Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

On the path of a quasistatic crack in Mode III

(revised version: June 2004)

by

Gerardo Oleaga

Preprint no.: 30 2004

ON THE PATH OF A QUASISTATIC CRACK IN MODE III

GERARDO E. OLEAGA

Abstract. We present an approach to study the path of a crack growing in a quasistatic regime in a brittle body. The propagation process is modelled by a sequence of discrete steps optimizing the elastic energy released. A detailed study of the Mode III case is presented. We obtain an explicit relationship between the optimal growing direction and the parameters defining the local elastic field around the tip. This allows to describe a simple algorithm to compute the crack configuration. A comparison with other models proposed for the same problem is provided as well.

1. Introduction

One of the basic problems in the field of Fracture Mechanics consists in the prediction of crack paths once the loading and initial configuration are given. In this article we study the evolution of a crack growing in a *quasistatic regime* under the linear elastic theory. By "quasistatic" we mean, roughly speaking, that the body-crack system is in a state of critical equilibrium. In terms of Griffith's classical approach (cf. [11]), this is achieved by the balance between the amount of mechanical energy that the body is "able" to release and the *surface energy* needed for crack advance. Our main result is an explicit formula linking the preferred direction of propagation in terms of some parameters defining the local field around the tip (cf. (4.17)). This relationship allowed us to describe a simple algorithm to compute the crack path in a Mode III field (cf. Algorithm 1). Before stating our main approach we will recall briefly the classical setting for the continuum theory of brittle fracture, to point out some of the difficulties that we must face.

To fix ideas, consider a two dimensional open set Ω containing an initially straight crack, with one of its tips in its interior (see Figure 1). When some loading is applied over $\partial\Omega$, elastic energy is stored in the body. Griffith reasoned essentially as follows: if we *virtually* extend the given crack at one of its tips *in the same straight line* while keeping the loading fixed, then we can compute the different values of the stored mechanical energy $E(l)$ (elastic + potential of applied forces) in terms of the crack extension length l. To produce this virtual extension we would have to release an amount of energy equivalent (or bigger) to the *surface energy* of the additional crack. This condition can be stated as:

$$-\Delta E \geq \kappa l, \tag{1.1}$$

where κ is a constant of the material measuring the surface energy per unit length and $\Delta E := E(l) - E(0)$ is the mechanical energy variation for the given loading. Inequality (1.1) should be valid for all small extensions $0 < \tilde{l} < l$ if we assume that the crack grows continously from a vanishingly small length. Then we must have

Date: May 13, 2004.
2000 *Mathematics Subject Classification.* 74R05, 74B05, 74G70.
Key words and phrases. Crack propagation, discrete model, singular fields, linear elasticity.
Supported by EU-Project "Front Singularities" University of Leipzig and the Max Planck Institute MIS. Partial support was also provided by the Spanish DGES project BFM2000-0605.

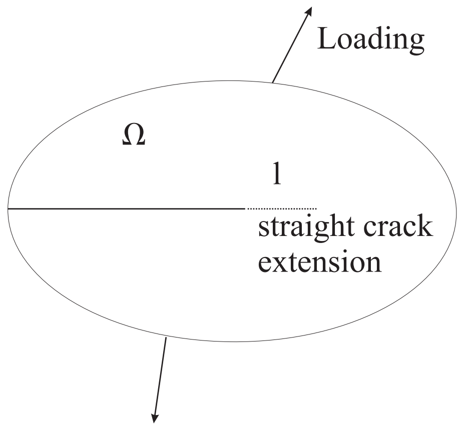

FIGURE 1. Griffith's approach

that:

$$G := \lim_{l \to 0^+} -\frac{\Delta E}{l} \geq \kappa, \tag{1.2}$$

to ensure that the crack propagation is possible. In this case we can say that there is enough stored energy available to "pay" for the growing crack. The quantity G is usually referred to as the *Energy Release Rate.* If G happens to be strictly greater that κ, there is an excess in the stored energy and other phenomena such as wave propagation, rate dependent dissipation could take place (cf. [8]). On the other hand, if $G = \kappa$, the crack is able to advance along the given path in a state of critical equilibrium, which we term as quasistatic propagation. Notice that one is tempted to write this condition as follows:

$$\frac{d}{dl}\left(E\left(l\right) + \kappa l\right) = 0.$$

This relationship erroneously suggests that the critical configuration of the crack is a *global minimizer* of the total energy $E + \kappa l$ (mechanic+surface energy). It is to be emphasized that this is far from the spirit of (1.2), which gives us a way to analyze the stability of the crack configuration, but no clues about its global shape.

In Linear Elastic Fracture Mechanics, the condition for critical growth is quantified by means of the so-called *stress intensity factors* (cf. [12]). These are key parameters that play a crucial role, giving the local strength of the field around the tip. Irwin [12] resolved the local field into three distinct two dimensional fields or "modes". The components of the near tip stress with reference to a rectangular coordinate system are expressed for each of these mode contributions as:

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} \Sigma_{ij}\left(\theta\right) + \sigma_{ij}^{(1)} + o\left(1\right) \quad r \to 0,$$

where r, θ are polar coordinates, $\theta = 0$ being the tangent direction to the curve defining the crack and $K = K_{\mathrm{I}}, K_{\mathrm{II}}$ or K_{III}, depending on the mode considered. The factor K reflects the influence of the geometrical configuration of the body and the details of the loading for each mode. Irwin showed that G is a *local quantity,* related to the stress intensity factors through the celebrated relationship:

$$G = \frac{1-\nu^2}{Y}\left(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2\right) + \frac{1}{2\mu}K_{\mathrm{III}}^2, \tag{1.3}$$

where Y denotes Young's elastic modulus, ν the Poisson ratio and μ is one of Lamé constants (elastic shear modulus). This important formula, combined with the critical growth condition (Griffith's criterion), gives one scalar equation for the crack propagation free boundary problem:

$$\frac{1-\nu^2}{E}\left(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2\right) + \frac{1}{2\mu}K_{\mathrm{III}}^2 = \kappa.$$

The meaning of this equality can be stated as follows: for a given configuration we measure the stress intensity factors and we compute G, if this value happens to be lower than κ, the crack cannot advance and the configuration is in a state of stable equilibrium. The crack remains until a change in the loading conditions increases G up to the critical value. This provides us with only one scalar relationship for crack motion. We need in principle two such equations to find the evolution of the curve.

Several criteria were introduced in order to provide the remaining equation. One of the most popular is the so called *symmetry principle* (cf. [10]) which states that along the path the following relationship should be satisfied:

$$K_{\mathrm{II}} = 0. \tag{1.4}$$

As stated by Cotterell and Rice in [6], this equation is in some sense a necessary condition if the crack path is assumed to be smooth. These authors studied the implications of (1.4) for paths slightly deviating from straightness, and also for kinks, when it is applied to the tip far from the corner. In a more general context, this criterium was also considered by Leblond [13], and Amestoy & Leblond [1] while studying the in-plane displacement modes. Some mathematical aspects of (1.4) are also addressed in the work of Friedman, Hu and Velázquez (see [9]), where the dependence of the stress intensity factors on crack path is analyzed in a rigorous setting. They obtained a system of ordinary differential equations for the evolution of K_{I} and K_{II} assuming a smooth class of crack extensions of parabolic type (without kinking).

The main drawbacks of relationship (1.4) could be summarized as follows:

- It is physically justified only for smooths paths. While studying a free boundary problem it is a big disadvantage to restrict the admissible solution curves from the outset.
- When the initial loading and crack configuration are such that $K_{\mathrm{II}} \neq 0$, it is not evident that the tip will jump in such a way that (1.4) will hold for the "new" tip around a smooth piece of the path. Moreover, other principles such as the "maximum energy release rate" need not necessarily coincide with $K_{\mathrm{II}} = 0$ at the incipient tip (cf. [6]).
- This principle cannot be applied in a pure Mode III setting, and it is not evident how to generalize its formulation.

A different approach was considered by some authors, valid for all modes of propagation, which avoids all the disadvantages mentioned (see [4], [7]). They considered the fracture process as a discrete evolution of fairly general sets. At each step these sets are found by minimizing the total energy of the crack-body system. The admissible competitors are constrained to contain the path defined in the previous step of the process. One disadvantage of this approach is that allows, in the set of admissible curves, some paths that may not be compatible with Griffith's condition (1.2). "Big cracks" may appear in a single step, without taking into account the details of the growing process. Some attempts to deal with the physical limitations of this model are presented by M. Buliga in [5]. To illustrate the situation we consider the following Figure. In this one-dimensional setting, we

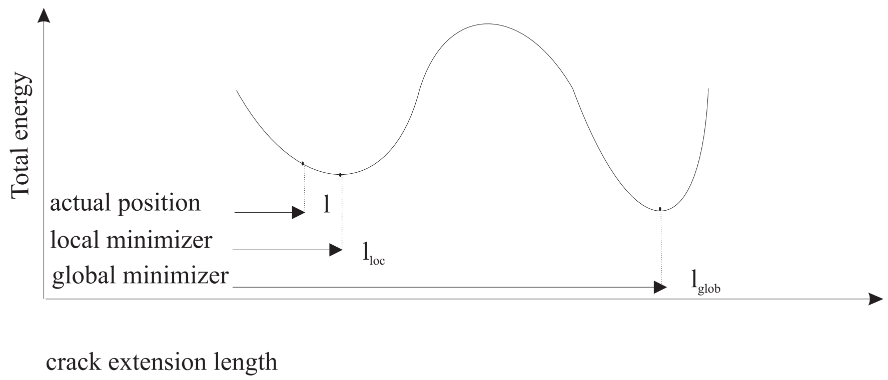

FIGURE 2. Global minimization

realize that when the crack grows from position l to position l_{glob}, it should cross an interval where the derivative of the total energy is positive. This means that there $\frac{dE}{dl} + \kappa > 0$ holds, and then $G < \kappa$. There would not be enough "driving force" to push the tip in that interval. On the other hand, even if the crack goes from l to l_{loc}, without breaking any basic physical law, one should take care about how strong is the local field. If $G - \kappa$ turns to be very large then the propagation process could be no more quasistatic, and other inertial or dissipative effects should be included in the model.

