infty) = 1$, $\psi(+\infty) = 0$. We refer to Figure 2 for a sketch of the qualitative behavior of a corresponding solution. Recall that the equation above implies an a priori bound in $C^{1,1}(\mathbb{R})$ for the solution u . For simplicity, we assume that Γ is monotone.

Theorem 3.1 implies that in the case $A = A(u_1) \geq 0$ there exists a unique weak monotone stationary solution (u, v) of (1.1) with $v(x) = \chi_{\mathbb{R}_-}(x)\Gamma^{-1}(u(x))$, that is, a solution u of

$$0 = u'' + f(u, \chi_{\mathbb{R}_-}\Gamma^{-1}(u)), \quad (5.1)$$

with $u(x) \rightarrow 0$ for $x \rightarrow -\infty$ and $u(x) \rightarrow 1$ for $x \rightarrow +\infty$.

We want to show that in the neighborhood of the monotone front we can find many non-monotone stationary solutions. This shows that the monotone front is not asymptotically stable in any L^p -norm, $p < \infty$.

Lemma 5.1 (Many non-monotone stationary solutions). *Assume $A > 0$. There exists $\delta_0 > 0$ such that for all $0 < \delta_1 < \delta_2 < \delta_0$ there exists a stationary solution of (1.1) with $v(x) = \chi(x)\Gamma^{-1}(u)$, that is, a solution u of*

$$u'' + f(u, \chi\Gamma^{-1}(u)) = 0, \quad (5.2)$$

where χ is the characteristic function of the set $(-\infty, 0) \cup (\delta_1, \delta_2)$.

Proof. We will construct a function

$$G : \mathbb{R}^5 \supset B_\rho(P) \rightarrow \mathbb{R}^3, \quad G(\delta_1, \delta_2, \bar{u}_0, \bar{u}_1, \bar{u}_2) \in \mathbb{R}^3,$$

where $P = (0, 0, u^*, u^*, u^*)$. At P there holds $G(P) = 0$. Our aim is to use the implicit function theorem such that $G(\delta_1, \delta_2, \bar{u}_0, \bar{u}_1, \bar{u}_2) = 0$ implies that equation (5.2) has a solution u for small δ_1, δ_2 .

We consider the two ($i = 0, 1$) autonomous equations

$$\begin{aligned} u'' + f_i(u) &= 0, & \text{for} \\ f_0(u) &:= f(u, 0), \quad f_1(u) := f(u, \Gamma^{-1}(u)). \end{aligned}$$

The two equations define two flow maps, $\Phi_x^i : (u(0), u'(0)) \mapsto ((\Phi_x^i)_1, (\Phi_x^i)_2) := (u(x), u'(x))$. Given $(\delta, \bar{u}) = (\delta_1, \delta_2, \bar{u}_0, \bar{u}_1, \bar{u}_2)$ we set

$$p := \left(-2 \int_0^{\bar{u}_0} f_1(s) ds \right)^{1/2},$$

which we expect to be the derivative of u in $x = 0$. We furthermore set

$$\begin{aligned} G_1(\delta, \bar{u}) &:= (\Phi_{\delta_1}^0)_1(\bar{u}_0, p) - \bar{u}_1, \\ G_2(\delta, \bar{u}) &:= (\Phi_{\delta_2 - \delta_1}^1)_1 \circ \Phi_{\delta_1}^0(\bar{u}_0, p) - \bar{u}_2, \\ G_3(\delta, \bar{u}) &:= \frac{1}{2} [(\Phi_{\delta_2 - \delta_1}^1)_2 \circ \Phi_{\delta_1}^0(\bar{u}_0, p)]^2 - \int_{(\Phi_{\delta_2 - \delta_1}^1)_1 \circ \Phi_{\delta_1}^0(\bar{u}_0, p)}^1 f_0(s) ds. \end{aligned}$$

By definition, $G(P) = 0$. If we find a point (δ, \bar{u}) with $G(\delta, \bar{u}) = 0$, we can construct a solution to (5.2) by gluing together the flows above. It remains to calculate the derivatives of G in the point P .

$$\begin{aligned} \partial_{\bar{u}} G_1(P) \cdot \langle \bar{w}_0, \bar{w}_1, \bar{w}_2 \rangle &= \bar{w}_0 - \bar{w}_1, \\ \partial_{\bar{u}} G_2(P) \cdot \langle \bar{w}_0, \bar{w}_1, \bar{w}_2 \rangle &= \bar{w}_0 - \bar{w}_2, \\ \partial_{\bar{u}} G_3(P) \cdot \langle \bar{w}_0, \bar{w}_1, \bar{w}_2 \rangle &= p \cdot \partial_{\bar{u}_0} p \cdot \bar{w}_0 + f_0(u^*) \cdot \bar{w}_0 \\ &= \partial_{\bar{u}_0} \frac{1}{2} |p|^2 + f_0(u^*) \cdot \bar{w}_0 \\ &= (-f_1(u^*) + f_0(u^*)) \cdot \bar{w}_0. \end{aligned}$$

By $f_1(u^*) < f_0(u^*)$ the 3×3 -matrix $\partial_{\bar{u}} G(P)$ is invertible. The implicit function theorem can be applied and provides the solutions. \square

Note that with the same proof one can construct solutions for any finite number of interfaces.

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