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by

John Andersson, Norayr Matevosyan, and Hayk Mikayelyan

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ON THE TANGENTIAL TOUCH BETWEEN THE FREE AND THE FIXED BOUNDARIES FOR THE TWO-PHASE OBSTACLE-LIKE PROBLEM

JOHN ANDERSSON, NORAYR MATEVOSYAN, HAYK MIKAYELYAN†

ABSTRACT. In this paper we consider the following two-phase obstacle-problem-like equation in the unit half-ball

$$\Delta u = \lambda_+\chi_{\{u>0\}} - \lambda_-\chi_{\{u<0\}},\ \ \lambda_\pm > 0.$$

We prove that the free boundary touches the fixed one in (uniformly) tangential fashion if the boundary data f and its first and second derivatives vanish at the touch-point.

1. Introduction

1.1. **The Problem.** The following two-phase analogue of classical obstacle problem was suggested by G. S. Weiss in [W2] and then considered by N.N. Uraltseva in [U] and H. Shahgholian, N.N. Uraltseva and G.S. Weiss in [SUW]. Study properties of a weak solution $u \in W^{1,2}(D)$ of

$$\Delta u = \lambda_+\chi_{\{u>0\}} - \lambda_-\chi_{\{u<0\}}, \tag{1}$$

in the domain D, such that $u - f \in W_0^{1,2}(D)$ for a given $f \in W^{1,2}(D)$. In our paper we always assume $\lambda_\pm > 0$, and we consider the cases $D = B_1$ and $D = B_1^+$, as well as the case of the so-called global solutions $D = \mathbb{R}^n_+$.

Obviously (1) is the Euler-Lagrange equation of the energy functional

$$J(u) = \int_D |\nabla u|^2 + 2\lambda_+ \max(u, 0) + 2\lambda_- \max(-u, 0)dx.$$

Note that if the boundary data f is non-negative (non-positive) then the solution u is so, too, and we arrive at classical obstacle problem (see [C]). In the two-phase case we do not have the property that the gradient vanishes on the free boundary Γ_u (see Section 1.2 for definition), as it was in the classical case; this causes difficulties.

Key words and phrases. Free boundary problems, two-phase obstacle problem, contact points.

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We consider the following problem: Let u be a weak solution of (1) in B_1^+, $0 \in \overline{\Gamma}_u$, $f := u|_\Pi \in C^{2,Dini}(B_1 \cap \Pi)$ and

$$f(0) = |\nabla f(0)| = |D^2 f(0)| = 0. \tag{2}$$

Then we prove that the free boundary of u approaches the fixed one at 0 tangentially. Under some growth assumptions we prove that this approach is uniform (Theorem B) and we show the necessity of this assumption with an example. From (2) it obviously follows that $\frac{|f(x')|}{|x'|^2} \leq \omega(|x'|)$ for some Dini modulus of continuity ω, i.e., the blow-up of f is zero

$$f_r(x') := \frac{f(rx')}{r^2} \to 0 \text{ as } r \to 0.$$

Let us recall the definition of $C^{2,Dini}(B_1 \cap \Pi)$; these are functions from $C^2(B_1 \cap \Pi)$ such that

$$|D^2 f(x) - D^2 f(y)| \leq \omega(|x-y|),$$

where $D^2 f$ is the Hessian of f and ω is a Dini modulus of continuity, i.e.,

$$\int_0^1 \frac{\omega(s)}{s} ds < \infty.$$

1.2. **Notations.** In the sequel we use following notations:

$\mathbf{R}^n_+$	$\{x \in \mathbf{R}^n : x_1 > 0\}$
$\mathbf{R}^n_-$	$\{x \in \mathbf{R}^n : x_1 < 0\}$,
$B(z,r)$	$\{x \in \mathbf{R}^n : \|x - z\| < r\}$,
B_r	$B(0,r)$,
B_r^+	$\mathbf{R}^n_+ \cap B_r$,
Π	$\{x \in \mathbf{R}^n : x_1 = 0\}$,
x'	$(x_2, \dots, x_n)$,
K_ϵ	$\{x \in \mathbf{R}^n_+ : x_1 > \epsilon\|x'\|\}$,
$\\|\cdot\\|_\infty$	canonical norm,
$e_1, \dots, e_n$	standard basis in $\mathbf{R}^n$,
$\nu,\ e$	arbitrary unit vectors,
$D_\nu,\ D_{\nu e}$	first and second directional derivatives,
$v^+,\ v^-$	$\max(v,0),\ \max(-v,0)$,
χ_D	characteristic function of the set D,
∂D	boundary of the set D,
Ω_u^+	$\{x \in D : u(x) > 0\}$,
Ω_u^-	$\{x \in D : u(x) < 0\}$,
Λ_u	$\{x \in B_1^+ : u(x) = \|\nabla u(x)\| = 0\}$,
Γ_u	$(\partial\Omega_u^+ \cup \partial\Omega_u^-) \cap D$, the free boundary,
$\mathcal{P}(\dots)$	see Definition 2.