We begin our approach to the problem by revising the concept of Energy-Release-Rate in Section 2. We describe a natural way to find the "optimal" growing direction using the spirit of Griffith's ideas. To be fair, this can be considered as a discrete version of the so-called *maximum energy release rate* criterium. The setting is applicable to all modes of propagation. In Section 3 we describe some well-known facts about Mode III fields, as well as other relationships that are useful for later purposes. In Section 4 we describe the discrete evolution of the path using the previous results, providing an algorithm for propagation. As we will see, two terms in the asymptotic expansion of the energy released with respect to crack length are needed in Mode III, while it is known that only the first one (ie., the "generalized" energy release rate) is needed for Modes I and II. The techniques applied are classical, including complex representation of the fields, Schwarz-Christoffel conformal map (cf. [14]) and elementary application of perturbation methods for algebraic equations (cf. [2]). We conclude in Section 5 with some open questions. We give the basic derivation of the conformal transforms for kinked slits in the Appendix.

2. Our basic approach

2.1. **The Energy-Release-Rate concept revised.** We begin by studying the quantity G defined in (1.2) in more detail. It is known that its value depends on the overall geometry of the body-crack and the loading conditions. But it is sometimes bypassed that it depends *on the trial paths* as well. In other words, instead of using a straight virtual extension as above, we can add at the tip an arbitrary piece of curve γ with a well defined arc-length parameter l. We can compute now G following the new path and study the stability of the configuration as well. We illustrate this in the following Figure.

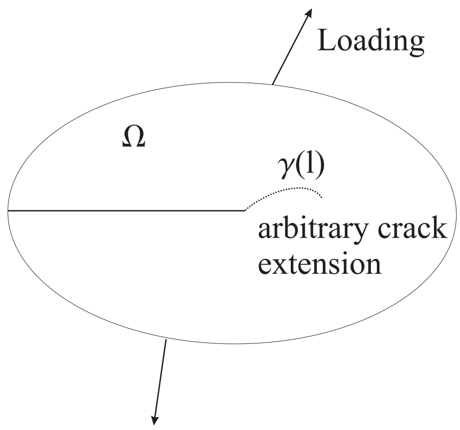

Figure 3. Energy released by an arbitrary virtual extension.

We consider now the following quantity:

$$G(\gamma) := -\lim_{l\to 0^+} \frac{\Delta E(l;\gamma)}{l}, \tag{2.1}$$

where $E(l;\gamma)$ defines the value of mechanical energy when the initial crack grows along the path γ an extension l. We omit explicit reference to the loading and initial configuration in the notation.

Now the following question arises. If the value of $G(\gamma)$ depends on the trial extension, we may have some "unstable" paths for which G is greater than κ, and some others that cannot be followed just because the energy to be released is less than the amount needed. A more precise statement should say that the configuration is in equilibrium if

$$G(\gamma) \leq \kappa \tag{2.2}$$

for all admissible extensions γ *of the crack*. A very simple consequence of this formulation is the following: If all the paths γ satisfying $G(\gamma) = \kappa$, are tangent to the *same* direction, then this will determine the unique way that the crack can follow inside this family. The critical path (or paths) γ^* must therefore satisfy the following additional condition:

$$G(\gamma^*) = \max_{\gamma} G(\gamma) \tag{2.3}$$

This simple argument allows for studying the crack growth direction *without further physical assumptions* or any other ad-hoc propagation law. In a different context, (2.3) is referred to as the *maximum energy release rate* condition.

This approach can be applied to the in-plane modes as well. The kinking angle could be in principle computed by applying condition (2.3). For smooth paths, some problems regarding crack shape were considered recently by M. Brokate and A. Khludnev in [3]. They studied the sensitivity of G with respect to the subsequent crack path. As we will see later, the knowledge of this functional is not enough to determine the prefered shape in the case of pure Mode III. Roughly speaking, *any smooth path* is compatible with (2.3) and we need to include one more term in the expansion for the energy, proportional to $l^{3/2}$, l being the length of the extended path.

2.2. **A discrete approach.** We model the process of crack advance by means of a sequence of single steps. Each step will add a path of small length l. Given an initial configuration we consider the quantity:

$$-\Delta E\left(l;\gamma\right)=-\left(E\left(l;\gamma\right)-E\left(0\right)\right).$$

Our aim is to estimate the optimal γ^* satisfying:

$$-\Delta E\left(l;\gamma^*\right)=\max_{\gamma}\left(-\Delta E\left(l;\gamma\right)\right),$$

for $l>0$ small and fixed.

A simple way to quantify this point of view is to consider an indexed family of paths covering a wide variety of behaviors. In our setting, we take the *kinked* configurations with kinking angle φ, $|\varphi|\leq\pi$, indexed by the parameter $\alpha:=\varphi/\pi$ (cf. Figure 4). One obvious reason for doing this is the relative simplicity of these paths which allow for explicit computations of the (asymptotically) preferred angles of propagation for $l\to0^+$. On the other hand, the small segments should approximate any rectifiable curve.

As we will see later, $-\Delta E\left(l;\alpha\right)$ (the parameter α is used now to indicate the trial path) can be expanded in fractional powers of l:

$$-\Delta E\left(l;\alpha\right)=G\left(\alpha\right)l+H\left(\alpha\right)l^{3/2}+O\left(l^2\right),$$

for a suitable function $H\left(\alpha\right)$. Our main objective is to estimate the optimal angle $\alpha\left(l\right)$ for $l\to0^+$, and in this way to establish a discrete crack propagation law in terms of the local parameters defining the elastic field. In the following we give a derivation of these results.

3. General facts about Mode III fields.

3.1. **The boundary value problem.** Consider a three dimensional body given by $\Omega\times\mathbb{R}:=\left\{(x_1,x_2,x_3):(x_1,x_2)\in\Omega,\quad x_3\in\mathbb{R}\right\}$, where Ω is a domain in $\mathbb{R}^2$ with piecewise smooth boundary. This setting is typical for Mode III, where the invariance of the set with respect to x_3 allows for some simplifications of the problem at hand. We recall the equilibrium equations of elasticity in the absence of body forces:

$$\sigma_{ij,j}=0,\quad 1\leq i,j\leq3, \tag{3.1}$$

where σ is the stress tensor and the summation convention is assumed. The tensor σ is linked to the *strain* tensor ε by the constitutive equations:

$$\sigma_{ij} = \lambda\delta_{ij}\mathrm{tr}\,(\varepsilon) + 2\mu\varepsilon_{ij}, \tag{3.2}$$

where tr means the trace and λ, μ are the so called Lamé coefficients. For small deformations, ε is given in terms of the displacement gradient as follows:

$$\varepsilon_{ij} := \frac{1}{2}\left(u_{i,j} + u_{j,i}\right).$$

The boundary conditions are given by:

$$u_i\left(x_1, x_2, x_3\right) = D_i\left(x_1, x_2\right), \quad \left(x_1, x_2\right) \in \partial_D\Omega,$$

in terms of the displacements, and by:

$$\sigma_{ij}n_j\left(x_1, x_2, x_3\right) = T_i\left(x_1, x_2\right), \quad \left(x_1, x_2\right) \in \partial_T\Omega,$$

for the normal stresses. Notice that $\partial_D\Omega \cup \partial_T\Omega = \partial\Omega$ and n is a unit normal to $\partial\Omega$.The fields D and T represent the prescribed displacements and tractions at the boundary. For pure Mode III they further satisfy:

$$D_i \equiv 0,\, T_i \equiv 0, \quad i = 1, 2.$$

Taking into account that the boundary conditions are independent of x_3, and due to the symmetry of the domain we can assume (by uniqueness) that the fields involved are independent of x_3 too. Then we can write:

$$\begin{aligned} \sigma_{i3} &\equiv \sigma_{3i} = \mu u_{3,i} \quad \text{for } i = 1, 2, \\ \sigma_{33} &= \lambda\left(u_{1,1} + u_{2,2}\right). \end{aligned} \tag{3.3}$$

The equilibrium equation for $i = 3$ in (3.1) is given by:

$$\sigma_{31,1} + \sigma_{32,2} = 0. \tag{3.4}$$

Using the constitutive relationship (3.2) we have that:

$$\mu\left(u_{3,11} + u_{3,22}\right) = 0, \tag{3.5}$$

stating that u_3 is a harmonic function in Ω. The condition for the *out of plane* component of the force is given by (notice that $n_3 \equiv 0$):

$$\sigma_{31}n_1 + \sigma_{32}n_2 = T_3, \text{ on } \partial_T\Omega,$$

and taking into account equation (3.3) we have:

$$u_{3,1}n_1 + u_{3,2}n_2 \equiv \frac{\partial u_3}{\partial n} = \frac{1}{\mu}T_3 \text{ on } \partial_T\Omega.$$

The field u_3 is then *uncoupled* from the other components.

