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by

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Abstract

Motivated by problems in electrostatics and vortex dynamics, we develop two general methods for constructing the Green's function for simply connected domains on the surface of the unit sphere. We prove a Riemann mapping theorem showing that such domains can be conformally mapped to the upper hemisphere. We then categorize all domains on the sphere for which the Green's function can be constructed by an extension of the classical method of images. We illustrate our methods by several examples, such as the upper-hemisphere, geodesic triangles, and latitudinal rectangles. We describe the point vortex motion in these domains, which is governed by a Hamiltonian determined by the Dirichlet Green's function.

1 Introduction

The need for constructing the Green's function for simply connected domains on general surfaces arises both for problems in electrostatics [21] and in two-dimensional vortex dynamics [19]. In electrostatics, one would like to construct equilibrium configurations of point charges on the surface of a conducting sphere interacting by the Coulomb law, for example. (This is sometimes referred to as the dual problem for stable molecules.) The surface in question may have a boundary on which one must prescribe the Dirichlet or Neumann boundary condition [1, 2, 6, 21]. The corresponding point vortex problem arises in atmospheric and oceanographic models where one is interested in the transport of vorticity in a two-dimensional layer [12, 13, 14, 19]. The solid boundary typically would represent some sort of obstruction to the flow. For example, in the oceanographic context it might represent solid surfaces or shore effects where the no fluid penetration condition $\mathbf{u} \cdot \mathbf{n} = 0$ must be enforced. (Here $\mathbf{n}$ is the unit normal vector on the boundary.) In both contexts, one starts by constructing a Green's function for the Dirichlet or Neumann Laplacian in the domain. They are also known as Green's functions of the first and second kinds for the Dirichlet or Neumann Laplacian in the domain.

Let Δ be the Laplace-Beltrami operator for the surface in question, let Ω be the domain and $\partial\Omega$ its boundary. The Dirichlet Green's function satisfies

$$\Delta G(\theta, \phi; \theta_\alpha, \phi_\alpha) = \delta \quad \text{in} \quad \Omega$$
$$G = 0 \quad \text{on} \quad \partial\Omega.$$

where δ is the Dirac delta function supported at $(\theta_\alpha, \phi_\alpha)$.

Methods for constructing the Green's function mostly rely on conformal mapping ideas [7, 16], the method of images, or the inversion [10, 11, 23]. The method of images, in one form or another, has a firm place in the standard bag of tricks of every mathematical physicist and applied mathematician. Accordingly, it has solid place in the physics and engineering literature. See, e. g., [23], §17, where it is called the simple reflection method. In his classical paper [11], Keller gave the method of images a conceptual formulation, thus putting it on a solid mathematical footing. According to [11], it is '... a method for constructing a Green's function for a part of space bounded by planes in terms of the corresponding Green's function for the full space'. In this formulation, 'space' is meant to be Euclidean, and the 'planes' are the Euclidean hyperplanes. Keeping the discussion general, Keller does not specify any relation between the known Green's function and the Green's function we want to construct. However, in the body of the paper he reveals that the desired relation is via the action of a Coxeter group, respecting the differential operator in question, and that the domain 'bounded by planes' is the fundamental domain of this group. Although the Riemann mapping method is more general than the image method, it relies on the explicit construction of a conformal transformation from the domain in question to a standard domain, such as the unit disc or the upper half plane.

In this paper we develop both the method of images and the Riemann mapping method for regions not restricted to the domains of Euclidean space 'bounded by planes'. For the image method, our description hinges on a group of symmetries. This group acts properly discontinuously on the space, preserving the differential operator in question. Thus, our framework is more general, and, at the same time, more concrete than the setting of [11]. We develop these methods with the application to the vortex motion on compact surfaces as our main focus. To understand the motion of vortices on such surfaces, one must first construct the Green's function of the first kind for the domain in question, then use it to construct the Hamiltonian governing the motion [19]. To keep the paper reasonably self-contained, we briefly describe the equations of vortex motion in a spherical domain with boundaries, both on the surface and in the stereographic projection. We prove a version of the Riemann mapping theorem for a simply connected domain on the sphere which shows that the upper-hemisphere plays a role similar to that of the unit disc in the euclidean plane. We fully categorize the domains on the sphere whose Green's function can be constructed via the image method.

Let $G_D(\theta, \phi; \theta_\alpha, \phi_\alpha)$ denote the Dirichlet Green's function for a closed, simply connected domain Ω on the surface of the sphere. It can be decomposed as

$$G_D = G + G_H^{(\alpha)} \tag{1.1}$$

where G is the Green's function for the sphere itself

$$G(\theta, \phi; \theta_\alpha, \phi_\alpha) = -\frac{1}{4\pi} \log\left[1 - \cos\theta\cos\theta_\alpha - \sin\theta\sin\theta_\alpha\cos(\phi - \phi_\alpha)\right]. \tag{1.2}$$

Note that the argument of the log is the chord distance between the points (θ, ϕ) and $(\theta_\alpha, \phi_\alpha)$. The term $G_H^{(\alpha)}$ is to be constructed, so that G_D vanishes on the boundary, $\partial\Omega$. Our goal is to construct G_D for general simply-connected domains Ω on the surface of the unit sphere. We first prove in §2 a version of the Riemann mapping theorem which allows us to conformally map general domains spherical domains to the upper hemisphere, as shown in figure 1. In §3, we work out the image method for the sphere. Then in §4 we flesh out the method through several explicit examples. There we construct the Hamiltonian governing point vortex motion in simply connected domains on the surface of the unit sphere. In the concluding §5 we briefly outline a few extensions and generalizations of our methods.

2 Conformal mappings and Green's functions

The Riemann mapping theorem is a powerful tool of complex analysis with many applications (see, e. g., [18]). We will apply it to the Green's function of the Dirichlet Laplacian of a spherical domain. Although, the setting is classical, we could not find a suitable treatment of this material in the literature. See, however, a remark in [5], Vol. 1, p. 377. We will show that the Riemann mapping theorem allows us to express the Green's function of a simply connected domain in the complex plane or the Riemann sphere in terms of the Green's function of the unit disc. The latter is well known. Let Ω be the domain in question, and let D be the unit disc. A conformal mapping $\phi : \Omega \to D$ yields an explicit expression for the Green's function of Ω. For the convenience of the reader, we establish the general setting, and provide the proofs in some detail. See [5, 18, 22, 23] for the general background on these concepts and the standard techniques.

