Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

Guan Pengfei, Xinan Ma, and Lin Changshou

Preprint no.: 38 2004

THE EXISTENCE OF CONVEX BODY WITH PRESCRIBED CURVATURE MEASURES

PENGFEI GUAN, CHANGSHOU LIN, AND XI-NAN MA

1. Introduction

Curvature measure is one of the basic notion in the theory of convex bodies. Together with surface area measures, they play fundamental roles in the study of convex bodies. They are closely related to the differential geometry and integral geometry of convex hypersurfaces. Let Ω is a bounded convex body in $\mathbf{R}^{n+1}$ with C^2 boundary M, the corresponding curvature measures and surface area measures of Ω can be defined according to some geometric quantities of M. Let $\kappa = (\kappa_1, \cdots, \kappa_n)$ be the principal curvatures of M at point x, let $W_k(x) = S_k(\kappa(x))$ be the k-th Weingarten curvature of M at x (where S_k the k-th elementary symmetric function). In particular, W_1 is the mean curvature, W_2 is the scalar curvature, and W_n is the Gauss-Kronecker curvature. The k-th curvature measure of Ω is defined as

$$\mathcal{C}_k(\Omega, \beta) := \int_{\beta \cap M} W_{n-k} dF_n,$$

for every Borel measurable set β in $\mathbf{R}^{n+1}$, where dF_n is the volume element of the induced metric of $\mathbf{R}^{n+1}$ on M. Since M is convex, M is star-shaped about some point. We may assume that the origin is inside of Ω. Since M and $\mathbf{S}^n$ is diffeomrphic through radial correspondence R_M. Then the k-th curvature measure can also be defined as a measure on each Borel set β in $\mathbf{S}^n$ (e.g., see [21]):

$$\mathcal{C}_k(M, \beta) = \int_{R_M(\beta)} W_{n-k} dF_n.$$

We note that $\mathcal{C}_k(M, \mathbf{S}^n)$ is the k-th quermasintegral of Ω. Similarly, if M is strictly convex, let $r_1, ..., r_n$ be the principal radii of curvature of M, $P_k = S_k(r_1, \cdots, r_n)$. The k-th surface area measure of Ω then can be defined as

$$\mathcal{S}_k(\Omega, \beta) := \int_{\beta} P_k d\sigma_n,$$

Research of the first author was supported in part by NSERC Discovery Grant. Research of the third author was supported by FokYingTung Eduction Foundation, the grants from MOC of China and NSFC No.10371041.

for every Borel set β in $\mathbf{S}^n$, where $d\sigma_n$ is standard volume element on $\mathbf{S}^n$

The Minkowski problem is the problem of prescribing n-th surface area measure on $\mathbf{S}^n$, the Christoffel problem concerns the prescribing the 1-th surface area measure (e.g., see [1, 15, 18, 6, 20, 8, 3]). The general problem of prescribing surface area measures is called the Christoffel- Minkowski problem, we refer [13] for an updated account. As for the curvature measures, the problem of prescribing C_0 is called the Alexandrov problem, which can be considered as a counterpart to Minkowski problem. The existence and uniqueness were obtained by Alexandrov [2]. The regularity of the Alexandrov problem in elliptic case was proved by Pogorelov [19] for $n = 2$ and by Oliker [17] for higher dimension case. The general regularity results (degenerate case) of the problem were obtained in [10]. Apparently, the existence problem for curvature measures of $\mathcal{C}_{n-k}$ for general case $k < n$ has not been touched (see also note 8 on P. 396 in [21]). The main purpose of this paper is to study the problem in a differential geometrical setting. The problem can be explicitly stated as follow.

Curvature measure problem: Given a C^2 positive function f on $\mathbf{S}^n$. For each $0 \leq k < n$, find a convex hypersurface M as a graph over $\mathbf{S}^n$, such that $\mathcal{C}_{n-k}(M,\beta) = \int_\beta f d\sigma$ for each Borel set β in $\mathbf{S}^n$, where $d\sigma$ is the standard volume element on $\mathbf{S}^n$.

The problem is equivalent to solve certain curvature equation on $\mathbf{S}^n$. If M is of class C^2, then

$$\mathcal{C}_{n-k}(M,\beta) = \int_{R_M(\beta)} S_k dF = \int_\beta S_k g d\sigma_n. \tag{1.1}$$

where g is the density of dF respect to standard volume element $d\sigma_n$ on $\mathbf{S}^n$. The problem of prescribing $(n-k)$-th curvature measure can be reduced to the following curvature equation

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n) = \frac{f(x)}{g(x)}, \quad 1 \leq k \leq n \quad \text{on} \quad \mathbf{S}^n \tag{1.2}$$

Here we first encounter a similar difficulty as for the Christoffel-Minkowski problem in [13]: the issue of convexity of solution when $k < n$. The another difficulty issue around equation (1.2) is the lack of some appropriate a priori estimates for admissible solutions. At a first glance, equation (1.2) is similar to the equation of prescribing Weingarten curvature equation in [5, 12]. But, the appearance of $g(x)$ (which implicitly involves the gradient of solution) make the matter very delicate. It seems that the estimates in [5] and [12] can not be obtained through similar way. Equation (1.2) was studied in an unpublished notes [11] of Yanyan Li and the first author. The uniqueness and C^1 estimates were established for admissible solutions in [11]. But C^2 estimates for equation (1.2) were missing (except for $k = 1$ and $k = n$, the first case follows from the theory of quasilinear equations, and later case was dealt in [17, 10]).

Since equation (1.2) is originated in geometric problem in the theory of convex bodies, the purpose of this paper is to find convex hypersurface M (as a graph over $\mathbf{S}^n$) satisfying equation (1.2). The followings are our main results.