Notation: From now on we will denote $u\left(x_1, x_2\right) \equiv u_3\left(x_1, x_2, x_3\right)$ for the out of plane displacement field. Moreover, we will drop the index 3 in the components of the stresses and boundary data, ie. we will write T for T_3 and D for D_3 (we recall that the other components of the boundary data are zero) and σ_j for σ_{3j}.

3.2. **The moving boundary value problem.** Let us assume that $(0,0) \in \Omega$. Consider a piecewise smooth Jordan curve Γ_0 in Ω as initial crack configuration such that one of its ends lies in $\partial\Omega$ and the other one is located at the origin. We impose now zero normal stress on the surface $\Gamma_0 \times \mathbb{R}$:

$$\sigma_1 n_1 + \sigma_2 n_2 = 0, \text{ on } \Gamma_0,$$

which in turn means (cf.(3.3)):

$$\mu u_{,1} n_1 + \mu u_{,2} n_2 \equiv \mu \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_0, \tag{3.6}$$

We then include the Neumann homogeneous type boundary condition for a growing crack Γ defining in this way a *moving boundary problem* for the out of plane field u $(x = (x_1, x_2))$:

$$u_{,11} + u_{,22} = 0, \qquad x \in \Omega \setminus \Gamma, \tag{3.7}$$

$$u(x) = D(x), \quad x \in \partial_D \Omega, \tag{3.8}$$

$$\frac{\partial u}{\partial n}(x) = T(x), \quad x \in \partial_T \Omega, \tag{3.9}$$

$$\frac{\partial u}{\partial n}(x) = 0 \qquad x \in \Gamma, \tag{3.10}$$

3.3. **Energy variation for a crack extension.** The main ingredient of our approach is an expression for the energy variations for different crack configurations. It is then necessary to consider solutions $u \in H^1(\Omega \setminus \Gamma)$, or in other words, with integrable energy.

Consider an extension of the curve Γ_0 given by a piecewise smooth curve Γ_l indexed by arc extension length, such that:

$$\Gamma_{l_1} \subset \Gamma_{l_2} \quad 0 \le l_1 \le l_2.$$

The mechanical energy functional per unit height for each $l \ge 0$ is given by:

$$E(l) := \frac{\mu}{2} \int_{\Omega_l} |\nabla u_l|^2 - \int_{\partial_T \Omega} T\, u_l, \tag{3.11}$$

where $\Omega_l := \Omega \setminus \Gamma_l$. We define the incremental quantities:

$$\begin{aligned} \Delta E &:= E(l) - E(0)\ , \\ v(x) &:= u_l(x) - u_0(x) \quad x \in \Omega_l \subset \Omega_0 \\ \Delta\Gamma &:= \Gamma_l \setminus \Gamma_0. \end{aligned}$$

Notice that $v = 0$ on $\partial_D\Omega$ and $\frac{\partial v}{\partial n} = 0$ on $\partial_T \Omega$. We have that:

$$\begin{aligned} \Delta E &= \mu \int_{\Omega_l} \nabla u_0 \cdot \nabla v + \frac{\mu}{2} \int_{\Omega_l} \nabla v \cdot \nabla v - \int_{\partial_T \Omega} T\, v \\ &= \frac{\mu}{2} \int_{\Delta\Gamma} \left(v^+ \frac{\partial u_0}{\partial n^+} + v^- \frac{\partial u_0}{\partial n^-} \right) , \end{aligned}$$

where we used the identities (3.7)-(3.10) and the fact that $\int_{\Delta\Gamma^\pm} v \frac{\partial u_l}{\partial n} = 0$. On each side of the slit $\Delta\Gamma$ we have that $\vec{n}^+ = -\vec{n}^-$, and then we can write:

$$\Delta E = \frac{\mu}{2} \int_{\Delta\Gamma} [v] \frac{\partial u_0}{\partial n^+} ,$$

where the integration is taken with respect to the arc-length parameter and $[v] := v^+ - v^-$ is the jump of the field v across the curve $\Delta\Gamma$. Recalling the fact that u_0 is continuous along $\Delta\Gamma$ we have $[v] = [u_l]$, and then we write the final expression for the mechanical energy increment in pure Mode III as follows:

$$\Delta E = \frac{\mu}{2} \int_{\Delta\Gamma} [u_l] \frac{\partial u_0}{\partial n^+}. \tag{3.12}$$

Notice that (3.12) is valid for finite extensions and for arbitrary shapes of $\Delta\Gamma$.

3.4. **The asymptotic field.** We study now the local field around the crack tip. This is acomplished by neglecting the effects of the curvature of the pre-existing crack and the particular geometry of the body, assuming that the tip is far from the boundary. For this purpose, let us take:

$$\Omega \equiv \mathbb{R}^2, \tag{3.13}$$

$$\Gamma_0 \equiv \{(x_1, 0) : x_1 \leq 0\}. \tag{3.14}$$

Given a configuration of $\Gamma_l \supset \Gamma_0$ we consider a field u_l satisfying the equilibrium equations:

$$u_{l,11} + u_{l,22} = 0, \qquad x \in \mathbb{R}^2 \setminus \Gamma_l, \tag{3.15}$$

$$\frac{\partial u_l}{\partial n}(x) = 0, \qquad x \in \Gamma_l. \tag{3.16}$$

Additionally, we require that u_l belongs to the class:

$$\mathcal{C}_l := \cap_{r>0} H^1(B_r(0) \setminus \Gamma_l), \tag{3.17}$$

where $B_r(0)$ is the disc of radius r, centered at the origin. This condition guarantees the existence of ΔE in (3.12). Nevertheless, the field u_l is still not well defined (for instance, any added constant satisfies the same conditions). Additionally, we must impose the matching condition with the initial equilibrium field:

$$\lim_{x\to\infty} |u_l(x) - u_0(x)| = 0, \tag{3.18}$$

uniformly in x, where u_0 is a given scalar field satisfying the equilibrium equation (3.15), the boundary condition (3.16) for $l = 0$, and belonging to the class $\mathcal{C}_0$ defined in (3.17). The uniqueness of the field u_l can be established easily if we transform the set $\mathbb{R}^2 \setminus \Gamma_l$ into the upper half plane by conformal mapping. Considering now two solutions u_l^1 and u_l^2, we have that the transformed difference $\Delta u = u_l^1 - u_l^2$ is harmonic in the upper half plane and satisfies Neumann homogeneous conditions on the real line. By Schwarz reflection we obtain a harmonic function in the whole plane with uniform limit 0 at infinity. This function is bounded, since it belongs to the class $\mathcal{C}_l$. The uniqueness then follows from Liouville theorem. Later on, we will give explicit conformal maps for the relevant crack extensions. Notice that we have the freedom to select u_0 inside a wide class of functions describing the local field. Once u_0 and Γ_l are selected, the whole family u_l is uniquely defined.

The basic crack configurations that we will use are given by *kinked paths* (cf. Figure 4 below). Given an angle φ, with $-\pi \leq \varphi \leq \pi$, and given $l \geq 0$ we take:

$$\Gamma_l := \Gamma_0 \cup \{(x_1, x_2) : x_1 = r\cos\varphi,\, x_2 = r\sin\varphi,\, 0 \leq r \leq l\}. \tag{3.19}$$

As above, we denote by $\Delta\Gamma$ the segment defining the extension $\Gamma_l \setminus \Gamma_0 = \{(x_1, x_2) : x_1 = r\cos\varphi,\, x_2 = r\sin\varphi,\, 0 \leq r \leq l\}$.

3.5. **Complex representation and expansion of the field around the tip.** **Notation:** We denote by $\mathbb{C}$ the set of complex numbers and $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ the extended complex plane. $\mathbb{H}$ is the upper half plane including the real line and $\mathbb{H}^+, \mathbb{H}^-$ are the $z \in \mathbb{C}$ such that $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ respectively.