The classical Laplacian $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ of the Euclidean plane is a special case of the Laplace-Beltrami operator of a Riemannian manifold. Let X be a manifold of n dimensions, and let $r = r_{ij}$ be a Riemannian metric tensor in local coordinates $x_1, \ldots, x_n$. The Laplace-Beltrami operator, $\Delta = \Delta(X, r)$ is a canonical, second order differential operator on functions on X (see, e. g., [20]). Let $\det r$ be the determinant of the $n \times n$ matrix $[r_{ij}]$, and denote by $[r^{ij}]$ the inverse matrix. Then in the local coordinates we have

$$\Delta f = \frac{1}{\sqrt{\det r}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(r^{ij}\sqrt{\det r}\frac{\partial f}{\partial x_i}). \tag{2.1}$$

If $\Omega \subset X$ is a domain, with a piecewise smooth boundary $\partial\Omega$, we denote by Δ_Ω the corresponding Dirichlet Laplacian. In our immediate applications, (X, r) is either $\mathbf{R}^2$ or the standard unit sphere $S^2 \subset \mathbf{R}^3$. However, it is instructive to put our exposition in a more general context. For typographical reasons, in the displayed formulas that follow we will suppress the subscripts Ω, r, etc if this does not lead to an ambiguity. Denote by $d_r x = (\det r) dx_1 \cdots dx_n$ the Riemannian volume form. Then Δ is a symmetric operator in the corresponding Hilbert space. More precisely, if f and g are functions on Ω, satisfying the Dirichlet boundary conditions, then

$$\int_\Omega f(x)(\Delta g(x))dx = \int_\Omega (\Delta f(x))g(x)dx. \tag{2.2}$$

The Green's function, $g_\Omega(x,y)$, is the kernel of the right inverse of Δ_Ω. It is symmetric, vanishes on $\Omega \times \partial\Omega$, and satisfies the differential equation $\Delta_\Omega g_\Omega(x,y) = \delta(x-y)$. To be more precise, the following integral identity holds

$$\int_\Omega f(x)(\Delta g(x,y))dx = f(y). \tag{2.3}$$

We will now specialize to the case of dimension two, i.e., to the Riemannian surfaces (X,r). Recall that X is a Riemann surface, if it has a system of local complex coordinates, z_i, and the transition functions are holomorphic. A Riemannian metric on X is conformal if the metric tensor locally has the form $\frac{1}{2}r(z)(dz^2 + d\bar{z}^2)$, where $r(\cdot) > 0$.[1]

Two Riemannian metrics, r_1 and r_2, on a manifold are conformally equivalent if they differ by a (necessarily positive) factor, $r_2 = \rho(\cdot)r_1$. We will also say that the metrics r_1 and r_2 belong to the same conformal class. In what follows we consider only Riemann surfaces and conformal metrics. The complex plane, $\mathbf{C}$, and the Riemann sphere S^2, with the canonical metrics, are our basic examples. An open subset of a Riemann surface with a conformal metric inherits both structures. We will now recall the basic notion of a conformal mapping.

Definition 1. *Let Z, U be Riemann surfaces. A mapping $\phi : Z \to U$ is* conformal *if in respective local coordinates it is given by $u = \phi(z)$, where $\partial\phi/\partial\bar{z} = 0$ identically, and $\partial\phi/\partial z \neq 0$ everywhere.*

An invertible conformal mapping $\phi : Z \to U$ is a *conformal equivalence*. In what follows we consider only them, and we suppress the adjective"invertible".

In general, the relation between the Laplace-Beltrami operators on a manifold corresponding to different metrics is complicated. In the case under consideration, it simplifies.

Proposition 1. *Let Z be a Riemann surface. Let $r, \tilde{r}$ be two conformal metrics on Z, and set $\tilde{r} = \rho(\cdot)r$. Denote by Δ and $\tilde{\Delta}$ the corresponding Laplace-Beltrami operators. Then*

$$\tilde{\Delta} = \rho^{-1}\Delta. \tag{2.4}$$

Proof. It suffices to show this relation locally. Let z be a local coordinate on Z. Then there is a function $r(z) > 0$ such that the metric r is given by $\frac{1}{2}r(z)(dz^2 + d\bar{z}^2)$. Substituting this into eq. (2.1), we get by a direct calculation the local expression of the Laplace-Beltrami operator:

$$(\Delta f)(z) = 4r(z)^{-1}\frac{\partial^2 f}{\partial z \partial \bar{z}}. \tag{2.5}$$

The operator $\tilde{\Delta}$ is given by eq.(2.5) as well, only now the factor preceeding $\partial^2/\partial z\partial\bar{z}$ is $4(\rho r)(z)^{-1}$. ■

[1] The factor $\frac{1}{2}$ is introduced for numerical convenience.

Differentiable mappings allow us to transfer functions, metrics, and differential operators from one manifold to another. Thus, let Z and U be Riemann surfaces, and let $\phi : Z \to U$ be a conformal mapping. If L is any differential operator on U, we can pull it back to Z via ϕ, obtaining a differential operator, $\phi^* L$, of the same order. Likewise, if L is a differential operator on Z, we can push it forward by ϕ, obtaining a differential operator, $\phi_* L$, on U. See, e. g., [24] for the background on this material. The metrics can be also pulled back and pushed forward, and we will use analogous notation for that. The operations of pulling back and pushing forward are mutual inverses. Indeed, replacing ϕ by its inverse, $\phi^{-1} : U \to Z$, we interchange them. We will study the pull-backs and the push-forwards of the Laplace-Beltrami operators.

Proposition 2. *Let Z and U be Riemann surfaces, endowed with the conformal metrics r_Z and r_U, respectively. Let Δ_Z and Δ_U be the corresponding Laplace-Beltrami operators. Let $\phi : Z \to U$ be a conformal mapping. Let $\phi^*(r_U)$ (resp. $\phi_*(r_Z)$) be the pull-back of the metric r_U (resp. the push-forward of r_Z). Then:*

1. The operator $\phi_* \Delta_Z$ is the Laplace-Beltrami operator on U, corresponding to the pushed forward metric $\phi_*(r_Z)$;

2. The operator $\phi^* \Delta_U$ is the Laplace-Beltrami operator on Z, corresponding to the pulled back metric $\phi^*(r_U)$.