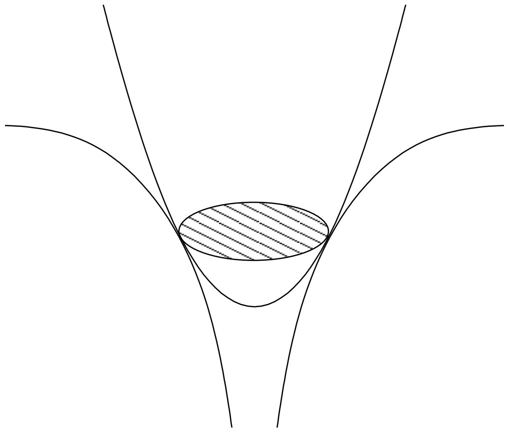

FIGURE 1. "Ball solutions"

1.3. **"Typical" examples.** We show here with some examples how the situation near a touch point between free and the fixed boundaries can look like.

Fix the ball B_R and consider the function $\frac{\lambda}{2n}|x|^2$, $\lambda > 0$. Let us take the radial fundamental solution of Laplace equation U multiplied with a constant C_R, such that $C_R \partial_r U(R) = \frac{\lambda}{n} R$. Then for some constant C the function

$$V(x) = \frac{\lambda}{2n}|x|^2 - C_R U(|x|) + C$$

is non-negative in $\mathbb{R}^n$, $\Delta V = \lambda - C_R \delta_0$ and $V = |\nabla V| = 0$ on ∂B_R (Figure 1).

Now we see how we can construct solutions of (1) in $B_R \backslash \{0\}$ and in $\mathbb{R}^n \backslash B_R$. So for instance in $\mathbb{R}^2$ we can illustrate some solutions considered in rectangles (Figure 2). The dashed curves denote free boundaries, $\pm$ denote regions $\Omega_u^{\pm}$ and 0 the region Λ_u. Figure 2 a) shows that the case when the solution does not have quadratic growth near the touch point is possible. In this case the blow-up of the solution is zero. Figure 2 b) shows that even if we have non-negative boundary data near touch point, the blow-up still can be negative and Figure 2 c) shows that the condition (2) is essential for the tangential touch.

Let us take boundary data f on ∂B_1^+ to be odd-symmetric with respect to x_2. Then the solution u will be odd-symmetric, too. From our results (see Section 2) follows that the set Λ_u is large near 0, where the free boundary touches the fixed boundary, as it is the case in the example from the Figure 2 a). So we do not have orthogonal touch, as it might be expected. The similar argument works also in higher dimensions for every plane-symmetric domain.

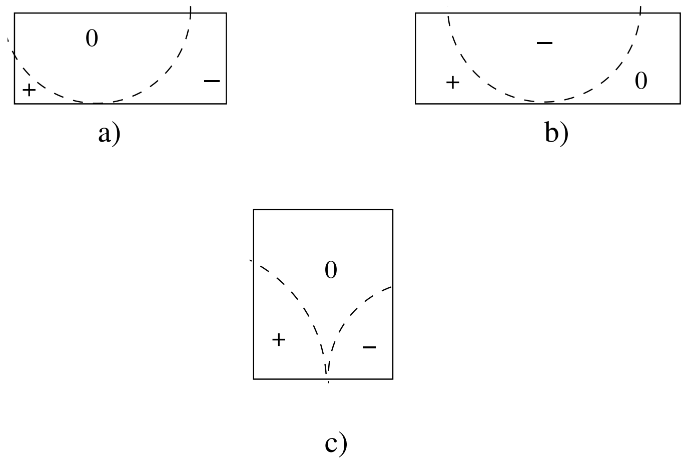

FIGURE 2. "Typical" examples

2. MAIN RESULTS

In this section we state two theorems. The first one says that if the boundary data of a solution of (1) satisfies condition (2), then the free boundary can approach the fixed one only tangentially. In the second theorem we assert that this approach is uniform for a certain class of functions.