Theorem 1.1. *Suppose $f(x) \in C^2(\mathbf{S}^n)$, $\quad f > 0$, $n \geq 2$, $1 \leq k \leq n-1$. If f satisfies the condition*

$$|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \text{is a strictly convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\}, \tag{1.3}$$

then there exists a unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ such that it satisfies (1.2).

When $k = 1$ or 2, the strict convex condition (1.3) can be weakened.

Theorem 1.2. *Suppose $k = 1$, or $\;$ 2 and $k < n$, and suppose $f(x) \in C^2(\mathbf{S}^n)$ is a positive function. If f satisfies*

$$|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})^{-\frac{1}{k}} \quad \text{is a convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\}, \tag{1.4}$$

then there exists unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ such that it satisfies equation (1.2).

Since the Alexandrov problem (Gauss curvature measure problem) has already been solved [2, 19, 17, 10], Theorem 1.2 yields solutions to two other important measures, the mean curvature measure and scalar curvature measure under convex condition (1.4). For the existence of convex solutions, some condition on f is necessary, see Remark 2.5.

The plan of the paper is as follows. In section 2, we derive uniqueness and C^1 bound of the solutions of (1.2). Theorem 1.1 will be proved in section 3. The novel feature if the C^2 estimates. Instead of obtaining an upper bound of the principal curvatures, we look for a lower bound of the principal curvatures by transforming (1.2) to a new equation of support function on $\mathbf{S}^n$ through Gauss map. Section 4 is devoted to the proof of Theorem 1.2. The key part is the C^2 estimates for the case $k = 2$, which we use a special structure of S_2. Then we establish a deformation lemma 4.3 as in [13, 12] to ensure the convexity of solutions in the process of applying the method of continuity.

Acknowledgment: Part of the work was done while the third author was visiting NCTS in National Tsinghua University in Taiwan, he would like to thank their warm hospitality.

2. Uniqueness and C^1 boundness

We first recall some relevant geometric quantities of a smooth closed hypersurface $M \subset \mathbb{R}^{n+1}$, which we suppose the origin is not contained in M.

$A, B, ...$ will be from 1 to $n+1$ and Latin from 1 to n, the repeated indices denote summation over the indices. Covariant differentiation will simply be indicated by indices.

Let M^n be a n-dimension closed hypersurface immersed in $\mathbb{R}^{n+1}$. We choose an orthonormal frame in $\mathbf{R}^{n+1}$ such that $\{e_1, e_2, ..., e_n\}$ are tangent to M and e_{n+1} is the outer normal. Let the corresponding coframe be denoted by $\{\omega_A\}$ and the connection forms by $\{\omega_{A,B}\}$. The pull back of their through the immersion are still denoted by $\{\omega_A\}$,$\{\omega_{A,B}\}$ in the abuse of notation. Therefore on M

$$\omega_{n+1} = 0.$$

The second fundamental form is defined by the symmetry matrix $\{h_{ij}\}$ with

$$\omega_{i,n+1} = h_{ij}\omega_j. \tag{2.1}$$

We recall the following fundamental formulas of a hypersurface in $\mathbb{R}^{n+1}$ (e.g., see [22]).

$$\begin{aligned}
&X_{ij} = -h_{ij}e_{n+1}, \quad \text{(Gauss formula)}\\
&(e_{n+1})_i = h_{ij}e_j, \quad \text{(Weigarten equation)}\\
&h_{ijk} = h_{ikj}, \qquad \text{(Codazzi formula)}\\
&R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} \quad \text{(Gauss equation)},
\end{aligned} \tag{2.2}$$

where R_{ijkl} is the curvature tensor. And we have the following formulas

$$\begin{aligned}
&h_{ijkl} = h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk},\\
&h_{ijkl} = h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi},\\
&(e_{n+1})_{ii} = \sum_{j=1}^{n} h_{iij}e_j - \sum_{j=1}^{n} h_{ij}^2 e_{n+1}.
\end{aligned} \tag{2.3}$$

Since M is starshaped with respect to origin, the position vector X of M can be written as $X(x) = \rho(x)x, \quad x \in \mathbf{S}^n$, where ρ is a smooth function on $\mathbf{S}^n$. Let $\{e_1, ..., e_n\}$ be smooth local orthonormal frame field on $\mathbf{S}^n$, let ∇ be the gradient on $\mathbf{S}^n$ and covariant differentiation will simply be indicated by indices. Then in term of ρ the metric of M is given by

$$g_{ij} = \rho^2\delta_{ij} + \rho_i\rho_j.$$

So the area factor

$$g = (\det g_{ij})^{\frac{1}{2}} = \rho^{n-1}(\rho^2 + |\nabla\rho|^2)^{\frac{1}{2}}.$$

The second fundamental form of M is

$$h_{ij} = (\rho^2 + |\nabla\rho|^2)^{-\frac{1}{2}}(\rho^2\delta_{ij} + 2\rho_i\rho_j - \rho\rho_{ij}).$$

and the unit outer normal of the hypersurface M in $\mathbb{R}^{n+1}$ is

$$\mathbf{N} = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}}. \tag{2.4}$$

The principal curvature $(\kappa_1, \kappa_2, ..., \kappa_n)$ of M are the eigenvalue of the second fundamental form respect to the metric and therefore are the solutions of

$$\det(h_{ij} - kg_{ij}) = 0.$$

Equation (1.2) can be expressed as a differential equations on the radial function ρ and position vector X respectively.

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n) = f\rho^{1-n}(\rho^2 + |\nabla\rho|^2)^{-1/2}, \quad \text{on} \quad \mathbf{S}^n, \tag{2.5}$$

where $f > 0$ is the given function. From (2.4) we have

$$< X, N >= \rho^2(\rho^2 + |\nabla\rho|^2)^{-1/2}.$$

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n)(X) = |X|^{-(n+1)} f(\frac{X}{|X|}) < X, N >, \quad \forall X \in M. \tag{2.6}$$

Definition 2.1. For $1 \le k \le n$, let Γ_k be a cone in $\mathbf{R}^n$ determined by

$$\Gamma_k = \{\lambda \in \mathbf{R}^n : \quad S_1(\lambda) > 0, ..., \quad S_k(\lambda) > 0\}.$$

A C^2 surface M is called k-admissible if at every point $X \in M$, $(\kappa_1, \kappa_2, ..., \kappa_n) \in \Gamma_k$.