Proof. It suffices to verify the first claim locally. Let z be a local coordinate in a neighborhood $Z_0 \subset Z$, and let u be a local coordinate in its image $U_0 = \phi(Z_0) \subset U$. Let f be a function in U_0. Then $F(z) = f(u(z))$ is the pull-back of f to Z_0. In order to determine the push-forward of the operator $\partial^2/\partial z \partial \bar{z}$ on Z_0, we compute $\partial^2 f(u(z))/\partial z \partial \bar{z}$. By a straightforward calculation

$$\phi_*(\frac{\partial^2}{\partial z \partial \bar{z}}) = |\phi'(z)|^2 \frac{\partial^2}{\partial u \partial \bar{u}}.$$

The pull-back of the metric $\frac{1}{2}(du^2 + d\bar{u}^2)$ on U_0 is $\frac{1}{2}|\phi'(z)|^2(dz^2 + d\bar{z}^2)$. Hence, the right hand side of the preceding equation (multiplied by 4) is the Laplace-Beltrami operator, corresponding to the push-forward of the metric $\frac{1}{2}(dz^2 + d\bar{z}^2)$. See equation (2.5). Using Proposition 1, we obtain the claim. ∎

Proposition 3. *Let r and $\tilde{r}$ be conformal metrics on a Riemann surface Z. Let Δ and $\tilde{\Delta}$ be the corresponding Laplace-Beltrami operators. Let $g(x, y)$ and $\tilde{g}(x, y)$ be the Green's functions of the operators Δ and $\tilde{\Delta}$ respectively. Then*

$$\tilde{g}(x, y) = g(x, y). \tag{2.6}$$

Proof. Let z be a local complex coordinate in Z. Then there are positive functions, $r(z)$ and $\tilde{r}(z)$, such that the metrics are given by $\frac{1}{2}r(z)(dz^2 + d\bar{z}^2)$ and $\frac{1}{2}\tilde{r}(z)(dz^2 + d\bar{z}^2)$ respectively. Set

$$\rho(z) = \frac{\tilde{r}(z)}{r(z)}.$$

Let d_r and $d_{\tilde{r}}$ be the volume forms on Z corresponding to the metrics r and $\tilde{r}$ respectively. Then

$$d_{\tilde{r}} = \rho(\cdot) d_r. \tag{2.7}$$

The identity eq.(2.3) for the operator $\tilde{\Delta}$ says that $\int_Z f(x)(\tilde{\Delta}\tilde{g}(x, y))d_{\tilde{r}}(x) = f(y)$. We will evaluate this integral, substituting $g(x, y)$ for $\tilde{g}(x, y)$. In view of Proposition 1 and eq. (2.7), we have

$$\int_Z f(x)(\tilde{\Delta} g(x,y))d_{\tilde{r}}(x) =$$
$$\int_Z f(x)(\rho^{-1}(x)\Delta g(x,y))\rho(x)d_r(x) =$$
$$\int_Z f(x)(\Delta g(x,y))d_r(x) = f(y).$$

By uniqueness of the Green's function, this implies the claim. ■

Theorem 1. *Let Z and U be Riemann surfaces, endowed with conformal metrics. Let Δ_Z and Δ_U be the corresponding Laplace-Beltrami operators. Let $g_Z(\cdot,\cdot)$ and $g_U(\cdot,\cdot)$ be the respective Green's functions. Suppose that $\phi : Z \to U$ is a conformal mapping. Then*

$$g_Z(x,y) = g_U(\phi(x), \phi(y)). \tag{2.8}$$

Proof. The right hand side of eq. (2.8) is the pull-back of the Green's function of the Laplace-Beltrami operator Δ_U. It is equal to the Green's function of the pull-back of Δ_U, the differential operator $\phi_*\Delta_U$. We leave the verification of this to the reader. (The claim is valid in a far greater generality.) Let r_U and r_Z be the metrics in question. By Proposition 2, the operator $\phi_*\Delta_U$ is the Laplace-Beltrami operator corresponding to the pull-back metric $\phi_*(r_U)$. Both r_Z and r_U are conformal metrics. Now Proposition 3 implies the claim. ■

Remark. Theorem 1 extends to all Riemannian surfaces, but it does not hold in dimensions greater than two.

By the Riemann mapping theorem, any bounded, simply connected domain, $\Omega \subset \mathbf{C}$ is conformally equivalent to the unit disc. Since the Green's function of the latter is known, Theorem 1 yields, in principle, the Green's function of any such Ω. The proposition below allows to extend this observation to spherical domains.

Proposition 4. *Let Ω be a simply connected domain in the standard sphere, such that the complement to Ω has a nonempty interior. Then Ω is conformally equivalent to the unit disc.*

Proof. Denote by $S \subset \mathbf{R^3}$ the unit sphere. Let $\sigma : S \to \mathbf{C}$ be the stereographic projection from the north pole, $(0,0,1)$. By assumption, the complement to Ω has an interior point, t_0. Let ρ be a rotation of $\mathbf{R^3}$, such that $\rho(t_0) = (0,0,1)$. Since the spherical domain $\rho \cdot \Omega$ is "bounded away" from $(0,0,1)$, the planar domain $Z = \sigma \cdot \rho \cdot \Omega \subset \mathbf{C}$ is bounded. Since $\sigma \cdot \rho : S \setminus \{t_0\} \to \mathbf{C}$ is a homeomorphism, the domain $Z \subset \mathbf{C}$ is simply connected. Let $\phi : Z \to D$ be a conformal mapping onto the unit disc, guaranteed by the Riemann mapping theorem. Then $\psi = \phi \cdot \sigma \cdot \rho : \Omega \to D$ is a homeomorphism. Rotations are isometries, hence conformal. The stereographic projection is also conformal. See, e. g., [18]. Since ψ is a composition of conformal mappings, it is conformal. ■

2.1 Example: The Green's function of the spherical disc

Our starting point is the Green's function of the unit disc, which is well known (see, e. g., [23]). Let $u, v \in D$. We have

$$g_D(u,v) = \frac{1}{2\pi} \log \frac{|u-v|}{|1-\bar{v}u|}. \tag{2.9}$$

Let $D(r)$ be the disc of radius r. The mapping $u \mapsto u/r$ sends $D(r)$ conformally onto D. Theorem 1 and eq. (2.9) imply

$$g_{D(r)} = \frac{1}{2\pi} \log \frac{|u-v|r}{|r^2 - \bar{v}u|}. \tag{2.10}$$

The unit sphere $\mathbf{S} \subset \mathbf{R}^3$ is given by $x^2 + y^2 + z^2 = 1$ in the cartesian coordinates in $\mathbf{R}^3$. We will also use the spherical coordinates $0 \le \theta \le \pi, 0 \le \phi \le 2\pi$. Then

$$x = \cos\phi \sin\theta, y = \sin\phi \sin\theta, z = \cos\theta.$$

The simplest domains in $\mathbf{S}$ are the spherical discs. We denote by $SD(\rho) \subset \mathbf{S}$ the spherical disc of radius ρ, where $0 < \rho < \pi$. If the disc $SD(\rho)$ is centered at the north pole, then

$$SD(\rho) = \{(\theta, \phi) : 0 \le \theta \le \rho\}.$$

The stereographic projection from the south pole is given by

$$(x, y, z) \mapsto \frac{x + iy}{1 + z}.$$

In the spherical coordinates we have

$$(\theta, \phi) \mapsto \tan \frac{\theta}{2} e^{i\phi}.$$

Denote by $g_{SD(\rho)}((\theta, \phi); (\theta', \phi'))$ the Green's function of the spherical disc of radius $\rho < \pi$. Then

$$g_{SD(\rho)}((\theta, \phi); (\theta', \phi')) = \frac{1}{2\pi} \log \frac{|\tan\frac{\theta}{2} e^{i\phi} - \tan\frac{\theta'}{2} e^{i\phi'}| \tan\frac{\rho}{2}}{|\tan^2\frac{\rho}{2} - \tan\frac{\theta}{2} \tan\frac{\theta'}{2} e^{i(\phi-\phi')}|}. \tag{2.11}$$