Theorem A. *Let u be a solution of (1) in B_1^+ with boundary data f on Π, condition (2) is satisfied and $0 \in \overline{\Gamma}_u$. Then the free boundary approaches Π at the point 0 tangentially.*

Corollary 1. *Let u be as in Theorem A, then one of the following expressions is true*

$$\frac{|\Omega_u^+ \cap B_r^+|}{|B_r^+|} \to 1, \ \frac{|\Omega_u^- \cap B_r^+|}{|B_r^+|} \to 1, \ \frac{|\Lambda_u \cap B_r^+|}{|B_r^+|} \to 1 \ \text{as } r \to 0.$$

Moreover, the first two cases are possible only if the condition (4), see below, is satisfied for some c_0 and r_0, and the third case if it fails to hold (see Lemma 8).

Definition 2. *Let ω be a Dini modulus of continuity and M, c_0, r_0 be positive constants. We define $\mathcal{P}(M, R, c_0, r_0)$ to be the class of solutions u of (1) in B_1^+, $\|u\|_{L^\infty(B_1^+)} \le M$, $0 \in \overline{\Gamma}_u$ such that the boundary data*

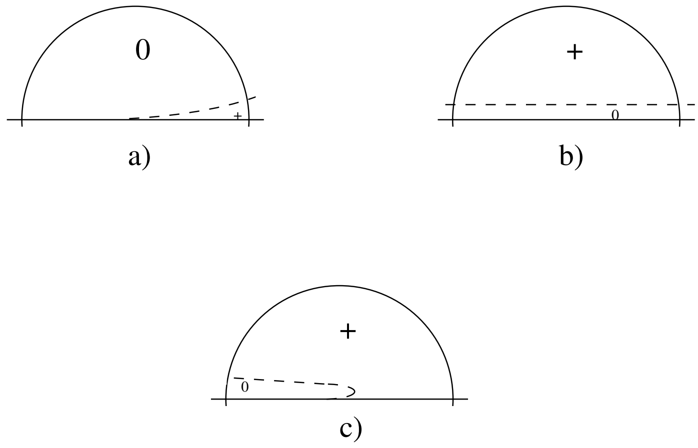

FIGURE 3. Non-uniform approach

$f = u|_\Pi \in C^{2,Dini}(B_1 \cap \Pi)$ *satisfies condition (2) and*

$$\|f\|_{C^2(\overline{B}_1 \cap \Pi)} \le R, \quad \int_0^1 \frac{\omega(s)}{s} ds \le R. \tag{3}$$

We assume, further

$$\sup_{B_r^+} |u| \ge c_0 r^2, \ \textit{for } 0 < r < r_0, \tag{4}$$

for all $u \in \mathcal{P}(M, R, c_0, r_0)$.

Remark 3. *If* u *solve (1) in* B_1^+, $0 \in \overline{\Gamma}_u$ *and* $u|_\Pi \equiv 0$, *then condition (4) is fulfilled with the constant* $c_0 = max(\lambda_\pm)C$, *for* $0 < r < 1$, *where* C *is a dimension dependent constant (see Lemma 6 and Corollary 7).*

Theorem B. *There exists a modulus of continuity* $\sigma(r)$ *and* $\tilde{r} > 0$ *such that if* $u \in \mathcal{P}(M, R, c_0, r_0)$ *then*

$$\Gamma_u \cap B_{\tilde{r}} \subset \{x : x_1 < |x'|\sigma(|x'|)\}.$$

In other words the free boundary of the functions from $\mathcal{P}$ *approaches* Π *at the point* 0 *uniformly tangentially.*

Here σ and $\tilde{r}$ depend on the dimension, $\lambda_\pm$, c_0, r_0, M and R.

Remark 4. *Since the main tool we use proving Theorems A and B is the blow-up argument, these results could be generalized for domains with smooth enough boundary.*

The following example shows that we do not have uniform tangential approach when the set Λ_u is large near the touch point, i.e., the condition (4) fails to hold (see Lemma 8). In the example the boundary data is positive, so we treat here the classical obstacle problem in $\mathbb{R}^2$. First let us take small positive boundary data on the right side from touch point (zero) to get a tiny positivity set as it is shown on the Figure 3 a). This can be done using the "ball solutions" above. Next consider the function $u(x) = \frac{\lambda_+}{2}((x_1-\epsilon)_+)^2$. We will get it as a solution if we take its boundary data on ∂B_1^+ (Figure 3 b)). Consider now the boundary data which is the sum of the boundary data of the previous two examples. It will look like it is shown on the Figure 3 c). So we see that when the 'tiny' positivity set in Figure 3 a) becomes smaller and ϵ in Figure 3 b) tends to zero, then for any $\tilde{r}$ and σ in the Figure 3 c) we get free boundary points in $B_{\tilde{r}} \cap \{x : x_1 > |x'|\sigma(|x'|)\}$, while the boundary data on Π remains bounded.