The following three lemmas had been proved in [11], for the completeness we provide the proofs here.

Lemma 2.2. *If M satisfies (2.6), then*

$$(\frac{\min_{\mathbf{S}^n} f}{C_n^k})^{1/(n-k)} \le \min_{\mathbf{S}^n} |X| \le \max_{\mathbf{S}^n} |X| \le (\frac{\max_{\mathbf{S}^n} f}{C_n^k})^{1/(n-k)}.$$

In particular, if M is convex and ρ is the radial function of M, then there is a constant C depending only on $\max f$ *and* $\min f$ *such that*

$$max_{\mathbf{S}^n} |\nabla\rho| \le C. \tag{2.7}$$

Proof: Let $B_R(o)$ be a ball of smallest radius so that $M \subset B_R(o)$, then at the maximum point X_1 of $|X|$, $R = |X_1|$. Through some geometrical considerations, we have

$$f(\frac{X_1}{|X_1|}) \ge C_n^k |X_1|^{n-k}.$$

This is

$$\max_{\mathbf{S}^n} |X| \le (\frac{\max_{\mathbf{S}^n} f}{C_n^k})^{1/(n-k)}.$$

The first half inequality can be shown in a similar way.

The gradient estimates follows from C^0 estimates and convexity. In fact, the gradient estimates for general admissible solutions are also true, which was proved in [11]. □

Set $F = S_k^{1/k}$, equation (2.5) is written as

$$F(\lambda) \equiv F(\lambda_1, ..., \lambda_n) = f^{1/k}\rho^{(1-n)/k}(\rho^2 + |\nabla\rho|^2)^{-1/(2k)} \equiv K(x, \rho, \nabla\rho).$$

The following is the uniqueness result of the problem.

Lemma 2.3. *Suppose* $1 \le k < n$, $\lambda(\rho_i) \in \Gamma_k$, $i = 1, 2$. *Suppose* ρ_1, ρ_2 *are solutions of (2.5). Then* $\rho_1 \equiv \rho_2$.

Proof Suppose the contrary, $\rho_2 > \rho_1$ somewhere on $\mathbb{S}^n$. Take $t \ge 1$ such that

$$t\rho_1 \ge \rho_2 \quad \text{on } \mathbb{S}^n, \qquad t\rho_1 = \rho_2 \quad \text{at some point } P \in \mathbb{S}^n.$$

Obviously, $\lambda(t\rho_1) = t^{-1}\lambda(\rho_1)$, and therefore $F(\lambda(t\rho_1)) = t^{-1}F(\lambda(\rho_1))$. It is clear that

$$\begin{aligned} &K(x, t\rho_1, \nabla(t\rho_1)) = t^{-n/k}K(x, \rho_1, \nabla\rho_1) \\ &= t^{-n/k}F(\lambda(\rho_1)) \le t^{-1}F(\lambda(\rho_1)) = F(\lambda(t\rho_1)). \end{aligned}$$

It follows that

$$F(\lambda(t\rho_1)) - K(x, t\rho_1, \nabla(t\rho_1)) \ge 0, \qquad F(\lambda(\rho_2)) - K(x, \rho_2, \nabla\rho_2) = 0.$$

Hence

$$L(t\rho_1 - \rho_2) \ge 0,$$

where L is the linearized operator. Applying the strong maximum principle, we have $t\rho_1 - \rho_2 \equiv 0$ on $\mathbb{S}^n$. Since $n > k$, from equation (2.5), we conclude that $t = 1$. □

The following lemma will also be used in this paper.

Lemma 2.4. *Let* L *denote the linearized operator of* $F(\lambda) - K(x, \rho, \nabla\rho)$ *at a solution* ρ *of (2.5),* w *satisfies* $Lw = 0$ *on* $\mathbb{S}^n$. *Then* $w \equiv 0$ *on* $\mathbb{S}^n$.

Proof Writing $F(x, \rho, \nabla\rho, \nabla^2\rho) \equiv F(\lambda)$, we have

$$F(x, t\rho, \nabla(t\rho), \nabla^2(t\rho)) = F(\lambda(t\rho)) = F(\lambda(\rho)/t).$$

Applying $\frac{d}{dt}\big|_{t=1}$, we have

$$F_{\nabla^2\rho}\nabla^2(\rho) + F_{\nabla\rho}\nabla\rho + F_\rho\rho = -\sum_i \lambda_i F_{\lambda_i} = -F.$$

It is easy to see that

$$K(x, t\rho, \nabla(t\rho)) = t^{-n/k} K(x, \rho, \nabla\rho).$$

Applying $\frac{d}{dt}\Big|_{t=1}$, we have

$$K_{\nabla\rho}\nabla\rho + F_\rho \rho = -n/kK(x, \rho, \nabla\rho).$$

It follows from and that

$$L\rho = -F(\lambda) + n/kK(x, \rho, \nabla) = (n/k - 1)K(x, \rho, \nabla\rho) > 0.$$

Set $w = z\rho$. We know that

$$0 = Lw = L(z\rho) \equiv L'z + zL\rho,$$

where $L'z = \rho F_{\nabla^2\rho}\nabla^2 z$+first order term in z. Notice that $L\rho > 0$, we derive from the maximum principle that $z \equiv 0$, namely, $w \equiv 0$. □

We conclude this section with the following remark.