Note that the stereographic projection sends $SD(\rho)$ into the disc $D(\cdot)$ of radius $\tan\frac{\rho}{2}$. Eq. (2.11) now follows from Theorem 1, eq. (2.10), and the preceding formulas. The spherical disc $SD(\pi/2)$ is a hemisphere. Denote by $\mathbf{S}_+$ the upper hemisphere, and let $g_{S_+}(\cdot;\cdot)$ be the corresponding Green's function. Specializing in eq. (2.11), we obtain

$$g_{S_+}((\theta, \phi); (\theta', \phi')) = \frac{1}{2\pi} \log \frac{|\tan\frac{\theta}{2} e^{i\phi} - \tan\frac{\theta'}{2} e^{i\phi'}|}{|1 - \tan\frac{\theta}{2} \tan\frac{\theta'}{2} e^{i(\phi-\phi')}|}. \tag{2.12}$$

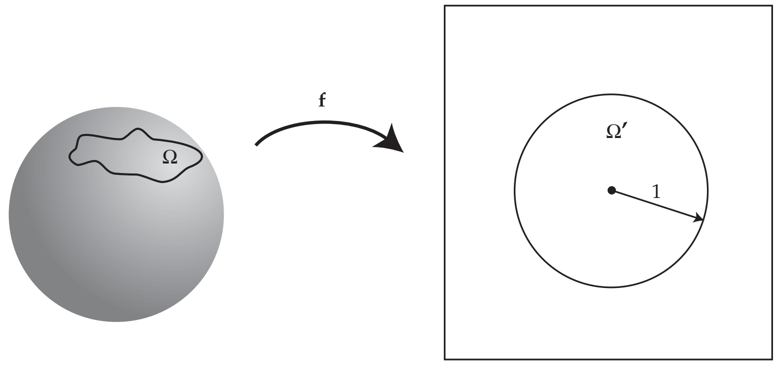

Figure 1: Conformal mapping of a simply connected spherical domain onto the unit disc

3 The method of images

3.1 General setting

The general purpose of this method is to express the Green's function of a differential problem via that of a simpler problem. As the name "method of images" suggests, the problems in question are related by a symmetry. In what follows we outline the general framework of the method of images. In the body of this section we will elaborate in the context of the Laplacian on the Euclidean sphere.

Let X be a manifold of arbitrary dimension, and let L be a differential operator on the space of functions on X. To simplify the exposition, we assume in what follows that L is a second order operator. Let $\Omega \subset X$ be a closed domain with a piecewise smooth boundary. We denote by L_D (resp. L_N) the operator on functions in Ω given by L with the Dirichlet (resp. Neumann) boundary conditions on $\partial\Omega$. Under certain conditions the method of images will allow us to express the Green's functions of L_D and L_N in terms of the Green's function of L. We will now formulate these assumptions on X, L, Ω.

Assumptions:

1. There is a group Γ of diffeomorphisms of X, such that the operator L is Γ-invariant. Let us say that the points $x_1, x_2 \in X$ are Γ-conjugate if there exists $g \in \Gamma$ such that $x_2 = g \cdot x_1$.

2. Any point $x \in X$ is Γ-conjugate to a unique $\omega \in \Omega$.

3. The boundary of Ω is a finite union of hypersurfaces: $\partial\Omega = \cup_{i=1}^{p} Y_i$ and the interior of the boundary is a disjoint union of their interiors: $\text{int}(\partial\Omega) = \cup_{i=1}^{p} H_i$. Then for any point $x \in \text{int}(\partial\Omega)$ the inward unit normal vector $\vec{n} = \vec{n}(x)$ is well defined.

4. For every $x \in \text{int}(\partial\Omega)$ there is a unique nontrivial element $g = g(x) \in \Gamma$, such that $g \cdot x = x$. We then have $g(x) \cdot \vec{n}(x) = -\vec{n}(x)$.

Let $G(x, y)$ denote the Green's function of L on X. We will use the notation $G_D(\cdot, \cdot)$ and $G_N(\cdot, \cdot)$ for the Dirichlet and the Neumann Green's function of L on Ω. The proposition below summarizes the method of images.

Proposition 5. *Suppose that the assumptions 1-4 are satisfied. Then* Γ *has a natural parity function, which takes values* ± 1. *We denote it by* $(-1)^{\ell(\gamma)}$. *Let* $\omega, \omega' \in \Omega$ *be arbitrary points. Then the following formulas hold:*

$$G_D(\omega, \omega') = \sum_{\gamma \in \Gamma} (-1)^{\ell(\gamma)} G(\omega, \gamma \cdot \omega'); \tag{3.1}$$

$$G_N(\omega, \omega') = \sum_{\gamma \in \Gamma} G(\omega, \gamma \cdot \omega'). \tag{3.2}$$

Proof. We begin with the geometric considerations. By our assumptions, there are p hypersurfaces $Y_i \subset X$, such that the subsets $H_i = \mathrm{int}(\partial\Omega) \cap Y_i \subset Y_i$ are open and disjoint. Hence if $x \in \mathrm{int}(\partial\Omega)$, then there is a unique index $1 \leq i \leq p$ such that $x \in H_i \subset Y_i$. We will consider for the moment only this hypersurface, and suppress the index in the discussion below. Let $g = g(x)$ be as in assumption 4. Since $x \in Y \cap g \cdot Y$, the hypersurfaces Y and $g \cdot Y$ intersect. If they intersect at x transversally, then $g \cdot H$ contains points from $\mathrm{int}(\Omega)$, which is impossible. If they intersect nontrivially, but tangentially, then there is a neighborhood of x having nonempty intersections with infinitely many regions $\gamma \cdot \Omega$, $\gamma \in \Gamma$. This is impossible, in view of the discreteness of Γ. Thus, the only possibility is $g \cdot Y = Y$. If $g : Y \to Y$ is a nonidentity mapping, then H contains points $y \neq g \cdot y$. Our assumptions preclude that. Therefore, g induces the identity transformation on Y.