Remark 5. *In order to get uniform tangential touch for a class of solutions we impose condition (4). This condition, however, can be replaced by the following one, which is considered in* [KKS] *for a different problem;*

$$\frac{|\Omega_u^+ \cap B_r^+|}{|B_r^+|} \geq c_0 > 0, \ \textit{for } r < r_0.$$

From Lemma 8 and Corollary 1 it follows that both conditions are equivalent in our case.

3. Technicalities

3.1. **Non-degeneracy.** In this section we introduce some (modified) results from [W2], [U] and [SUW] as well as prove growth estimates at the boundary (Lemmas 8 and 9).

Lemma 6. *Let u solve (1) in B_1. There exists a dimension dependent constant C such that $\|f^\pm\|_\infty < \lambda_\pm C$ implies $0 \notin \overline{\Omega_u^\pm}$.*

Proof. Consider the "+"-case. Due to the comparison principle a similar argument is true (and well-known) for obstacle problems, i.e. it is known in our case if the boundary data f is non-negative or non-positive. Let us now consider the related (one-phase) obstacle problem in B_1 with boundary data f^+, denote its solution by v. It is enough to show that $\Omega_u^+ \subset \Omega_v^+$. Consider the function $w = u - v$ in Ω_u^+. We have $w|_{\partial\Omega_u^+} \leq 0$ and $\Delta w \geq 0$ in Ω_u^+, hence $u - v \leq 0$ and we are done. □

Corollary 7. *Let u be the solution of (1), $x_0 \in \overline{\Omega}^\pm$ and $B_{r_0}(x_0) \subset D$. Then*

$$\sup_{\partial B_r(x_0)} u^\pm \geq \lambda_\pm C r^2, \ \textit{for } r < r_0. \tag{5}$$

Here the constant C is the same as in the previous lemma.
In other words if $x_0 \notin int\, \Lambda_u$ and $B_{r_0}(x_0) \subset D$, then

$$\sup_{\partial B_r(x_0)} |u| \geq \min(\lambda_\pm) C r^2, \ \textit{for } r < r_0. \tag{6}$$

Proof. Let us restrict the function u to $B_r(x_0)$ and scale

$$u_r(x) = \frac{u(rx + x_0)}{r^2}.$$

u_r is then a solution of (1) in B_1 with boundary data $u_r|_{\partial B_1}$. Since $0 \in \overline{\Omega}^+$ we must have

$$\sup_{\partial B_1} u_r \geq \lambda_+ C,$$

which in turn implies (5). □

Lemma 8. *Let u be the solution of (1) in B_1^+ and suppose for given constants c_0, r_0, we have*

$$\frac{|\Omega_u^+ \cap B_r^+|}{|B_r^+|} \geq c_0 > 0, \ \textit{for } r < r_0.$$

Then there exists a constant c depending on, c_0, $\lambda_\pm$ and the dimension such that

$$\sup_{B_r^+} |u| \geq c r^2, \ \textit{for } r < r_0. \tag{7}$$

The same is also true for Ω^-.

Proof. Let us denote $\tilde{B}_r^+ = B_r^+ \cap \{x_1 > \epsilon r\}$. We can fix an $\epsilon > 0$ such that

$$\frac{|\Omega_u^+ \cap \tilde{B}_r^+|}{|\tilde{B}_r^+|} \geq \frac{c_0}{2}, \ \text{for } r < r_0.$$

Hence for each $r > 0$ there exists $x_r \in \Omega_u^+ \cap \tilde{B}_{\frac{r}{2}}^+$. Applying previous corollary to the ball $B_{d_r}(x_r)$, where d_r is the e_1 component of x_r we get that

$$\sup_{B_r^+} |u| \geq \sup_{B_{d_r}(x_r)} |u| \geq \lambda_+ C d_r^2 \geq \epsilon \lambda_+ \frac{C}{4} r^2.$$

We proved the lemma with $c = \epsilon \lambda_+ \frac{C}{4}$. □

In the proof of the next lemma we use the technique from [A] (Lemma 5), see also [CKS]. A similar estimate in the interior was proved by Uraltseva in [U].