Remark 2.5. Large part of the study of curvature measures have been carried on for convex bodies. There are some generalizations of these curvature measures to other class of sets in R^{n+1} (e.g., [7]). From differential geometric point of view, the notion of $(n-k)-th$ curvature measure can be easily extended to k-convex bodies. Since for $k < n$, admissible solution of (1.2) is not convex in general. By Lemma 2.3, for $k < n$, the prescribing curvature measure equation (1.2) has no convex solution for most of f. This means some condition must be imposed on f for the existence of convex solutions. We believe that for any smooth positive function f, equation (1.2) always has an admissible solution. This is already prove for $k = n$ in [2, 19, 17, 10]. It is also true $k = 1$ by the standard quasilinear elliptic theory and C^1 estimates in [11]. For $1 < k < n$, one needs to establish C^2 a priori estimates for admissible solution of equation (1.2), which is still an open problem.

3. Proof of Theorem 1.1

In this section we prove C^2 estimates for equation (1.2) under the convexity of solution. For the mean curvature measure case ($k = 1$), a gradient bound is enough for a C^2 a priori bound by the standard theory of quasilinear elliptic equations. For the rest of this section, we assume $k > 1$.

For the C^2 estimates for admissible solutions of (1.2), it is equivalent to estimate the upper bounds of principal curvatures. If the hypersurface is strictly convex, it is simple to observe that a positive lower bound on the principal curvatures implies an upper bound of the principal curvatures. This follows from equation (1.2) and the Newton-Maclaurin inequality,

$$S_n^{\frac{1}{n}}(\lambda) \le [\frac{S_k}{C_n^k}]^{\frac{1}{k}}(\lambda).$$

This is the starting point of our approach here. To achieve such a lower bound, we shall use the inverse Gauss map and consider the equation for the support function of the hypersurface. The role of the Gauss map here should be compared with the role of the Legendre transformation on the graph of convex surface in a domain in $\mathbf{R}^n$. Since M is curved and compact, the Gauss map fits into the picture neatly. This way, we can make use some special features of the support function. We note that a lower bound on the principal curvature is an upper bound on the principal radii. And the principal radii are exactly the eigenvalues of the spherical hessians of the support function. Therefore, we are led to get a C^2 bound on the support function of M.

Let $X : M \to \mathbf{R}^{n+1}$ be a closed strictly convex smooth hypersurface in $\mathbf{R}^{n+1}$. We may assume the X is parametrized by the inverse Gauss map

$$X : \quad \mathbf{S}^n \quad \to \quad \mathbf{R}^{n+1}.$$

The support function of X is defined by

$$u(x) =< x, X(x) >, \quad \text{at} \quad x \in \mathbf{S}^n.$$

Let $e_1, e_2, ..., e_n$ be a smooth local orthonormal frame field on $\mathbf{S}^n$, we know that the inverse second fundamental form of X is

$$h_{ij} = u_{ij} + u\delta_{ij},$$

and the metric of X is

$$g_{ij} = \sum_{l=1}^{n} h_{il}h_{jl}.$$

The principal radii of curvature are the eigenvalues of matrix

$$W_{ij} = u_{ij} + u\delta_{ij}.$$

Equation (2.5) can be rewritten as an equation on support function u.

$$F(W_{ij}) = [\frac{\det W_{ij}}{S_{n-k}(W_{ij})}]^{\frac{1}{k}}(x) = G(X)u^{-\frac{1}{k}} \quad \text{on} \quad \mathbf{S}^n, \tag{3.1}$$

where X is position vector of hypersurface, and

$$G(X) = |X|^{\frac{n+1}{k}} f^{-\frac{1}{k}}(\frac{X}{|X|}).$$

Equation (3.1) is similar to the equation in [9], where a problem of prescribing Weingarten curvature was considered. The position function and the support function have the following explicit form.

$$X(x) = \sum_{i=1}^{n} u_i e_i + ux, \quad \text{on} \quad x \in \mathbf{S}^n.$$

It follows from some straightforward computations,

$$X_l = u_{il}e_i + u_i(e_i)_l + u_l x + u x_l = u_{il}e_i - x u_i \delta_{il} + u_l x + u e_l = W_{il} e_i, \tag{3.2}$$

$$\sum_{l=1}^{n} X_{ll} = \sum_{i,l=1}^{n} [W_{ill} e_i + W_{il}(e_i)_l]$$

$$= \sum_{i=1}^{n} [\sum_{l=1}^{n} W_{ll}]_i e_i + \sum_{i,l=1}^{n} W_{il}(-x\delta_{il}) = \sum_{i=1}^{n} [\sum_{l=1}^{n} W_{ll}]_i e_i - x \sum_{l=1}^{n} W_{ll}. \tag{3.3}$$

The following is a key lemma.

Lemma 3.1. *If $G(X)$ is strictly convex function in $\mathbf{R}^{n+1} \setminus \{o\}$, then*

$$\max(\Delta u + nu) \le C, \tag{3.4}$$

where the constant C depends only on $n, \max_{\mathbf{S}^n} f, \min_{\mathbf{S}^n} f$ and $|\nabla f|_{C^0}$ and $|\nabla^2 f|_{C^0}$. In turn,

$$|\nabla^2 \rho| \le C. \tag{3.5}$$

Proof: Since we already obtained C^1 bound in Lemma 2.2, to get (3.5), we only need to prove (3.4). Let

$$H = \sum_{l=1}^{n} = \Delta u + nu$$

and assume the maximum of H attains at some point $x_o \in \mathbf{S}^n$. We choose an orthonormal frame $e_1, e_2, ..., e_n$ near x_o such that $u_{ij}(x_o)$ is diagonal (so is $W_{ij} = u_{ij} + u\delta_{ij}$ at x_o). The following formula for commuting covariant derivatives are elementary:

$$(\Delta u)_{ii} = \Delta(u_{ii}) + 2\Delta u - 2nu_{ii}.$$

So we have

$$H_{ii} = (\Delta u)_{ii} + nu_{ii} = \Delta(W_{ii}) - nW_{ii} + H. \tag{3.6}$$

Let $F^{ij} = \frac{\partial F(W)}{\partial W_{ij}}$. At x_o the matrix F^{ij} is positive definite, diagonal. Setting the eigenvalues of W_{ij} at x_o as $\lambda(W_{ij}) = (\lambda_1, \lambda_2, ..., \lambda_n))$,

$$F^{ii} = \frac{1}{k}(\frac{S_n}{S_{n-k}})^{\frac{1}{k}}[\frac{S_{n-1}(\lambda|i)}{S_{n-k}} - \frac{S_n S_{n-k-1}(\lambda|i)}{{S_{n-k}}^2}].$$

