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Balls Have the Worst Best Sobolev  
Inequalities

by

*Francesco Maggi and Cédric Villani*

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# BALLS HAVE THE WORST BEST SOBOLEV INEQUALITIES

F. MAGGI AND C. VILLANI

ABSTRACT. Using transportation techniques in the spirit of Cordero-Erausquin, Nazaret and Villani [7], we establish an optimal non-parametric trace Sobolev inequality, for arbitrary locally Lipschitz domains in  $\mathbb{R}^n$ . We deduce a sharp variant of the Brézis-Lieb trace Sobolev inequality [4], containing both the isoperimetric inequality and the sharp Euclidean Sobolev embedding as particular cases. This inequality is optimal for a ball, and can be improved for any other bounded, Lipschitz, connected domain. We also derive a strengthening of the Brézis-Lieb inequality, suggested and left as an open problem in [4]. Many variants will be investigated in a companion paper [10].

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## 1. MOTIVATIONS AND MAIN RESULTS

The main purpose of this paper is to establish various sharp trace Sobolev inequalities in domains of  $\mathbb{R}^n$  and discuss their connection with other inequalities. Before stating our results, let us recall briefly some of the background.

**1.1. Optimal trace Sobolev inequalities.** Sobolev inequalities are among the most famous and useful functional inequalities in analysis. They express a strong integrability and/or regularity property for a function  $f$  in terms of some integrability property for some derivatives of  $f$ . The most basic example is the following: for

each  $n \geq 2$  and  $p \in [1, n)$  there is a finite constant  $S_n(p)$  such that for all measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  vanishing at infinity,

$$(1) \quad \|f\|_{L^{p^*}(\mathbb{R}^n)} \leq S_n(p) \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad p^* := \frac{np}{n-p}.$$

Here as in all the sequel of the paper,  $\mathbb{R}^n$  is equipped with the  $n$ -dimensional Lebesgue measure, and, by definition,  $\|\nabla f\|_{L^p(\mathbb{R}^n)}$  is the  $L^p(\mathbb{R}^n)$  norm of the function  $|\nabla f|$ , where  $|\cdot|$  stands for a given norm on  $\mathbb{R}^n$ , say the Euclidean norm. Following Lieb and Loss [9], we say that  $f$  vanishes at infinity if

$$\forall a > 0, \quad \mathcal{L}^n[\{x \in \mathbb{R}^n; |f(x)| \geq a\}] < +\infty,$$

where  $\mathcal{L}^n$  is the  $n$ -dimensional Lebesgue measure. This vanishing condition is about optimal: if it is not fulfilled,  $f$  cannot belong to any Lebesgue space, although  $\nabla f$  may very well lie in  $L^p(\mathbb{R}^n)$  (such is the case if  $f$  coincides with a nonzero constant out of a set of finite measure).

We shall not try to review the gigantic literature studying the many variants of Sobolev inequalities involving higher-order or fractional derivatives, exponents  $p \geq n$ , functions on Riemannian manifolds, or on open domains of  $\mathbb{R}^n$ . This latter variant is the one we shall focus on in the present work: throughout the paper, we shall consider functions defined on an open set (domain)  $\Omega \subset \mathbb{R}^n$ . We shall not necessarily require  $\Omega$  to be bounded, but we shall always assume that it is locally Lipschitz. By this we mean that in the neighborhood of any  $x_0 \in \partial\Omega$ , the set  $\Omega$  may be written as the epigraph of a well-chosen Lipschitz function, in a well-chosen coordinate system [8, p. 127]. In more concrete words, this means that the boundary  $\partial\Omega$  is locally Lipschitz and that  $\Omega$  lies “on one side” of its boundary ( $(0, 1) \cup (1, 2)$ , for instance, does not satisfy this assumption, but  $(0, 1) \cup (2, 3)$  does).

If  $\Omega$  has finite Lebesgue measure, the condition that  $f$  vanish at infinity becomes void, and in particular does not prevent  $f$  to be a nonzero constant; so, an additional condition should be imposed to control the integrability of  $f$  in terms of that of its gradient. One possibility is to supplement the  $L^p(\Omega)$  bound on  $\nabla f$  with some  $L^q(\Omega)$  bound for  $f$ ; if  $q < p^*$ , the resulting inequality is still of interest. A classical choice is  $q = p$ , in accordance with the definition of the Sobolev space  $W^{1,p}(\Omega)$  as the space of  $L^p$  functions on  $\Omega$  whose gradient also lies in  $L^p$ .

Another possibility, which is the one we shall adopt here, is to introduce an  $L^q$  bound on the *trace* of  $f$  on  $\partial\Omega$ . This leads to a “trace Sobolev inequality”:

$$(2) \quad \|f\|_{L^{p^*}(\Omega)} \leq A \|\nabla f\|_{L^p(\Omega)} + C \|f\|_{L^q(\partial\Omega)}.$$

Here, for simplicity we wrote  $\|f\|_{L^q(\partial\Omega)}$  in place of  $\|\operatorname{tr} f\|_{L^q(\partial\Omega)}$ , where  $\operatorname{tr}$  is the trace operator, whose definition is recalled in the Appendix. Moreover,  $\partial\Omega$  is implicitly equipped with the  $(n-1)$ -dimensional Hausdorff measure  $\mathcal{H}^{n-1}$ , as in all the rest of the paper.

For a given domain  $\Omega$ , there is in general a range of admissible parameters  $q$  in inequality (2). However, there is a rather natural choice, namely

$$(3) \quad q = p^\sharp := \frac{np^*}{n-1} = \frac{(n-1)p}{n-p}.$$

Indeed, this is the only value for which the second term in the right-hand side of (2) has the same homogeneity as the other terms. In particular, this choice is the only hope to obtain inequalities which hold true unconditionally on  $\Omega$ . By the way (see the Appendix),  $p^\sharp$  is the maximum exponent  $q$  such that

$$\nabla f \in L^p(\Omega) \implies f \in L_{\text{loc}}^q(\partial\Omega).$$

Throughout the present paper, this will be our choice for  $q$ . The resulting inequality,

$$(4) \quad \|f\|_{L^{p^*}(\Omega)} \leq A\|\nabla f\|_{L^p(\Omega)} + C\|f\|_{L^{p^\sharp}(\partial\Omega)},$$

will be called the **homogeneous trace Sobolev inequality**. It does hold true for all domains  $\Omega$  with locally Lipschitz boundary; we do not need to give a precise reference for this fact, since the present paper will include a complete proof. In certain cases (mainly for  $p=2$  and  $\Omega$  bounded) the homogeneous trace Sobolev inequality has been studied in depth in a classical paper by Brézis and Lieb [4] from 1985.

When  $\Omega$  is the whole Euclidean space  $\mathbb{R}^n$ , the value of the optimal constant  $S_n(p)$ , and the optimal functions in inequality (1) have been identified long ago, independently by Aubin [1], Rodemich (unpublished) and Talenti [13]. A rather short proof (very different from those of the above-mentioned authors) can be found in [7], together with additional references. Apart from this particular case, the picture is not so neat and many problems are still open. This is true in particular for trace Sobolev inequalities in open sets of  $\mathbb{R}^n$ , and even more so for Sobolev inequalities on Riemannian manifolds. These questions have remained popular to this date, because they provide some of the simplest instances in which one can thoroughly study the influence of the geometry on the solution of certain analytical variational problems.

One among several main results obtained by Brézis and Lieb can be stated as follows: *For any bounded Lipschitz open domain  $\Omega \subset \mathbb{R}^n$  there exists a finite constant*

$C_n(\Omega)$  such that, for all measurable functions  $f : \Omega \rightarrow \mathbb{R}$ ,

$$\|f\|_{L^{2^*}(\Omega)} \leq S_n(2)\|\nabla f\|_{L^2(\Omega)} + C_n(\Omega)\|f\|_{L^{2^\sharp}(\partial\Omega)}.$$

The norm used above is the Euclidean norm. The important point is that the same constant  $S_n(2)$  which works for the Euclidean case, also works in the present context of trace Sobolev inequality. The proof is clever but not so difficult; it uses the properties of harmonic functions, and therefore seems to apply only to the case  $p = 2$ .

Some natural questions were left open in that paper: What about the constant  $C_n(\Omega)$ ? Can one generalize the above inequality to other values of  $p$ ? To our knowledge these questions have remained open since then. A more subtle problem suggested by Brézis and Lieb was the possible validity of the stronger inequality

$$\|f\|_{L^{2^*}(\Omega)}^2 \leq S_n(2)^2\|\nabla f\|_{L^2(\Omega)}^2 + C'_n(\Omega)^2\|f\|_{L^{2^\sharp}(\partial\Omega)}^2.$$

Also this problem seems to have remained open, apart from the case of the Euclidean ball, solved a few years ago by Carlen and Loss [6], and independently by Zhu [15]. Both papers used some specific properties of the case  $p = 2$ , for which there is a conformal symmetry, so that trace Sobolev inequalities in a half-space can be transformed into trace Sobolev inequalities in the ball. Carlen and Loss actually exploited the conformal symmetry in a systematic and beautiful way, thanks to the principle of “competing symmetries” which they had developed earlier in [5].

**1.2. Isoperimetry.** The Isoperimetric Theorem states that, among all domains in  $\mathbb{R}^n$  with given volume (or measure), balls have the smallest surface. This is expressed by the **isoperimetric inequality**: if  $\Omega$  has finite measure,

$$(5) \quad \frac{|\partial\Omega|}{|\Omega|^{\frac{n-1}{n}}} \geq \frac{|S^{n-1}|}{|B^n|^{\frac{n-1}{n}}},$$

where  $B^n := B_1(0)$  is the unit ball in  $\mathbb{R}^n$ , and  $S^{n-1} = \partial B^n$  is the unit sphere. The symbol  $|\cdot|$  here stands for the  $n$ -dimensional Lebesgue measure  $\mathcal{L}^n$  when applied to  $\Omega$  and  $B^n$ , and for the  $(n-1)$ -dimensional Hausdorff measure  $\mathcal{H}^{n-1}$  when applied to  $\partial\Omega$  and  $S^{n-1}$ . This inequality is insensitive to the choice of the norm, and equality in it holds only when  $\Omega$  is a ball. We call the quantity  $|\partial\Omega|/|\Omega|^{\frac{n-1}{n}}$  the **isoperimetric ratio**  $\text{IPR}(\Omega)$ ; so the Isoperimetric Theorem states that balls (and only balls) have the smallest isoperimetric ratio.

It is well-known that Sobolev inequalities are intimately related to isoperimetric inequalities. This relation is implicit in, for instance, Talenti’s proof of (1), based on

the Coarea Formula; it can be made even more explicit by considering the optimal Sobolev inequality for  $p = 1$  in  $\mathbb{R}^n$ , in the form

$$\|f\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq S_n(1) \|\nabla f\|_{TV(\mathbb{R}^n)}, \quad S_n(1) = \frac{1}{n|B^n|^{1/n}} = \frac{1}{\text{IPR}(B_n)},$$

where  $TV$  stands for the total variation norm. When  $\Omega$  is, say, a bounded Lipschitz domain, then the formula

$$\|1_\Omega\|_{TV} = |\partial\Omega|$$

is an immediate consequence of the Gauss–Green Theorem on  $\Omega$  and of the definition of total variation of a measure so the sharp Sobolev inequality for  $p = 1$  reduces to the isoperimetric inequality when one plugs in  $f := 1_\Omega$ .

**1.3. New trace Sobolev inequalities.** In this paper, we shall establish several new Sobolev inequalities with trace. We shall give a new proof of the Brézis–Lieb inequality, and at the same time improve and generalize it in several ways:

- by getting rid of the dependence of the second constant on the domain  $\Omega$ : our inequality will be valid on any locally Lipschitz domain, bounded or not, with an explicit constant  $C$  independent of  $\Omega$ , and in fact it will be optimal with respect to *both* constants  $A$  and  $C$ , considered separately;

- by generalizing it to all values of  $p \in [1, n)$ ;

- by generalizing it to all norms on  $\mathbb{R}^n$ , not necessarily Euclidean;

- by proving the stronger variant evoked by Brézis and Lieb, where the norms are raised to adequate powers.

To summarize, we shall derive the following two inequalities:

$$(6) \quad \|f\|_{L^{p^*}(\Omega)}^p \leq S_n^p(p) \|\nabla f\|_{L^p(\Omega)}^p + C_n^p(p) \|f\|_{L^{p^\sharp}(\partial\Omega)}^p;$$

$$(7) \quad \|f\|_{L^{p^*}(\Omega)} \leq S_n(p) \|\nabla f\|_{L^p(\Omega)} + T_n^{-1}(p) \|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

In the above the norm may or may not be the Euclidean norm, but the constant  $S_n(p)$  will always be the optimal constant for the corresponding Sobolev inequality in  $\mathbb{R}^n$ , and the constants  $C_n(p)$  and  $T_n^{-1}(p)$  will be independent of  $\Omega$ . We shall not give an explicit bound for  $C_n(p)$ , but it would be very easy to deduce such a bound from our arguments. As for  $T_n(p)$ , on the other hand,

$$T_n(p) = \frac{\|1_{B^n}\|_{L^{p^\sharp}(B^n)}}{\|1_{B^n}\|_{L^{p^*}(S^{n-1})}} = \left( \frac{|S^{n-1}|^{1/(n-1)}}{|B^n|^{1/n}} \right)^{\frac{n-p}{p}} = (n|B^n|^{1/n})^{1/p^\sharp} = S_n(1)^{-1/p^\sharp},$$

where the norm defining the unit ball (and the unit sphere) here is dual to the norm used in the definition of  $\|\nabla f\|_{L^p}$ . With such a constant  $T_n(p)$ , there is equality in (7) when  $f = 1_{B^n}$ , and it follows that

- inequality (7) contains the isoperimetric inequality as a particular case, for *all* values of  $p$  (not just  $p = 1$  as in the classical interpretation);
- the constant  $T_n^{-1}(p)$  is optimal in (7), in the class of constants which do not depend on  $\Omega$ .

The fact that our results hold unconditionally on the norm is certainly not the most important point here; by the way, it only makes sense when one is interested in sharp constants, since all norms on  $\mathbb{R}^n$  are equivalent. Yet it has the merit to show that the Euclidean structure appearing in the original Sobolev problem is really not crucial – nor are its links with harmonic functions.

**1.4. Generalized isoperimetry.** We defined above the isoperimetric ratio of a domain  $\Omega \subset \mathbb{R}^n$  with finite measure. We shall now introduce a quantity which can be thought of as a functional variant of that isoperimetric ratio.

Let  $\Omega$  be a locally Lipschitz domain in  $\mathbb{R}^n$ ; we define  $\text{IPR}(p, \Omega)$  as the largest admissible constant  $R$  in the inequality

$$\|f\|_{L^{p^*}(\Omega)} \leq S_n(p) \|\nabla f\|_{L^p(\Omega)} + R^{-1/p^\sharp} \|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

When  $|\Omega| < \infty$ , by plugging the constant function 1 in the definition of  $\text{IPR}(p, \Omega)$ , we see that

$$\text{IPR}(p, \Omega) \leq \text{IPR}(\Omega).$$

On the other hand, it follows from (7) that

$$\text{IPR}(p, \Omega) \geq \text{IPR}(p, B^n) = \text{IPR}(B^n),$$

so  $\text{IPR}(p, \cdot)$  achieves its minimum on the ball, just as the classical isoperimetric ratio. We shall show that, if  $p > 1$  and  $\Omega$  is bounded, connected, and *is not a ball*, then actually  $\text{IPR}(p, \Omega) > \text{IPR}(p, B^n)$ . This shows that, in some sense, *only balls have the worst optimal trace Sobolev inequalities*.

This rigidity theorem looks quite similar to the usual discussion of equality cases for the isoperimetric equality (and its proof will actually use it), but there are some differences to be noted. First, the functional isoperimetric ratio  $\text{IPR}(p, \cdot)$  can be defined for domains with infinite measure. Next, it can be much larger than the classical isoperimetric ratio, and in some sense it is much more sensitive to the local geometry of  $\Omega$ . A first trivial remark in this direction is that if  $\Omega$  is not connected, and has a connected component which is a ball (no matter how small), then  $\text{IPR}(p, \Omega) = \text{IPR}(p, B^n)$ . Even in the class of connected domains, our rigidity



theorem does not apply to unbounded domains, or to the case  $p = 1$ : for both cases we shall construct simple counter-examples showing that  $\text{IPR}(p, \Omega) = \text{IPR}(p, B^n)$  does not force  $\Omega$  to be a ball. Roughly speaking, it is sufficient that  $\Omega$  has portions of its boundary looking very much like the boundary of a ball.

**1.5. Non-parametric Sobolev inequalities.** The proofs of both (6) and (7) will rest on a more general inequality:

$$(8) \quad \frac{\|\nabla f\|_{L^p(\Omega)}}{\|f\|_{L^{p^*}(\Omega)}} \geq \Phi \left( \frac{\|f\|_{L^{p^\sharp}(\partial\Omega)}}{\|f\|_{L^{p^*}(\Omega)}} \right),$$

where  $\Phi : [0, \infty) \rightarrow \mathbb{R}$  is a nonincreasing function, whose graph will be given explicitly as a parametric curve, depending only on  $n$  and  $p$ . For any value of  $p$ , this single inequality contains as limit cases the sharp homogeneous trace Sobolev inequality (6) and the sharp isoperimetric inequality (5); indeed, these two inequalities can be recovered by studying the behavior of  $\Phi(T)$  close to  $T = 0$  and  $\Phi = 0$  respectively. As for inequality (7), it expresses the fact that  $\Phi$  lies above the line passing through its two extreme points: a consequence of the *concavity* of  $\Phi$ .