Returning to the full set of hypersurfaces Y_i, $1 \leq i \leq p$, intersecting $\mathrm{int}(\partial\Omega)$, we conclude from the preceding argument that for each index i there is an element $\sigma_i \in \Gamma$ such that for any $x \in H_i$ we have $\sigma_i(x) = x$. The element σ_i restricts to the identity on Y_i, but $\sigma_i \cdot \vec{n}(x) = -\vec{n}(x)$. Thus, the differential of σ_i is the reflection of the tangent space $T_x X$ about the codimension one subspace $T_x Y_i \subset T_x X$. Therefore σ_i reverses the orientation of X, we have $\sigma_i^2 = 1$, and Y_i is the set of fixed points of σ_i. It is customary to call diffeomorphisms satisfying the conditions above the *reflections*. Thus, $\sigma_i \in \Gamma$ is a *reflection of* X *about* Y_i, for $1 \leq i \leq p$.

The elements σ_i, $1 \leq i \leq p$, generate Γ, thus Γ is a *reflection group* of transformations. As an abstract group, Γ is a *Coxeter group*. It means that all relations between the generators of Γ follow from the p equations $\sigma_i^2 = 1$, and from the $p(p-1)/2$ equations

$$(\sigma_i \sigma_j)^{n_{ij}} = 1, \; 1 \leq i < j \leq p. \tag{3.3}$$

The exponents n_{ij} in eq.(3.3) may take any natural values between 2 and ∞. By convention, $n_{ij} = \infty$ means that $\sigma_i \sigma_j \in \Gamma$ is an element of infinite order. We refer the reader to [4] for the classical material about reflection groups, including the facts above, and to [8, 9] for more recent developments. We will use the standard terminology of the subject [4, 9]. For instance, the regions $g \cdot \Omega$, $g \in \Gamma$, are the *Weyl chambers*, the closed sets $Y_i \cap \Omega$, $1 \leq i \leq p$, are the *walls* of the *fundamental chamber* Ω, etc.

We conclude the geometric part of our proof by introducing the *length and the parity functions* on Γ. Every element $\gamma \in \Gamma$ has several representations as a word in the generators σ_i. Then $\ell(\gamma)$ is the minimum of lengths of these representations. The length function has a geometric interpretation in terms of the decomposition of X into Weyl chambers [8]. A word $\gamma = \sigma_{i_1} \cdots \sigma_{i_n}$ is reduced if $n = \ell(\gamma)$. Although $\ell(\gamma)$ is well defined, γ may have several reduced representations. Also note that $\ell(gh) \neq \ell(g)\ell(h)$, in general. The parity $(-1)^{\ell(\gamma)} = \pm 1$ is the unique multiplicative function on Γ which takes values -1 on the generators and 1 on the identity element. A different, but equivalent way to define the parity function is to set $(-1)^{\ell(\gamma)} = -1$ if γ is orientation reversing, and $(-1)^{\ell(\gamma)} = 1$ if γ preserves orientation. If Γ is a linear reflection group [4], then $(-1)^{\ell(\gamma)} = \det(\gamma)$.

By assumption 1, the Green's function of L is Γ-invariant, i.e., $G(\gamma \cdot x, \gamma \cdot y) = G(x, y)$ for any γ. Let $G_D(x, y)$, $G_N(x, y)$ be the expressions introduced by eqs.(3.1), (3.2), viewed as functions on $X \times X$. More precisely

$$G_D(x, y) = \sum_{\gamma \in \Gamma} (-1)^{\ell(\gamma)} G(x, \gamma \cdot y); \quad G_N(x, y) = \sum_{\gamma \in \Gamma} G(x, \gamma \cdot y). \tag{3.4}$$

Then both $G_D(x, y)$, $G_N(x, y)$ are symmetric in their variables:

$$G_D(y, x) = G_D(x, y), \; G_N(y, x) = G_N(x, y).$$

Fix the second variable, and consider $G_D(x, y)$, $G_N(x, y)$ as functions of the first variable, which varies over X. A straightforward calculation shows that $G_D(x, y)$ and $G_N(x, y)$ are Γ-equivariant:

$$G_D(\gamma \cdot x, y) = (-1)^{\ell(\gamma)} G_D(x, y); \; G_N(\gamma \cdot x, y) = G_N(x, y). \tag{3.5}$$

More precisely, eq.(3.5) means that $G_N(x, y)$ (resp. $G_D(x, y)$) is Γ-invariant (resp. Γ-anti-invariant) as a function of the first variable. Therefore, both $G_D(\cdot, y)$, $G_N(\cdot, y)$ may be viewed as functions on the quotient space $\Omega = X/\Gamma$. By the same token, without loss of generality, the parameter y may be restricted to Ω. Thus, we identify $G_D(\cdot, \cdot)$, $G_N(\cdot, \cdot)$ with functions on $\Omega \times \Omega$. The functions $G_D(\omega, \omega')$, $G_N(\omega, \omega')$ are obtained from $G(x, y)$ by averaging over Γ, and the operator L is Γ-invariant. Hence $G_D(\cdot, \cdot)$, $G_N(\cdot, \cdot)$ satisfy the same relations with respect to L as $G(\cdot, \cdot)$, i.e., the relations that the Green's function satisfies.

It remains to establish that $G_D(\omega, \omega')$ (resp. $G_N(\omega, \omega')$) satisfies the Dirichlet (resp. Neumann) boundary conditions. This is a consequence of the anti-invariance (resp. invariance) of the averaged functions. See eq.(3.5). Consider $G_D(\omega, \omega')$. If $\omega \in \partial(\Omega)$, then there is a generating reflection σ such that $\sigma \cdot \omega = \omega$. By eq.(3.5), $G_D(\omega, \omega') = -G_D(\omega, \omega')$, thus $G_D(\omega, \omega')$ vanishes on $\partial(\Omega)$. The proof that the normal derivative of $G_N(\omega, \omega')$ vanishes on $\partial(\Omega)$ is analogous, and we leave it to the reader. ■

Remark 1. In order to use the method of images, we need the following:

1. The Green's function of the differential operator L on X be explicitly known;
2. The operator L be invariant under a particular group of diffeomorphisms of X;
3. The domain Ω be somewhat special.

A classical framework for the method of images is the Euclidean Laplacian on $\mathbf{R}^n$, in particular on $\mathbf{R}^2$. Another possibility is the Laplacian, Δ, on a sphere, in particular, on the unit sphere $S^2 \subset \mathbf{R}^3$.