Lemma 9. *Let u solve (1) in B_1^+, $\|u\| \leq M$ and assume its boundary data $f = u|_\Pi$ and the Dini modulus of continuity ω satisfy conditions (2) and (3). Then there exists a constant $C = C(M, R)$ such that*

$$\sup_{B_r^+} |u - D_{e_1} u(0) x_1| \leq C r^2, \ \ 0 < r < \frac{1}{2}.$$

Proof. Let us denote by

$$S_j(u) := \sup_{B^+_{2^{-j}}} |u - D_{e_1}u(0)x_1|$$

and $\mathbb{M}(u) := \{j : S_j(u) \le 4S_{j+1}(u)\}$. We want to show that $S_j(u) \le C2^{-2j}$. First let us show this for all $j \in \mathbb{M}(u)$. The proof is done by contradictory argument: assume there exist a sequence $\{u_j\}$ of solutions of (1) in B^+_1 such that

$$S_{k_j}(u_j) \ge j2^{-2k_j},$$

for some $k_j \in \mathbb{M}(u_j)$. Denoting by $w_j(x) := u_j(x) - D_{e_1}u_j(0)x_1$ and by

$$\tilde{w}_j(x) := \frac{w_j(2^{-k_j}x)}{S_{k_j+1}(u_j)},$$

we get

$$\|\Delta \tilde{w}_j\|_\infty \le \max(\lambda_\pm)\frac{2^{-2k_j}}{S_{k_j+1}(u_j)} \le \max(\lambda_\pm)\frac{2^{-2k_j}}{\frac{1}{4}S_{k_j}(u_j)} \le \max(\lambda_\pm)\frac{4}{j} \to 0.$$

We also have

$$\sup_{B^+_{\frac{1}{2}}} |\tilde{w}_j| = 1. \tag{8}$$

The condition $(D_e f(x'))^\pm \le |x'|\omega(|x'|)$, for any unit vector $e \in \Pi$, implies that

$$\sup_{B^+_r} |D_e w_j| \le Cr, \tag{9}$$

where C depends on M and R. To see this we should consider harmonic functions $v^\pm_j$ in $B^+_{\frac{1}{2}}$ with the same boundary data as $(D_e w_j)^\pm$. Inequality (9) then follows from the subharmonicity of $(D_e w_j)^\pm$ (see [U]) and standard estimates on Green's function for the half-ball (see [Wi]). From (9) we have

$$\sup_{B^+_r} |D_e \tilde{w}_j| \le \frac{4Cr}{j}, \tag{10}$$

A subsequence of $\tilde{w}_j$ converges in $C^1(B^+_{\frac{1}{2}})$ to a harmonic function u_0. Due to (10) we get $D_e u_0 = 0$ for all $e \in \Pi$, thus $u_0 = ax_1$. On the other hand $D_{e_1}\tilde{w}_j(0) = 0$ and by C^1-convergence (up to Π) the same holds for u_0. Hence $u_0 \equiv 0$ which contradicts (8).

Next let us show that $S_j(u) \le 4C2^{-2j}$ for all j. Suppose j is the first integer for which the inequality fails to hold, then

$$S_{j-1}(u) \le 4C2^{-2(j-1)} \le 4S_j(u),$$

i.e. $j-1 \in \mathbb{M}(u)$ and

$$S_j(u) \le S_{j-1}(u) \le C2^{-2(j-1)} = 4C2^{-2j},$$

a contradiction. □

3.2. **Monotonicity formulae.** Here we introduce two monotonicity formulae in the following two lemmas, which play crucial role in our proofs. The first one was presented by H. W. Alt, L. A. Caffarelli and A. Friedman in [ACF] and was developed then in [CKS]. The second one is due to G. Weiss [W1], [SUW]. In [A] Andersson adapted it to the half-space case and our representation is analogous. See also [M] for the formula in parabolic case.

Lemma 10. (ACF monotonicity formula)
Let h_1, h_2 be two non-negative continuous subsolutions of $\Delta u = 0$ in B_R. Assume further that $h_1 h_2 = 0$ and that $h_1(0) = h_2(0) = 0$. Then the following function is non-decreasing in $r \in (0, R)$

$$\varphi(r) = \frac{1}{r^4}\left(\int_{B_r} \frac{|\nabla h_1|^2 dx}{|x|^{n-2}}\right)\left(\int_{B_r} \frac{|\nabla h_2|^2 dx}{|x|^{n-2}}\right). \tag{11}$$

More exactly, if any of the sets spt$(h_j) \cap \partial B_r$ digresses from a spherical cap by a positive area, then either $\varphi'(r) > 0$ or $\varphi(r) = 0$.