To be just a bit more explicit, assume that we can prove

$$\Phi(T) \geq S_n^{-1}(p) \left[ 1 - \left( \frac{T}{T_n(p)} \right)^p \right];$$

then, from the definition of  $\Phi$  follows at once the inequality

$$\|f\|_{L^{p^*}(\Omega)} \leq S_n(p) \|\nabla f\|_{L^p(\Omega)} + T_n^{-1}(p) \|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

Let us now explain why inequality (8) is “completely optimal”. For any locally Lipschitz domain  $\Omega \subset \mathbb{R}^n$  and  $T \geq 0$ , define

$$\Phi_\Omega^{(p)}(T) := \inf \left\{ \|\nabla f\|_{L^p(\Omega)}; \quad \|f\|_{L^{p^*}(\Omega)} = 1, \|f\|_{L^{p^\sharp}(\partial\Omega)} = T \right\}.$$

By definition  $\Phi = \Phi_\Omega^{(p)}$  is the biggest *curve* such that the inequality (8) is always satisfied for functions  $f$  on  $\Omega$ ; it defines what one may call a **non-parametric optimal (trace) Sobolev inequality**, in the sense that no a priori form for  $\Phi_\Omega$  (power, sum of powers, etc.) is assumed. It is quite hard to understand exactly what geometrical properties of  $\Omega$  the function  $\Phi_\Omega$  reflects; one can already remark that  $\Phi_\Omega$  is invariant under rotations, translations or dilations, and that

$$\Phi_\Omega^{(p)}(T_0) = 0, \quad T_0 := \text{IPR}(\Omega)^{1/p^\sharp},$$

as soon as  $\Omega$  has finite measure (consider the constant function  $f := |\Omega|^{-1/p^*}$ ). Our main result can be recast as

$$\Phi_{\Omega}^{(p)}(T) \geq \Phi_n^{(p)}(T), \quad \forall T \in [0, T_n(p)],$$

where  $\Phi_n^{(p)}$  is the curve associated with  $B^n$ , and  $T_n(p)$  is the abscissa where  $\Phi_n^{(p)}$  vanishes, i.e.  $\text{IPR}(B^n)^{1/p^\sharp}$ .

**1.6. Optimal transportation.** At least as interesting as our main results is the way we obtain them: by a transportation argument.

The optimal transportation problem, also known as Monge-Kantorovich minimization problem, made its way in partial differential equations thanks to Brenier's seminal 1987 work [2]. After McCann's PhD Thesis, optimal transportation tools were used by a number of authors (such as Alesker, Ball, Barthe, Caffarelli, Carlen, Carrillo, Cordero-Erausquin, Gangbo, Dar, McCann, Milman, Naor, Nazaret, Otto, Schmuckenschläger, and the second author) to study various classes of functional inequalities with geometric content. An account of these works, together with a lot of related material, can be found in the second author's book [14], especially Chapters 6 and 9.

In particular, links between mass transportation and optimal Sobolev inequalities were first explored in a recent paper by Cordero-Erausquin, Nazaret and Villani [7]. In that work, a direct and simple proof of (1) was derived from a mass transportation approach. The present paper can be considered as the sequel of that earlier work; the very same strategy which was used there will be adapted here to form the core of our analysis.

Just as in [7], the crucial point here will not be the minimization property of the optimal transportation, but the fact that it provides a simple, well-behaved **monotone change of variables** between two given probability densities.

It is rather striking to see how this idea is efficient in solving a problem for which classical methods have already been tried and failed. For instance, it is quite difficult to figure out how to use the classical tools of rearrangement, unless the domain  $\Omega$  is symmetric and the values at the boundary are assumed to be constant. There is however a price to pay: just as the proof in [7], the arguments in the present paper will probably appear somewhat mysterious and non-natural to many a reader, in contrast with the standard tools of calculus of variations underlying most of the theory of sharp Sobolev inequalities.

On the other hand, the fact that our method works so well for this particular, quite specific problem does not imply that it should be so efficient for other problems. Various methods have been developed to tackle related problems in calculus of

variations, and it is probably easy to find problems where transportation arguments do not work, while more conventional tools still apply.

In a nutshell, our strategy is the following: normalize the  $L^{p^*}$  norm of  $f$  to 1, so that  $|f|^{p^*}$  defines a probability density; and then transport it to  $|g|^{p^*}$ , where  $g$  is an optimizer for the Sobolev inequality in the whole space, *truncated* on a ball and normalized to have unit  $L^{p^*}$  norm. For each such  $g$ , the method of [7] can be applied and yields a Sobolev-like inequality with a trace term; yet *none* of these inequalities is strong enough for our purpose (none of them, for instance, does imply the Brézis-Lieb inequality, even in the non-sharp form considered in [4]). It is only the collection of all these inequalities, which makes the job.

**1.7. Main theorems.** Our most striking results are summarized in the following next three theorems. In all three statements, we choose once for all an arbitrary norm  $|\cdot|$  in  $\mathbb{R}^n$  ( $n \geq 2$ ), an exponent  $p \in [1, n)$ , and we define

$$p' := \frac{p}{p-1}, \quad p^* := \frac{np}{n-p}, \quad p^\# := \frac{(n-1)p}{n-p}.$$

The ball  $B^n$  and the sphere  $S^{n-1}$  are defined with respect to the chosen norm. We use the dual norm, defined by

$$|y|_* := \sup\{x \cdot y; |x| \leq 1\},$$

to define the  $L^p$  norm of vector-valued functions such as  $\nabla f$ .

**Theorem 1 (optimal non-parametric trace Sobolev inequality).** *Let  $\Omega$  be a locally Lipschitz open domain in  $\mathbb{R}^n$ . For all  $T \in [0, \infty)$ , define*

$$(9) \quad \Phi_\Omega^{(p)}(T) := \inf \left\{ \|\nabla f\|_{L^p(\Omega)}; \|f\|_{L^{p^*}(\Omega)} = 1, \|f\|_{L^{p^\#}(\Omega)} = T \right\},$$

where the infimum is taken over all locally integrable functions  $f$  defined on  $\Omega$  and vanishing at infinity. Let also

$$\Phi_n^{(p)} := \Phi_{B^n}^{(p)},$$

$$(10) \quad T_n(p) := \frac{\|1_{B^n}\|_{L^{p^\#}(B^n)}}{\|1_{B^n}\|_{L^{p^*}(S^{n-1})}} = \left( \frac{|S^{n-1}|^{\frac{1}{n-1}}}{|B^n|^{\frac{1}{n}}} \right)^{\frac{n-p}{p}} = (n|B^n|^{1/n})^{1/p^\#} = S_n(1)^{-1/p^\#},$$

Then the ball  $B^n$  has the lowest  $\Phi$ -curve:

$$\Phi_\Omega^{(p)} \geq \Phi_n^{(p)} \quad \text{on } [0, T_n(p)].$$

Moreover, this particular curve is given for  $p > 1$  by the parametric curve  $G = \Phi_n^{(p)}(T)$ , where  $G, T$  (to be thought of as gradient norm and trace norm, respectively) depend on  $t \in [0, +\infty]$  as follows:

$$(11) \quad T(t) := \left[ \frac{|S^{n-1}|^{1/(n-1)} t^n}{(1+t^{p'})^n \int_0^t \frac{s^{n-1} ds}{(1+s^{p'})^n}} \right]^{1/p^*}$$

$$(12) \quad G(t) := |S^{n-1}|^{1/n} \left( \frac{n-p}{p-1} \right) \frac{\left( \int_0^t \frac{s^{n-1+p'} ds}{(1+s^{p'})^n} \right)^{1/p}}{\left( \int_0^t \frac{s^{n-1} ds}{(1+s^{p'})^n} \right)^{1/p^*}}.$$

In particular,  $T(0) = T_n(p)$ ,  $G(0) = 0$ , and  $T(+\infty) = 0$ ,  $G(+\infty) = S_n^{-1}(p)$ , where  $S_n(p)$  is the optimal constant in the Sobolev inequality (1).

For  $p = 1$ , the curve  $\Phi_n^{(1)}$  is just the straight line with slope  $-1$ , joining  $(0, a_n)$  to  $(a_n, 0)$ , where  $a_n = S_n^{-1}(1) = T_n(1)$ .

As a consequence, if one defines  $\tilde{\Phi}_n^{(p)} := \Phi_n^{(p)} 1_{[0, T_n(p)]}$ , then the following Sobolev-type inequality holds true: For all locally integrable functions  $f : \Omega \rightarrow \mathbb{R}$ , vanishing at infinity,

$$\|\nabla f\|_{L^p(\Omega)} \geq \|f\|_{L^{p^*}(\Omega)} \tilde{\Phi}_n^{(p)} \left( \frac{\|f\|_{L^{p^\sharp}(\partial\Omega)}}{\|f\|_{L^{p^*}(\Omega)}} \right).$$

Moreover, if  $p > 1$ ,  $\Omega$  is connected and  $f$  achieves equality in the above inequality, then  $\Omega$  is a ball  $B_r(x_0)$  and  $f$  takes the form

$$(13) \quad f(x) = \frac{\pm 1}{(a+b|x-x_0|^{p'})^{\frac{n-p}{p}}}, \quad x \in B_r(x_0),$$

for some  $a, b, r > 0$ .

*Remarks.* (i) Depending on the values of  $p$  and  $n$ , the integrals in (11) and (12) can sometimes be evaluated explicitly into rational functions. Here is an example: for  $p = 2, n = 3$ , then

$$T(t) = \sqrt{2} t^{1/2} (1+t^2)^{-1/6} (\text{Arctan}(t)(1+t^2)^2 - t + t^3)^{-1/6}.$$

In the general case, it is possible to express them in terms of hypergeometric functions, but it is not clear that this remark is useful at all.

- (ii) The study of trace Sobolev inequalities on the ball performed by Carlen and Loss [6] actually includes a study of  $\Phi_n^{(2)}$ . Even though the notation and presentation is somewhat different from ours (which can be explained in part by the fact that we discovered this reference after completing the present study), the reader should have no trouble making the connection between these two approaches.

**Theorem 2 (doubly optimal homogeneous trace Sobolev inequality).** *Let  $\Omega$  be a locally Lipschitz open domain in  $\mathbb{R}^n$ . Let  $S_n(p)$  be the optimal constant in the Sobolev inequality (1), and let  $T_n(p)$  be defined by (10). Then, for all locally integrable functions  $f : \Omega \rightarrow \mathbb{R}$ , vanishing at infinity,*

$$(14) \quad \|f\|_{L^{p^*}(\Omega)} \leq S_n(p) \|\nabla f\|_{L^p(\Omega)} + T_n^{-1}(p) \|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

Moreover, if  $p > 1$  and  $\Omega$  is a bounded, connected Lipschitz domain which is not a ball, then there exists a constant  $C(p, \Omega) < T_n^{-1}(p)$  such that for all integrable functions  $f : \Omega \rightarrow \mathbb{R}$ ,

$$(15) \quad \|f\|_{L^{p^*}(\Omega)} \leq S_n(p) \|\nabla f\|_{L^p(\Omega)} + C(p, \Omega) \|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

**Theorem 3 ( $p$ -power trace Sobolev inequality).** *Let  $\Omega$  be a locally Lipschitz open domain in  $\mathbb{R}^n$ , and let  $S_n(p)$  be the optimal constant in the Sobolev inequality (1). Then there is a constant  $C = C_n(p)$ , depending only on  $n$  and  $p$ , such that for all locally integrable functions  $f : \Omega \rightarrow \mathbb{R}$ , vanishing at infinity,*

$$(16) \quad \|f\|_{L^{p^*}(\Omega)}^p \leq S_n^p(p) \|\nabla f\|_{L^p(\Omega)}^p + C^p \|f\|_{L^{p^\sharp}(\Omega)}^p.$$

Theorem 1 is the key to the other results. Indeed, to establish inequalities (14) and (16), it will be sufficient to establish properties of  $\Phi_n^{(p)}$ , and this we shall do by using its explicit parametric form. A simple study of  $\Phi_n^{(p)}$  close to  $T = 0$  will lead us to conclude that

$$\forall T \in [0, T_n(p)], \quad \Phi(T) \geq S_n^{-1}(p) [1 - (CT)^p]^{1/p};$$

this at once implies (16). An intricate computation will reveal that  $\Phi_n^{(p)}$  is concave (a property which by the way is not true for all domains, as we shall see later), and, as a consequence,

$$\forall T \in [0, T_n(p)], \quad \Phi(T) \geq S_n^{-1}(p) [1 - T/T_n(p)];$$

this at once implies (14). The rigidity statement contained in Theorem 2 will be proven separately; the proof rests on classical functional inequalities, plus the *strict* concavity of  $\Phi_n^{(p)}$  for  $p > 1$ .

The optimal functions (13) will play a crucial role in the proof of (1), since they are optimizers in the main inequality; we shall transport an arbitrary function onto such a minimizer. The case  $p = 1$  is somewhat degenerate and will be treated separately by a simpler argument. Although the curve  $\Phi_n^{(p)}$  does converge to  $\Phi_n^{(1)}$  as  $p \rightarrow 1$ , there are no minimizers for  $p = 1$  (apart from the two end-points), while for  $p > 1$  there are minimizers for all values of  $T \in [0, T_n(p)]$ ... except for the end-point at  $T = 0$ ! In fact, when  $p \rightarrow 1$  in the parametric expression of  $\Phi_n^{(p)}$ , the first end-point corresponds to  $t \leq 1$ , the second one to  $t > 1$ , and the whole line corresponds to  $t = 1^+$ .

The most relevant information about the curves  $\Phi$  is summarized in Figure 1.

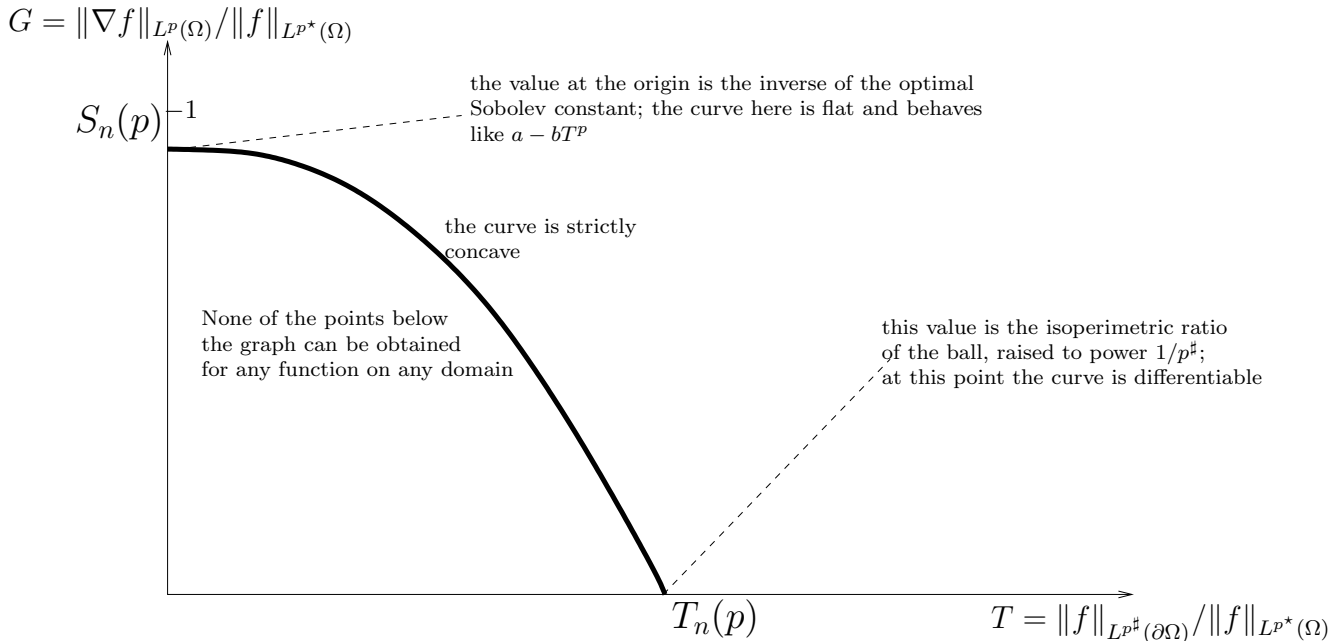


FIGURE 1. Typical shape of  $\Phi_n^{(p)}$  on  $[0, T_n(p)]$  for  $p > 1$

A bit more should be said about the case  $p = 1$ . As we saw, in that particular case the isoperimetric inequality can be read at the level of either the horizontal or

the vertical part of the graph. This can be understood intuitively by the fact that approximate minimizers are obtained by choosing the indicator function of the whole ball, and then redefine it to be 0 on a very small neighborhood of a given portion of the boundary. The loss in the trace norm will be exactly compensated by the gain in the total variation of the gradient; this one-for-one trade property explains the fact that the line has unit slope. The inequality which we are considering here is the following generalization of the isoperimetric inequality: For all functions  $f \in BV(\Omega)$ , satisfying

$$\|f\|_{L^{\frac{n}{n-1}}(\Omega)} \geq |B^n|^{\frac{n-1}{n}},$$

one has

$$(17) \quad \|\nabla f\|_{TV(\Omega)} + \|f\|_{L^1(\partial\Omega)} \geq |S^{n-1}|.$$

If  $f$  varies in the class of indicator functions, we are once again facing the usual isoperimetric inequality. As a matter of fact, inequality (17) can be established directly from the usual Sobolev inequality (1): define  $\tilde{f} = f1_\Omega$  on  $\mathbb{R}^n$ , then

$$\|\tilde{f}\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = \|f\|_{L^{\frac{n}{n-1}}(\Omega)};$$

on the other hand, it is a simple exercise about traces of functions with bounded variations [8, p. 177] that

$$\|\nabla \tilde{f}\|_{TV(\mathbb{R}^n)} = \|\nabla f\|_{TV(\Omega)} + \|f\|_{L^1(\partial\Omega)}.$$

Then we can apply (1) in the form

$$\|\tilde{f}\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq S_n(1) \|\tilde{f}\|_{TV(\mathbb{R}^n)}$$

and get (17).