3.2 Reflections on the sphere

Recall that the geodesics of S^2 are the big circles. A domain $\Omega \subset S^2$ is a spherical polygon (or simply a polygon) if $\partial\Omega$ is a finite union of geodesic segments. According to our definition, a hemisphere is also a polygon. It is the only polygon without vertices. If Ω is a simple polygon, and it is not a hemisphere, then Ω is uniquely determined by its vertices $v_1, \dots, v_n$, listed in the counterclockwise order. We then say that Ω is a simple n-gon. The geodesic segments $a_i = v_i v_{i+1} : 1 \le i \le n$ are its sides. If $n = 2$, Ω is a bi-gon. For $n = 3$, Ω is a triangle, etc. See figure 2 for an example of a n-gon. For any polygon Ω we denote by Γ_Ω the group of isometries of S^2 generated by the reflections in the sides of Ω. The spherical triangle $\Delta(\alpha, \beta, \gamma)$ with the angles α, β, γ exists if and only if $\alpha + \beta + \gamma > \pi$. If $\Delta(\alpha, \beta, \gamma)$ exists, then it is unique, up to isometry.

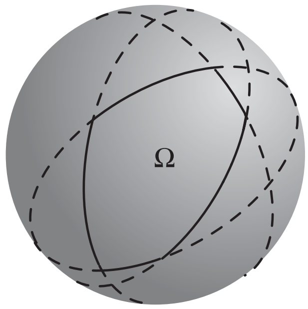

Figure 2: Great circle n-gon on the sphere.

Proposition 6. *Let Ω be a spherical polygon, and let $\Gamma = \Gamma_\Omega$. Then the triple Δ, Ω, Γ satisfies the "Assumptions" of the method of images if and only if Ω is one of the following polygons: 1. It is a hemisphere;*

2. It is the bi-gon with the angle π/n, where $n = 2, 3, \ldots$;

3. It is one of the following triangles:

i) The triangle $\Delta(\pi/2, \pi/2, \pi/n)$, where $n = 2, 3, \ldots$;

ii) The triangle $\Delta(\pi/2, \pi/3, \pi/3)$;

iii) The triangle $\Delta(\pi/2, \pi/3, \pi/4)$;

iv) The triangle $\Delta(\pi/2, \pi/3, \pi/5)$.

Proof. It follows from our assumptions that Ω tiles the sphere under reflections. Hence each vertex angle of Ω is π/n, where n is an integer. Next, we investigate which spherical polygons have angles of this type. If Ω has no vertices, then we are in the case 1. Since a polygon cannot have one vertex, the next possibility is that Ω is a bi-gon. Then we recover case 2. Suppose now that Ω is a triangle, whose angles are $\pi/p, \pi/q, \pi/r$. A necessary and sufficient condition for the existence of a triangle like that is

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \tag{3.6}$$

Obviously, at least one of the angles is $\pi/2$. If two angles are equal to $\pi/2$, we recover the case 3(i). If only one angle is $\pi/2$, then without loss of generality $r = 2$, and $p, q > 2$. Eq. (3.6) becomes $1/p + 1/q > 1/2$. It is clear that p and q cannot be both greater than 3. Hence, without loss of generality, $q = 3$. Eq. (3.6) then yields $p < 6$. This leaves three possibilities: $p = 3, 4, 5$, which correspond to the cases 3(ii), 3(iii), and 3(iv) respectively. Next, we show that Ω cannot have more than 3 vertices. Suppose, for instance, that Ω

is a quadrilateral, with angles $\pi/p, \pi/q, \pi/r, \pi/s$. Analogously to eq. (3.6), the existence condition is

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}>2. \tag{3.7}$$

Since this inequality is not satisfied even with the smallest possible values $p=q=r=s=2$, we conclude that eq. (3.7) has no solutions. The same argument works if Ω has more than four vertices. It remains to check that these polygons do tile the sphere. In cases 1, 2, and 3(i) it is trivial. In the remaining cases it is verified directly. ■

Remark 2. It is instructive to understand the groups Γ corresponding to the polygons, listed in the proposition above. Thus, for the case 2, $\Gamma = D_n$, the dihedral group of order $2n$. It is the group of symmetries of the regular n-gon. In the case 3(i), Γ is an extension of D_n by $\mathbf{Z}/2\mathbf{Z}$. It is the group of symmetries of the solid, obtained by taking the regular pyramid whose base is the regular n-gon, and joining to it its reflection about the base. In the remaining 3 cases, Γ is the group of symmetries of the appropriate Platonic solid. The solids are the cube, the icosahedron, and the dodecahedron respectively. It is especially easy to compute $|\Gamma|$, because it is equal to the number of triangles in the corresponding tiling. To compute that number we use the well known identity

$$\text{area}(\Delta(\alpha,\beta,\gamma)) = \alpha+\beta+\gamma-\pi. \tag{3.8}$$

This immediately yields that the orders of the group Γ in the cases 3(ii), 3(iii), and 3(iv) are 24, 48, and 120 respectively.

4 Applications to point vortex systems

To demonstrate the utility of the previous formulations, we apply the method to point vortex motion on a unit sphere with solid boundaries in order to compute the vortex trajectories inside closed simply connected domains of various shapes. We first review the basic facts connecting the Green's function with the associated dynamics of the vortices and fluid particles. For more on this, see Chapter 3 of [19].

Assuming we know the Dirichlet Green's function, G_D, as in the notation of eq. (1.1) for a single source located at $\mathbf{x}_\alpha \in \Omega$, the Green's function for the case of an arbitrary number of distinct sources located at $\mathbf{x}_\alpha \in \Omega$, $\alpha = 1,\ldots,N$ can be constructed via linear superposition

$$G_D^{(N)}(\mathbf{x};\mathbf{x}_1,\ldots,\mathbf{x}_N) = \sum_{\alpha=1}^{N} G_D(\mathbf{x};\mathbf{x}_\alpha) \equiv \sum_{\alpha=1}^{N} G(\mathbf{x};\mathbf{x}_\alpha) + \sum_{\alpha=1}^{N} G_H^{(\alpha)}(\mathbf{x}). \tag{4.1}$$

Then, the streamfunction governing the fluid motion at an arbitrary point $\mathbf{x}$ is given by

$$\psi = \sum_{\alpha=1}^{N} \Gamma_\alpha G_D(\mathbf{x};\mathbf{x}_\alpha).$$

¿From this, one computes the motion of a vortex located at $\mathbf{x} = \mathbf{x}_\beta$ by subtracting off the singular contribution generated from the term $\alpha = \beta$, giving rise to the *non-singular* streamfunction

$$\psi_\beta \equiv \left(\psi + \frac{\Gamma_\beta}{2\pi}\log\|\mathbf{x}-\mathbf{x}_\beta\|\right)|_{\mathbf{x}=\mathbf{x}_\beta}$$

and the velocity field is then obtained in the usual way as

$$\dot{\mathbf{x}}_\beta \equiv (\dot{x}_\beta, \dot{y}_\beta) = \left(\frac{\partial \psi_\beta}{\partial y_\beta}, -\frac{\partial \psi_\beta}{\partial x_\beta} \right).$$