The following facts are true (e.g., see [9]).

$$\sum_{i=1}^n F^{ii}W_{ii} = F, \quad \sum_{i=1}^n F^{ii} \geq (C_n^{n-k})^{-\frac{1}{k}}.$$

Now at x_o, we have

$$H_i = 0, \qquad H_{ij} \leq 0 \tag{3.7}$$

Through this section the repeated upper indices denote summation over the indices, and our calculation will do at x_o. Using the above calculations we have

$$\begin{aligned} 0 \quad &\geq \sum_{i,j=1}^n F^{ij}H_{ij} = \sum_{i=1}^n F^{ii}H_{ii} = \sum_{i=1}^n F^{ii}\Delta(W_{ii}) - n\sum_{i=1}^n F^{ii}W_{ii} + H\sum_{i=1}^n F^{ii} \\ &\geq \sum_{i=1}^n F^{ii}\Delta(W_{ii}) - nF + (C_n^{n-k})^{-\frac{1}{k}}H. \end{aligned} \tag{3.8}$$

From the equation (3.1)

$$F^{ij}W_{ijl} = [G(X)u^{-\frac{1}{k}}]_l, \quad F^{ij}W_{ijll} + F^{ij,st}W_{ijl}W_{stl} = [G(X)u^{-\frac{1}{k}}]_{ll}.$$

From the concavity of F, we get

$$\sum_{i=1}^n F^{ii}\Delta(W_{ii}) \geq \sum_{l=1}^n [G(X)u^{-\frac{1}{k}}]_{ll},$$

combining this with (3.8) we have the following inequality at x_o

$$\sum_{l=1}^n [G(X)u^{-\frac{1}{k}}]_{ll} - nF + (C_n^{n-k})^{-\frac{1}{k}}H \leq 0. \tag{3.9}$$

Now we treat the term $[G(X)u^{-\frac{1}{k}}]_{ll}$,in the following the repeated indices on α, β denote summation over the indices from $1, 2, ...n+1$. Denote $G_\alpha = \frac{\partial G}{\partial X^\alpha}$, $G_{\alpha\beta} = \frac{\partial^2 G}{\partial X^\alpha \partial X^\beta}$.

$$\begin{aligned} &[G(X)u^{-\frac{1}{k}}]_l = G_\alpha X_l^\alpha u^{-\frac{1}{k}} + G(X)(-\frac{1}{k})u^{-\frac{1}{k}-1}u_l, \\ &\sum_{l=1}^n [G(X)u^{-\frac{1}{k}}]_{ll} = G_{\alpha\beta}X_l^\alpha X_l^\beta u^{-\frac{1}{k}} + G_\alpha X_{ll}^\alpha u^{-\frac{1}{k}} \\ &-\frac{2}{k}G_\alpha X_l^\alpha u^{-\frac{1}{k}-1}u_l + \frac{1}{k}(\frac{1}{k}+1)G(X)u^{-\frac{1}{k}-2}|Du|^2 - \frac{1}{k}G(X)u^{-\frac{1}{k}-1}u_{ll}. \end{aligned}$$

Using (3.2) and (3.3), it follows that at x_o

$$\begin{aligned} &\sum_{l=1}^n [G(X)u^{-\frac{1}{k}}]_{ll} = G_{\alpha\beta}e_l^\alpha e_l^\beta W_{ll}^2 u^{-\frac{1}{k}} - [G_\alpha x^\alpha u^{-\frac{1}{k}} + \frac{1}{k}G(X)u^{-\frac{1}{k}-1}]H \\ &-\frac{2}{k}(G_\alpha e_l^\alpha u_l W_{ll})u^{-\frac{1}{k}-1} + \frac{1}{k}(\frac{1}{k}+1)G(X)u^{-\frac{1}{k}-2}|Du|^2 + +\frac{n}{k}G(X)u^{-\frac{1}{k}}. \end{aligned} \tag{3.10}$$

Using (3.10), at x_o (3.9) becomes

$$
\begin{aligned}
&G_{\alpha\beta}e_l^\alpha e_l^\beta W_{ll}^2 u^{-\frac{1}{k}} - [G_\alpha x^\alpha u^{-\frac{1}{k}} + \frac{1}{k}G(X)u^{-\frac{1}{k}-1}]H - nF + (C_n^{n-k})^{-\frac{1}{k}}H \\
&-\frac{2}{k}(G_\alpha e_l^\alpha u_l W_{ll})u^{-\frac{1}{k}-1} + \frac{1}{k}(\frac{1}{k}+1)G(X)u^{-\frac{1}{k}-2}|Du|^2 + +\frac{n}{k}G(X)u^{-\frac{1}{k}} \leq 0.
\end{aligned} \tag{3.11}
$$

If $G(X)$ is strictly convex in $\mathbf{R}^{n+1} \setminus \{o\}$, then exist a uniform constant $c_o > 0$ such that

$$\sum_{\alpha\beta=1}^{n} G_{\alpha,\beta}e_l^\alpha e_l^\beta \geq c_o, \quad l = 1, 2, ...n.$$