**1.8. Organization of the paper.** The proofs of Theorems 1, 2 and 3 will stretch from Sections 2 to 4.

In Section 2, we shall establish a “mother inequality”, based on a transportation argument. From the point of view of  $n$ -dimensional analysis, this is the key section; even though the scheme of the proof is quite simple, there are a few technical difficulties, partly caused by our will to treat general assumptions. The core of the proof is the short Subsection 2.2 (a bit more than one page), most of the rest is just technique.

From the mother inequality we later deduce Theorem 1 in Section 3. Theorems 2 and 3 are then obtained from a study of  $\Phi$ , as sketched above. The rigidity statement is established separately in Section 4.

We made the choice to present rather detailed proofs, and in particular did not assume preliminary knowledge of the homogeneous trace Sobolev inequality, even in non-sharp form. In the Appendix we recall all the needed facts about the trace operator, starting from scratch. For the convenience of the reader, we have used one single source, namely the book of Evans and Gariepy [8], as a reference source for technical details about Sobolev functions; and one other source, namely the second author's book [14] as a reference source for optimal transportation theory. Thus, all the proofs in the present paper can be fully completed with just the help of the above-mentioned paper [7] and the two textbooks [8, 14].

**1.9. Further remarks and open problems.** Before starting the proofs, we make a few final comments about the results, and sketch some open problems. Many extensions of our method will be considered in our companion paper [10].

First, Theorem 3 is in fact a result about the behavior of  $\Phi_n^{(p)}(T)$  close to  $T = 0$ . It is natural to ask whether the same behavior holds true for any  $\Phi_\Omega$ , with the same optimal constant  $C$ , or whether on the contrary this behavior forces  $\Omega$  to be a ball. The following inequality (which can be derived with just a little more effort than we spent to show  $\Phi_\Omega^{(p)}(0) = S_n^{-1}(p)$ ) seems to indicate that, in some sense,  $\Phi_\Omega$  always behave like the ball near  $T = 0$ : *If  $B_R \subset \Omega$ , then for every  $s \in (0, 1)$ ,*

$$\left[1 - \left(1 - \frac{|B_R|}{|\Omega|}\right) s^{p^*}\right]^{1/p^*} \Phi_n^{(p)}(\text{IPR}(B^n)^{1/p^\sharp} s) \geq \Phi_\Omega^{(p)}(\text{IPR}(\Omega)^{1/p^\sharp} s).$$

Similarly, it is natural to ask about rigidity theorems stated in terms of  $\Phi_\Omega$ : for instance, it is tempting to conjecture that if  $\Phi_\Omega^{(p)}(T) = \Phi_n^{(p)}(T)$  for some  $T \in (0, T_n(p)]$ , then  $\Omega$  is a ball. The value  $T = 0$  should be taken out from such a statement, since a simple scaling argument proves  $\Phi_\Omega^{(p)}(0) = S_n^{-1}(p)$  for all domains  $\Omega$ . Also, as in Theorem 2, one should assume that  $p > 1$ , and that  $\Omega$  is bounded and connected. Indeed, we shall present an unbounded connected domain which has the same  $\Phi^{(p)}$  curve as the ball, for all values of  $p$ ; and a bounded connected domain whose  $\Phi^{(1)}$  curve coincides with that of the ball on a non-trivial interval  $[0, a]$ . Once all these possible causes for failure have been removed, we would bet that the conjecture is true: if the equality  $\Phi_\Omega^{(p)}(T) = \Phi_n^{(p)}(T)$  holds true for *some*  $T \neq 0$  and *some*  $p > 1$ , and  $\Omega$  is bounded, Lipschitz, connected, then  $\Omega$  is a ball. Would the infimum in the definition of  $\Phi_\Omega^{(p)}(T)$  be attained automatically, this conjecture would be a simple consequence of the end of Theorem 1.



Another question of interest is whether one can merge inequalities (14) and (16) into a single stronger inequality such as

$$(18) \quad \frac{\Phi_n^{(p)}(T)}{S_n^{-1}(p)} + \left( \frac{T}{T_n(p)} \right)^p \geq 1.$$

This only amounts to a fine study of the function  $\Phi_n^{(p)}$ , which is just a function of one real variable... yet this does not at all appear as a simple matter, and we did not succeed in establishing or disproving this inequality. The proof should be quite more subtle than our proof for the concavity of  $\Phi_n^{(p)}$ , which is already quite tricky. Numerical plots using the explicit expressions did seem to support the validity of (18).

Finally, here is a list of the questions which will be addressed in [10]:

- (i) It is natural to ask whether the trace Sobolev inequality can be significantly reinforced if one has additional information about the domain  $\Omega$ , preventing it to be a ball. In some particular cases, like half-spaces or angular sectors, one can guess the optimizers, and show that *the trace term can be omitted* in the Sobolev inequality. We say that a domain  $\Omega$  is a “gradient domain” if the following Sobolev inequality holds true:  $\|f\|_{L^{p^*}(\Omega)} \leq A\|\nabla f\|_{L^p(\Omega)}$ . Obviously such domains should have infinite measure, but this condition is not sufficient. In [10] we shall derive a simple sufficient criterion by a variant of our method. Exterior angular sectors satisfy our criterion, and for such domains our method yields optimal constants.
- (ii) It is interesting to ask what happens if one drops the requirement  $T \leq T_n(p)$  and allows  $T$  to take all values in  $\mathbb{R}_+$ . Since the set of all admissible values of  $\|\nabla f\|_{L^p(\Omega)}/\|f\|_{L^{p^*}(\Omega)}$  in terms of  $\|f\|_{L^{p^\sharp}(\partial\Omega)}/\|f\|_{L^{p^*}(\Omega)}$  is made of vertical lines, a seemingly equivalent question is to consider *all* the possible values of  $\|f\|_{L^{p^*}(\Omega)}$  in terms of  $\|\nabla f\|_{L^p(\Omega)}$  and  $\|f\|_{L^{p^\sharp}(\partial\Omega)}$ . In the study of this problem, the Sobolev inequality enters in competition with the trace inequality expressing the domination of  $\|f\|_{L^{p^\sharp}(\partial\Omega)}$  by  $\|\nabla f\|_{L^p(\Omega)}$  and  $\|f\|_{L^{p^*}(\Omega)}$  (Theorem A.3 in the Appendix). This extension does not look immediate at all; in [10] we shall present some partial results about this problem.
- (iii) Since it was shown in [7] that some Gagliardo-Nirenberg interpolation inequalities could be treated in exactly the same way as sharp Sobolev inequalities, it is natural to expect that one can adapt the proofs in the present paper to produce sharp Gagliardo-Nirenberg inequalities with trace term. We shall present such extensions in [10].

- (iv) As limit cases of these sharp inequalities we shall derive new trace inequalities of Faber-Krahn and of logarithmic Sobolev type.
- (v) The ideas developed in this paper can be adapted also to deal with the so-called critical case  $p = n$ , for which the usual Sobolev inequality should be replaced by the Moser-Trudinger inequality. Although the underlying philosophy is just the same as in the case  $p < n$ , these limit inequalities need a bit more care. This adaptation will be considered in detail in [10].

## 2. MOTHER INEQUALITY

Let  $\Omega$  be a locally Lipschitz open domain in  $\mathbb{R}^n$ , and  $p \in (1, n)$ . We denote by  $W_{\text{loc}}^{1,1}(\overline{\Omega})$  the space of functions which are locally integrable in  $\Omega$ , and whose distributional gradient  $\nabla f$  is integrable on every *bounded* subset of  $\Omega$ . It is important to distinguish this space from the larger space  $W_{\text{loc}}^{1,1}(\Omega)$ , made of functions such that  $\nabla f$  is integrable on every *compact* subset of  $\Omega$ . As we recall in the Appendix, functions in  $W_{\text{loc}}^{1,1}(\overline{\Omega})$  admit a trace (which somehow justifies our unusual notation with  $\overline{\Omega}$ ), while functions in  $W_{\text{loc}}^{1,1}(\Omega)$  do not necessarily.

For simplicity we shall use the symbol  $f$  for the trace of a function  $f$ . We also use the notation

$$p' = \frac{p}{p-1}, \quad p^* = \frac{np}{n-p}, \quad p^\sharp = \frac{(n-1)p}{n-p}.$$

We let  $|\cdot|$  be an arbitrary norm on  $\mathbb{R}^n$ ,  $|\cdot|_*$  its dual norm, and write

$$\|\nabla f\|_{L^p(\Omega)} = \left( \int_{\Omega} |\nabla f(x)|_*^p dx \right)^{1/p}.$$

The goal of this section is to establish the following two results:

**Proposition 4 (mother inequality).** *Let  $\Omega$  be a locally Lipschitz open domain in  $\mathbb{R}^n$ , and  $p \in [1, n)$ . Let  $f$  and  $g$  be two nonnegative functions in  $W_{\text{loc}}^{1,1}(\overline{\Omega})$  and  $L_{\text{loc}}^1(\mathbb{R}^n)$ , respectively. Assume that  $g$  is supported in a ball of radius  $R$  (possibly infinite), centered at  $y_0 \in \mathbb{R}^n$ . Further assume that*

$$\|f\|_{L^{p^*}(\Omega)} = \|g\|_{L^{p^*}(\mathbb{R}^n)} = 1.$$

*Then*

$$(19) \quad n \|g\|_{L^{p^\sharp}(\mathbb{R}^n)}^{p^\sharp} \leq p^\sharp \left( \int_{\mathbb{R}^n} g(y)^{p^*} |y - y_0|^{p'} dy \right)^{1/p'} \|\nabla f\|_{L^p(\Omega)} + R \|f\|_{L^{p^\sharp}(\partial\Omega)}^{p^\sharp}.$$

**Proposition 5 (homogeneous trace Sobolev inequality).** *Let  $\Omega$  be a locally Lipschitz open domain in  $\mathbb{R}^n$ , and  $p \in [1, n)$ . Then, there are constants  $A$  and  $C$ , only depending on  $n$  and  $p$ , such that for all functions  $f \in W_{\text{loc}}^{1,1}(\overline{\Omega})$ , vanishing at infinity,*

$$(20) \quad \|f\|_{L^{p^*}(\Omega)} \leq A\|\nabla f\|_{L^p(\Omega)} + C\|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

*In particular,  $f$  automatically belongs to  $L^{p^*}(\Omega)$  if its gradient belongs to  $L^p(\Omega; \mathbb{R}^n)$  and its trace belongs to  $L^{p^\sharp}(\partial\Omega)$ .*

It seems clear that inequality (20) follows from inequality (19). Indeed, suppose that  $g$  is any fixed nonnegative function such that  $\int g^{p^*} = 1$ ,  $\int g(y)|y|^{p'} dy < +\infty$  and  $g$  is supported in a bounded set. Then (19) transforms into

$$(21) \quad 0 < K \leq C_1\|\nabla f\|_{L^p(\Omega)} + C_2\|f\|_{L^{p^\sharp}(\partial\Omega)}^{p^\sharp},$$

where  $f$  is an arbitrary nonnegative function on  $\Omega$  with  $\|f\|_{L^{p^*}} = 1$ . A cheap argument allows one to deduce (20) from (21): If  $\|f\|_{L^{p^\sharp}(\partial\Omega)} \leq 1$ , then (since  $p^\sharp \geq 1$ ) (21) implies

$$K \leq C_1\|\nabla f\|_{L^p(\Omega)} + C_2\|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

Therefore, the inequality

$$K \leq C_1\|\nabla f\|_{L^p(\Omega)} + \max(K, C_2)\|f\|_{L^{p^\sharp}(\partial\Omega)}$$

holds true independently of the value of  $\|f\|_{L^{p^\sharp}(\partial\Omega)}$ , as soon as  $\|f\|_{L^{p^*}} = 1$ . By homogeneity, (20) holds true for all nonnegative functions  $f$  which lie in  $L^{p^*}(\Omega)$ . To remove the nonnegativity restriction, we note that for any function  $f \in W_{\text{loc}}^{1,1}(\Omega)$ , one has  $\nabla|f| = (\text{sgn } f)(\nabla f)$  [8, p. 130], so that

$$\|f\|_{L^{p^*}(\Omega)} = \||f|\|_{L^{p^*}(\Omega)}, \quad \|\nabla f\|_{L^p(\Omega)} = \|\nabla|f|\|_{L^p(\Omega)}, \quad \|f\|_{L^{p^\sharp}(\Omega)} = \||f|\|_{L^{p^\sharp}(\Omega)}.$$

Here we used the fact that the trace operator commutes with the absolute value operator (Theorem A.2 in the Appendix).

It seems that we just established Proposition 5 from Proposition 4, and so we just have to prove the latter. There are however two loopholes in that line of reasoning. The first is that we proved (20) by *assuming* that  $\|f\|_{L^{p^*}(\Omega)}$  is finite. It seems intuitive that this restriction can be removed by a density argument, but one should be careful when handling such arguments in a possibly unbounded domain. The second loophole is that actually, to establish (4) in full generality, we shall be led to use inequality (20) as a helpful technical tool! Yet the whole reasoning is not doomed: it will appear that a certain level of generality for inequality (19) will be

enough to prove inequality (20) in a more general context, which in turn will enable to prove (19) in a still more general context, etc. So both Propositions 4 and 5 will be established *at the same time*, in several runs.

**2.1. Preparations.** Let  $f$  and  $g$  be as in Proposition 4, and let  $F := 1_\Omega f^{p^*}$ ,  $G := g^{p^*}$ ; by assumption, both  $F$  and  $G$  are probability densities. Without loss of generality, we assume  $y_0 = 0$  in Proposition 4. An easy truncation and monotonicity argument in inequality (19) (replace  $g$  by  $C(R)1_{B_R(0)}g$  and let  $R \rightarrow \infty$ ) allows to deduce the case  $R = \infty$  from the case  $R < +\infty$ . Therefore, in all the sequel of this section we assume that  $g$  is compactly supported ( $R < +\infty$ ).

By Brenier's Theorem [3], or rather the slightly more general version established by McCann (see [12] or [14, Corollary 2.30]) there exists a proper lower semicontinuous convex function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  such that

$$(\nabla\varphi)\#(F dx) = G dx.$$

Here the symbol “ $\#$ ” is used for the operation of measure-theoretical push-forward (recall that if  $\psi : X \rightarrow Y$  and  $\mu$  is a measure on  $X$  then  $\psi\#\mu(A) := \mu(\psi^{-1}(A))$  for every  $A \subset Y$ ). Moreover,  $\nabla\varphi$  is the gradient of  $\varphi$ , which in view of Rademacher's theorem [8, p. 81] can be understood either as the distributional, or as the classical gradient of  $\varphi$ : it is a locally bounded measurable function defined almost everywhere in the interior  $U$  of the convex set  $\text{Dom}(\varphi)$  where  $\varphi$  is finite.

*Remark.* The existence of  $\nabla\varphi$  can be proven by elementary techniques which do not involve Sobolev inequalities of any kind.

As proven by McCann (see [12] or [14, Theorem 4.8]), the Monge-Ampère equation

$$(22) \quad F(x) = G(\nabla\varphi(x)) \det \nabla_A^2 \varphi(x)$$

holds true  $F(x) dx$ -almost everywhere; in this context it is just a change of variable formula. Here  $\nabla_A^2 \varphi$  stands for the Aleksandrov Hessian of  $\varphi$ , which is nothing but the absolutely continuous part of the distributional Hessian of  $\varphi$  in  $U$ . This result is strongly based on Aleksandrov's Theorem [8, p. 242] about the almost everywhere second differentiability of convex functions.

Since  $G$  is compactly supported ( $R < +\infty$ ),  $\varphi$  can be assumed to be Lipschitz, and, in particular, finite everywhere (so  $U = \mathbb{R}^n$ ). Let us give a short proof. First,  $\nabla\varphi$  takes its values in the support of  $G$  (see [14, Theorem 2.12 (ii)]); the theorem is stated there only in the case when  $F$  has finite moments of second order, but the general case can be proven just the same, or deduced by approximation). As a

consequence,  $\partial\varphi(U) \subset \overline{\nabla\varphi(U)} \subset B_R(0)$ . Now consider the Lipschitz convex function

$$\tilde{\varphi}(x) = \sup_{|y| \leq R} [x \cdot y - \varphi^*(y)].$$

It is clear that  $\tilde{\varphi} \leq \varphi^{**} = \varphi$ . For any  $x \in U$ , there exists  $y \in \partial\varphi(x) \subset B_R(0)$ , and then  $x \cdot y - \varphi^*(y) = \varphi(x)$ , which shows that  $\tilde{\varphi}(x) \geq \varphi(x)$ . The conclusion is that  $\tilde{\varphi}$  coincides with  $\varphi$  on  $U$ ; as a consequence,  $\nabla\tilde{\varphi}$  also coincides with  $\nabla\varphi$  on  $U$ . But the complementary of  $U$  is of zero measure for  $F dx$ , because it is made of the complementary of  $\text{Dom}(\varphi)$ , which is negligible, and of the boundary of this set, which has zero Lebesgue measure. We infer that

$$\nabla\tilde{\varphi}\#(F dx) = \nabla\varphi\#(F dx).$$

Thus, replacing  $\varphi$  by  $\tilde{\varphi}$  if necessary, we may assume that  $\varphi$  is  $R$ -Lipschitz.

**2.2. Basic transportation argument.** We can now start the core of the proof, in the same manner as in [7]. The main technical point lies in the justification of an integration by parts, which will be temporarily left apart.

Let  $f$  be a nonnegative locally integrable function on  $\Omega$ , vanishing at infinity, such that  $\nabla f \in L^p(\Omega)$ ,  $f \in L^{p^\sharp}(\partial\Omega)$  and  $\|f\|_{L^{p^\sharp}(\Omega)} = 1$ . We also consider a nonnegative function  $g$ , supported in the ball  $B_R(0)$ ,  $R < +\infty$ , such that  $\|g\|_{L^{p^\sharp}(\mathbb{R}^n)} = 1$ . We define  $F := f^{p^\sharp}$  and  $G := g^{p^\sharp}$ .