The Hamiltonian governing the motion of a single point vortex located at $\mathbf{x}_\alpha$ is simply

$$\mathcal{H}(\mathbf{x}_\alpha) = \frac{\Gamma_\alpha}{2} G_H^{(\alpha)}(\mathbf{x}_\alpha),$$

with canonical equations of motion

$$\dot{x}_\alpha = \frac{\partial \mathcal{H}}{\partial y_\alpha}; \qquad \dot{y}_\alpha = -\frac{\partial \mathcal{H}}{\partial x_\alpha}.$$

This construction for a given domain Ω on the surface of a sphere with a point vortex of strength Γ placed at position $(\theta_v, \phi_v) \in \Omega$ is carried out via the following sequence of steps:

1. Map Ω stereographically to Ω', hence $(\theta_v, \phi_v) \mapsto (r_v, \phi_v)$, where $r_v = \tan(\theta_v/2)$;

2. Consider the planar problem in Ω' which we solve via the standard *planar* image method. Streamlines are given by

$$\| \frac{f(z) - f(z_v)}{f(z) - f^*(z_v)} \| = \text{const.}$$

   where $z = R\exp(i\phi)$ and $\xi = f(z)$ conformally maps the domain Ω' to the upper-half plane;

3. If we denote the location of the positive image vortices by ρ_j and the negative ones η_j, then the vortex Hamiltonian is given by

$$\mathcal{H}_v = -\frac{\Gamma^2}{4\pi} \log \left(|\frac{df}{dz}|^2_{z=z_v} | \frac{1}{f(z) - f^*(z_v)} |^2 \right) - \frac{\Gamma^2}{4\pi} \log \left(\frac{(1+|z_v|^2)(1+|\eta_0|^2)\cdots(1+|\eta_m|^2)}{(1+|\rho_1|^2)(1+|\rho_2|^2)\cdots(1+|\rho_m|^2)} \right)$$

The method, of course, hinges on our ability to correctly place the image vortices in step 2. We demonstrate this construction in the following examples and refer the reader to [19] for more discussions of the method and applications.

Example 1. Upper hemisphere: This is the generic case to be treated first. Take Ω to be the upper hemisphere bounded by the latitude $\theta = \theta_0 = 90^o$. The planar equivalent of this domain is a circular region, hence one uses the inversion method to obtain the image system, as described, for example, in [19]. The image system consists of just one vortex of strength $-\Gamma$ at the inverse point $r_i = r_0^2/r_v$; $\phi_i = \phi_v$. The governing Hamiltonian is then

$$\mathcal{H} \quad = \quad \frac{\Gamma^2}{4\pi} \log \left(\frac{(1+|z_v|^2)(1+|z_i|^2)}{|z_v - z_i|^2} \right),$$

which yields vortex paths

$$\frac{(1+r^2)(r^2+r_0^4)}{(r_0^2 - r^2)^2} = \text{const.}$$

Vortex paths are shown in figure 3.

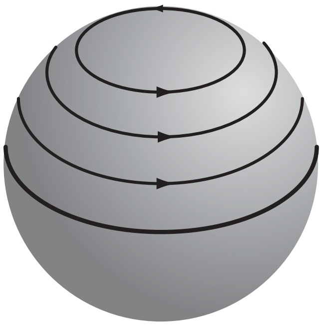

Figure 3: Vortex paths in upper hemisphere.

Example 2. Wedge-like domains: Here we take Ω to be the sector bounded by the longitudes 0 and π/m, $m \in N$. Then Ω' is a wedge of angle π/m. The analytic function $f(z) = z^m$ maps Ω' to the upper half plane. There are $(2m-1)$ image vortices, with positive vortices located at

$$\rho_\alpha = z_v \exp\left(\frac{2i\pi\alpha}{m}\right), \qquad 1 \leq \alpha \leq m-1.$$

The negative ones are located at

$$\eta_\beta = z_v^* \exp\left(\frac{2i\pi\beta}{m}\right), \qquad 1 \leq \beta \leq m.$$

The vortex paths, as shown in figure 4 are obtained from

$$\frac{(1+r^2)^2}{r^2 \sin^2 m\phi} = \text{const.}$$

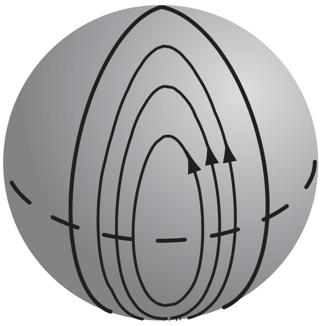

Figure 4: Vortex paths in a wedge-like domain

Example 3. Geodesic triangles: Take Ω to be the half-sector bounded by the longitudes 0 and π/m, $m \in N$. Ω' is the corresponding sector of a circle in the stereographic plane. It is a standard result [16] that the function

$$f(z) = \left[\frac{(1+z^m)}{(1-z^m)}\right]^2$$

maps the sector to the upper-half plane. There are exactly $(4m-1)$ image vortices located at the following locations: $(m-1)$ positive vortices located at the points $\rho_\alpha = z_v \exp(2\pi i\alpha/m)$,

$1 \leq \alpha \leq m-1$; m positive vortices located at $\rho_\beta = z_v \exp(2\pi i\beta/m)$, $1 \leq \beta \leq m$; m negative vortices located at $\eta_\alpha = z_v^* \exp(2\pi i\alpha/m)$, $1 \leq \alpha \leq m$; and m negative vortices located at $\eta_\beta = 1/z_v^* \exp(2\pi i\beta/m)$, $1 \leq \beta \leq m$. The Hamiltonian for the vortex motion is then obtained by first computing

$$|\frac{df}{dz}|_{z=z_v} = \frac{4m|z_v|^{m-1}|z_v^m+1|}{|(1-z_v^m)^3|}.$$

This gives

$$\mathcal{H} = -\frac{\Gamma^2}{4\pi}\log\left[\frac{16m^2|z_v|^{2m-2}|z_v^m+1|^2}{|(1-z_v^m)^3|^2} \times \frac{(1+|z_v|^2)^2}{|((1+z_v^m)/(1-z_v^m))^2-((1+z_v^{*m})/(1-z_v^{*m}))^2|^2}\right].$$

This yields vortex paths

$$\frac{[(1+r^{2m})^2-4r^{2m}\cos^2 m\phi](1+r^2)^2}{r^2(r^{2m}-1)^2\sin^2 m\phi} = \text{const.}$$

which are shown on figure 5.