Lemma 11. (Weiss' monotonicity formula)
Let u solve (1) in B_R^+ and $u|_{\Pi\cap B_R} = 0$. Then the function

$$\Phi(r) = r^{-n-2}\int_{B_r\cap\mathbb{R}^n_+}(|\nabla u|^2 + 2\lambda_+ u^+ + 2\lambda_- u^-) - r^{-n-3}\int_{\partial B_r\cap\mathbb{R}^n_+} 2u^2 d\mathcal{H}^{n-1} \tag{12}$$

is non-decreasing for $r \in (0, R)$. Moreover, if $\Phi(\rho) = \Phi(\sigma)$ for any $0 < \rho < \sigma < R$ then Φ is homogeneous of degree two in $(B_\sigma \setminus B_\rho) \cap \mathbb{R}^n_+$.

The proof is analogous to the proof of the Lemma 1 in [A].

4. Global solutions

In this section we will classify all solutions of (1) in the $\mathbb{R}^n_+$ with zero boundary data and quadratic growth. We will see that only possible cases are

$$u(x) = \pm\frac{\lambda_\pm}{2}((x_1 - a)_+)^2,\ \ a \geq 0 \ \text{ or } \ u(x) = \pm\frac{\lambda_\pm}{2}x_1^2 \pm \alpha x_1,\ \ \alpha \geq 0. \tag{13}$$

The proofs of next two lemmas adapt the proofs of analogous results from [SU] for our case.

Let us first prove that u is two-dimensional.

Lemma 12. *Let u solve (1) in $\mathbb{R}^n_+$ with boundary data $u|_\Pi = 0$. Then the function u is two-dimensional, i.e., in some system of coordinates*

$$u(x) = u(x_1, x_2),$$

where the e_1 direction is normal to Π.

Proof. Let us take any e orthogonal to e_1 and consider functions $(D_e u)^{\pm}$. In [U] Uraltseva proved that these functions are subharmonic. Note that they will remain so if we extend them by zero to $\mathbb{R}^n_-$. Now we can apply ACF monotonicity formula to $(D_e u)^{\pm}$. For $r < s$ we have

$$\varphi(r, D_e u) \leq \varphi(s, D_e u) \leq \lim_{s\to\infty} \varphi(s, D_e u) =: C_e.$$

In [U] is shown that the second derivatives of u are bounded, thus we can find a sequence $u_{r_j} = \frac{u(r_j x)}{r_j^2} \to u_\infty$, uniformly on compact subsets and in $(W^{2,p}_{loc} \cap C^{1,\alpha}_{loc})(\mathbb{R}^n_+ \cup \Pi)$, for any $1 < p < \infty$ and $0 < \alpha < 1$. We have now

$$C_e = \lim_{r_j\to\infty} \varphi(sr_j, D_e u) = \lim_{r_j\to\infty} \varphi(s, D_e u_{r_j}) = \varphi(s, D_e u_\infty), \ \ \forall s > 0.$$

From $\{x_1 < 0\} \subset \{D_e u = 0\}$ follows that $\varphi(r, D_e u_\infty) \equiv 0$ or $\varphi'(r, D_e u_\infty) > 0$ for all $r > 0$, thus $C_e = 0$ and we get $D_e u \geq 0$ or $D_e u \leq 0$.

For $e_2 \in \Pi$ assume $D_{e_2} u \geq 0$ and let $e_3 \in \Pi$ be orthogonal to e_2. Consider unit vector $e(\phi) = \cos\phi\, e_2 + \sin\phi\, e_3 \in \Pi$, $\phi \in [0, \pi]$. From the C^1-continuity we have that the sets $\{\phi : \Omega^{\pm}_{D_{e(\phi)}u} \neq \emptyset\}$ are relatively open in $[0, \pi]$. On the other hand they are both non-empty and have empty intersection; this means that there exists $\phi_0 \in (0, \pi)$ such that $D_{e(\phi_0)} u \equiv 0$. Rotating coordinates we can get $D_{e_2} u \geq 0$ and $D_{e_3} u \equiv 0$. Repeating this with e_k, $k = 4, \ldots, n$, we get that u is two dimensional. □

We prove now the main result of this section under the assumption of homogeneity.

Proposition 13. *Let u be homogeneous of degree two and solve (1) in $\mathbb{R}^n_+$ with boundary data $u|_\Pi = 0$. Then either $u(x) = \frac{\lambda_+}{2} x_1^2$ or $u(x) = -\frac{\lambda_-}{2} x_1^2$.*

Proof. We can consider only two-dimensional functions u. So let us rewrite u in radial coordinates as

$$u(x) = u(r, \theta) = r^2 \phi(\theta), \ \ r \in [0, \infty), \ \theta \in [0, \pi].$$

Then we get the ODE

$$\phi'' + 4\phi = \lambda_+ \chi_{\{\phi > 0\}} - \lambda_- \chi_{\{\phi < 0\}}$$

in the interval $[0, \pi]$ with boundary data $\phi(0) = \phi(\pi) = 0$. It can be checked that the only solutions of this ODE are $\phi(\theta) = \pm\frac{\lambda_\pm}{2} \sin^2\theta$. □

Lemma 14. *Let u solve (1) in $\mathbb{R}^n_+$ with boundary data $u|_\Pi = 0$ and be quadratically bounded at infinity. Then u is one of the representations in (13).*

Proof. If the function u is non-negative or non-positive, then the result we want to prove follows from Theorem B in [SU]. So let us show that

u does not change the sign. We do this by contradiction; assume that $u^{\pm}$ are both non-trivial.