By playing with the exponents and using the definition of push-forward, we can write

$$\int_{\mathbb{R}^n} g^{p^\sharp} = \int_{\mathbb{R}^n} g^{-p^\sharp/n}(y) g^{p^\sharp}(y) dy = \int_{\Omega} g^{-p^\sharp/n}(\nabla\varphi(x)) f^{p^\sharp}(x) dx.$$

By (22) and the arithmetic-geometric inequality,

$$\begin{aligned} \int_{\Omega} g^{-p^\sharp/n}(\nabla\varphi(x)) f^{p^\sharp}(x) dx &= \int_{\Omega} (\det \nabla_A^2 \varphi(x))^{1/n} f^{p^\sharp(1-1/n)}(x) dx \\ &\leq \frac{1}{n} \int_{\Omega} \Delta_A \varphi(x) f^{p^\sharp(1-1/n)}(x) dx, \end{aligned}$$

where  $\Delta_A \varphi$  is the trace of  $\nabla_A^2 \varphi$ .

Since  $\nabla_A^2 \varphi$  is bounded from above by the distributional Hessian  $\nabla^2 \varphi$ , it follows that its trace is bounded from above by the distributional Laplacian  $\Delta \varphi$  of  $\varphi$ . This and integration by parts formally imply the inequality

$$(23) \quad \int_{\Omega} \Delta_A \varphi f^{p^\sharp} \leq -p^\sharp \int_{\Omega} \nabla \varphi \cdot f^{p^\sharp-1} \nabla f + \int_{\partial\Omega} \nabla \varphi \cdot \sigma f^{p^\sharp},$$

where  $\sigma$  is the outer normal vector to  $\Omega$ . We postpone the justification of this integration by parts to the next subsection. Taking formula (23) for granted, let us go on with the argument.

By Hölder's inequality and the definition of push-forward, the first term can be estimated as follows:

$$(24) \quad -p^\sharp \int_{\Omega} \nabla \varphi \cdot f^{p^\sharp-1} \nabla f \leq p^\sharp \|\nabla f\|_{L^p(\Omega)} \left( \int_{\Omega} f^{p^\sharp}(x) |\nabla \varphi(x)|^{p'} dx \right)^{1/p'}$$

$$= p^\sharp \|\nabla f\|_{L^p(\Omega)} \left( \int_{\mathbb{R}^n} g^{p^\sharp}(y) |y|^{p'} dy \right)^{1/p'}.$$

The other term on the right-hand side of (23) is easily dealt with: since  $\nabla \varphi$  takes its values in  $B_R(0)$ ,

$$\int_{\partial\Omega} (\nabla \varphi \cdot \sigma) f^{p^\sharp} \leq R \int_{\partial\Omega} f^{p^\sharp} = R \|f\|_{L^{p^\sharp}(\partial\Omega)}^{p^\sharp}.$$

The conclusion is

$$(25) \quad \int_{\Omega} \Delta \varphi f^{p^\sharp} \leq p^\sharp \|\nabla f\|_{L^p(\Omega)} \left( \int_{\mathbb{R}^n} g^{p^\sharp}(y) |y|^{p'} dy \right)^{1/p'} + R \|f\|_{L^{p^\sharp}(\partial\Omega)}^{p^\sharp}.$$

To summarize: We have proven Proposition 4, taking the integration by parts formula (23) for granted. And as explained in the beginning of this section, this implies Proposition 5 if in the statement of that Proposition we admit  $f \in L^{p^\sharp}(\Omega)$ .

These technical holes will be filled up in the next subsection. The reader who does not care about technical details may safely skip this bit and start again at Section 3.

**2.3. Integration by parts and related justifications.** We shall now justify (23), and at the same time complete the proofs of Propositions 4 and 5. To do so, we shall use approximation arguments and regularization in the same spirit as in [7, section 4]. With respect to that source, the main difference lies in the presence of a boundary term; there are also minor simplifications due to the assumption  $R < +\infty$ . As we already warned, this will be a slightly technical business, which can be skipped at first reading.

We divide the argument in four steps, by increasing order of generality.

In the **first step**, we assume that

$$\Omega \text{ is bounded and } f \in C^1(\overline{\Omega}).$$

Since  $\Delta_A \varphi$  is bounded above by the distributional Laplacian  $\Delta \varphi$ , which is a nonnegative measure, we can write

$$\int_{\Omega} \Delta_A \varphi f^{p^\sharp} \leq \int_{\Omega} \Delta \varphi f^{p^\sharp}.$$

For each  $k \in \{1, \dots, n\}$ ,  $\partial \varphi / \partial x_k$  is of bounded variation in  $\Omega$ ;  $\Omega$  is bounded and has Lipschitz boundary; and  $f$  is of class  $C^1$ ; so we are in a position to apply the integration by parts formula for  $BV$  functions [8, Theorem 1, p. 177]:

$$\int_{\Omega} \frac{\partial}{\partial x_k} \frac{\partial \varphi}{\partial x_k} f^{p^\sharp} = - \int_{\Omega} \frac{\partial \varphi}{\partial x_k} \frac{\partial f^{p^\sharp}}{\partial x_k} + \int_{\partial \Omega} \frac{\partial \varphi}{\partial x_k} \sigma_k f^{p^\sharp} d\mathcal{H}^{n-1},$$

where  $\sigma_k$  is the  $k^{\text{th}}$  component of the unit normal vector  $\sigma$ . Here again, we make no notational difference between  $\partial \varphi / \partial x_k$  and its trace on  $\partial \Omega$ . Summing up all these formulae, we obtain

$$\int_{\Omega} \Delta \varphi f^{p^\sharp} = - \int_{\Omega} \nabla \varphi \cdot \nabla (f^{p^\sharp}) + \int_{\partial \Omega} \nabla \varphi \cdot \sigma f^{p^\sharp}.$$

Since  $f$  is smooth, we can apply the chain-rule to rewrite the right-hand side as

$$-p^\sharp \int_{\Omega} f^{p^\sharp-1} \nabla \varphi \cdot \nabla f + \int_{\partial \Omega} \nabla \varphi \cdot \sigma f^{p^\sharp},$$

which proves (23).

**Remark 6.** Before turning to the second step, we note that all the rest of the argument in the previous subsection can now be completed, and in particular we now have a proof of Proposition 5 in the case when  $\Omega$  is bounded and  $f \in C^1(\Omega)$ . This will be useful in the Appendix and in the next steps.

In the **second step**, we relax the assumption of smoothness for  $f$ , but we still assume that

$$\Omega \text{ is bounded.}$$

In particular, since  $\Omega$  is bounded and  $p^* \geq p$ , we know that  $f$  lies in  $L^p(\Omega)$ , and therefore in the Sobolev space  $W^{1,p}(\Omega)$ . Since  $\Omega$  is bounded and Lipschitz, we can approximate  $f$  by a sequence  $f_k \in C^\infty(\overline{\Omega})$ , such that  $f_k \rightarrow f$  in  $W^{1,p}(\Omega)$  as  $k \rightarrow \infty$  (see [8, p. 127]). Up to extracting a subsequence, we may also assume that  $f_k$  converges to  $f$  almost everywhere. Moreover, the continuity of the trace operator from  $W^{1,p}(\Omega)$  to  $L^{p^\sharp}(\partial \Omega)$  (see Theorem A.3 in the Appendix; our proof uses the first step above) implies that  $f_k$  converges to  $f$  in  $L^{p^\sharp}(\partial \Omega)$ . Finally, the density of  $C^\infty(\overline{\Omega})$  in  $W^{1,p}(\Omega)$  and the previously mentioned continuity property of the trace operator also imply that the homogeneous trace Sobolev inequality (20) holds true

for all  $f \in W^{1,p}(\Omega)$ ; from this it follows that  $f_k \rightarrow f$  in  $L^{p^*}(\Omega)$ . To sum up,  $f_k$  converges to  $f$  almost everywhere, in  $W^{1,p}(\Omega)$ ,  $L^{p^*}(\Omega)$ , and  $L^{p^\sharp}(\partial\Omega)$ .

From Step 1 we know that each  $f_k$  satisfies

$$\int_{\Omega} \Delta\varphi f_k^{p^\sharp} \leq - \int_{\Omega} \nabla\varphi \cdot f_k^{p^\sharp-1} \nabla f_k + \int_{\partial\Omega} \nabla\varphi \cdot \sigma f_k^{p^\sharp}.$$

It remains to pass to the limit in this inequality as  $k \rightarrow \infty$ .

Since  $f_k \rightarrow f$  almost everywhere, Fatou's lemma implies

$$(26) \quad \int_{\Omega} \Delta_A\varphi f^{p^\sharp} \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \Delta_A\varphi f_k^{p^\sharp}.$$

Since  $f_k$  converges to  $f$  in  $L^{p^*}(\Omega)$ ,  $f_k^{p^\sharp-1}$  converges to  $f^{p^\sharp-1}$  in  $L^{p^*/(p^\sharp-1)} = L^{p'}(\Omega)$ ; on the other hand,  $\nabla f_k$  converges to  $\nabla f$  in  $L^p(\Omega)$ . Combining this information with the boundedness of  $\nabla\varphi$ , we get

$$(27) \quad \int_{\Omega} f_k^{p^\sharp-1} \nabla\varphi \cdot \nabla f_k \xrightarrow{k \rightarrow \infty} \int_{\Omega} f^{p^\sharp-1} \nabla\varphi \cdot \nabla f.$$

Finally, the boundedness of  $\nabla\varphi$  implies the boundedness of its trace on  $\partial\Omega$  (this is a consequence of the nonnegativity of the trace operator), and we conclude that

$$(28) \quad \int_{\partial\Omega} \nabla\varphi \cdot \sigma f_k^{p^\sharp} \xrightarrow{k \rightarrow \infty} \int_{\partial\Omega} \nabla\varphi \cdot \sigma f^{p^\sharp}.$$

The combination of (26), (27) and (28) yields

$$\int_{\Omega} \Delta\varphi f^{p^\sharp} \leq -p^\sharp \int_{\Omega} f^{p^\sharp-1} \nabla\varphi \cdot \nabla f + \int_{\partial\Omega} \nabla\varphi \cdot \sigma f^{p^\sharp}.$$

This concludes the proof of Proposition 4 for bounded  $\Omega$ . It also follows that Proposition 5 holds true as soon as  $\Omega$  is bounded and  $f \in L^{p^*}(\Omega)$ , with constants depending neither on  $\Omega$  nor on  $\|f\|_{L^{p^*}(\Omega)}$ .

In the **third step** of the justification, we relax the assumption of boundedness of the domain.

For that we introduce a smooth cut-off function  $\chi$  on  $\mathbb{R}^n$ , with values in  $[0, 1]$ , identically equal to 1 on  $B_1(0)$  and identically equal to 0 on  $\mathbb{R}^n \setminus B_2(0)$ , and we set

$$f_\varepsilon(x) := f(x)\chi(\varepsilon x); \quad \Omega_\varepsilon := \Omega \cap B_{3/\varepsilon}(0).$$



Applying the result of Step 2, we have

$$(29) \quad \int_{\Omega_\varepsilon} \Delta\varphi f_\varepsilon^{p^\sharp} \leq -p^\sharp \int_{\Omega_\varepsilon} f_\varepsilon^{p^\sharp-1} \nabla\varphi \cdot \nabla f_\varepsilon + \int_{\partial\Omega_\varepsilon} \nabla\varphi \cdot \sigma f_\varepsilon^{p^\sharp}.$$

Our problem is now to pass to the limit in this equation as  $\varepsilon \rightarrow 0$ . For the term on the left-hand side, this can be done by using just the monotone convergence theorem, while for the last term on the right-hand side, it suffices to use the dominated convergence theorem (recall that  $f \in L^{p^\sharp}(\partial\Omega)$  and  $\nabla\varphi \in L^\infty$ ). The first term on the right-hand side will require more work. Before handling it, we shall establish a few bounds on  $f_\varepsilon$ .

Since  $f_\varepsilon$  is identically 0 close to the boundary of  $B_{3/\varepsilon}(0)$ , the trace of  $f_\varepsilon$  vanishes on  $\partial\Omega_\varepsilon \cap B_{3/\varepsilon}(0)$ . So the trace of  $f_\varepsilon$  is only nonzero on  $\partial\Omega \setminus B_{3/\varepsilon}(0)$ , and there it is bounded by the trace of  $f$ . It follows that

$$\|f_\varepsilon\|_{L^{p^\sharp}(\partial\Omega_\varepsilon)} \leq \|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

Using the  $L^{p^*}$  bound on  $f$ , we now show that  $\nabla f_\varepsilon$  converges to  $\nabla f$ . By the formula  $\nabla f_\varepsilon - \nabla f = \nabla f(1 - \chi_\varepsilon) + f(\nabla\chi_\varepsilon)$ , we can write

$$\|\nabla f_\varepsilon - \nabla f\|_{L^p(\Omega)}^p \leq C \left( \int_{\Omega} 1_{|x| \geq \varepsilon^{-1}} |\nabla f(x)|^p dx + \varepsilon^p \int_{\Omega} 1_{\varepsilon^{-1} \leq |x| \leq 2\varepsilon^{-1}} f(x)^p dx \right).$$

Since  $|\nabla f|^p \in L^1(\Omega)$ , the first term goes to 0 by dominated convergence. On the other hand we can use Hölder's inequality to bound the second term as follows:

$$\begin{aligned} \varepsilon^p \int_{\Omega_\varepsilon} 1_{\varepsilon^{-1} \leq |x| \leq 2\varepsilon^{-1}} f^p(x) dx &\leq \varepsilon^p \left( \int_{\Omega_\varepsilon} f^{p^*} \right)^{p/p^*} \left( \int_{\Omega_\varepsilon \cap A} 1_{\varepsilon^{-1} \leq |x| \leq 2\varepsilon^{-1}} dx \right)^{p/n} \\ &\leq (|B_2(0) \setminus B_1(0)| \|f\|_{L^{p^*}})^p. \end{aligned}$$

This proves our claim.

Now, by dominated convergence,  $f_\varepsilon 1_{\Omega_\varepsilon}$  converges to  $f$  in  $L^{p^*}(\Omega)$ , and therefore  $f_\varepsilon^{p^\sharp-1} 1_{\Omega_\varepsilon}$  converges to  $f^{p^\sharp-1}$  in  $L^{p'}(\Omega)$ . Since also  $\nabla\varphi$  is bounded, we conclude that

$$\int_{\Omega_\varepsilon} f_\varepsilon^{p^\sharp-1} \nabla\varphi \cdot \nabla f_\varepsilon \longrightarrow \int_{\Omega} f^{p^\sharp-1} \nabla\varphi \cdot \nabla f.$$

This concludes the argument and allows us to pass to the limit in (29) as  $\varepsilon \rightarrow 0$ :

$$\int_{\Omega} \Delta\varphi f^{p^\sharp} = -p^\sharp \int_{\Omega} f^{p^\sharp-1} \nabla\varphi \cdot \nabla f + \int_{\partial\Omega} \nabla\varphi \cdot \sigma f^{p^\sharp}.$$

As a consequence of this step, the integration by parts formula, and Proposition 4 as well, are fully justified. As for Proposition 5, it is proven under the additional assumption that  $f$  lies in  $L^{p^*}$ , without boundedness assumption on  $\Omega$ .

We now come to **Step 4** of the argument, in which we relax the a priori assumption that  $f \in L^{p^*}(\Omega)$  in Proposition 5. Let  $\Omega$  be an arbitrary locally Lipschitz domain in  $\mathbb{R}^n$ , and  $f$  a locally integrable measurable function, such that  $\nabla f \in L^p(\Omega)$  and  $f \in L^{p^\sharp}(\partial\Omega)$ . Whenever  $0 < a < M < +\infty$ , we define

$$T_{a,M}(f) := (f - a)1_{a \leq f \leq M} + (M - a)1_{f > M}.$$

By the chain-rule for Sobolev functions [8, p. 130], we know that  $\nabla T_{a,M}(f) = (\nabla f)1_{a \leq f \leq M}$ , and that  $T_{a,M}$  commutes with the trace operator (see Theorem A.2 in the Appendix). Moreover,  $T_{a,M}$  does not increase any  $L^q$  norm. We deduce that

$$(30) \quad \|\nabla T_{a,M}(f)\|_{L^p(\Omega)} \leq \|\nabla f\|_{L^p(\Omega)}; \quad \|T_{a,M}(f)\|_{L^{p^\sharp}(\partial\Omega)} \leq \|f\|_{L^{p^\sharp}(\partial\Omega)}.$$

Now note that  $T_{a,M}(f)$  is bounded above by  $M$ , and vanishes outside of the set of finite measure  $\{x; f(x) \geq a\}$ ; in particular it lies in  $L^{p^*}(\Omega)$ . A priori the  $L^{p^*}$  norm of  $T_{a,M}(f)$  depends on  $M$ , but the bounds (30) and the trace Sobolev inequality established at the end of Step 3 implies that this is not the case. We can now let  $M$  go to infinity, then  $a$  go to 0, and use the monotone convergence theorem to conclude that  $f$  is bounded in  $L^{p^*}$ . Another application of Step 3 concludes the proof. The proofs of both Propositions 4 and 5 are now complete.

### 3. STUDY OF $\Phi_n^{(p)}$

The purpose of this section is to establish Theorems 1, 2 and 3, apart from the rigidity theorem contained in Theorem 2. From Proposition 5 and the explanations given in Subsection 1.7, we see that it is sufficient to

(i) prove the inequality (8):

$$\|\nabla f\|_{L^p(\Omega)} \geq \|f\|_{L^{p^*}(\Omega)} \tilde{\Phi}_n^{(p)} \left( \frac{\|f\|_{L^{p^\sharp}(\partial\Omega)}}{\|f\|_{L^{p^*}(\Omega)}} \right),$$

where

$$\tilde{\Phi}_n^{(p)} := 1_{[0, T_n(p)]} \Phi_n^{(p)}.$$

(ii) show that  $\Phi_n^{(p)}$  is a well-defined function (not just a parametric curve) on the interval  $[0, T_n(p)]$ ;

(iii) prove that  $\Phi_n^{(p)}(0) = S_n^{-1}(p)$  and  $\Phi_n^{(p)}(T_n(p)) = 0$ ;

(iv) show that

$$\forall T \in [0, T_n(p)], \quad \Phi_n^{(p)}(T) \geq S_n^{-1}(p)[1 - (CT)^p]^{1/p};$$

(v) establish the concavity of  $\Phi_n^{(p)}$  (strict for  $p > 1$ ) on the interval  $[0, T_n(p)]$ .