Consider now equations (3.3) and (3.4) for the stress components:

$$\sigma_{1,1} = -\sigma_{2,2}, \tag{3.20}$$

$$\sigma_{1,2} = \sigma_{2,1}. \tag{3.21}$$

These are the Cauchy-Riemann equations for the scalar functions σ_1 and $-\sigma_2$.There exists a complex function η, analytic in $\Omega \setminus \Gamma$, such that:

$$\mu\eta'(\zeta) = \sigma_1 - i\sigma_2, \quad \zeta = x_1 + ix_2, \tag{3.22}$$

and moreover:

$$\operatorname{Re} \eta(\zeta) = u. \tag{3.23}$$

For $\Gamma = \Gamma_0$ $(\Omega \equiv \mathbb{C})$ we can apply the conformal map

$$\zeta = f_0(z) := -z^2, \tag{3.24}$$

which sends $z \in \mathbb{H}^+$ into $\mathbb{C} \setminus \Gamma_0$. We denote by h_0 the composition of η_0 and f_0, to emphasize the change of domain:

$$h_0(z) := \eta_0\left(-z^2\right), \quad \operatorname{Im} z > 0. \tag{3.25}$$

Now the boundary conditions take the form:

$$\operatorname{Im} h_0'(z) = 0, \quad \text{for } \operatorname{Im}(z) = 0.$$

Applying Schwarz's symmetry principle (cf. [14]) we can extend analytically h_0' to the lower half plane by means of:

$$h_0'(z) = \overline{h_0'(\bar{z})}. \tag{3.26}$$

Notice that (3.17) rules out any singularity around the origin. Therefore, h_0 admits an expansion with real coefficients:

$$h_0(z) = \sum_{n=0} c_n z^n, \quad c_n \in \mathbb{R},\, z \in \mathbb{C} \tag{3.27}$$

and the following expansion is valid in the original domain:

$$\eta_0(\zeta) = \sum_{n=0} c_n (-\zeta)^{n/2}, \quad c_n \in \mathbb{R},\, \zeta \in \mathbb{C} \setminus \Gamma_0. \tag{3.28}$$

Taking real part and then polar coordinates this expansion turns to be:

$$u_0 = \operatorname{Re}\left(\sum_{n=0} c_n \left(\sqrt{-\zeta}\right)^n\right) = c_0 - c_1 r^{1/2} \sin(\theta/2) - c_2 r \cos(\theta) + \ldots \tag{3.29}$$

The coefficient for $r^{1/2}\sin(\theta/2)$ is (up to a multiplicative constant) the so-called *stress intensity factor* for Mode III loading:

$$c_1 = -\frac{1}{\mu}\sqrt{\frac{2}{\pi}} K_{\mathrm{III}}\,.$$

This term contains the leading behavior of the field as $r \to 0$.

The asymptotic relationship (3.29) is the typical expression for the Mode III field around the tip (cf. Freund [8]) of a smooth crack. It is usually obtained by dimensional arguments and asymptotic analysis. In our case, the basic crack shapes considered are the ones defined in (3.19), where the length scale is given by the kink extension l. If we re-scale the domain with respect to this parameter, the shape of the kink will remain in the limit. Therefore, the expansion (3.29) is not accurate near the corner of the path (at distance l from the tip). In order to compute the energy increments (cf.(3.12)) we need a precise value of the field along the whole segment $\Delta\Gamma$. By using the same argument as above, we introduce a *new expansion* using a family of conformal maps, indexed by the parameter l (see Appendix A):

$$f_l(z) := -(z - a(l))^{1-\alpha} (z - b(l))^{1+\alpha}, \tag{3.30}$$

where $\alpha := \varphi/\pi$ ($-1 \le \alpha \le 1$) and:

$$a(l) := -\sqrt{l}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}}, \quad b(l) := \sqrt{l}\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{2}}, \tag{3.31}$$

The function given in (3.30) sends the upper half plane $\mathbb{H}^+$ into the set $\mathbb{R}^2\backslash\Gamma_l$, where the curves Γ_l are defined in (3.19). Moreover $f_l(\mathbb{R}) = \Gamma_l$ and $f_l([a(l), b(l)]) = \Delta\Gamma$. See Figure 4 below.

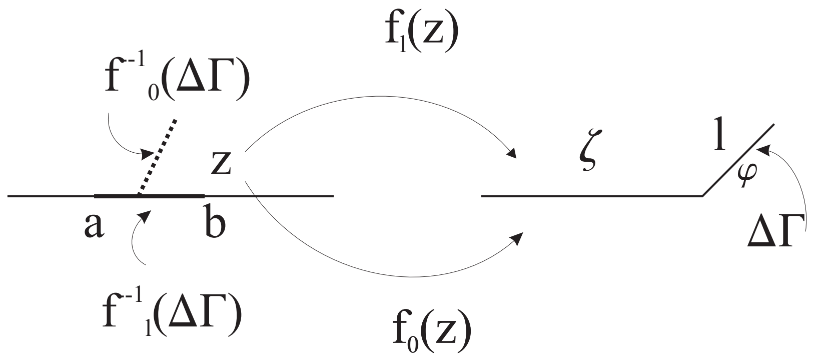

FIGURE 4. The mapping properties of f_l and f_0.

Under the same arguments that lead to the expansion (3.27), we obtain for $\Gamma = \Gamma_l$ that the solution u_l is the real part of some complex function η_l. Moreover, writing:

$$\zeta = f_l(z),$$

we define:

$$h_l(z) := \eta_l(f_l(z)), \quad \mathrm{Im}(z) > 0. \tag{3.32}$$

Elementary properties of conformal mapping combined with the Neumann homogeneous conditions satisfied by $\mathrm{Re}\,\eta_l(\zeta)$, give:

$$\mathrm{Im}\, h_l'(z) = 0, \quad \text{for } \mathrm{Im}(z) = 0.$$

By the symmetry principle the extended function

$$h_l'(z) = \overline{h_l'(\bar{z})}, \quad z \in \mathrm{Im}\, z < 0,$$

is analytic in the whole plane. Therefore, h_l admits an expansion of the form:

$$h_l(z) = \sum_{n=0} c_n(l)\, z^n, \quad c_n(l) \in \mathbb{R}. \tag{3.33}$$

In the original domain we must have the following identity:

$$\eta_l(\zeta) = \sum_{n=0} c_n(l) \left(f_l^{-1}(\zeta)\right)^n, \quad c_n \in \mathbb{R}, \zeta \in \mathbb{C} \setminus \Gamma_l. \tag{3.34}$$

Notice that for $l = 0$ and for any φ, we have that $a(0) = b(0) = 0$, then:

$$f_0(z) = -z^2,$$

and we recover the expansion (3.28) for η_0. We prove now the following simple result.

Proposition 1. *The field u_l satisfying (3.15)-(3.18) is given by the real part of the non negative powers in the Laurent representation of $g_l(z) := \eta_0(f_l(z))$ around the origin, namely:*

$$u_l(f_l(z)) = \operatorname{Re} \frac{1}{2\pi i} \int_C \frac{g_l(\zeta)}{\zeta - z} dz\,, \tag{3.35}$$

where C is a curve enclosing z and the interval $I_l := [a(l), b(l)]$ is defined in (3.31).

Proof. The real function given by:

$$v_l(x) := u_l(x) - u_0(x)$$

is harmonic in $\mathbb{R}^2 \setminus \Gamma_l$. By the matching condition (3.18) it goes to zero uniformly at infinity. By (3.16) satisfies homogeneous Neumann boundary conditions on Γ_0 and

$$\frac{\partial v_l}{\partial n}(x) = -\frac{\partial u_0}{\partial n}(x) \quad x \in \Delta\Gamma.$$

If we look at this function through the map $f_l(z)$, we have that $v_l(f_l(z))$ is harmonic in the upper half plane, it goes to zero at infinity, satisfies homogeneous Neumann boundary conditions on $\mathbb{R} \setminus I_l$ and non homogeneous Neumann conditions on I_l. It can therefore be extended by symmetry to a harmonic function in $\mathbb{C} \setminus I_l$. By analytic completion of the Poisson formula for the upper half plane we have that:

$$v_l(f_l(z)) = \operatorname{Re} p_l(z)\,,$$

where

$$p_l(z) := \frac{1}{\pi} \int_{a(l)}^{b(l)} \frac{r_l(t)}{t - z} dt,$$

$$r_l(x) = -\int_{a(l)}^{x} \operatorname{sig}(t) \frac{\partial u_0}{\partial n}(f_l(t)) \left|f_l'(t)\right| dt \quad a(l) \le x \le b(l)\,,$$

where $\operatorname{sig}(t) = -1$ for $t < 0$ and $\operatorname{sig}(t) = +1$ for $t > 0$. Notice that $r_l(b(l)) = 0$. The function p_l is holomorphic in $\mathbb{C}^* \setminus I_l$ (recall that $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$) and $p_l(\infty) = 0$. It therefore admits a Laurent expansion for $|z| > R > \max\{-a(l), b(l)\}$ with *strictly negative* powers of z. Using the notation introduced in (3.32), we have that:

$$h_l(z) = g_l(z) + p_l(z)\,,$$

where $g_l(z) := \eta_0(f_l(z))$. As h_l is analytic in $\mathbb{C}$ (cf. 3.33) we must have that g_l and p_l share the negative powers in z (with opposite sign) in their respective Laurent expansions around $z = 0$. Therefore, if the curve C encloses z and I_l, we must have that:

$$h_l(z) = \frac{1}{2\pi i} \int_C \frac{g_l(\zeta)}{\zeta - z} d\zeta, \tag{3.36}$$

and (3.35) follows. □

Corollary 1. *The coefficients $c_n(l)$ in (3.33) satisfy the following relationship:*

$$c_n(l) = c_n + (n+1)\, c_{n+1} b_0(l) + O(l), \tag{3.37}$$

where $b_0(l) = \sqrt{l} b_0(1)$ is the constant term in the Laurent expansion for $\chi_l := f_0^{-1} \circ f_l$ as $z \to \infty$.