Consider the shrink down of u; $\tilde{u} := \lim_{j\to\infty} u_j$, where $u_j(x) = \frac{u(r_j x)}{r_j^2}$, $r_j \to \infty$. It is homogeneous of degree two. To see this we need to use Weiss' monotonicity formula

$$\Phi(s, \tilde{u}) = \lim_{j\to\infty} \Phi(s, u_j) = \lim_{j\to\infty} \Phi(sr_j, u) = \Phi(\infty, u).$$

Thus $\tilde{u}$ equals to one of $\pm\frac{\lambda_\pm}{2}x_1^2$ by Proposition 13 above. Assume for definiteness that we have the "+"-sign.

This means that for any $\delta > 0$ there exists R_δ such that

$$\Omega_u^- \setminus B_{R_\delta}^+ \subset \{x;\ x_1 < \delta|x_2|\}. \tag{14}$$

Let us now take the barrier function

$$U(x_1, x_2) = x_1^4 + x_2^4 - 6x_1^2 x_2^2 + C.$$

For large enough C we have $\Omega_u^- \Subset \Omega_U^+$. Since u is quadratically bounded we get from the comparison principle that $u^-(x) \le \varepsilon U(x)$ for any $\varepsilon > 0$, thus $\Omega_u^- = \emptyset$. □

5. Proofs

Proof of the Theorem A. We consider here only the case when (4) fails to hold. From Lemma 9 follows that $D_{x_1}u(0) = 0$ and

$$\sup_{B_r^+} |u| \le c_0 r^2, \text{ for } r < r_0. \tag{15}$$

Now assume that we do not have tangential touch at 0, i.e., there is an $\epsilon > 0$ and a sequence $x^j \in K_\epsilon \cap \Gamma_u$, $x^j \to 0$. Repeating the proof of Lemma 8 we obtain

$$\sup_{B_{2d_j}^+} |u| \ge C d_j^2, \text{ for } r < r_1, \tag{16}$$

where $d_j = |x^j|$. Consider now the blow up sequence

$$\tilde{u}_j(x) = \frac{u(2d_j x)}{4d_j^2},$$

which is bounded by (15). Therefore there is a subsequence converging in $C^{1,\alpha}$ to a global solution u_0 with zero boundary data, which is non-trivial (due to (16)). As in the proof of Lemma 14, using Weiss' monotonicity formula we get that u_0 is homogeneous of degree 2, this implies that $u_0(x) = \pm\frac{\lambda_\pm}{2}x_1^2$ which contradicts the fact that $x^j \in K_\epsilon$.

□

Proof of the Theorem B. The proof is done by contradictory argument. Assume there exist an $\epsilon > 0$, functions u_j satisfying the conditions of the theorem and a sequence $x^j \to 0$ such that $x^j \in K_\epsilon \cap \Gamma_{u_j}$. Let us consider the blow-up sequence

$$\tilde{u}_j(x) = \frac{u_j(d_j x)}{\sup_{B^+_{d_j}} |u_j|},$$

where $d_j := |x^j|$. We have that

$$\Delta \tilde{u}_j = \frac{d_j^2}{\sup_{B^+_{d_j}} |u_j|} \Delta u_j.$$

Two cases are possible: either

$$\frac{d_j^2}{\sup_{B^+_{d_j}} |u_j|} \to 0 \tag{17}$$

for some subsequence or

$$\frac{d_j^2}{\sup_{B^+_{d_j}} |u_j|} \nrightarrow 0 \tag{18}$$

for all subsequences.