These points will be addressed one after another. As we explained before, we only consider the case  $p \in (1, n)$ , since the case  $p = 1$  can be treated directly.

**3.1. Construction of  $\Phi_n^{(p)}$ .** Let, on  $\mathbb{R}^n$ ,

$$(31) \quad w_{a,b}(x) := \frac{1}{(a + b|x|^{p'})^{\frac{n-p}{p}}}.$$

Recall that the norm used in (31) and the norm used in  $\|\nabla f\|_{L^p}$  are, by assumption, dual to each other. For any choice of  $a, b > 0$ , the function  $w_{a,b}$  is optimal in the Sobolev inequality set in the whole space (this fact is well-known in the Euclidean case; a proof in the general case can be found in [7]). When  $\Omega = \mathbb{R}^n$ , the invariance of the Sobolev inequality under dilations, translations and multiplications makes it useless to consider the whole family of functions  $w_{a,b}$ ; but in general we do not have so much invariance and it will be important to allow various choices of  $a, b$ . We define  $g_{a,b}$  as the normalized truncation of  $w_{a,b}$  to the unit ball  $B^n$ :

$$(32) \quad g_{a,b} := \frac{1_{B^n} w_{a,b}}{\|w_{a,b}\|_{L^{p^*}(B^n)}}.$$

We now apply Proposition 4 with an arbitrary function  $f$  in  $L^{p^*}(\Omega)$  and  $g = g_{a,b}$ . Let

$$(33) \quad G_{a,b} := \|\nabla g_{a,b}\|_{L^p(\Omega)}, \quad Y_{a,b} := \left( \int_{B^n} g_{a,b}(x)^{p^*} |x|^{p'} dx \right)^{1/p'}, \quad T_{a,b} := \left( \int_{S^{n-1}} g_{a,b}^{p^\sharp} \right)^{1/p^\sharp}.$$

With that choice (19) can be restated as

$$n \int_{B^n} g_{a,b}^{p^\sharp} \leq Y_{a,b} \frac{\|\nabla f\|_{L^p(\Omega)}}{\|f\|_{L^{p^*}(\Omega)}} + \frac{1}{p^\sharp} \left( \frac{\|f\|_{L^{p^\sharp}(\partial\Omega)}}{\|f\|_{L^{p^*}(\Omega)}} \right)^{p^\sharp}.$$

On the other hand, it is easy to check that  $g_{a,b}$  is optimal in this inequality. In fact, an application of the divergence theorem and direct calculation yield the identities

$$n \int_{B^n} g_{a,b}^{p^\sharp} = -p^\sharp \int_{B^n} y \cdot g_{a,b}^{p^\sharp-1}(y) \nabla g_{a,b} dy + \int_{S^{n-1}} g_{a,b}^{p^\sharp};$$

$$(34) \quad - \int_{B^n} y \cdot g_{a,b}^{p^\sharp-1}(y) \nabla g_{a,b} dy = G_{a,b} Y_{a,b}.$$

Note how, in fact, formula (34) suggest an optimality condition, determining the optimizers of the Sobolev inequality, in the form of a first-order, integrable ordinary differential equation.

Moreover, if  $\Omega$  is connected, the study of cases of equality in this application of inequality (19) can be performed in exactly the same way as in [7, Proposition 6] (this point is not crucial to the rest of the paper, so we do not insist on it). We conclude to the following

**Proposition 7.** *Let  $\Omega$  be a locally Lipschitz domain in  $\mathbb{R}^n$ . For every locally integrable  $f : \Omega \rightarrow \mathbb{R}$  vanishing at infinity with  $\nabla f \in L^p(\Omega)$  we have, with the notation above,*

$$(35) \quad Y_{a,b} G_{a,b} + \frac{T_{a,b}^{p^\sharp}}{p^\sharp} \leq Y_{a,b} \frac{\|\nabla f\|_{L^p(\Omega)}}{\|f\|_{L^{p^\sharp}(\Omega)}} + \frac{1}{p^\sharp} \left( \frac{\|f\|_{L^{p^\sharp}(\partial\Omega)}}{\|f\|_{L^{p^\sharp}(\Omega)}} \right)^{p^\sharp};$$

Moreover, if  $\Omega$  is connected, then equality holds in (35) if and only if there exist  $\lambda \in \mathbb{R}$ ,  $x_0 \in \mathbb{R}^n$ ,  $r > 0$  such that

$$(36) \quad \Omega = B_r(x_0), \quad f(x) = \lambda w_{a,b} \left( \frac{x - x_0}{r} \right).$$

**Remark 8.** If in the above argument we consider  $\Omega = \mathbb{R}^n$  and choose  $\mathbb{R}^n$  in place of  $B^n$  and  $w_{a,b}(x)/\|w_{a,b}\|_{L^{p^\sharp}(\mathbb{R}^n)}$  in place of  $g_{a,b}$  then we prove as in [7] the sharp Sobolev inequality on  $\mathbb{R}^n$ , showing in particular that

$$\frac{\|\nabla w_{a,b}\|_{L^p(\mathbb{R}^n)}}{\|w_{a,b}\|_{L^{p^\sharp}(\mathbb{R}^n)}} = S_n^{-1}(p).$$

Let us now assume  $\|f\|_{L^{p^\sharp}(\Omega)} = 1$ , and use the shorthands  $G := \|\nabla f\|_{L^p(\Omega)}$ ,  $T := \|f\|_{L^{p^\sharp}(\partial\Omega)}$ . The preceding inequality can be rewritten as

$$(37) \quad G \geq G_{a,b} + \frac{1}{p^\sharp Y_{a,b}} \left( T_{a,b}^{p^\sharp} - T^{p^\sharp} \right) =: \Phi_{a,b}(T), \quad a > 0, b > 0.$$

In particular, it is clear that

$$T = T_{a,b} \implies G \geq G_{a,b}.$$

Therefore, the definition of  $\Phi_n^{(p)}$  as  $\Phi_{B^n}^{(p)}$  implies  $\Phi_n^{(p)}(T_{a,b}) \geq G_{a,b}$ . Since on the other hand the function  $w_{a,b}$  leads rise to the point  $(T_{a,b}, G_{a,b})$  in the  $(T, G)$  diagram, we

conclude that

$$\Phi_n^{(p)}(T_{a,b}) = G_{a,b}.$$

From (37) we see that there is another representation of  $\Phi = \Phi_n^{(p)}$ :

$$\Phi_n^{(p)}(T) = \sup_{a,b>0} \Phi_{a,b}(T), \quad \forall T \in \{T_{a,b} : a, b > 0\},$$

where the family of curves  $\Phi_{a,b}$  is defined by

$$\Phi_{a,b}(T) = A_{a,b} - C_{a,b}T^{p^\sharp},$$

$$A_{a,b} := \left[ G_{a,b} + \frac{T_{a,b}^{p^\sharp}}{p^\sharp Y_{a,b}} \right], \quad C_{a,b} = \frac{1}{p^\sharp Y_{a,b}}.$$

See Figure 2 for a qualitative picture of what goes on.

**3.2. Identification of  $\Phi_n^{(p)}$ .** We shall now identify the function  $\Phi_n^{(p)}$  constructed above, completing the proof of Theorem 1. It will be useful to define the variable

$$t = t(a, b) := \left( \frac{b}{a} \right)^{1/p'} \in (0, \infty),$$

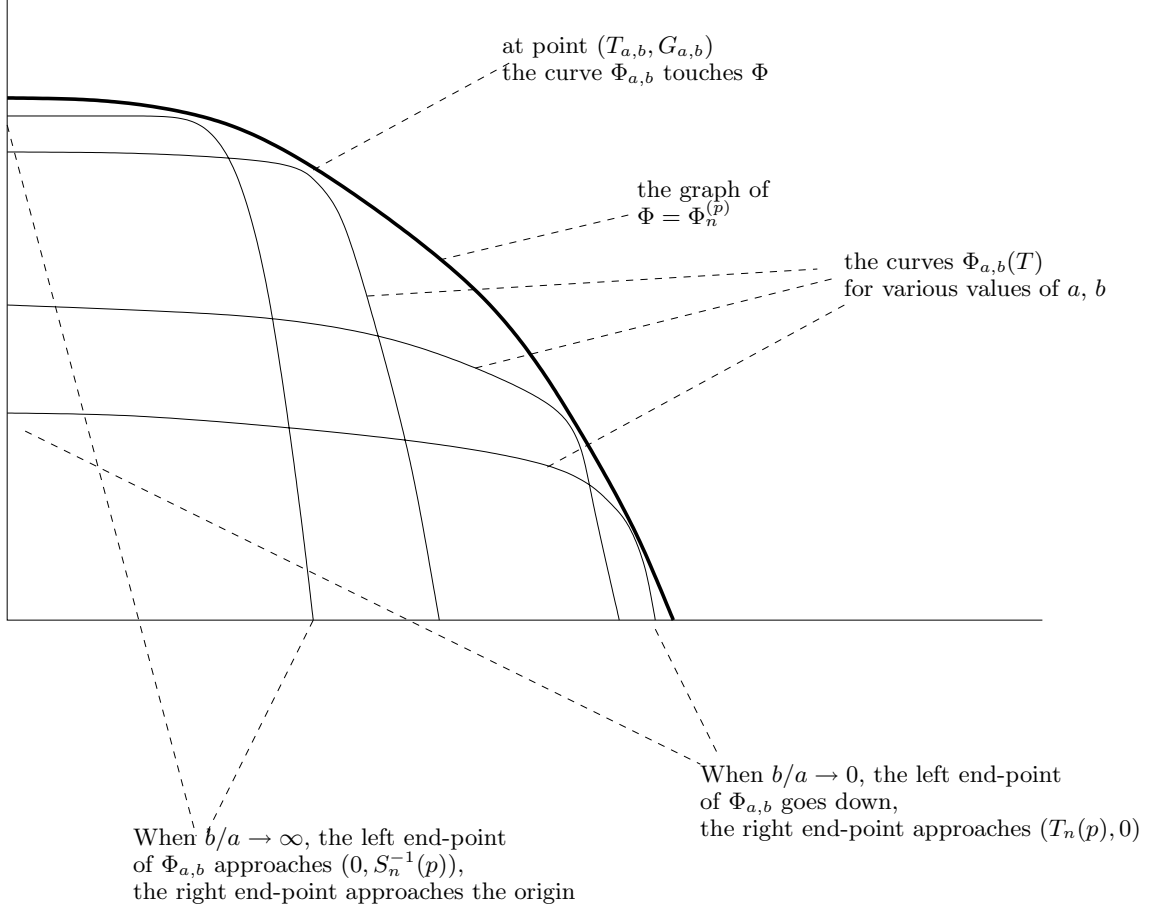
and the functions

$$\begin{aligned} h(t) &:= t^{n-1}(1+t^{p'})^{-n}, \\ \varphi(t) &:= \int_0^t s^{n-1}(1+s^{p'})^{-n} ds = \int_0^t h(s) ds, \\ \psi(t) &:= \int_0^t s^{n+p'-1}(1+s^{p'})^{-n} ds = \int_0^t s^{p'} h(s) ds. \end{aligned}$$

By simple calculations,

$$(38) \quad \|w_{a,b}\|_{L^{p^*}(B^n)} = \frac{1}{(at)^{n/p^*}} \|w_{1,1}\|_{L^{p^*}(tB^n)};$$

$$(39) \quad \|\nabla w_{a,b}\|_{L^p(B^n)} = \frac{1}{(at)^{n/p^*}} \|\nabla w_{1,1}\|_{L^p(tB^n)};$$

FIGURE 2.  $\Phi_n^{(p)}$  as an envelope

$$(40) \quad G_{a,b} = \frac{\|\nabla w_{1,1}\|_{L^p(tB^n)}}{\|w_{1,1}\|_{L^p(tB^n)}}$$

$$(41) \quad = |S^{n-1}|^{1/n} \left( \frac{n-p}{p-1} \right) \frac{\psi(t)^{1/p}}{\varphi(t)^{1/p^*}} =: G(t);$$

$$(42) \quad T_{a,b} = \left( \frac{|S^{n-1}|^{1/(n-1)} t h(t)}{\varphi(t)} \right)^{1/p^*} =: T(t);$$

$$(43) \quad Y_{a,b} = \frac{1}{t} \left( \frac{\psi(t)}{\varphi(t)} \right)^{1/p'} =: Y(t).$$

Now, the conclusions of the previous subsection translate into  $\Phi_n^{(p)}(T(t)) = G(t)$ . Moreover,  $\Phi_n^{(p)}(T) = \sup_{0 < t < \infty} \Phi_{[t]}(T)$ , where

$$\Phi_{[t]}(T) := G(t) + \frac{1}{p^\sharp Y(t)} (T(t)^{p^\sharp} - T^{p^\sharp}).$$

**3.3. End-points.** The curve  $\Phi_n^{(p)}$  crosses the  $G$  axis at  $S_n^{-1}(p)$ . This in fact is a property shared by all Lipschitz domains  $\Omega$ :  $\Phi_\Omega^{(p)}(0) = S_n^{-1}(p)$ . This can be easily proved as follows: let us fix  $x_0 \in \Omega$ ,  $r > 0$  such that  $x_0 + 2rB^n \subset \Omega$ ,  $\psi \in C_c^\infty(x_0 + 2rB^n)$  with  $\psi(x_0 + rx) = 1$  for every  $x \in B^n$  and  $0 \leq \psi \leq 1$ . For every  $a, b > 0$  we consider

$$f(x) = \psi(x) w_{a,b}(x - x_0), \quad \forall x \in \Omega.$$

Then  $\int_{\partial\Omega} f^{p^\sharp} = 0$  and

$$\frac{\|\nabla w_{a,b}\|_{L^p(rB^n)}}{\|w_{a,b}\|_{L^{p^*}(2rB^n)}} \leq \frac{\|\nabla f\|_{L^p(\Omega)}}{\|f\|_{L^{p^*}(\Omega)}} \leq \frac{\|\nabla w_{a,b}\|_{L^p(2rB^n)}}{\|w_{a,b}\|_{L^{p^*}(rB^n)}}.$$

By changing variables and letting  $(b/a)^{1/p'} = t \rightarrow \infty$  we discover that  $\Phi_\Omega^{(p)}(0) \leq S_n^{-1}(p)$ . On the other hand, by the Sobolev inequality on  $\mathbb{R}^n$  the reverse inequality holds too, and thus  $\Phi_\Omega^{(p)}(0) = S_n^{-1}(p)$ .

We next check that  $\Phi_n^{(p)}$  crosses the  $T$  axis precisely at  $T_n(p)$ :

$$\lim_{t \rightarrow 0} T(t) = n^{1/p^*} |S^{n-1}|^{1/p^\sharp - 1/p^*} = T_n(p).$$

Finally, since  $\varphi(+\infty) < +\infty$ , clearly  $T(+\infty) = 0$ .

To summarize: the two end-points of the parametric curve  $\Phi_n^{(p)}$  as  $t \rightarrow 0$  and  $t \rightarrow \infty$  are, respectively,  $(T_n(p), 0)$  and  $(0, S_n^{-1}(p))$ .

**3.4. Study of a parametric curve.** Since  $T$  is continuous, its range contains the whole of  $[0, T_n(p)]$ . We shall now get a more precise description by proving the following properties of  $G$ ,  $T$  and  $Y$  as functions of  $t$ :

**Proposition 9.** *With the notation above,  $G(t)$  is strictly increasing as a function of  $t \in (0, +\infty)$ , while  $T(t)$  and  $Y(t)$  are strictly decreasing; moreover,*

$$\begin{aligned} G(0) &= 0, & G(\infty) &= S_n^{-1}(p); \\ T(0) &= T_n(p), & T(\infty) &= 0; \\ Y(0) &= [n/(n+p')]^{1/p'}, & Y(\infty) &= 0. \end{aligned}$$

Furthermore, the following formulas hold:

$$(44) \quad G'(t) = G(t) \left\{ \frac{1}{p} \frac{\psi'(t)}{\psi(t)} - \frac{1}{p^*} \frac{\varphi'(t)}{\varphi(t)} \right\};$$

$$(45) \quad T'(t) = \frac{T(t)}{p^*} \left\{ \frac{(th(t))'}{th(t)} - \frac{\varphi'(t)}{\varphi(t)} \right\};$$

$$(46) \quad Y'(t) = Y(t) \left\{ \frac{1}{p'} \frac{\psi'(t)}{\psi(t)} - \frac{1}{p'} \frac{\varphi'(t)}{\varphi(t)} - \frac{1}{t} \right\}.$$

This proposition implies in particular that  $\Phi_n^{(p)}$  is the graph of a smooth function  $T \mapsto G(T)$ .

*Proof.* The limit values of  $T$  have been already discussed in the previous subsection. Formulas (44),(45) and (46) follow easily from a direct computation.

*Monotonicity of  $G$ :* Since  $\varphi'(t) = h(t)$ ,  $\psi'(t) = t^{p'} h(t)$ ,  $\psi(t) < t^{p'} \varphi(t)$  and  $p^* > p$ , we deduce

$$\frac{1}{p} \frac{\psi'(t)}{\psi(t)} - \frac{1}{p^*} \frac{\varphi'(t)}{\varphi(t)} > \frac{h(t)}{p^* \varphi(t) \psi(t)} \left\{ t^{p'} \varphi(t) - \psi(t) \right\} > 0$$

and thus  $G'(t) > 0$ , so  $G$  is increasing.