Proof. The function $g_l(z)$ may be written as follows (cf. (3.24) and (3.25) above):

$$g_l(z) = \eta_0(f_l(z)) = h_0\left(f_0^{-1} \circ f_l(z)\right).$$

We denote by χ_l the function $f_0^{-1} \circ f_l$, which sends the upper half plane in a one to one way to the upper half plane minus the set $f_0^{-1}(\Delta\Gamma)$. Notice that χ_l is real on $\mathbb{R} \setminus I_l$ and therefore can be extended by symmetry to an analytic univalent function on $\mathbb{C} \setminus I_l$. Moreover it admits an expansion of the form (cf. (3.30)):

$$\chi_l(z) = \sqrt{(z-a(l))^{1-\alpha}(z-b(l))^{1+\alpha}} = z + b_0(l) + \frac{b_1(l)}{z} + \frac{b_2(l)}{z^2} \ldots \quad z \to \infty, \tag{3.38}$$

Due to the following scaling property of χ_l:

$$\chi_l(z) = \sqrt{l}\chi_1\left(z/\sqrt{l}\right), \tag{3.39}$$

we have that:

$$b_n(l) = l^{\frac{n+1}{2}} b_n(1). \tag{3.40}$$

We use now the formula for the coefficients in a Laurent expansion (cf. [14]), taking into account the series for h_0 given in (3.27) and (3.36):

$$c_n(l) = \frac{1}{2\pi i}\int_C \frac{g_l(\zeta)}{\zeta^{n+1}} d\zeta = \frac{1}{2\pi i}\int_C \frac{h_0(\chi_l(\zeta))}{\zeta^{n+1}} d\zeta = \frac{1}{2\pi i}\sum_{j=0}^{\infty} c_j \int_C \frac{\chi_l^j(\zeta)}{\zeta^{n+1}} d\zeta$$

From (3.38) we obtain:

$$c_n(l) = \frac{1}{2\pi i}\sum_{j=0}^{\infty} c_j \int_C \frac{d\zeta}{\zeta^{n+1}}\left(\zeta + \sum_{k=0}^{\infty}\frac{b_k(l)}{\zeta^k}\right)^j.$$

We see at once that:

$$\int_C \frac{d\zeta}{\zeta^{n+1}}\left(\zeta + \sum_{k=0}^{\infty}\frac{b_k(l)}{\zeta^k}\right)^j = 0 \quad \text{for } j < n,$$

and then we can write:

$$c_n(l) = \frac{1}{2\pi i}\sum_{j=n}^{\infty} c_j \int_C \frac{d\zeta}{\zeta^{n+1}}\left(\zeta + \sum_{k=0}^{\infty}\frac{b_k(l)}{\zeta^k}\right)^j.$$

For $j = n$ the contribution inside the parentheses is given by ζ^n, giving the value c_n. For $j = n+1$ we have the contribution of $\zeta^n b_0(l)$ giving rise to $(n+1)\, c_{n+1} b_0(l)$ (cf. (3.40)). The next one gives, for $j = n+2$, the term $c_{n+2}\left(\binom{n+2}{2} b_0^2(l) + (n+2)\, b_1(l)\right)$, which is of order l. It is easy to see that further terms give corrections of order $l^{3/2}$ and higher. □

3.6. **A complex-variable formula for the energy variation.** It is useful to express the energy increment in (3.12) by means of a complex integral in the transformed domain z. Let us recall the complex representation of the stresses (3.22) and displacements (3.23):

$$\sigma_1^l - i\sigma_2^l = \mu\eta_l'(\zeta), \quad u_l(\zeta) = \operatorname{Re}\eta_l(\zeta).$$

Notice the superscript l indicating the state of stress corresponding to displacement u_l. If we rotate the axes an angle θ, the tangential and normal components of the stress change as follows:

$$\sigma_t^l - i\sigma_n^l = \mu\eta_l'(\zeta)\, e^{i\theta}.$$

This can be easily seen by applying the force balance to a triangle with an appropriate orientation. It is worth to mention that σ_t^l represents the out of plane stress component (ie., in the x_3 direction) when we take a face with normal in direction θ. Similarly, σ_n^l is the out of plane stress component when we consider a face with normal in the direction of $\theta + \pi/2$. Our purpose is to give an expression for the line integral:

$$\Delta E = \frac{1}{2}\int_{\Delta\Gamma} [u_l]\left(\mu\frac{\partial u_0}{\partial n^+}\right) = \frac{\mu}{2}\int_{\Delta\Gamma} [\operatorname{Re}\eta_l]\ \operatorname{Im}\left(\eta_0'\, e^{i\varphi}\right). \tag{3.41}$$

Where n^+ is the exterior normal to $\Delta\Gamma^+$, ie. with argument $\varphi - \pi/2$.

We apply now the conformal map f_l as in (3.32). Recalling the definition of g_l:

$$g_l(z) := (\eta_0 \circ f_l)(z), \tag{3.42}$$

then:

$$\mu g_l'(z) = \mu\eta_0'(f_l(z))\, f_l'(z) = \left(\sigma_t^0 - i\sigma_n^0\right)(f_l(z))\,|f_l'(z)|,$$

where now t and n are the directions given by the conformal transformation of directions z_1 and z_2 respectively. When z approaches the real axis outside the interval $I_l := [a(l), b(l)]$ (ie. $z_2 \to 0^+$, $z_1 \in \mathbb{R}\backslash I_l$), σ_t^0, σ_n^0 represent the tangential and normal stress components along the initial crack Γ_0. Therefore, taking into account that $\sigma_n^0 \to 0$ on Γ_0, we have that:

$$\mu g_l'(z) = \sigma_t^0(f_l(z))\,|f_l'(z)| \in \mathbb{R} \quad \text{for } z \in \mathbb{R}\backslash I_l. \tag{3.43}$$

Let us now change the integration variable in (3.41) by means of $\zeta = f_l(z)$:

$$\begin{aligned}\Delta E &= \frac{\mu}{2}\int_{a(l)}^{0} \operatorname{Re} h_l\ \operatorname{Im}\left(\eta_0'\, e^{i\varphi}\right)|f_l'|\, dz - \frac{\mu}{2}\int_0^{b(l)} \operatorname{Re} h_l\ \operatorname{Im}\left(\eta_0'\, e^{i\varphi}\right)|f_l'|\, dz \\ &= \frac{\mu}{2}\int_{a(l)}^{b(l)} \operatorname{Re} h_l\ \operatorname{Im} g_l'\, dz.\end{aligned}$$

where we recall that $h_l = \eta_l \circ f_l$ (cf.(3.32) and (3.42)), $f_l([a(l), 0]) = \Delta\Gamma^+$ and $f_l([0, b(l)]) = \Delta\Gamma^-$ (cf.(3.31)). Notice also that $f_l'(z) = e^{i\varphi}|f_l'(z)|$ for $a(l) < z < 0$, and $f_l'(z) = e^{i(\varphi+\pi)}|f_l'(z)|$ for $0 < z < b(l)$.

Using the boundary conditions we have that:

$$h_l(z) \in \mathbb{R} \quad \text{for } z \in \mathbb{R}. \tag{3.44}$$

On the other hand we can extend the function g_l to the lower half plane:

$$g_l(z) = \overline{g_l(\bar{z})}, \quad z \in \operatorname{Im} z < 0. \tag{3.45}$$

By (3.43) this function is real on $\mathbb{R}\backslash I_l$ and we have that the extended g_l is *sectionally holomorphic,* ie. it is analytic in the plane with the exception of the segment I_l (cf.(3.31)).

Using now (3.44), (3.45) and the fact that the extended $h_l(z)$ is analytic in $\mathbb{C}$ and real for $z \in \mathbb{R}$, we obtain:

$$\Delta E = \frac{\mu}{2} \int_{a(l)}^{b(l)} h_l \, \mathrm{Im}\, g_l' \, dz = \frac{\mu i}{4} \int_C h_l \, g_l' \, dz \, ,$$

where C is a closed curve in the complex plane surrounding the real interval I_l. After integrating by parts we find that:

$$\Delta E = \frac{\mu}{4i} \int_C h_l' \, g_l \, dz \tag{3.46}$$

Notice that for $l = 0$, g_0 reduces to h_0, the integrand being an analytic function and the result is therefore zero, as it were expected.