Since $\sum_{l=1}^n W_{ll}^2 \geq \frac{H^2}{n}$, we obtain $H(x_o) \leq C$. □

Proof of existence theorem I: For any positive function $f \in C^2(\mathbf{S}^n)$, for $0 \leq t \leq 1$ and $1 \leq k \leq n-1$, set $f_t(x) = [1 - t + tf^{-\frac{1}{k}}(x)]^{-k}$. We consider the equation

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n)(x) = f_t(x)\rho^{1-n}(\rho^2 + |\nabla\rho|^2)^{-1/2}, \quad \text{on} \quad \mathbf{S}^n, \tag{3.12}$$

where $n \geq 2$. We find the hypersurface in the class of strictly convex surface. Let $I = \{t \in [0,1]$: such that (3.12) is solvable$\}$. Since $\rho = [C_n^k]^{-\frac{1}{n-2}}$ is a solution for $t = 0$, I is not empty. By Lemma 2.2 and Lemma 3.5, $\rho \in C^{1,1}(\mathbf{S}^n)$ and is bound below. That is equation (3.12) is elliptic. By the Evans-Krylov theorem $\rho \in C^{2,\alpha}(\mathbf{S}^n)$ and

$$||\rho||_{C^{2,\alpha}(\mathbf{S}^n)} \leq C, \tag{3.13}$$

Where C depends only on $n, \max_{\mathbf{S}^n} f, \min_{\mathbf{S}^n} f$ and $|\nabla f|_{C^0}$ and $|\nabla^2 f|_{C^0}$, and α. The a priori estimates guarantee I is closed. The openness is from Lemma 2.4 and the implicit function theorem. So we have the existence. The uniqueness of the solution for $t \in [0,1]$ is from Lemma 2.3. This complete the proof of Theorem 1.1. □

Remark 3.2. We suspect the strict convexity condition (1.3) can be weakened. For the cases $k = 1, 2$, this is verified in Theorem 1.2. The proof of Theorem 1.2 is different from the proof of Theorem 1.1 in this section. Due to the weakened condition, we are not able to obtained a positive lower bound for the principal curvatures directly. Instead, we will use special structure of the elementary symmetric function S_2 to get an upper bound of principal curvatures for convex solutions of (1.2). Since the convexity of solutions is not guaranteed for equation (1.2) when $k < n$, we will use condition (1.4) and a deformation lemma to prove the existence of strictly convex solution of (1.2) as in [13] in the next section.

4. Proof of Theorem 1.2

In this section, we will first prove the C^2 estimate for the scalar curvature case under the convexity assumption of the solution. We shall make use of some explicit structure of S_2.

We consider the following prescribed scalar curvature measure equation

$$S_2(\lambda\{h_{ij}\})(X) = |X|^{-(n+1)} f(\frac{X}{|X|}) < X, N >, \quad \forall X \in M. \tag{4.1}$$

Now we state the mean curvature estimate for the above equation on the convexity of solution surface.

Lemma 4.1. *Let f be a C^2 positive function on $\mathbf{S}^n$ and M be a starshaped hypersurface in $\mathbb{R}^{n+1}$ respect to the origin, if M is a convex solution surface of equation (4.1) and for the function $\rho = |X|$ on $\mathbf{S}^n$ the following estimates hold*

$$\|\rho\|_{C^2} \leq C, \tag{4.2}$$

where the constant C depends only on $n, k, \min_{\mathbf{S}^n} f$ and $\|f\|_{C^2}$.

Proof: C^1 estimates were already obtained in Lemma 2.2 in the section 2. We only need to get an upper bound of the mean curvature H.

Let

$$F(X) = f(\frac{X}{|X|}), \quad \phi(X) = |X|^{-(n+1)} F(X), \tag{4.3}$$

then the equation (4.1) becomes

$$S_2(\kappa_1, \kappa_2, ..., \kappa_n)(X) = \phi(X) < X, e_{n+1} >, , \quad \text{on} \quad M, \tag{4.4}$$

Assume the function $P = H + \frac{a}{2}|X|^2$ attains its maximum at $X_o \in M$, where a is a constant will be determined later. Then at X_o we have

$$P_i = H_i + a < X, e_i >= 0, \tag{4.5}$$

$$P_{ii} = H_{ii} + a[1 - h_{ii} < X, e_{n+1} >]. \tag{4.6}$$

Let $F^{ij} = \frac{\partial S_2(\lambda\{h_{ij}\})}{\partial h_{ij}}$, and choose a suitable orthonormal frame $\{e_1, e_2, ..., e_n\}$ in a neighborhood of $X_o \in M$ such that at X_o the matrix $\{h_{ij}\}$ is diagonal. Then at X_o, the matrix $\{F^{ij}\}$ is also diagonal and positive definitive. At X_o

$$\sum_{ij=1}^{n} F^{ij} P_{ij} = \sum_{i=1}^{n} F^{ii} H_{ii} + a \sum_{i=1}^{n} F^{ii} - a < X, e_{n+1} > \sum_{i=1}^{n} F^{ii} h_{ii} \leq 0, \tag{4.7}$$

from this inequality we shall obtain the mean curvature estimate.

In what follows, all the calculations will be done at $x_o \in M$.

First we deal with the term $\sum_{i=1}^n F^{ii}H_{ii}$. From (4.5) and (2.3), we have

$$\begin{aligned}\sum_{i=1}^n F^{ii}H_{ii} &= \sum_{i=1}^n F^{ii}(\sum_{j=1}^n h_{jjii}) = \sum_{i=1}^n F^{ii}\sum_{j=1}^n(h_{iijj} + h_{ii}h_{jj}^2 - h_{jj}h_{ii}^2)\\ &= \sum_{ij=1}^n F^{ii}h_{iijj} + |A|^2\sum_{i=1}^n F^{ii}h_{ii} - H\sum_{i=1}^n F^{ii}h_{ii}^2,\end{aligned}$$

where $|A|^2 = \sum_{i=1}^n h_{ii}^2$.

