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Synchronization of Networks with Prescribed Degree Distributions

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Abstract

We show that the degree distributions of graphs do not suffice to characterize the synchronization of systems evolving on them. We prove that, for any given degree sequence satisfying certain conditions, there exists a connected graph having that degree sequence for which the first nontrivial eigenvalue of the graph Laplacian is arbitrarily close to zero. Consequently, complex dynamical systems defined on such graphs have poor synchronization properties. The result holds under quite mild assumptions, and shows that there exists classes of random, scale-free, regular, small-world, and other common network architectures which impede synchronization. The proof is based on a construction that also serves as an algorithm for building non-synchronizing networks having a prescribed degree distribution.

Index Terms—Synchronization, networks, graph theory, Laplacian, degree sequence

1 Introduction

Many network arhitectures encountered in nature or used in applications are described by their degree distributions. In other words, if $P(d)$ denotes the fraction of vertices having d incident edges, then the shape of the function $P(d)$ distinguishes certain network classes. For example, in the classical random graphs studied by Erdös and Rényi [1,2], $P(d)$ has a binomial

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distribution, which converges to a Poisson distribution for large network sizes. The degree distribution of regular networks, where every vertex has the same degree k, are given by the delta function $P(d) = \delta(d - k)$, whose small perturbations by random rewiring introduce small-world effects [3, 4]. Many common networks, such as the World-Wide Web [5], the Internet [6], and, networks of protein interactions [7] have been shown to have approximate power-law degree distributions, and have been termed *scale-free* [8]. The power grid has exponentially distributed vertex degrees [3]. Some social networks have distributions similar to a power-law, possibly with some deviations or truncations at the tails [9]. A recent survey is given in [10]. These examples demonstrate the recent widespread effort in classifying common large networks according to their degree distributions.

In many applications, the vertices of networks have internal dynamics, and one is interested in the time evolution of a dynamical system defined on the network. A natural question is then what, if anything, the degree distribution can say about the dynamics on the network. Asked differently, to what extent does a classification according to degree distributions reflect itself in a classification according to different qualitative dynamics? The question is all the more significant since many large networks are constructed randomly. Hence, in essence every realization of, say, a power-law distribution is a different network. On the one hand, it is conceivable that the dynamics on different realizations could be sufficiently different. On the other hand, it is known that these realizations can have certain common characteristics; for instance, small average distances and high local clustering found in small-world networks [10]. Consequently, the relation between the degree distribution and the dynamics is not trivial. The present work studies this relation in the context of a specific but important dynamical behavior of networked systems, namely synchronization.

It is well-known that the synchronization of diffusively-coupled systems on networks is crucially affected by the network topology. In particular, the so-called spectral gap, or the smallest nontrivial eigenvalue λ_1, of a discrete Laplacian operator plays a decisive role for chaotic synchronization: Larger values of λ_1 enable synchronization for a wider range of parameter values, in both discrete and continuous-time systems [11–13], and also in the presence of transmission delays [14]. Here we shall prove that, given any degree distribution satisfying certain mild assumptions, a connected network having that distribution can be constructed in such a way that λ_1 is inversely proportional to the number of edges in the graph. Hence, there exist large networks with these degree distributions which are arbitrarily poor synchronizers. The proof is based on ideas from degree sequences of graph theory, and in particular makes use of the relation between λ_1 and the Cheeger constant of the graph. It applies to the class of distributions that include the classical random (Poisson), exponential, and power-law distributions, as well as nearest-neighbor-coupled networks and their small-world variants, and many others. Thus, we establish that the degree distribution of a network does not suffice to characterize the synchronizability of complex dynamics on its vertices.

Our proof is constructive in nature. Hence, from another perspective, it can be viewed as an algorithm for designing non-synchronizing networks having a prescribed degree distribution. This may have implications for engineering systems which should essentially operate in asynchrony, such as the Internet, where synchronized client requests can cause congestion in the data traffic through servers or routers, or neuronal networks where synchronization can be sign of pathology. Furthermore, the non-synchronizing networks that we construct can actually have quite small diameters or average distances. This proves that the informal arguments which refer to efficient information transmission via small distances as a mechanism for synchronization are ill-founded. For a related discussion based on numerical studies of small-world networks, see [12, 15]. Here we give a rigorous proof for a much larger class of distributions.

2 Degree sequences

In the following we consider finite graphs without loops or multiple edges. As mentioned above, the degree distribution $P(d)$ gives the fraction of vertices having d incident edges. A related notion is that of a *degree sequence,* which is a list of nonnegative integers $\pi = (d_1, \ldots, d_n)$ where d_i is the degree of the ith vertex. We also denote the largest and smallest degrees by $d_{\max}$ and $d_{\min}$, respectively. For each graph such a list is well-defined, but not every list of integers corresponds to a graph. A sequence $\pi = (d_1, \ldots, d_n)$ of nonnegative integers is called *graphic* if there exists a graph G with n vertices for which $d_1, \ldots, d_n$ are the degrees of its vertices. G is then referred to as a *realization* of π. A characterization of graphic degree sequences is given by the following.

Lemma 1 ([16, 17]) *For $n > 1$, the nonnegative integer sequence π with n elements is graphic if and only if π' is graphic, where π' is the sequence with $n-1$ elements obtained from π by deleting its largest element $d_{\max}$ and subtracting 1 from its $d_{\max}$ next largest elements. The only 1-element graphic sequence is $\pi_1 = (0)$.*

Often one is interested in connected graphs, and the following result gives the correspondence between degree sequences and connected realizations (see e.g. [18]).