4. The crack path.

4.1. **An expansion for ΔE.** To clarify the dependence on l, we write (3.46) as follows (cf.(3.42)):

$$\Delta E = \frac{\mu}{4i} \int_C h_l'(z) \, h_0(\chi_l(z)) \, dz. \tag{4.1}$$

This shows more explicitly the role of the conformal maps $\chi_l = f_0^{-1} \circ f_l : \mathbb{H} \mapsto \mathbb{H}$. Let us insert the expansions (3.27) and (3.33) in (4.1):

$$\Delta E = \frac{\mu}{4i} \int_C \left(\sum_{j=1} j \, c_j(l) \, z^{j-1} \right) \left(\sum_{k=0} c_k \, (\chi_l(z))^k \right) dz. \tag{4.2}$$

For $k = 0$ in the second sum, the integrand is holomorphic and there is no contribution to the integral. Using the scale invariance property $\chi_l(z) = \sqrt{l}\chi_1\left(z/\sqrt{l}\right)$ (cf. (3.39)) we find:

$$\Delta E = \frac{\mu}{4i} \int_C \left(\sum_{j=1} j \, c_j(l) \, z^{j-1} \right) \left(\sum_{k=1} l^{k/2} c_k \left(\chi_1\left(z/\sqrt{l}\right) \right)^k \right) dz.$$

After the change of variables $z = \sqrt{l}w$, this turns to be:

$$\Delta E = \frac{\mu}{4i} \int_C \left(\sum_{j=1} l^{\frac{j}{2}} j \, c_j(l) \, w^{j-1} \right) \left(\sum_{k=1} l^{\frac{k}{2}} c_k \, (\chi_1(w))^k \right) dw. \tag{4.3}$$

Notice that we can keep the same integration path if C encloses the segment $[a(1), b(1)]$ with positive orientation.

4.2. **The energy release rate.** We look now for the first term in the expansion in powers of l. Taking j, $k = 1$ we see that the lower exponent for l has the value 1, and together with (3.38), (3.37) and (3.40) gives:

$$\begin{aligned}\Delta E &= \frac{l\mu}{4i} c_1 c_1 (l) \int_C \chi_1 (w)\, dw + O\left(l^{3/2}\right)\\ &= \frac{l\mu}{4i} c_1^2 \int_C \left(b_0 (1) + \frac{b_1 (1)}{w} + \dots\right) dw + O\left(l^{3/2}\right),\end{aligned}$$

where C is a suitable closed path. The coefficient $b_1 (1)$ is computed explicitly from (3.30) and (3.31), giving the value:

$$b_1 (1) = -\frac{1}{2} \left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha}. \tag{4.4}$$

where we recall that $\alpha = \varphi/\pi$, and φ is the kinking angle. After integrating, we have that:

$$\Delta E = -\frac{l\mu\pi}{4} c_1^2 \left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha} + O\left(l^{3/2}\right). \tag{4.5}$$

This expression shows us immediately the preferred direction of motion. Under Griffith's model (cf. [11]) we assume that there is a constant κ, depending on the material properties, which measures the amount of energy per unit length that we must give in order to create a unit extension of the crack. For small l, the amount of elastic energy per unit length that the body can give is determined by the linear term in the expansion for ΔE, whose negative value is the Energy Release Rate given in (1.2):

$$G (\alpha) = -\lim_{l\to 0^+} \frac{\Delta E}{l} = \frac{\mu\pi}{4} c_1^2 \left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha}. \tag{4.6}$$

Notice that $G (\alpha) \geq 0$ for every $-1 < \alpha < 1$. Starting with $c_1 = 0$, and increasing slowly the loading to raise the stress around the tip (and therefore enlarging $|c_1|$) there will be a minimum critical value of $|c_1|$ which will make $G (\alpha) = \kappa$ for some angle α. This threshold will be reached when the universal function:

$$A (\alpha) := \left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha} \tag{4.7}$$

has its *maximum* value, which is attained for $\alpha = 0$ independently of the applied loading. The critical value of $|c_1|$ turns to be:

$$|c_1|_{\text{crit}} = \sqrt{\frac{4\kappa}{\mu\pi}}. \tag{4.8}$$

If $|c_1| < |c_1|_{\text{crit}}$ the motion of the tip is not possible, since there is not enough elastic energy available to pay for the surface energy. If $|c_1| > |c_1|_{\text{crit}}$ there is an excess of energy stored in the body and if the crack starts to grow it will generate dynamic effects (sound waves). If we remain in a quasistatic growth regime, we must have $|c_1| \approx |c_1|_{\text{crit}}$ and there is only one direction satisfying the balance of energy, namely $\alpha = 0$. This shows that the motion of a Mode III crack will always follow the tangent to the initial configuration in a first approximation. In other words, for a pure out of plane loading, it would not be possible to generate a kink in the path in a quasistatic regime. As we already mentioned, this is in contrast with the in-plane modes where a kink is predicted for $K_{\text{II}} \neq 0$ (cf. for instance [6] and [13]), K_{II} being the so-called stress intensity factor for Mode II loading. A plot of $A(\alpha)$ is shown in Figure (5).

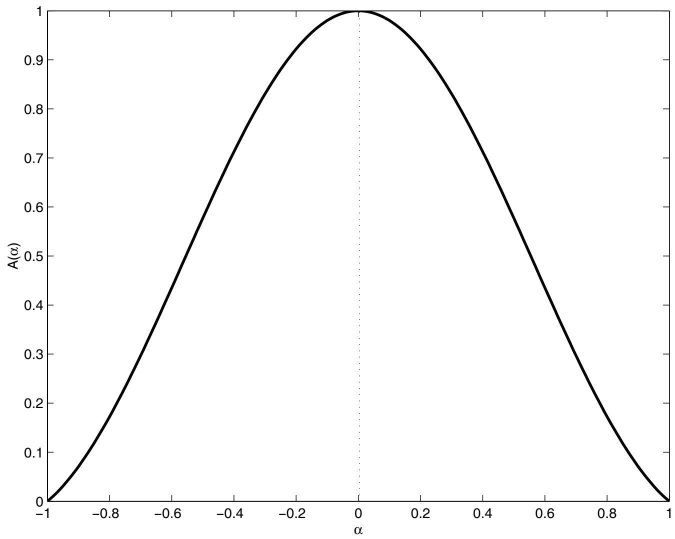

FIGURE 5. The function $A(\alpha)$.

It turns out that the linear term in the expansion of ΔE gives little information about the initial shape of the growing crack. Moreover, any smooth configuration of Γ would satisfy the principle of "maximum energy release rate" at each point. We must therefore look for the next term, which is proportional to $l^{3/2}$. We note on pass that (4.6) can also be obtained by means of Irwin type formulae once the stress intensity factor for the kinked configuration is available. This factor was already computed by some authors, using integral equation methods and conformal mapping techniques (cf. for instance, formula 4.16 in [17]).

4.3. **The second term in the expansion.** Consider (4.3) once more, and notice that the next power for l is $3/2$, given by all the pairs j,k satisfying $j+k=3$ and from $j,k=1$ from the first correction term of the coefficient $c_1(l)=c_1+\sqrt{l}2c_2b_0(1)+O(l)$ (cf. (3.37) and (3.40)).

Case 1 ($j=1,k=1$ in (4.3)).

$$\frac{\mu}{4i}\int_C l\;c_1(l)\;c_1\,\chi_1(w)\;dw=-l\,G(\alpha)+l^{3/2}\mu\pi c_1c_2\,b_0(1)\,b_1(1)+O\left(l^2\right).$$

We can compute explicitly $b_0(1)$ *from (3.30) and (3.31) obtaining:*

$$b_0(1)=-\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{\alpha}{2}}\frac{2\alpha}{\sqrt{1-\alpha^2}}. \tag{4.9}$$

Together with (4.4) this gives (cf. (4.7)):

(4.10)

$$\frac{\mu}{4i}\int_C l\;c_1(l)\;c_1\,\chi_1(w)\;dw=-l\,G(\alpha)+l^{3/2}\mu\pi c_1c_2\,\frac{\alpha}{\sqrt{1-\alpha^2}}A(\alpha)^{\frac{3\alpha}{2}}+O\left(l^2\right).$$

Case 2 ($j = 1$, $k = 2$)**.** *The contribution to the integral is given by the expression:*

$$\frac{\mu}{4i}\int_C l^{\frac{3}{2}} c_1(l)\ c_2\ (\chi_1(w))^2\ dw = \frac{\mu}{2i} l^{\frac{3}{2}} c_1 c_2 \int_C \frac{b_2(1) + b_0(1)\, b_1(1)}{w}\, dw + O\left(l^2\right)$$

where we used again (3.38). The coefficient $b_2(1)$ *is computed explicitly from (3.30), (3.31), giving the value:*

$$b_2(1) = -\frac{1}{3}\frac{\alpha}{\sqrt{1-\alpha^2}}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{3\alpha}{2}}. \tag{4.11}$$

From (4.4), (4.9) and (4.11) we have:

$$b_2(1) + b_0(1)\, b_1(1) = \frac{2}{3}\frac{\alpha}{\sqrt{1-\alpha^2}}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{3\alpha}{2}}.$$

We then obtain the contribution (cf. (4.7)):

$$\frac{\mu}{4i}\int_C l^{\frac{3}{2}} c_1(l)\ c_2\ (\chi_1(w))^2\ dw = \frac{2\mu\pi}{3} l^{\frac{3}{2}} c_1 c_2 \frac{\alpha}{\sqrt{1-\alpha^2}} A(\alpha)^{\frac{3}{2}} + O\left(l^2\right) \tag{4.12}$$

Case 3 ($j = 2$, $k = 1$ in (4.3))**.**

$$\begin{aligned}\frac{\mu}{4i}\int_C l^{3/2} 2\, c_1 c_2(l)\ w\,\chi_1(w)\ dt &= \frac{\mu}{2i} l^{3/2} c_1 c_2 \int_C w\,\chi_1(w)\ dw + O\left(l^2\right)\\ &= \mu\pi l^{3/2} c_1 c_2\, b_2(1) + O\left(l^2\right).\end{aligned}$$

Using (4.11) we obtain:

$$\frac{\mu}{4i}\int_C l^{3/2} 2\, c_1 c_2(l)\ w\,\chi_1(w)\ dw = -\frac{\mu\pi}{3} l^{3/2} c_1 c_2 \frac{\alpha}{\sqrt{1-\alpha^2}} A(\alpha)^{\frac{3\alpha}{2}} + O\left(l^2\right) \tag{4.13}$$

We now gather the contributions given by (4.10), (4.12), (4.13) which together with (4.5) gives the expression:

$$\Delta E = -\frac{\mu\pi}{4} c_1^2 A(\alpha)\, l + \frac{4\mu\pi}{3} c_1 c_2 B(\alpha)\ l^{\frac{3}{2}} + O\left(l^2\right), \tag{4.14}$$

where we defined the universal function B as follows:

$$B(\alpha) := \frac{\alpha}{\left(1-\alpha^2\right)^{1/2}} A(\alpha)^{\frac{3}{2}}. \tag{4.15}$$

A plot of $B(\alpha)$ for $-1 < \alpha < 1$ is shown in Figure 6.