*Limit values of  $G$ :* By Remark 8, formula (40) and the dominated convergence theorem,  $G(t) \rightarrow S_n^{-1}(p)$  as  $t \rightarrow \infty$ . On the other hand, as  $t \rightarrow 0$ ,  $\varphi$  and  $\psi$  admit the Taylor expansions  $\varphi(t) = t^n/n + o(t^n)$  and  $\psi(t) = t^{n+p'}/(n+p') + o(t^{n+p'})$ ; so that

$$G(t) = c(n, p) \frac{\frac{t^{(n+p')/p}}{(n+p')^{1/p}} + o(t^{(n+p')/p})}{\frac{t^{n/p^*}}{n^{1/p^*}} + o(t^{n/p^*})},$$

and thus  $G(t) \rightarrow 0$  as  $t \rightarrow 0$ , since  $(n+p')p^* > np$ .

*Monotonicity of  $T$ :* To establish it, it suffices to prove  $(th(t))'\varphi(t) - th(t)^2 < 0$ . Since

$$(47) \quad h'(t) = \frac{h(t)}{t} \left\{ n - 1 - np' \frac{t^{p'}}{1+t^{p'}} \right\},$$

it suffices to check that

$$(48) \quad n\varphi(t) \left\{ 1 - p' \frac{t^{p'}}{1+t^{p'}} \right\} < th(t), \quad \forall t > 0.$$



An integration by parts and the use of (47) show that

$$(49) \quad n\varphi(t) = th(t) + np' \int_0^t \frac{s^{p'} h(s)}{1 + s^{p'}} ds$$

Thus (48) holds if

$$np' \int_0^t \frac{s^{p'} h(s)}{1 + s^{p'}} ds \leq np' \frac{t^{p'}}{1 + t^{p'}} \varphi(t),$$

which is true since  $t \mapsto t^{p'}(1 + t^{p'})^{-1}$  is increasing.

*Limit values of Y:* Trivially  $Y(+\infty) = 0$ . By Taylor's formula,

$$Y(t)^{p'} = \frac{1}{t^{p'}} \frac{\frac{t^{n+p'}}{n+p'} + o(t^{n+p'})}{\frac{t^n}{n} + o(t^n)},$$

so that  $Y(0) = [n/(n + p')]^{1/p'}$ .

*Monotonicity of Y:*  $Y$  is strictly decreasing if

$$t^{p'} h(t) t \varphi(t) < \psi(t) (th(t) + p' \varphi(t)),$$

an inequality which can be rewritten as

$$(50) \quad \frac{t^{p'} \varphi(t)}{\psi(t)} < 1 + \frac{p' \varphi(t)}{th(t)}.$$

Since  $(t^{p'} \varphi(t))' = p' t^{p'-1} \varphi(t) + t^{p'} h(t)$ , we have

$$(51) \quad t^{p'} \varphi(t) = \psi(t) + p' \int_0^t s^{p'-1} \varphi(s) ds.$$

Plugging this into (50), we are led to ask whether

$$\frac{p' \int_0^t s^{p'-1} \varphi(s) ds}{\psi(t)} < \frac{p' \varphi(t)}{th(t)},$$

or equivalently whether

$$1 > \left\{ \int_0^t s^{p'} \frac{\frac{1}{s} \int_0^s h(r) dr}{\frac{1}{t} \int_0^t h(r) dr} ds \right\} \left\{ \int_0^t s^{p'} \frac{h(s)}{h(t)} ds \right\}^{-1}.$$

This is true since the function

$$t \mapsto \frac{1}{th(t)} \int_0^t h(r) dr$$

is strictly increasing. Indeed, its first derivative is

$$\frac{th(t)^2 - (th(t))'\varphi(t)}{t^2h(t)^2},$$

which is positive for every  $t > 0$ , as we noticed while we were studying the monotonicity of  $T$ .  $\square$

**3.5. Concavity of  $\Phi_n^{(p)}$ .** Let us now prove that  $\Phi_n^{(p)}$  is concave on  $[0, T_n(p)]$ . In the sequel, we shall use the shorthand  $\Phi = \Phi_n^{(p)}$  to alleviate notation.

Since  $G(t) = \Phi(T(t))$  and all functions involved are smooth, we know that

$$\frac{d\Phi}{dT}(T(t)) = \frac{G'(t)}{T'(t)}.$$

We also know that  $\Phi(T) \geq \Phi_{[t]}(T)$ , with equality at  $T = T(t)$  and where  $\Phi_{[t]}$  is concave and smooth. Thus, if  $\Phi$  is concave, then it must be that

$$\frac{d\Phi}{dT}(T(t)) = \frac{d\Phi_{[t]}}{dT}(T(t)) = -\frac{T(t)^{p^\sharp-1}}{Y(t)}.$$

Our proof will split into two steps:

*Step 1:* We shall show that

$$\frac{G'(t)}{T'(t)} = -\frac{T(t)^{p^\sharp-1}}{Y(t)};$$

*Step 2:* We shall show that

$$\frac{d}{dt} \left( \frac{T(t)^{p^\sharp-1}}{Y(t)} \right) < 0.$$

From Step 1 we deduce that

$$\frac{d^2\Phi}{dT^2}(T(t)) = -\frac{1}{T'(t)} \frac{d}{dt} \left( \frac{T(t)^{p^\sharp-1}}{Y(t)} \right),$$

and then by Step 2 we deduce the (strict) concavity of  $\Phi$  on  $[0, T_n(p)]$ .

*Proof of Step 1:* Recall (34),

$$(52) \quad p^\sharp G(t)Y(t) + T(t)^{p^\sharp} = n \int_{B^n} g_{a,b}^{p^\sharp}, \quad t = (b/a)^{1/p'}.$$

On one hand, if  $tr = s$ ,

$$\int_{B^n} w_{a,b}^{p^\sharp} = |S^{n-1}| \int_0^1 \frac{r^{n-1} dr}{(a + br^{p'})^{n-1}} = \frac{|S^{n-1}|}{a^{n-1} t^n} \int_0^t \frac{s^{n-1} ds}{(1 + s^{p'})^{n-1}},$$

so that by (38)

$$n \int_{B^n} g_{a,b}^{p^\sharp} = \frac{n \int_{B^n} w_{a,b}^{p^\sharp}}{\|w_{a,b}\|_{L^{p^\sharp}(B^n)}^{p^\sharp}} = \frac{n |S^{n-1}|^{1/n}}{t \varphi(t)^{1-1/n}} \int_0^t \frac{s^{n-1} ds}{(1 + s^{p'})^{n-1}}.$$

An integration by parts reveals that

$$\int_0^t \frac{s^{n-1} ds}{(1 + s^{p'})^{n-1}} = \frac{t}{n} \left\{ h(t)(1 + t^{p'}) + p'(n-1) \frac{\psi(t)}{t} \right\},$$

so that, by (52) we end up with

$$(53) \quad p^\sharp G(t)Y(t) + T(t)^{p^\sharp} = \frac{|S^{n-1}|^{1/n}}{\varphi(t)^{1-1/n}} \left\{ h(t)(1 + t^{p'}) + p'(n-1) \frac{\psi(t)}{t} \right\}.$$

What we wish to prove is

$$(54) \quad G'(t)Y(t) + T(t)^{p^\sharp-1} T'(t) = 0;$$

by differentiating (53) we find that our goal is achieved if we can prove

$$(55) \quad p^\sharp G(t)Y'(t) = \frac{d}{dt} \left\{ \frac{|S^{n-1}|^{1/n}}{\varphi(t)^{1-1/n}} \left\{ h(t)(1 + t^{p'}) + p'(n-1) \frac{\psi(t)}{t} \right\} \right\}.$$

On one hand by (41), (43) and (46) we deduce that

$$\begin{aligned} p^\sharp G(t)Y'(t) &= p^\sharp |S^{n-1}|^{1/n} \left( \frac{n-p}{p-1} \right) \frac{\psi(t)^{1/p}}{\varphi(t)^{1/p^\sharp}} \frac{1}{t} \left( \frac{\psi(t)}{\varphi(t)} \right)^{1/p'} \left\{ \frac{1}{p'} \frac{\psi'(t)}{\psi(t)} - \frac{1}{p'} \frac{\varphi'(t)}{\varphi(t)} - \frac{1}{t} \right\} \\ &= (n-1)p' |S^{n-1}|^{1/n} \frac{\psi(t)}{t \varphi(t)^{1-1/n}} \left\{ \frac{1}{p'} \frac{\psi'(t)}{\psi(t)} - \frac{1}{p'} \frac{\varphi'(t)}{\varphi(t)} - \frac{1}{t} \right\}. \end{aligned}$$

On the other hand, a rather lengthy computation shows that

$$\begin{aligned} &\frac{d}{dt} \left\{ \frac{|S^{n-1}|^{1/n}}{\varphi(t)^{1-1/n}} \left\{ h(t)(1 + t^{p'}) + p'(n-1) \frac{\psi(t)}{t} \right\} \right\} = \\ &\frac{|S^{n-1}|^{1/n}}{t} (n-1)p' \frac{\psi(t)}{\varphi(t)^{1-1/n}} \left\{ \frac{h(t)}{p' \psi(t)} (1 + t^{p'}) \left( 1 - \frac{th(t)}{n\varphi(t)} \right) - \frac{1}{n'} \frac{h(t)}{\varphi(t)} - \frac{1}{t} \right\}. \end{aligned}$$

In particular (55) holds if and only if

$$\frac{1}{p'} \frac{\psi'(t)}{\psi(t)} - \frac{1}{p'} \frac{\varphi'(t)}{\varphi(t)} = \frac{h(t)}{p'\psi(t)} (1 + t^{p'}) \left( 1 - \frac{th(t)}{n\varphi(t)} \right) - \frac{1}{n'} \frac{h(t)}{\varphi(t)}.$$

We multiply both sides by  $\varphi(t)\psi(t)/h(t)$  and then apply (49) to the term  $(1 - [th(t)/n\varphi(t)])$  to deduce that our thesis is equivalent to

$$\frac{t^{p'}}{p'} \varphi(t) + \left( \frac{1}{n'} - \frac{1}{p'} \right) \psi(t) = (1 + t^{p'}) \int_0^t \frac{s^{p'} h(s)}{1 + s^{p'}} ds.$$

Then, by (51), we can reduce to ask whether

$$(56) \quad \int_0^t s^{p'-1} \varphi(s) ds + \frac{1}{n'} \psi(t) = (1 + t^{p'}) \int_0^t \frac{s^{p'} h(s)}{1 + s^{p'}} ds.$$

Since

$$\frac{d}{dt} \left\{ (1 + t^{p'}) \int_0^t \frac{s^{p'} h(s)}{1 + s^{p'}} ds \right\} = p' t^{p'-1} \int_0^t \frac{s^{p'} h(s)}{1 + s^{p'}} ds + t^{p'} h(t),$$

we deduce that

$$(1 + t^{p'}) \int_0^t \frac{s^{p'} h(s)}{1 + s^{p'}} ds = p' \int_0^t s^{p'-1} \int_0^s \frac{r^{p'} h(r)}{1 + r^{p'}} dr ds + \psi(t).$$

Thus (56) is equivalent to  $0 = \int_0^t s^{p'-1} \zeta(s) ds$  where

$$\zeta(s) := p' \int_0^s \frac{r^{p'} h(r)}{1 + r^{p'}} dr + \frac{sh(s)}{n} - \int_0^s h(r) dr.$$

We substitute in here the explicit form of  $h(s)$ , to find that

$$\zeta(s) = p' \int_0^s \frac{r^{n+p'-1}}{(1 + r^{p'})^{n+1}} dr + \frac{s^n}{n(1 + s^{p'})^n} - \int_0^s \frac{r^{n-1}}{(1 + r^{p'})^n} dr.$$

But

$$\frac{d}{ds} \left\{ \frac{s^n}{n(1 + s^{p'})^n} \right\} = \frac{s^{n-1}}{(1 + s^{p'})^n} - \frac{s^{n+p'-1}}{(1 + s^{p'})^{n+1}}$$

and so  $\zeta \equiv 0$  and the proof of Step 1 is finished.

*Proof of Step 2:* Since  $T(t)^{p^\sharp-1}/Y(t) = (|S^{n-1}|^{1/(n-1)} t^{p'+1} h(t)/\psi(t))^{1/p'}$  we just need to prove that  $(t^{p'+1} h(t)/\psi(t))' < 0$ , i.e.,

$$t^{p'+1} h(t) t^{p'} h(t) > [p' t^{p'} h(t) + t^{p'} (th(t))'] \psi(t).$$

By (47) we have

$$(th(t))' = nh(t) \left\{ 1 - p' \frac{t^{p'}}{1 + t^{p'}} \right\},$$

so that we have reduced to ask whether

$$t^{p'+1}h(t)t^{p'}h(t) > \psi(t)t^{p'}h(t) \left\{ n + p' - np' \frac{t^{p'}}{1 + t^{p'}} \right\},$$

or equivalently whether

$$(57) \quad t^{p'+1}h(t) > \psi(t) \left\{ n + p' - np' \frac{t^{p'}}{1 + t^{p'}} \right\}.$$

But, by (47)

$$\begin{aligned} \psi(t) &= \frac{t^{p'+1}}{p'+1}h(t) - \int_0^t \frac{s^{p'+1}}{p'+1}h'(s) ds \\ &= \frac{t^{p'+1}}{p'+1}h(t) - \int_0^t \frac{s^{p'}}{p'+1}h(s) \left\{ n - 1 - np' \frac{s^{p'}}{1 + s^{p'}} \right\} ds, \end{aligned}$$

so that

$$(p' + n)\psi(t) = t^{p'+1}h(t) + np' \int_0^t s^{p'}h(s) \frac{s^{p'}}{1 + s^{p'}} ds$$

Thus (57) becomes

$$-np' \int_0^t s^{p'}h(s) \frac{s^{p'}}{1 + s^{p'}} ds > -np'\psi(t) \frac{t^{p'}}{1 + t^{p'}}$$

which is true since  $t \mapsto t^{p'}(1 + t^{p'})^{-1}$  is increasing. This remark concludes the proof of Step 2 and of the concavity of  $\Phi$ .

**3.6. Behavior at the left end-point.** Now we wish to show the existence of a constant  $C_n(p)$  such that

$$\Phi_n^{(p)}(T) \geq S_n^{-1}(p)(1 - (C_n(p)T)^p)^{1/p}$$

on  $[0, T_n(p)]$ . Equivalently, we wish to check that for any value of  $t \in (0, \infty)$ ,

$$G(t) \geq S_n^{-1}(p)(1 - (C_n(p)T(t))^p)^{1/p},$$

which can be rewritten as

$$(58) \quad \left( \frac{G(t)}{G(\infty)} \right)^p + (C_n(p)T(t))^p \geq 1.$$

Let  $\Delta$  be the line joining  $(0, S_n^{-1}(p))$  and  $(T_n(p), 0)$ . For any  $T_0 > 0$  we can find  $C$  so large that the curve  $G = S_n^{-1}(p)[1 - CT^p]^{1/p}$  lies below  $\Delta$  (and therefore below the graph of  $\Phi_n^{(p)}$  for  $T_0 \leq T$ ). So all we have to do is prove (58) for  $T \leq T_0$ , i.e. for  $t$  large enough.

Recall that

$$\left[ \frac{G(t)}{G(\infty)} \right]^p = \left( \frac{\int_0^t s^{p'} h(s) ds}{\int_0^\infty h(s) ds} \right)^{p/p^*},$$

where both integrals admit a finite limit as  $t \rightarrow \infty$ . It follows that, as  $t \rightarrow \infty$ ,

$$\left[ \frac{G(t)}{G(\infty)} \right]^p = 1 - O\left( \int_t^\infty s^{p'} h(s) ds + \int_t^\infty h(s) ds \right).$$

Elementary integral estimates imply

$$\left[ \frac{G(t)}{G(\infty)} \right]^p \geq 1 - K(n, p)t^{-(n-p)/(p-1)}, \quad t > \theta_1(n, p)$$

for some constants  $K(n, p)$  and  $t(n, p)$  (which would be easy to compute explicitly).

Secondly,

$$\begin{aligned} T(t)^{p/p^*} &\geq \left( \frac{|S^{n-1}|^{1/(n-1)} t h(t)}{\varphi(\infty)} \right)^{1-p/n} \geq L(n, p) \left( t^{n-np'} \frac{t^{np'}}{(1+t^{p'})^n} \right)^{1-p/n} \\ &\geq L(n, p) t^{-(n-p)/(p-1)}, \quad \forall t > \theta_2(n, p), \end{aligned}$$

for some constants  $L(n, p)$  and  $t'(n, p)$  (which again could easily be evaluated explicitly).

All in all, we deduce that

$$\left( \frac{G(t)}{G(\infty)} \right)^p + (CT(t))^p \geq 1 + [CL(n, p) - K(n, p)] t^{-(n-p)/(p-1)}, \quad \forall t > \max(\theta_1(n, p), \theta_2(n, p)),$$

and to conclude the proof of (58) for  $t$  large enough it is sufficient to choose

$$C_n(p) := \frac{K(n, p)}{L(n, p)}.$$

**Remark 10.** The bound which we just proved is sharp about the behaviour of  $\Phi$  near  $T = 0$ . Indeed, from the proof it is easy to check that the power  $p$  in (58) can be replaced by no power  $q > p$ . On the other hand, to get a sharp estimate on the behavior of  $\Phi$  near  $G = 0$ , it is sufficient to consider  $\Phi_{[0]}$ :

$$\Phi_n^{(p)}(T) \geq \frac{1}{p^\# Y(0)} \left\{ (T_n^{(p)})^{p^\#} - T^{p^\#} \right\}.$$

In particular, the graph of  $\Phi$  is flat close to  $T = 0$ , but it is not vertical close to  $G = 0$ .

#### 4. RIGIDITY AND COUNTEREXAMPLES

The Sobolev inequality (14) cannot be improved, in the sense that both constants appearing in its right-hand side are the smallest possible ones for this inequality to apply to all domains.