Then we treat the term $\sum_{ij=1}^n F^{ii}h_{iijj}$. Differentiate equation (4.4) twice, by (2.2),

$$\begin{aligned}\sum_{ij=1}^n F^{ii}h_{iijj} &= \sum_{j=1}^n[\phi(X) < X, e_{n+1} >]_{jj} + \sum_{j,k\neq l} h_{jkl}^2 - \sum_{j,k\neq l} h_{jkk}h_{jll}\\ &= \Delta\phi < X, e_{n+1} > +2\sum_{j=1}^n \phi_j h_{jj} < X, e_j > +\phi\sum_{j=1}^n < X, e_{n+1} >_{jj}\\ &+ \sum_{j,k\neq l} h_{jkl}^2 - \sum_{j,k,l} h_{jkk}h_{jll} + \sum_{j,k} h_{jkk}^2.\end{aligned}$$

Now use (2.2) and (2.3), we have

$$\begin{aligned}&\sum_{i=1}^n < X, e_{n+1} >_{ii} = \sum_{i,l=1}^n [h_{il} < X, e_l >]_i = \sum_{i=1}^n[\sum_{l=1}^n h_{iil} < X, e_l > +h_{ii} - h_{ii}^2 < X, e_{n+1} >]\\ &= \sum_{l=1}^n H_l < X, e_l > +H - |A|^2 < X, e_{n+1} > = -a\sum_{i=1}^n < x, e_i >^2 +H - |A|^2 < X, e_{n+1} > .\end{aligned}$$

In turn, by equation (4.4) and 4.5) we have the following estimate

$$\begin{aligned}&\sum_{ij=1}^n F^{ii}h_{iijj} \geq -|A|^2 S_2(h_{ij}) + \phi H + \Delta\phi < X, e_{n+1} >\\ &+2\sum_{j=1}^n \phi_j h_{jj} < X, e_j > -a\phi\sum_{i=1}^n < x, e_i >^2 -a^2\sum_{i=1}^n < x, e_i >^2 .\end{aligned} \tag{4.8}$$

It is easy to compute that

$$\begin{aligned}&\sum_{i=1}^n F^{ii} = (n-1)H, \quad \sum_{i=1}^n F^{ii}h_{ii} = 2S_2(h_{ij}),\\ &\sum_{i=1}^n F^{ii}h_{ii}^2 = HS_2(h_{ij}) - 3S_3(h_{ij}), \quad |A|^2 = H^2 - 2S_2(h_{ij}).\end{aligned} \tag{4.9}$$

Combining the (4.7)-(4.9), at x_o we get the following

$$
\begin{aligned}
& a(n-1)H + \phi H + 2\sum_{i=1}^{n} \phi_i h_{ii} < X, e_i > + \Delta\phi < X, e_{n+1} > + 3HS_3(h_{ij}) \\
& \leq 2S_2(h_{ij})^2 + 2a < X, e_{n+1} > S_2(h_{ij}) + [a\phi + a^2] \sum_{i=1}^{n} < X, e_i >^2 . \qquad (4.10)
\end{aligned}
$$

Let F_A, F_{AB} be the ordinary Euclidean differential in $\mathbf{R}^{n+1}$, use (2.2), we compute

$$
\begin{aligned}
\phi_i &= -(n+1)|X|^{-(n+3)} < X, e_i > F(X) + |X|^{-(n+1)} \sum_{A=1}^{n+1} F_A X_i^A \\
\Delta\phi &= \sum_{i=1}^{n} \phi_{ii} = H[(n+1)|X|^{-(n+3)} < X, e_{n+1} > F - |X|^{-(n+1)} \sum_{A=1}^{n+1} F_A e_{n+1}^A] \\
& -2(n+1)|X|^{-(n+3)} \sum_{i=1}^{n} \sum_{A=1}^{n+1} < X, e_i > F_A X_i^A - n(n+1)|X|^{-(n+3)} F \\
& +|X|^{-(n+1)} \sum_{A,B=1}^{n+1} \sum_{i=1}^{n} F_{AB} X_i^A X_i^B + (n+1)(n+3)|X|^{-(n+5)} F \sum_{i=1}^{n} < X, e_i >^2 .
\end{aligned}
$$

Now we use the convexity of the solution, we have

$$S_3(h_{ij}) \geq 0, \quad 0 \leq h_{ii} \leq H.$$

If a is suitable large, we get the following mean curvature estimate

$$\max H \leq C(n, \max_{\mathbf{S}^n} f, \min_{\mathbf{S}^n} f, |\nabla f|_{C^0}, |\nabla^2 f|_{C^0}). \qquad (4.11)$$

This finishes the proof of the Lemma. □

Since C^2 estimates in Lemma 4.1 only valid for convex solutions, in order to carry on the method of continuity, we need to show the convexity is preserved during the process.

Theorem 4.2. *Suppose M is a convex hypersurface and satisfies equation (2.6) for $k < n$ with the second fundamental form $W = \{h_{ij}\}$ and $|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})$ is convex in $\mathbb{R}^{n+1} \setminus \{0\}$, then W is positive definite.*

We now use Theorem 4.2 to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is the same as in the proof of Theorem 1.1 by the method of continuity, here we make use of Theorem 4.2. The openness and uniqueness have already treated in the proof of Theorem 1.1. The closeness follows from a priori estimates in Lemma 2.2 and quasilinear elliptic theory in the case of $k = 1$ and the a priori estimates in Lemma 4.1 in the case of $k = 2$, and the preservation of convexity in Theorem 4.2. □

The proof of Theorem 4.2 relies on the following deformation lemma. A similar lemma was proved for spherical hessian equations in [13] and for curvature equations in [12] (only with different homogeneity on the right side of the equation). This type of lemma is a generalization of corresponding results in [4, 14] for semilinear equations on domains in $\mathbf{R}^n$.