Notice that B is zero for $\alpha = 0$. This is compatible with the fact that ΔE has a second derivative with respect to l for $\alpha = 0$ fixed (ie. for a straight motion). This is also a consequence (through Irwin's relationship) of the differentiability of the stress intensity factor for rectilinear paths. On the other hand, we can see that the energy released is bigger when the sign of the term of power $l^{3/2}$ is negative. For a positive $c_1 c_2$ the tip would release more energy if it goes to the left. If the relative signs of c_1 and c_2 are such that $c_1 c_2 < 0$, then it is better to choose a positive angle. We will make this statements more precise in the following Section.

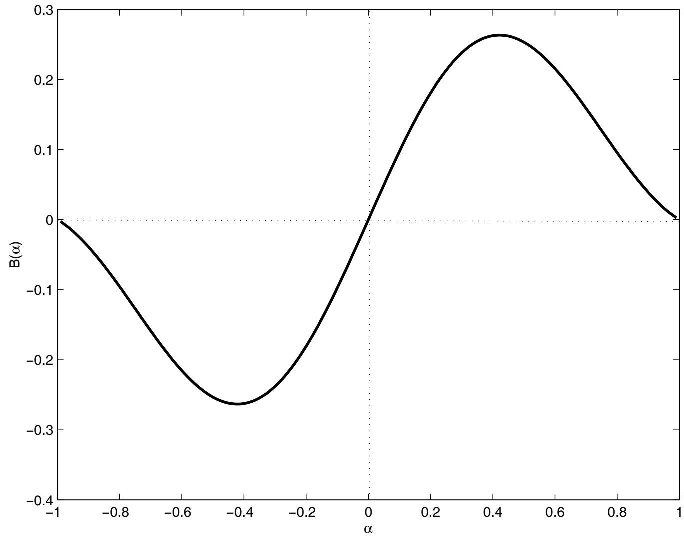

FIGURE 6. The universal function $B(\alpha)$ in the term of order $l^{3/2}$.

4.4. **The approximate shape of the growing crack.** Consider now the approximate expression for the Energy increment:

$$-\frac{\Delta E}{\mu\pi} \approx \frac{c_1^2}{4} A(\alpha)\, l - \frac{4c_1c_2}{3} B(\alpha)\, l^{3/2} \quad l \to 0^+. \tag{4.16}$$

For a small (but finite) l we would like to find the angle $\alpha(l)$ that maximizes the right hand side. We know that this angle should be near zero because the first term has its maximum in that direction. With the aid of the term of order $l^{3/2}$ we will find the asymptotic correction to the straight direction. To this end, consider the expansions of the analytic functions A and B around $\alpha = 0$:

$$\begin{aligned} A(\alpha) &= 1 - 2\alpha^2 + O\left(\alpha^4\right), \\ B(\alpha) &= \alpha - \frac{5}{2}\alpha^3 + O\left(\alpha^4\right) \end{aligned}$$

The derivatives are then given by:

$$\begin{aligned} A'(\alpha) &= -4\alpha + O\left(\alpha^3\right), \\ B'(\alpha) &= 1 + O\left(\alpha^2\right). \end{aligned}$$

We then gather the lower powers of α in (4.16) that cancel the following expression to the lowest order:

$$\frac{c_1^2}{4}\left(-4\alpha + O\left(\alpha^3\right)\right) l - \frac{4c_1c_2}{3}\left(1 + O\left(\alpha^2\right)\right) l^{3/2}.$$

We see that $l\alpha$ should be balanced with $l^{3/2}$ giving as a result:

$$c_1^2\alpha + \frac{4c_1c_2}{3}\, l^{1/2} = o\left(l^{1/2}\right),$$

or in other way:

$$\lim_{l\to 0^+} \frac{\alpha}{l^{1/2}} = -\frac{4}{3}\frac{c_2}{c_1}. \tag{4.17}$$

As we argued before this shows that the sign of the initial angle is determined by the quotient $\frac{c_2}{c_1}$. In terms of the original coordinates (x_1, x_2), we have that:

$$x_1 = l\cos(\pi\alpha) \approx l, \quad \frac{x_2}{x_1} = \tan(\pi\alpha) \approx \pi\alpha.$$

We see that (4.17) is compatible with the following shape for the starting crack:

$$x_2 \approx -\frac{4\pi}{3}\frac{c_2}{c_1}(x_1)^{3/2}, \quad x_1 > 0. \tag{4.18}$$

4.5. **A discrete algorithm of propagation.** Following the results obtained in the previous Sections, we describe briefly an algorithm for the pure Mode III propagation of a quasistatic crack. This approach takes full account of the near tip field and selects the shape according to the discrete version of the maximum energy release rate criterium. On the other hand it does not violate condition (1.2) during the propagation process, unless the crack is already at rest.

Consider a domain $\Omega \subset \mathbb{R}^2$, an initial configuration of the crack Γ_0, smooth near its end tip, and boundary data indexed by a parameter $t \geq 0 : (T(t), D(t))$, where T indicates the normal stress, and D is the given displacement as in (3.8)-(3.9). Given a natural number N, we proceed to divide the time interval into a finite sequence of time steps:

$$0 \leq t_j \leq t^{\max}, \quad 0 \leq j \leq N.$$

We should keep in mind that t is not a "physical time" parameter, and there is nothing in the model connecting the time and length scales. Ideally, the time steps should be selected in such a way that the "energy excess" in a single step j, should be of the order of $t_{j+1} - t_j$. We will go back to this point later.

Given a value of $l > 0$, the propagation is modelled as a discrete process, each step consisting of a kink of length l. The angle of kinking is selected according to the previous results. In each time-step, we can have several kink-steps, depending on the possibility of reaching a stable equilibrium point. We proceed to describe the algorithm.

Definition 1. *We say that a crack configuration Γ_j, at "time" step j, is in* stable equilibrium *with respect to the loading $L_j := (T_j, D_j)$ if the following inequality holds:*

$$c_1^2(j) < |c_1|^2_{crit}.$$

where $|c_1|^2_{crit}$ is given in (4.8) and $c_1(j) = c_1(\Gamma_j, L_j)$ is the coefficient of the expansion (3.29), computed for the field u at step j, corresponding to the crack Γ_j and loading conditions L_j.

The passage from step j to $j+1$, for $j < N$, is computed as follows:

Algorithm 1. *Assume that Γ_j is in equilibrium with respect to the loading L_j. Then:*

1. *Change the loading to $L = (T_{j+1}, D_{j+1})$,* keeping the crack configuration $\Gamma = \Gamma_j$ fixed.
2. *Compute $c_1(\Gamma, L)$.*

a): *If* $c_1^2(\Gamma, L) < |c_1|^2_{crit}$ *then we are still in an equilibrium point. If* $j+1 < N$ *put* $j = j+1$, $\Gamma_{j+1} = \Gamma_j$ *and proceed to step 1) again.*

b): *If* $c_1^2(\Gamma, L) \geq |c_1|^2_{crit}$ *, then compute* $c_2(\Gamma, L)$ *(cf. (3.29)). Add to* Γ *a single kink of length* l *and angle* $\varphi = -\frac{4\pi}{3}\frac{c_2}{c_1}l^{1/2}$ *(cf. (4.17)). Renovate* Γ*:*

$$\Gamma = \Gamma \cup kink(l, \varphi)\,.$$

If the moving tip of Γ *lies outside* Ω *stop the propagation. In other case go to step 2).*

Observe that the complete breakdown of the body could be achieved before all the steps of loading are carried out. This is consistent with the physics, as soon as we realize that the physical time scale is very long compared with Δt. In spite of this, we should take care that the excess of energy in step 2,b)

$$c_1^2(\Gamma, L) - |c_1|^2_{crit}$$

is not very large, otherwise the quasistatic propagation assumption is violated. Is easy to add a single step in the algorithm controlling this excess. Notice that the propagation process should be performed in a situation such that $c_1^2(\Gamma, L) \gtrsim |c_1|^2_{crit}$. If a stable equilibrium is reached we change the loading keeping the crack fixed. If the driving force is equal or greater than the resistance we create more crack surface until a new equilibrium is reached. The underlying description is common to any quasistatic process in mechanics.