Let us consider the first case. From Lemma 9 it follows that

$$-Cr^2 + |D_{e_1} u_j(0)| r \le \sup_{B^+_r} |u_j| \le Cr^2 + |D_{e_1} u_j(0)| r. \tag{19}$$

This together with (17) gives that $|D_{e_1} u_j(0)| d_j^{-1} \to \infty$, so we can assume

$$|D_{e_1} u_j(0)| > j d_j. \tag{20}$$

From here, and (19) we obtain

$$\left| \frac{\sup_{B^+_{d_j}} |u_j|}{d_j |D_{e_1} u_j(0)|} - 1 \right| \le \frac{C}{j} \to 0.$$

We arrive at

$$\sup_{B^+_r} |\tilde{u}_j| = \frac{\sup_{B^+_{rd_j}} |u_j|}{\sup_{B^+_{d_j}} |u_j|} \le \frac{Cr^2 d_j^2 + |D_{e_1} u_j(0)| r d_j}{\sup_{B^+_{d_j}} |u_j|} \to r. \tag{21}$$

There is a subsequence of $\tilde{u}_j$ converging to a function u_0 in $C^{1,\alpha}$, that is harmonic in $\mathbb{R}^n_+$ (due to (17)), linearly bounded (due to (21)) and has zero boundary data at Π. Extending u_0 by odd reflection to $\mathbb{R}^n_-$ and using Liouville's theorem we get that $u_0(x) = D_{e_1} u_0(0) x_1$ which contradicts the fact of existence of zeros in K_ϵ.

In the case (18) we can without loss of generality assume

$$\frac{d_j^2}{\sup_{B^+_{d_j}} |u_j|} \to d > 0. \tag{22}$$

We have then that a subsequence of $\tilde{u}_j$ converges to a function u_0 in C^1 and (22) implies that u_0 is a global solution with $d\lambda_\pm$ instead of $\lambda_\pm$ and zero boundary data. Condition (4) and Lemma 14 give us that u_0 is strictly positive or negative in $\mathbb{R}^n_+$, which contradicts the fact that $x^j \in K_\epsilon \cap \Gamma_{u_j}$. More precisely, functions $\tilde{u}_j$ vanish at $\tilde{x}_j := d_j^{-1} x_j \in K_\epsilon \cap \Gamma_{u_j} \cap \partial B_1$, thus we can always choose the subsequence of $\tilde{u}_j$ in such a way that the corresponding subsequence $\tilde{x}_j \to x_0 \in K_\epsilon \cap \Gamma_{u_j} \cap \partial B_1$ and then $u_0(x_0) = 0$. $\square$

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References

[ACF] H.W.Alt, L.A. Caffarelli, A. Friedman *Variational problems with two phases and their free boundaries* Trans. of AMS, Vol. 282, Nr. 2, 431-461

[A] J. Andersson *On the regularity of a free boundary near contact points with a fixed boundary* Preprint

[C] L.A. Caffarelli *The obstacle problem revisited* J. Fourier Analysis and Appl. 4 (1998), no. 4-5, 383-402

[CKS] L.A. Caffarelli, L. Karp, H. Shahgholian *Regularity of a free boundary with application to the Pompeiu problem* Ann. of Math. (2) 151 (2000), no. 1, 269-292

[KKS] A.L. Karakhanyan, C.E. Kenig, H. Shahgholian *The behavior of the free boundary near the fixed boundary for a minimization problem* Preprint

[M] N. Matevosyan *Tangential touch between free and fixed boundaries* Ph.D. thesis, KTH Stckholm, 2003

[SU] H. Shahgholian, N.N. Uraltseva, *Regularity properties of a free boundary near contact points with the fixed boundary*, Duke math. J. 116 (2003), no. 1, 1-34

[SUW] H. Shahgholian, N.N. Uraltseva, G.S. Weiss *Global solutions of an obstacle-problem-like equation with two phases* (to appear in Monatshefte für Mathematik)

[U] N.N. Uraltseva *two-phase obstacle problem*, Journal of Mathematical Sciences, Vol. 106, No. 3, 2001

[W1] G.S. Weiss *Partial regularity for a weak solutions of an elliptic free boundary problem* Commun. in PDE, 23(3&4), 439-455 (1998)

[W2] G.S. Weiss *An obstacle-problem-like equation with two phases: pointwise regularity of the solution and an estimate of the Hausdorff dimension of the free boundary* Interfaces and Free Boundaries, Vol. 3(2001), 121-128

[Wi] K.-O. Widman *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations* Math. Scand. 21, 1967, 17-37 (1968)

John Andersson, Institutionen för Matematik, Kungliga Tekniska Högskolan, 100 44 Stockholm, Sweden
E-mail address: johnan@e.kth.se

Norayr Matevosyan, Johann Radon Institut für Angewandte Mathematik, Österreichische Akademie der Wissenschaften, Altenbergerstrasse 69, 4040 Linz, Austria
E-mail address: norayr.matevosyan@oeaw.ac.at

Hayk Mikayelyan, Mathematisches Institut, Universität Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany
E-mail address: hayk@math.uni-leipzig.de