The goal of this section is to complete the proof of Theorem 2 by proving that under certain conditions, inequality (14) can be improved when it is applied to a domain which is not a ball. In this study we shall be led to use some slightly more advanced material about Sobolev inequalities, taken from Maz'ja's book [11].

**4.1. Rigidity.** Let  $\Omega$  be a connected, bounded, Lipschitz domain and let  $p \in (1, n)$ . Let  $\Delta_0$  be the straight line joining the endpoints  $(0, S_n^{-1}(p))$  and  $(T_n(p), 0)$ ; from the concavity of  $\Phi_n^{(p)}$  we know that the graph of  $\Phi_n^{(p)}$  lies above  $\Delta_0$ , and therefore so does  $\Phi_\Omega^{(p)}$ .

We assume that  $\Omega$  is not a ball, and wish to prove that there exists a straight line  $\Delta$ , with endpoints  $(0, S_n^{-1}(p))$  and  $(a, 0)$ , with  $a > T_n(p)$ , such that the curve  $\Phi_\Omega^{(p)}$  lies above  $\Delta$ .

Since  $\Omega$  is bounded and Lipschitz, its boundary can be described by a finite number of Lipschitz charts, and therefore it is easy to check the following regularity estimate: there exists a finite number  $C$  such that, for all open sets  $U \subset \Omega$ ,

$$\frac{|U|^{(n-1)/n}}{|\partial U|} \leq C.$$

Then the following Poincaré estimate with sharp exponents [11, p. 168] holds true: there exists a constant  $P = P(\Omega)$  such that for all functions  $f : \Omega \rightarrow \mathbb{R}$ ,

$$(59) \quad \|f - \langle f \rangle\|_{L^{p^*}(\Omega)} \leq P \|\nabla f\|_{L^p(\Omega)},$$

where  $\langle f \rangle = |\Omega|^{-1} \int_\Omega f$  stands for the average value of  $f$  on  $\Omega$ . Assume that  $\|f\|_{L^{p^*}(\Omega)} = 1$ , then we deduce

$$\|\langle f \rangle\|_{L^{p^*}(\Omega)} \geq 1 - P \|\nabla f\|_{L^p(\Omega)}.$$

In other words,

$$(60) \quad |\langle f \rangle| \geq \frac{1 - P \|\nabla f\|_{L^p(\Omega)}}{|\Omega|^{1/p^*}}.$$

Next, as a consequence of the continuity of the trace with sharp exponents (Theorem A.3 in the Appendix), there is a constant  $R = R(\Omega)$  such that

$$\|f - \langle f \rangle\|_{L^{p^\sharp}(\partial\Omega)} \leq R (\|f - \langle f \rangle\|_{L^{p^*}(\Omega)} + P\|\nabla f\|_{L^p(\Omega)}) \leq R(1 + P)\|\nabla f\|_{L^p(\Omega)}.$$

Hence

$$\begin{aligned} \|f\|_{L^{p^\sharp}(\partial\Omega)} &\geq \|\langle f \rangle\|_{L^{p^\sharp}(\partial\Omega)} - R(1 + P)\|\nabla f\|_{L^p(\Omega)} \\ &= |\langle f \rangle| |\partial\Omega|^{1/p^\sharp} - R(1 + P)\|\nabla f\|_{L^p(\Omega)}. \end{aligned}$$

Combining this with (60), we deduce that

$$\|f\|_{L^{p^\sharp}(\partial\Omega)} \geq \left( \frac{|\partial\Omega|^{1/p^\sharp}}{|\Omega|^{1/p^*}} \right) - C\|\nabla f\|_{L^p(\Omega)},$$

where  $C := R(1 + P) + P|\partial\Omega|^{1/p^\sharp}$ . This estimate can be rewritten as

$$\|f\|_{L^{p^\sharp}(\partial\Omega)} \geq \text{IPR}(\Omega)^{1/p^\sharp} - C\|\nabla f\|_{L^p(\Omega)},$$

where  $\text{IPR}(\Omega)$  stands for the usual isoperimetric ratio,  $|\partial\Omega|/|\Omega|^{(n-1)/n}$ .

Since  $\Omega$  is not a ball, the rigidity part of the isoperimetric theorem implies  $\text{IPR}(\Omega) > \text{IPR}(B^n)$ , so

$$\text{IPR}(\Omega)^{1/p^\sharp} > \text{IPR}(B^n)^{1/p^\sharp} = T_n(p).$$

In conclusion, we have shown that there exists  $\delta > 0$  such that for all  $f : \Omega \rightarrow \mathbb{R}$  with  $\|f\|_{L^{p^*}(\Omega)} = 1$ ,

$$\|f\|_{L^{p^\sharp}(\partial\Omega)} \geq T_n(p) + \delta - C\|\nabla f\|_{L^p(\Omega)}.$$

Setting  $\varepsilon := \delta/(2C)$ , we deduce that

$$(61) \quad \|\nabla f\|_{L^p(\Omega)} \leq \varepsilon \implies \|f\|_{L^{p^\sharp}(\partial\Omega)} \geq T_n(p) + \delta/2.$$

This shows that the curve  $\Phi_\Omega^{(p)}$  stays a positive distance away from the line  $\Delta_0$  when  $\|\nabla f\|_{L^p}$  is smaller than  $\varepsilon$ .

From the fact that  $\Phi_n^{(p)}$  is *strictly* concave and flat at the origin, it follows that we can find a line  $\Delta'$ , passing through the same first end-point as  $\Delta_0$ , with a higher slope, such that  $\Phi_n^{(p)}$  stays above  $\Delta'$  for  $\|\nabla f\|_{L^p} \geq \varepsilon$ . Without loss of generality, we can assume that  $\Delta'$  crosses the horizontal axis with abscissa at most  $T_n(p) + \delta/2$ . Then estimate (61) shows that the whole curve  $\Phi_\Omega^{(p)}$  stays above  $\Delta'$ . See Figure 3 for a qualitative illustration of the proof.



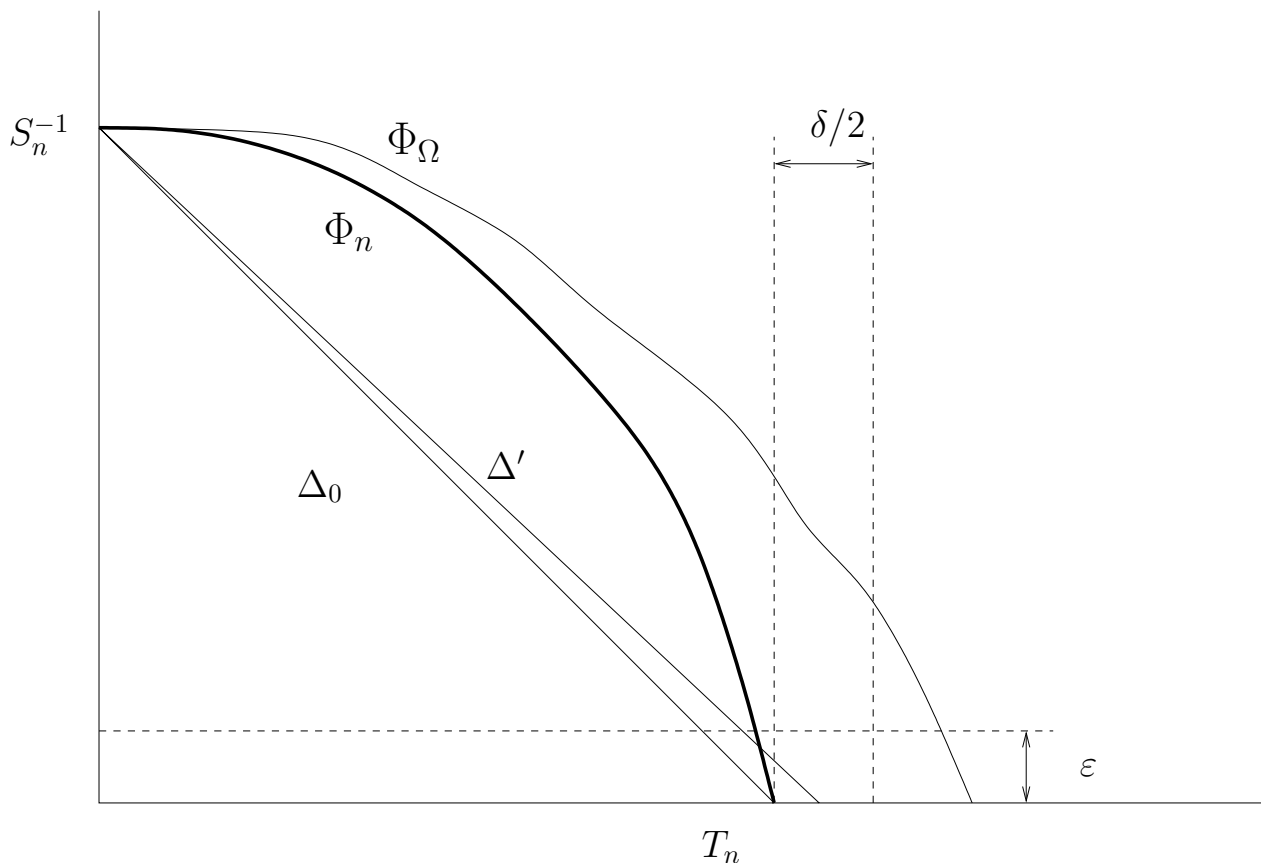


FIGURE 3. Proof of the rigidity theorem

In the rest of this section, we consider various counter-examples showing that the assumptions made in the previous proof are all useful.

**4.2. Non-connected counterexample.** Let  $U$  be an arbitrary Lipschitz open domain, disjoint from  $B^n$ , and consider  $\Omega = U \cup B^n$ . Considering functions defined on  $B^n$ , it is easy to show that  $\Phi_\Omega^{(p)} \leq \Phi_{B^n}^{(p)}$ , for all values of  $p$ . Since the reverse inequality is also true, we conclude that  $\Phi_\Omega^{(p)} = \Phi_{B^n}^{(p)}$ ; in particular the Sobolev inequality (14) cannot be improved.

**4.3. Unbounded domain.** Now we shall exhibit an unbounded, locally Lipschitz, connected domain  $\Omega$  with finite measure, which differs from a ball but has the same function  $\Phi$ . The construction is inspired from an example in Maz'ja's book [11, p. 165]. For simplicity we consider  $p \in (1, 2)$  and  $n = 2$ , but the construction applies in more generality. The proof itself is not very enlightening, and it will probably be sufficient for the reader to have a look at Figure 4, which gives an idea of the construction of  $\Omega$ .

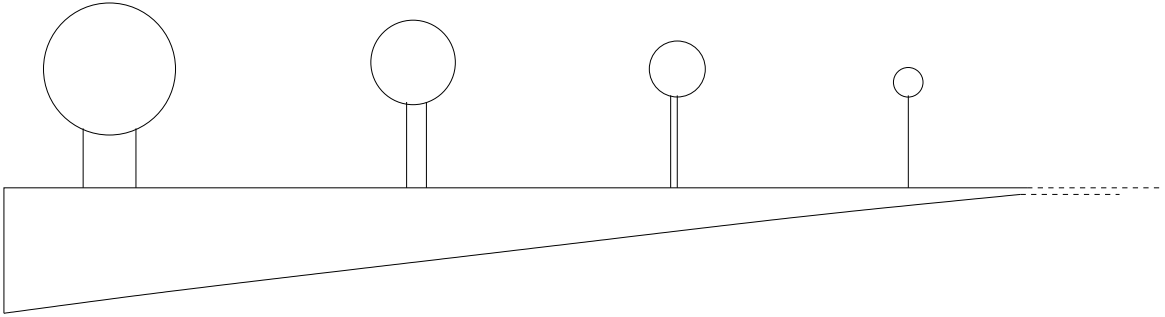


FIGURE 4. A sequence of shrinking mushrooms

Here follow some details. Let

$$\theta \in (0, \pi/2), \quad \alpha > 0, \quad \rho > 0,$$

be given. Define

$$\beta := \rho \sin \theta; \quad \gamma := \rho \cos \theta.$$

Consider  $\Omega[\rho, \alpha, \theta]$  defined as the union of the following four sets:

$$\begin{aligned} A &:= \rho B^2 \cap (-\infty, \gamma) \times \mathbb{R}, \\ R &:= (\gamma, \gamma + \alpha) \times (-\beta, \beta), \\ P^+ &:= (\gamma, \gamma + \alpha) \times (\beta, \beta + \alpha), \\ P^- &:= (\gamma, \gamma + \alpha) \times (-\beta - \alpha, -\beta). \end{aligned}$$

We now iterate the construction: Let  $\theta_k = \alpha_k = \rho_k^2$ , where

$$\max\{\rho_k, \alpha_k + \beta_k\} < 1, \quad \sum_{k=0}^{\infty} \rho_k^2 < \infty.$$

We define  $\Omega$  as the interior of the closure of

$$\bigcup_{k=0}^{\infty} \left\{ \left( (-\gamma_k, 2k) + \Omega[\rho_k, \alpha_k, \theta_k] \right) \cup (0, \alpha_k) \times (-1, 1) \right\}.$$

Of course  $\Omega$  is open and connected. Since every compact set in  $\mathbb{R}^2$  meets at most a finite number of elements from the union defining  $\Omega$ , it follows that the boundary of  $\Omega$  is locally representable by  $(n-1)$  dimensional Lipschitz maps. Furthermore  $\Omega$  has finite measure:

$$|\Omega| \leq \sum_{k=0}^{\infty} 2\alpha_k + \pi\rho_k^2 < \infty.$$

Let us check that  $\text{IPR}(p, \Omega) = \text{IPR}(p, B^2)$ .

**Lemma 11.** *Let  $\rho, \alpha, \theta$  be given and let  $\Omega[\rho, \alpha, \theta]$  be defined as above. Let  $f = f[\rho, \alpha, \theta]$  be defined on  $\Omega[\rho, \alpha, \theta]$  by*

$$f \equiv \left( \frac{1}{\pi\rho^2} \right)^{(2-p)/2p} =: c(\rho, p) \quad \text{on } A;$$

*$f = 0$  on  $\{\gamma + \alpha\} \times [-\beta, \beta]$ ; and  $f$  is radially symmetric decreasing on  $P^+$  (resp.  $P^-$ ) with center  $(\gamma, \beta)$  and symmetric profile*

$$\ell(t) := \frac{c(\rho, p)}{\alpha} t, \quad t \in [0, \alpha].$$

*Then  $f$  is a Lipschitz function and*

- (i)  $\|f\|_{L^{p^*}(\Omega[\rho, \alpha, \theta])} \rightarrow 1$  as  $\theta \rightarrow 0$ ,  $\theta/\rho \rightarrow 0$ ,  $\alpha/\rho \rightarrow 0$ ;
- (ii)  $\|\nabla f\|_{L^p(\Omega[\rho, \alpha, \theta])} \rightarrow 0$  as  $\alpha/\rho \rightarrow 0$ ,  $\theta\rho/\alpha \rightarrow 0$ ;
- (iii)  $\|f\|_{L^{p^\sharp}(\partial\Omega[\rho, \alpha, \theta])}^{p^\sharp} \rightarrow 2\sqrt{\pi} = (T_2^{(p)})^{1/p^\sharp}$  as  $\theta \rightarrow 0$ ,  $\alpha/\rho \rightarrow 0$ .

*Proof.* By explicit computations,

$$\int_A f^{p^*} = \frac{|A|}{\pi\rho^2} = 1 + O(\theta),$$

$$\int_R f^{p^*} = 2\beta \int_0^\alpha \left( \frac{c(\rho, p)}{\alpha} \right)^{p^*} t^{p^*} dt = \frac{2\rho\alpha \sin \theta}{(p^* + 1)\pi\rho^2}$$

and

$$\int_{P^+ \cup P^-} f^{p^*} = 2\frac{\pi}{2} \int_0^\alpha \left( \frac{c(\rho, p)}{\alpha} \right)^{p^*} t^{p^*+1} dt = \frac{\alpha^2}{(p^* + 2)\rho^2}.$$

This shows (i).

Next,

$$\int_R |\nabla f|^p = 2\beta\alpha \left( \frac{c(\rho, p)}{\alpha} \right)^p = \frac{2\rho \sin \theta}{\alpha^{p-1} (\pi\rho^2)^{(2-p)/2}} = \frac{2 \sin \theta}{\pi^{(2-p)/2}} \frac{\rho^{p-1}}{\alpha^{p-1}};$$

$$\int_{P^+ \cup P^-} |\nabla f|^p = 2 \frac{\pi}{4} \alpha^2 \left( \frac{c(\rho, p)}{\alpha} \right)^p = \frac{\pi^{p/2}}{2} \frac{\alpha^{2-p}}{\rho^{2-p}}.$$

This shows (ii).

Finally,

$$\begin{aligned} \int_{\partial\Omega[\rho, \alpha, \theta]} f^{p^\sharp} &= 2(\pi - \theta)\rho c(\rho)^{p^\sharp} + 2 \int_0^\alpha \left( \frac{c(\rho, p)}{\alpha} \right)^{p^\sharp} t^{p^\sharp} dt \\ &= \frac{2(\pi - \theta)\rho}{\sqrt{\pi\rho^2}} + \frac{2\alpha}{(p^\sharp + 1)(\pi\rho^2)^{1/2}} \\ &= \frac{2(\pi - \theta)}{\sqrt{\pi}} + \frac{2\alpha}{(p^\sharp + 1)\sqrt{\pi\rho}}. \end{aligned}$$

This concludes (iii), and the proof of the lemma.  $\square$

With this lemma at hand, it is easy to conclude. Assume that  $\theta_k = \alpha_k = \rho_k^2$  and  $\rho_k \rightarrow 0$ . By normalizing the function  $f_k$  constructed above, one can define on  $\Omega_k := \Omega[\theta_k, \alpha_k, \rho_k]$  a function  $g_k$  such that  $\|g_k\|_{L^{p^*}(\Omega_k)} = 1$ ,  $\|\nabla g_k\|_{L^p(\Omega_k)} \rightarrow 0$  and  $\|g_k\|_{L^{p^\sharp}(\partial\Omega_k)} \rightarrow T_2^{(p)}$ .