5. Open questions and final remarks.

We addressed the study of the path followed by a propagating crack in a quasistatic regime. The problem is handled from a discrete algorithm, keeping a "local" point of view on the optimal direction of propagation and Griffith's growing condition (cf. 1.2). We can mention some open questions and related problems.

The approach is well suited for numerical study. The angle condition of step 2 b) in the algorithm is easier to handle than the symmetry principle (cf. [16]). The corresponding numerical method should need an efficient way to compute the coefficients c_1 and c_2 in (3.29). This may be achieved by the use of appropriate J integrals (cf. [15]).

The well possedness of the scheme, as well as the analysis of further conditions on the "time" step and the size of the kink ensuring convergence for both parameters tending to zero are also matter of future research. The regularity of the limiting curve, for $l, \Delta t \to 0$ is interesting as well. According to our results, at any point of the limiting curve where $c_2 \neq 0$ the curvature of the path turns to be infinite (cf. 4.18). It would be interesting to establish if $c_2 = 0$ is a kind of "continuum limit condition" for the crack path, analogous to the symmetry principle for the mixed modes.

In this article we adressed Mode III propagation with some detail. Even though the main ideas are applicable to the in plane modes as well, the mathematical difficulties are considerably greater. The out-of-plane field is easier to handle, in spite of the fact that it was necessary to find more detailed information about the expansion of the energy released than the one needed for the in-plane modes.

Acknowledgements. This research was carried out while the author was visiting the Max Planck Institute for Mathematics in the Sciences during the period october-november 2002 and the academic year 2003-2004. The author wishes to express his deep gratitude to Prof. Stephan Luckhaus for enlightening discussions during the preparation of this manuscript as well as to professors Miguel A. Herrero and José Fernández Sáez for valuable comments about the final work.

References

[1] M. Amestoy, J.B. Leblond (1992), "Crack paths in plane situations II: Detailed form of the expansions of the stress intensity factors", Int. J. Solids Structures Vol. 29, Nr. 4, pp. 465-501.

[2] C.M. Bender and S.A. Orszag (1978). **Advanced Mathematical Methods for Scientists and Engineers**, International Edition, Mc Graw-Hill Book Co. Singapore.

[3] M. Brokate and A. Khludnev (2004) "On crack propagation shapes in elastic bodies", Z. angew. Math. Phys. 55, pp. 318-329.

[4] M. Buliga (1999). "Energy minimizing brittle crack propagation", Journal of Elasticity 52, pp. 201-238.

[5] M. Buliga (2001). "Brittle crack propagation based on an optimal energy balance", Rev. Roum. des Math. Pures et Appl., 45, 2, pp 201-209.

[6] B. Cotterell and J.R. Rice (1980) "Slightly curved or kinked cracks", International Journal of Fracture, Vol. 16, No. 2, pp. 155-169.

[7] G.A. Francfort and J.J. Marigo (1998), "Revisiting brittle fracture as an energy minimization problem", Journal of the Mechanics and Physics of Solids, Vol. 46, No. 8, pp. 1319-1342.

[8] L.B. Freund (1998). **Dynamic Fracture Mechanics**, 2d edition. Cambridge University Press, Cambridge.

[9] A. Friedman, B. Hu, J.J.L.Velázquez (2000), "The evolution of the stress intensity factors in the propagation of two dimensional cracks", Euro. Jnl. of Applied Mathematics, Vol.11, pp. 453-471.

[10] R.V. Goldstein and R.L. Salganik (1974). "Brittle fracture of solids with arbitrary cracks", International Journal of Fracture 10, pp. 507.

[11] A.A. Griffith (1920). The phenomenon of rupture and flow in solids, Phylosophical Transactions of the Royal Society (London) A 221, pp. 163-198.

[12] G.R. Irwin (1957). Analysis of stresses and strains near the end of a crack transversing a plate, Journal of Applied Mechanics 24, pp. 361-364.

[13] J.B.Leblond (1989), "Crack paths in plane situations I: General form of the expansion of the stress intensity factors", Int. J. Solids Structures Vol. 25, No. 11, pp. 1311-1325.

[14] Z. Nehari (1975), **Conformal Mapping,** Dover Publications New York.

[15] J.R. Rice (1968). Mathematical analysis in the mechanics of fracture, in **Fracture**, Vol. 2, ed. H. Liebowitz. New York: Academic, pp. 191-311.

[16] T.J. Stone and I. Babuska (1998). "A numerical method with a posteriori error estimation for determining the path taken by a propagating crack", Computer Methods in applied mechanics and engineering 160 pp. 245-271.

[17] Chien H. Wu (1978), "Elasticity problems of a slender Z-crack", Journal of Elasticity, Vol. 8, No. 2.

Appendix A. The basic conformal map for kinked configurations

In order to transform the upper half plane into the plane with a semi infinite slit and a small kink around the origin we will apply a Schwarz-Christoffel-type of transformation. If ζ is the variable for the kinked configuration and z is the one corresponding to the upper half plane, we look for a conformal map $\zeta = f_l(z)$ that satisfies:

$$d\zeta = f_l' dz = k\ (z-a)^{-\varphi/\pi}\, z\ (z-b)^{\varphi/\pi}\ dz \quad a < 0 < b. \tag{A.1}$$

This means that for $z \in \mathbb{R}$ and $dz > 0$, when z goes from a^- to a^+ we have a jump in the argument of $z-a$ of magnitude $-\pi$. Then, the exponent adjusts the jump of $d\zeta$ to the angle φ. When we cross 0, we have another change in the argument of $d\zeta$ of amount $-\pi$. As z grows we are going back to zero, in the ζ plane. We must select the value of b in such a way that the next turn is made in this point, with an angle φ. In this way, for $z > b$, ζ must lie in the real negative axis, with $d\zeta < 0$ for $dz > 0$. We also have to select the constant k in order to start with a positive $d\zeta$ for $dz > 0$, to eventually end with a negative one. The values of k, a and b depend on l and φ. We compute this parameters in the following paragraphs.

Let us call:

$$\alpha := \frac{\varphi}{\pi}\,.$$

It is easy to check that

$$\int^z (\zeta - a)^{-\alpha}\,\zeta\,(\zeta - b)^{\alpha}\,d\zeta = \frac{1}{2}\,(z-a)^{1-\alpha}\,(z-b)^{1+\alpha}\,,$$

for real constants $a < b$ satisfying:

$$\frac{a}{b} = -\frac{(1-\alpha)}{(1+\alpha)}. \tag{A.2}$$

From (A.1) we obtain the following expression for f_l:

$$f_l(z) = \frac{k}{2}\,(z-a)^{1-\alpha}\,(z-b)^{1+\alpha}\,, \tag{A.3}$$

where we also imposed that $f_l\,(a) = 0$. Notice that we have then $f_l\,(b) = 0$, ie. the two values a and b correspond to the kink corner located at the origin. The kink tip corresponds to the origin in the z plane, ie. we must have that $f_l\,(0) = le^{i\varphi}$.

We now proceed to compute constants a, b for a given length of the kink l. From (A.3) and (A.2), we should have that:

$$f_l(0) \equiv l\,e^{i\varphi} = -\frac{k}{2}e^{i\varphi}\,(-a)^{1-\alpha}\,(b)^{1+\alpha}\,,$$

and we obtain:

$$l = -\frac{k}{2}\,(-ab)\left(-\frac{b}{a}\right)^{\alpha}\,.$$

We have then the following two equations for a and b:

$$\frac{a}{b} = -\frac{1-\alpha}{1+\alpha}, \qquad ab = \frac{2l}{k}\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha}.$$

Multiplying both:

$$a^2 = -\frac{2l}{k}\left(\frac{1-\alpha}{1+\alpha}\right)^{1+\alpha} \to a = -\sqrt{-\frac{2l}{k}}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}} \qquad (a<0). \tag{A.4}$$

And we have then that:

$$b = \sqrt{-\frac{2l}{k}}\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{2}}\,. \tag{A.5}$$

For $l = 0$, we want to recover the transformation:

$$f_0(z) = -z^2, \tag{A.6}$$

and then we can fix a value for k. We have from (A.4) and (A.5) that $a = 0$ and $b = 0$ for $l = 0$. Going back to (A.3) we have that:

$$f_0(z) = \frac{k}{2}z^2\,,$$

and then we obtain (cf. (A.6)):

$$k = -2. \tag{A.7}$$

We summarize the results in the following formulae:

$$f_l(z) = -\left(z - a\left(l\right)\right)^{1-\alpha}\left(z - b\left(l\right)\right)^{1+\alpha}, \tag{A.8}$$

$$a\left(l\right) = -\sqrt{l}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}}, \tag{A.9}$$

$$b\left(l\right) = \sqrt{l}\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{2}}. \tag{A.10}$$

Departamento de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid (Spain).

Current address: Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig (Germany).

E-mail address: `oleaga@mat.ucm.es`