Lemma 4.3. *Assume M_o is a piece of C^4 hypersurface M, M is the solution of equation (2.6) and the matrix $W = \{h_{ij}\}$ is semi-positive definite. Suppose there is a positive constant $C_o > 0$, such that for a fixed integer $(n-1) \geq l \geq k, \forall X \in M_o, S_l(W(X)) \geq C_o$. Let $\phi(X) = S_{l+1}(W(X))$ and let $\tau(X)$ be the largest eigenvalue of $\{-(F^{-\frac{1}{k}})_{X_A X_B}(X, e_{n+1})\}$, where the differential is ordinary differential in $\mathbb{R}^{n+1}$. Then, there are constant C depending only $||X||_{C^3}, ||F||_{C^2}$ and C_o, the following differential inequality holds at each point $X \in M_o$,*

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \leq k(n-l)F^{\frac{k+1}{k}} S_l(W)\tau < X, e_{n+1} > +C(|\nabla\phi| + \phi),$$

where $F^{\alpha\beta} = \frac{\partial S_k(W)}{\partial w_{\alpha\beta}}$.

Proof. A proof was already given in [12] for the following prescribed curvature equation

$$S_k(\kappa_1, \kappa_2, ..., \kappa_n)(X) = F(X), \quad \text{on} \quad M. \tag{4.12}$$

Since we are treating a different homogeneity here, we will make a minor change in the last step of the proof in [12]. We follow the same notation as in [12] (see also [13]). For any $z \in M_o$, let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the eigenvalue of W at z. Since $S_l(W) \geq C_o > 0$ and $M \in C^3$, for any $z \in M$, there is a positive constant $C > 0$ depending only on $||X||_{C^3}$, $||F||_{C^2}$, n and C_o, such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l \geq C$. Let $G = \{1, 2, ..., l\}$ and $B = \{l+1, ..., n\}$. As $\phi = S_{l+1}(W)$ and $\phi_\alpha = \sum_{i,j} S^{ij} h_{ij\alpha}$, there is $C > 0$ such that

$$C\phi(z) \geq \sum_{i\in B} h_{ii}(z), \quad C(\phi(z) + |\phi_\alpha(z)|) \geq \sum_{i\in B} h_{ii\alpha}(z). \tag{4.13}$$

By (2.21) in [12], there is $c > 0$ such that

$$\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \leq cS_l(G)\sum_{i\in B}[f_{ii} - \frac{k+1}{k}\frac{f_i^2}{f}]. \tag{4.14}$$

Since

$$f(X, e_{n+1}) = F(X) < X, e_{n+1} >,$$

use (2.2), so for $\forall \quad i \in \{1, 2, ..., n\}$,

$$f_i = \sum_{A=1}^{n+1} F_{X_A} e_i^A < X, e_{n+1} > + F(X) h_{ii} < X, e_i >,$$

$$f_{ii} = \sum_{A,C=1}^{n+1} F_{X_A X_C} e_i^A e_i^C < X, e_{n+1} > + \sum_{A=1}^{n+1} F_{X_A} X_{ii}^A < X, e_{n+1} >$$
$$+2 \sum_{A=1}^{n+1} F_{X_A} e_i^A h_{ii} < X, e_i > + F(X) [\sum_{j=1}^{n} h_{iij} < X, e_j > + h_{ii} - h_{ii}^2 < X, e_{n+1} >].$$

From (2.2) and (4.13), for $i \in B$ we get

$$f_i = \sum_{A=1}^{n+1} F_{X_A} e_i^A < X, e_{n+1} >, \quad f_{ii} = \sum_{A,C=1}^{n+1} F_{X_A X_C} e_i^A e_i^C < X, e_{n+1} > .$$

It follows that for $\forall \quad i \in B$,

$$f_{ii} - \frac{k+1}{k} \frac{f_i^2}{f} \leq C \sum_{A,C=1}^{n+1} [F_{AC} - \frac{k+1}{k} \frac{F_A F_C}{F}] e_i^A e_i^C < x, e_{n+1} > . \tag{4.15}$$

So the lemma follows from (4.14) and (4.15). The proof of the Lemma is complete. □

Proof of Theorem 4.2. By the Evans-Krylov theorem and Schauder theorem, $X, e_{n+1} \in C^{4,\alpha}$. If $W = \{h_{ij}\}$ is not of full rank at some point x_o, then there is $n-1 \geq l \geq k$ such that $S_l(W(x)) > 0$, $\forall x \in M$ and $\phi(x_o) = S_{l+1}(W(x_o)) = 0$. By Lemma 4.3 and the condition on F,

$$\sum_{\alpha,\beta}^{n} F^{\alpha\beta} \phi_{\alpha\beta}(X) \leq C_1 |\nabla \phi(X)| + C_2 \phi(X). \tag{4.16}$$

The strong minimum principle implies $\phi = S_{l+1}(W) \equiv 0$. On the other hand, M is starshape respect to origin, so $< X, e_{n+1} >> 0$, where $<,>$ is ordinary inner product in $\mathbb{R}^{n+1}$. Since M is compact, there is a point $x \in M$ such that all the principal curvatures of M at x is positive. This is a contradiction. □

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Department of Mathematics, McMaster University, Hamilton, ON. L8S 4K1, Canada.
E-mail address: guan@math.mcmaster.ca

Department of Mathematics, National Chung Cheng University, Chia-Yi, Taiwan.
E-mail address: cslin@math.ccu.edu.tw

Department of mathematics, East China Normal University, Shanghai, 200062, China.
E-mail address: xnma@math.ecnu.edu.cn