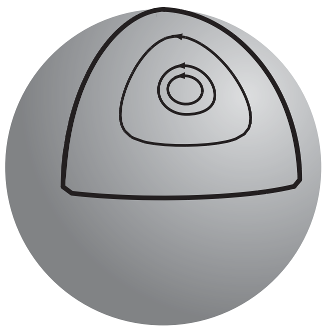

Figure 5: Vortex paths in a geodesic triangle.

Example 4. Spherical rectangles: As a final example, we consider the rectangular region Ω bounded by the longitudes 0 and ϕ_1 and the latitudes θ_1 and θ_2. Then Ω' is the annular sector bounded by the circular arcs of radii $r_1 = \tan(\theta_1/2)$ and $r_2 = \tan(\theta_2/2)$ and the radial lines $\phi = 0$ and $\phi = \phi_1$. Ω' maps to the rectangle

$$\Omega'' = \{(x,y)|\log r_1 \leq x \leq \log r_2; 0 \leq y \leq \phi_1\}$$

under the map $u = \log z$, where u is the plane of the rectangle Ω''. The rectangle is mapped to the lower half plane by the map

$$w = \mathcal{P}(u - \log r_1)$$

where $\mathcal{P}(u) = \mathcal{P}(u; g2; g3)$ is the Weierstrass $\mathcal{P}$ function and $g2$ and $g3$ are the Weierstrass invariants which are related to the half periods $\omega = \log(r_2/r_1)$ and $\omega' = i\phi_1$. The annular sector Ω' is mapped to the lower half plane by the function $f(z) = \mathcal{P}\log(z/r_1)$. As used in [17], the image system for a vortex in a rectangle consists of a doubly-infinite lattice whose corners are occupied by image vortices and the vortex Hamiltonian is given by

$$\mathcal{H} = -\frac{\Gamma^2}{2\pi}\log|\frac{(1+r_v^2)\mathcal{P}\log(z_v/r_1)}{\mathcal{P}\log(z_v/r_1) - \mathcal{P}\log(z_v^*/r_1)}|.$$

Vortex paths in this spherical rectangle are shown in figure 6.

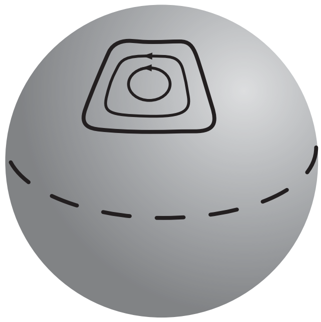

Figure 6: Vortex paths in a latitudinal rectangle.

5 Discussion

There are certainly many ways one could extend the results presented in this paper. First, the so-called 'method of inversion' [11] is distinctly different from the image method and can be used on different sorts of domains, most notably the unit disc as carried out in example 2.1. It should, in principle, be possible to construct a general formulation of the method of inversion as is done in this paper for the image method. Another potential extension of the ideas described here is to use Schwarz-Christoffel mappings for closed polygonal domains on the sphere, as is standard in the plane [7], in order to construct Green's functions for much more general simply-connected regions on the sphere made up of geodesic segments connected with fixed latitude segments. Arbitrary simply connected domains with smooth boundaries could then be approximated by these polygonal regions.

Bibliography

[1] R. Alexander, 'On the sum of distances between N points on a sphere', *Acta Math. Acad. Sci. Hungar.*, 23, 443-448, 1972.

[2] B. Bergersen, D. Boal, P. Palffy-Muhoray, 'Equilibrium configurations of particles on a sphere: the case of logarithmic interactions', *J. Phys. A: Math. Gen*, 27, 2579-2586, 1994.

[3] G. D. Birkhoff, **Collected Mathematical Papers**, AMS, New York, 1950.

[4] N. Bourbaki, **Groupes et algèbres de Lie**, Chapitres IV-VI. Hermann 1968.

[5] R. Courant and D. Hilbert, **Methods of Mathematical Physics**, Interscience Publishers Inc., New York, 1953.

[6] T. Erber, G.M. Hockney, 'Equilibrium configurations of N equal charges on a sphere', *J. Phys. A: Math. Gen.*, 24, L1369-L1377, 1991.

[7] M. Embree, L.N. Trefethen, 'Green's functions for multiply connected domains via conformal mapping', *SIAM Review*, vol. 41, no. 4, 745-761, 1999.

[8] E. Gutkin. Geometry and combinatorics of groups generated by reflections, *L'Enseignément Mathématique*, **32** 95–110, 1986.

[9] J. Humphreys, **Reflection Groups and Coxeter Groups**, Cambridge 1990.

[10] J.D. Jackson, **Classical Electrodynamics**, Wiley, 1963.

[11] J. Keller, 'The scope of the image method', *Comm Pure Appl. Math.* **6** (1953), 505-512.

[12] R. Kidambi, P.K. Newton, 'Motion of three point vortices on a sphere', *Physica D*, 116, 143-175, 1998.

[13] R. Kidambi, P.K. Newton, 'Point vortex motion on a sphere with solid boundaries', *Phys. Fluids*, vol. 12, no. 3, 581-588, March 2000.

[14] R. Kidambi, P.K. Newton, 'Streamline topologies for integrable vortex motion on a sphere', *Physica D* 140, 95-125, 2000.

[15] Y. Kimura, 'Vortex motion on surfaces with constant curvature', *Proc. Roy. Soc. Lond. A 455*, 245-259, 1999.

[16] H. Kober, **Dictionary of Conformal Representations**, Dover, 1952.

[17] I. Kunin, F. Hussain, X. Zhou, 'Dynamics of a pair of vortices in a rectangle', *Int. J. Eng. Sci.*, **32**(11), 1835–1844.

[18] J.E. Marsden, M.J. Hoffman, **Basic Complex Analysis**, W.H. Freeman and Co., 2nd Ed., 1987.

[19] P.K. Newton, **The N-Vortex Problem:** *Analytical Techniques*, Springer-Verlag, Applied Math. Sci. Vol. 145, May 2001.

[20] S. Rosenberg, **The Laplacian on a Riemannian Manifold**, Cambridge 1997.

[21] E.B. Saff, A.B.J. Kuijlaars, 'Distributing many points on a sphere', *Math. Intelligencer 19*(1), 5-11, 1997.

[22] M. Shubin, editor, **Partial Differential Equations I.**, Encyclopedia of Mathematical Sciences Vol. 30, Springer-Verlag, 1992.

[23] A. Sommerfeld, **Partial Differential Equations in Physics**, Academic Press, 1949.

[24] M. Spivak, **Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus**, Perseus Publishing, 1965.