We now do the same construction on  $\Omega$ , constructing such a function  $g_k$  in

$$(-\gamma_k, 2k) + \Omega[\rho_k, \alpha_k, \theta_k],$$

letting  $g_k \equiv 0$  in the rest of  $\Omega$ . Plugging these functions in the definition of  $\Phi_\Omega$  shows that there is a sequence of points in that curve which converges to  $(T_2^{(p)}, 0)$ . So the Sobolev inequality (14) cannot be improved for that domain  $\Omega$ .

This shows that  $\text{IPR}(p, \Omega) = \text{IPR}(p, B^2)$ . Slightly more careful computations show that actually the whole curve  $\Phi_\Omega$  coincides with  $\Phi_2^{(p)}$ .

Of course in that example the Lipschitz norm of the charts used to describe the boundary of  $\Omega$  do blow up as  $k \rightarrow \infty$ . If  $\Omega$  is allowed to have infinite measure, it is easy to construct a similar example in which these Lipschitz norms are uniformly bounded (make the balls larger and larger, instead of smaller and smaller).

**4.4. Non-Lipschitz counterexample.** It is rather easy to adapt the previous counterexample into another one for which  $\Omega$  is bounded but not Lipschitz: for this make the subdomains  $\Omega_k$  shrink very rapidly, in such a way that you can pile up an infinity of them in finite volume. This is more in the spirit of the counterexample in Maz'ja [11, p.165], consisting of a bounded open set for which the Poincaré inequality (59) fails. Note that the Poincaré inequality was precisely one of the ingredients which we used to prove the rigidity statement for  $\text{IPR}(p, \Omega)$ .

**4.5. Counterexample for  $p = 1$ .** Assume  $\Omega$  is bounded, Lipschitz and connected, but now  $p = 1$ . Recall that in that situation, the curve  $\Phi_n^{(1)}$  is a straight line with slope  $-1$  which crosses the vertical axis at the point  $(0, S_n^{-1}(1))$ . Improving the inequality (2) for this domain means finding a straight line with slope  $-k$ ,  $k < 1$ , passing through  $(0, S_n^{-1}(1))$  also, and lying below the graph of  $\Phi_\Omega$ . It is clear that this is impossible if  $\Phi_\Omega^{(1)}$  touches  $\Phi_n^{(1)}$  for some value of  $T > 0$ .

To construct such a domain, consider the union  $\Omega$  of two overlapping balls  $B_1$  and  $B_2$  with unit radius, and let  $f := 1_{B_1}$ . Both  $T_0 := \|f\|_{L^1(\partial\Omega)}$  and  $G_0 := \|\nabla f\|_{TV(\Omega)}$  are nonzero, and they add up to the surface of the unit sphere, so that  $(T_0, G_0)$  lies on the graph of  $\Phi_n^{(1)}$ . The function  $f$  can be approximated in  $BV$  by functions in  $W^{1,1}(\Omega)$ , so  $(T_0, G_0)$  also lies on the graph of  $\Phi_\Omega^{(1)}$ . This shows that the inequality (2) cannot be improved for  $\Omega$ .

Note that in this example, actually  $\Phi_\Omega^{(1)}$  coincides with  $\Phi_n^{(1)}$  on the interval  $[0, T_0]$ , but later departs from that curve. In particular,  $\Phi_\Omega^{(1)}$  is not concave. By perturbation, it can be deduced that  $\Phi_\Omega^{(p)}$  is not concave either if  $p > 1$  is small enough.

#### APPENDIX: THE TRACE OPERATOR

Throughout the paper, we used some properties of the trace operator, which are recalled in the sequel. The introduction of the trace is a slight variation of [8, section 4.3]. In the statements below,  $\Omega$  is a locally Lipschitz domain in  $\mathbb{R}^n$ , with (almost everywhere) outside unit normal vector  $\sigma$ . The restriction operator associated to  $\Omega$  is the operator mapping a function  $f \in C(\overline{\Omega})$  to its restriction to  $\partial\Omega$ . Finally, we define  $BV_{\text{loc}}(\overline{\Omega})$  (resp.  $W_{\text{loc}}^{1,1}(\overline{\Omega})$ ) as the space of functions in  $L_{\text{loc}}^1(\Omega)$  whose distributional derivative defines a finite measure (resp. integrable function) on each bounded subset of  $\Omega$ . Note carefully that these spaces differ from  $BV_{\text{loc}}(\Omega)$ ,  $W_{\text{loc}}^{1,1}(\Omega)$ , for which the definition is the same apart from the replacement of “bounded” by “compact”. The slightly unusual notation involving  $\overline{\Omega}$  is somehow justified by the fact that functions in these spaces admit a trace, as recalled below.

**Definition A.1 (trace).** (i) Let  $f \in BV_{\text{loc}}(\overline{\Omega})$ . Then there exists  $g \in L^1_{\text{loc}}(\partial\Omega)$ , uniquely defined  $\mathcal{H}^{n-1}$ -almost everywhere on  $\partial\Omega$ , such that for all compactly supported function  $\varphi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ ,

$$(62) \quad \int_{\partial\Omega} g(\varphi \cdot \sigma) d\mathcal{H}^{n-1} = \int_{\Omega} f(\nabla \cdot \varphi) + \int_{\Omega} \varphi \cdot \nabla f.$$

This function  $g$  is said to be “the” trace of  $f$  on  $\partial\Omega$  and denoted  $\text{tr } f$ .

(ii) The mapping  $f \mapsto \text{tr } f$  defines on  $BV_{\text{loc}}(\Omega)$  a nonnegative, linear extension of the restriction operator, called the trace operator.

Apart from an elementary localization argument, the various statements in this theorem can all be found in [8, sections 4.3 and 5.3], except for the fact that equation (62) characterizes the trace. This last property is equivalent to the following statement: if

$$\int_{\partial\Omega} g(\varphi \cdot \sigma) d\mathcal{H}^{n-1} = 0$$

for all compactly supported  $C^1$  vector field, then  $g = 0$ . Let us sketch a proof. As in [8, p. 133], we may use a localization argument to reduce to the case in which  $\Omega$  is bounded and the boundary of  $\Omega$  intersects the support of  $f$  only in a Lipschitz graph defined by an equation of the form  $x_n = \gamma(x_1, \dots, x_{n-1})$ , and  $\sigma \cdot e_n > 0$ , where  $e_n$  stands for the last coordinate axis. With the help of  $\gamma$ , it is easy to extend  $\sigma$  to a Lipschitz map  $\overline{\sigma}$ , defined almost everywhere on the whole of  $\Omega$ , and without loss of generality we can assume that  $\overline{\sigma} \cdot e_n > 0$  on a neighborhood  $U$  of the support of  $f$ . Whenever  $\chi$  is a Lipschitz function on  $U$ , so is  $\psi := \chi/\sigma \cdot e_n$ , and we can find a sequence  $(\psi_k)_{k \in \mathbb{N}}$  of mappings in  $C^1(\overline{U})$  converging uniformly to  $\psi$ . Choosing  $\varphi := \psi_k e_n$ , we get

$$\int_{\partial\Omega} g \psi_k (e_n \cdot \sigma) d\mathcal{H}^{n-1} = 0,$$

and by passing to the limit as  $k \rightarrow \infty$  we recover  $\int_{\partial\Omega} g \chi d\mathcal{H}^{n-1} = 0$ . Since  $\chi$  is an arbitrary Lipschitz function, it follows by a routine approximation argument that  $g \equiv 0$ , which was our goal.

The additional properties which we used within the proofs of the present paper are summarized in the next two theorems. We use the notation  $\beta(f) = \beta \circ f$ .

**Theorem A.2 (the trace commutes with composition).** *Let  $f \in W^{1,1}_{\text{loc}}(\Omega)$  and let  $\beta$  be a Lipschitz function, such that  $\beta(0) = 0$  and  $\beta'$  has a finite number of discontinuities. Then  $\text{tr } [\beta(f)] = \beta(\text{tr } f)$ .*

**Theorem A.3 (sharp continuity of the trace).** *Let  $f \in L^1(\Omega)$  be such that  $\nabla f \in L^p(\Omega)$  for some  $p \in [1, n)$ . Then  $\text{tr } f \in L^p_{\text{loc}}(\partial\Omega)$ . More precisely, if  $\Omega$  is bounded, then there exists a constant  $C = C(\Omega)$  such that*

$$\|\text{tr } f\|_{L^p(\partial\Omega)} \leq C(\Omega)(\|f\|_{L^1(\Omega)} + \|\nabla f\|_{L^p(\Omega)}).$$

- Remarks A.4.**
- (i) The statement in [8, p. 133] is  $\|\text{tr } f\|_{L^p(\partial\Omega)} \leq C(\Omega)(\|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)})$  for bounded  $\Omega$ . For  $p = 1$ , this is just the same as above: the trace operator is continuous from  $W^{1,1}(\Omega)$  to  $L^1(\Omega)$ . In fact it is also continuous from  $BV(\Omega)$  to  $L^1(\Omega)$  [8, p. 177].
  - (ii) If  $\Omega$  is unbounded but has finite measure, the function 1 lies in  $W^{1,1}(\Omega)$  but its trace does not lie in  $L^1(\partial\Omega)$ . This shows that we can only hope for a local statement in Theorem A.3.
  - (iii) The assumption in Theorem A.2 that  $\beta$  has only a finite number of discontinuities is really not essential; we make it only for simplicity, and because we will not need more. The key remark is that if  $f \in W^{1,1}(\Omega)$ , then  $\nabla f = 0$  almost everywhere on  $f^{-1}(E)$  for every  $E \subset \mathbb{R}$  of null measure.

*Proof of Theorem A.2.* By an standard localization argument, we only need to consider the case when  $\Omega$  is bounded and  $f \in W^{1,1}(\Omega)$ . If  $f$  is continuous, the conclusion is obvious. In the general, case, there exist a sequence  $(f_k)_{k \in \mathbb{N}}$  of functions in  $W^{1,1}(\Omega) \cap C(\overline{\Omega})$  converging to  $f$  in  $W^{1,1}$  norm (see [8, p. 127]). Let  $\varphi$  be a compactly supported  $C^1(\Omega; \mathbb{R}^n)$  function; for each  $k$ , we write

$$\int_{\partial\Omega} \beta(f_k)(\varphi \cdot \sigma) d\mathcal{H}^{n-1} = \int_{\Omega} \beta(f_k)(\nabla \cdot \varphi) + \int_{\Omega} \varphi \cdot \beta'(f_k) \nabla f_k.$$

Since  $f_k$  converges to  $f$  in  $W^{1,1}$  norm, the continuity of the trace operator  $W^{1,1}(\Omega) \rightarrow L^1(\partial\Omega)$  implies that  $\text{tr } f_k$  converges to  $\text{tr } f$  in  $L^1(\partial\Omega)$ . Since  $\beta$  is Lipschitz and  $\beta(0) = 0$ , it follows that  $\beta(\text{tr } f_k)$  converges to  $\beta(\text{tr } f)$  in  $L^1(\partial\Omega)$ , so

$$\int_{\partial\Omega} \beta(f_k)(\varphi \cdot \sigma) d\mathcal{H}^{n-1} \longrightarrow \int_{\partial\Omega} \beta(\text{tr } f)(\varphi \cdot \sigma) d\mathcal{H}^{n-1}.$$

Similarly, the convergence of  $f_k$  to  $f$  in  $L^1(\Omega)$  implies

$$\int_{\Omega} \beta(f_k)(\nabla \cdot \varphi) \longrightarrow \int_{\Omega} \beta(f)(\nabla \cdot \varphi).$$

Finally,  $\beta'(f_k)$  converges in  $w * -L^\infty(\Omega)$  to  $\beta'(f)$  and  $\nabla f_k$  converges to  $\nabla f$  in  $L^1(\Omega)$ , so that

$$\int_{\Omega} \varphi \cdot \beta'(f_k) \nabla f_k \longrightarrow \int_{\Omega} \varphi \cdot \beta'(f) \nabla f.$$

By the chain-rule for Sobolev functions, this is also

$$\int_{\Omega} \varphi \cdot \nabla \beta(f).$$

All in all, we have shown

$$\int_{\partial\Omega} \beta(\operatorname{tr} f)(\varphi \cdot \sigma) d\mathcal{H}^{n-1} = \int_{\Omega} \beta(f)(\nabla \cdot \varphi) + \int_{\Omega} \varphi \cdot \nabla \beta(f),$$

and this implies the desired conclusion.  $\square$

*Proof of Theorem A.3.* The first statement follows from the second by an easy localization argument, so we assume from the beginning that  $\Omega$  is bounded. As we recalled above, the result for  $p = 1$  is established in [8], so we assume  $p > 1$ . Let  $\beta_M(f) = f1_{|f| \leq M} + M1_{f > M} - M1_{f < -M}$ . Since  $\beta_M(\operatorname{tr} f) = \operatorname{tr}(\beta_M(f))$ , the theorem will be proven if we can show an  $L^{p^\sharp}$  bound on  $\operatorname{tr}(\beta_M(f))$ , independently of  $M$ . So we just have to prove the theorem in the case when  $f$  is bounded.

In that case,  $f \in W^{1,p}(\Omega)$  and we can approximate  $f$  by a family  $(f_k)_{k \in \mathbb{N}}$  of functions in  $W^{1,p}(\Omega) \cap C^\infty(\overline{\Omega})$  (see [8, p. 127]). Again, by continuity of the trace operator,  $f_k$  converges  $\mathcal{H}^{n-1}$ -almost everywhere to  $f$  on  $\partial\Omega$  and Fatou's lemma implies

$$\|f\|_{L^{p^\sharp}(\partial\Omega)} \leq \liminf_{k \rightarrow \infty} \|f_k\|_{L^{p^\sharp}(\partial\Omega)}.$$

So we just have to prove the theorem when  $f$  is smooth, say  $C^1(\overline{\Omega})$ .

In that case, we apply the  $W^{1,1} \rightarrow L^1$  continuity result to the family  $f^{p^\sharp}$ , to get

$$\int_{\partial\Omega} f^{p^\sharp} \leq C \left( \int_{\Omega} f^{p^\sharp} + \int_{\Omega} |\nabla(f^{p^\sharp})| \right).$$

Here and below, the symbol  $C$  stands for various unrelated constants depending only on  $\Omega$ . By the chain-rule and Hölder's inequality,

$$\begin{aligned} \int_{\Omega} |\nabla(f^{p^\sharp})| &= \int_{\Omega} f^{p^\sharp-1} |\nabla f| \\ &\leq \|f^{p^\sharp-1}\|_{L^{p'}(\Omega)} \|\nabla f\|_{L^p(\Omega)} = \|f\|_{L^{p^*}(\Omega)}^{p^\sharp-1} \|\nabla f\|_{L^p(\Omega)}. \end{aligned}$$



Since  $p > 1$ ,  $p^\sharp > 1$  and for all  $\delta > 0$  we can find a constant  $C_\delta$  such that

$$\left( \int_{\Omega} |\nabla(f^{p^\sharp})| \right)^{1/p^\sharp} \leq \delta \|f\|_{L^{p^*}(\Omega)} + C_\delta \|\nabla f\|_{L^p(\Omega)}.$$

On the other hand, by elementary Lebesgue interpolation, up to increasing  $C_\delta$  we can write

$$\|f\|_{L^{p^\sharp}(\Omega)} \leq \delta \|f\|_{L^{p^*}(\Omega)} + C_\delta \|f\|_{L^1(\Omega)}.$$

Putting the above inequalities together, we deduce that for all  $\eta > 0$  there is a constant  $C_\eta$  such that

$$\left( \int_{\partial\Omega} f^{p^\sharp} \right)^{1/p^*} \leq \eta \|f\|_{L^{p^*}(\Omega)} + C_\eta (\|f\|_{L^1(\Omega)} + \|\nabla f\|_{L^p(\Omega)}).$$

Applying the trace Sobolev inequality (20) (which was proven for smooth functions without any use of the continuity of the trace operator), we deduce

$$\|f\|_{L^{p^\sharp}(\partial\Omega)} \leq \eta S (\|f\|_{L^{p^\sharp}(\partial\Omega)} + \|\nabla f\|_{L^p(\Omega)}) + C_\eta (\|f\|_{L^1(\Omega)} + \|\nabla f\|_{L^p(\Omega)}).$$

If we choose  $\eta$  so small that  $S\eta < 1/2$ , this implies

$$\|f\|_{L^{p^\sharp}(\partial\Omega)} \leq 2(C_\eta + 1)(\|f\|_{L^1(\Omega)} + \|\nabla f\|_{L^p(\Omega)}),$$

which was our goal.  $\square$

**Remark A.5.** Another way to conclude the theorem without using the trace Sobolev inequality would be to use the Poincaré-type inequality

$$(63) \quad \|f - \langle f \rangle_\Omega\|_{L^{p^*}(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)},$$

where

$$\langle f \rangle_\Omega := \frac{1}{|\Omega|} \int_{\Omega} f;$$

note indeed that  $|\langle f \rangle| \leq C \|f\|_{L^1(\Omega)}$ . Inequality (63) is established in [8, p. 141] when  $\Omega$  is a ball. The general case can be found in Maz'ja's book [11, p. 168], but it is more involved, which is why we preferred to go through the trace Sobolev inequality in the argument above.

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FRANCESCO MAGGI

DIP. MAT. "U. DINI", UNIVERSITÀ DI FIRENZE

CURRENT ADDRESS:

MPI-MIS

INSELSTR. 22-26, D 04103 LEIPZIG,  
GERMANY

EMAIL: maggi@mis.mpg.de

CÉDRIC VILLANI

UMPA, ENS LYON  
46 ALLÉE D'ITALIE  
69364 LYON CEDEX 07  
FRANCE

E-MAIL: cvillani@umpa.ens-lyon.fr