Lemma 2 *A graphic sequence π with n elements has a connected realization if and only if the smallest element of π is positive and the sum of the elements of π is greater or equal than $2(n-1)$.*

One of the simplest degree sequences belong to k-regular graphs, in which each vertex has precisely k neighbors. Here, the degree sequence is given by

$$\pi = (k, \ldots, k). \tag{1}$$

There is a simple criterion for such constant sequences to be graphic (see e.g. [18]).

Lemma 3 *A sequence $\pi = (k, \ldots, k)$ with n elements where $k < n$ is graphic if and only if n or k is even.*

The construction of *small-world* networks starts with a k-regular graph with k much smaller than the graph size n, e.g., a large circular arrangement of vertices which are coupled to their near neighbors. Then a small number of edges c are added between randomly selected pairs of vertices [4]. When $c \ll n$, the degree of a vertex typically increases by at most one, which yields the degree sequence

$$\pi = (k+1, \dots, k+1, k, \dots, k) \tag{2}$$

where $2c$ vertices have degree $k+1$. In a variant of the model [3], randomly selected edges are *replaced* by others, so the degree of a vertex may increase or decrease by one, and the degree sequence becomes

$$\pi = (k+1, \dots, k+1, k, \dots, k, k-1, \dots, k-1) \tag{3}$$

where the number of vertices having degree $k+1$ or $k-1$ are each equal to c. More generally, there is also the possibility of having some vertex degrees increase or decrease by more than one, in which case the sequences (2) and (3) are modified accordingly, while the sum of the degrees remain equal to $nk + 2c$ and nk, respectively.

In addition to these well-known graph types, we shall also consider more general sequences, which we define as follows.

Definition A sequence π with largest element $d_{\max}$ is called a *full sequence* if each integer d satisfying $1 \le d \le d_{\max}$ is an element of π, and the sum of the elements of π is even.

We also give a criterion for full sequences to be graphic.

Lemma 4 *Let $\pi = (d_{\max}, \dots, d_{\max}, d_{\max} - 1, \dots, d_{\max} - 1, \dots, 1, \dots, 1)$ be a full sequence with n elements and $d_{\max} \le n/2$. Then π is graphic.*

Proof. We prove by induction on the number of elements of π. This is trivial for the full sequence with two elements. Suppose that every full sequence with at most $n \ge 2$ elements and largest element not larger than $n/2$ is graphic. Let π be a full sequence with $n+1$ elements and largest element $d_{\max} \le (n+1)/2$. We look at the sequence π' that is defined in Lemma 1. It is easy to see that π' is a full sequence. Let $d'_{\max}$ be the largest element of π'. By the definition of π', $d'_{\max} \le d_{\max}$. We claim that

$$d'_{\max} \le n/2. \tag{4}$$

For if $d'_{\max} > n/2$, then

$$n/2 < d'_{\max} \le d_{\max} \le (n+1)/2.$$

This implies

$$d'_{\max} = d_{\max} = (n+1)/2 \tag{5}$$

so the number of $d_{\max}$'s in π is at least $d_{\max} + 2$. Since π is a full sequence, the number of its elements is then

$$n + 1 \geq (d_{\max} + 2) + (d_{\max} - 1),$$

implying $d_{\max} \leq n/2$, which contradicts (5). Thus, (4) holds. Then π' is graphic by induction, and by Lemma 1 π is graphic. ■

The reason for introducing the concept of full sequences is that many common graph types have full degree sequences. For instance, large random graphs of Erdős-Rényi have their vertex degrees distributed according to the Poisson distribution

$$P(d) \sim \frac{\mu^d e^{-\mu}}{d!}, \quad d \geq 1. \tag{6}$$

Since such networks are randomly constructed, (6) is understood to hold in the limit as the network size increses while μ is kept constant. In scale-free networks, the degree distribution follows a power law

$$P(d) \sim d^{-\beta} \tag{7}$$

for some $\beta > 1$. The exponential distribution obeys

$$P(d) \sim e^{-\mu d}$$

with $\mu > 0$. Of course, in any finite graph these distributions are truncated from the right since the maximum degree is finite, and then one has $P(d) > 0$ for $1 \leq d \leq d_{\max}$. Furthermore, the sum of the vertex degrees is always even since it is twice the number of edges in the graph. Therefore, large finite graphs approximated by these distributions, or more generally by any distribution satisfying $P(d) > 0$ for $1 \leq d \leq d_{\max}$, have realizations with full degree sequences[1]. In the next section, we prove that the regular and small-world sequences (1)-(3), full degree sequences, as well as their variants where some degrees may be missing, have a realization which is a poor synchronizer. Furthermore, the synchronizability of this realization worsens with increasing graph size.

3 Constructing networks with small spectral gap

The synchronization of dynamical systems is usually studied in diffusively coupled equations, which in discrete-time may have the form

$$x_i(t+1) = f(x_i(t)) + \kappa \left[\frac{1}{d_i} \sum_{\substack{j \\ j \sim i}} f(x_j(t)) - f(x_i(t)) \right], \ i = 1, \ldots, n. \tag{8}$$

[1]In a particular realization some degrees may actually be missing; we also address this possibility in Section 3.

In (8), x_i denotes the state of the ith unit, which is viewed as a vertex of a graph, and has d_i neighbors to which it is coupled. The notation $i \sim j$ denotes that units i and j are coupled, which is represented in the underlying graph by an edge. Furthermore, f is a differentiable function and $\kappa \in \mathbf{R}$ quantifies the coupling strength. A solution $(x_1(t), \ldots, x_n(t))$ of (8) is said to synchronize if $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all i, j. The synchronization manifold $\mathcal{M}$ defined by the conditions $x_1 = \cdots = x_n$ is an invariant manifold for (8), and clearly the trajectories starting in $\mathcal{M}$ synchronize. In case $\mathcal{M}$ is an attracting set, the system (8) is said to synchronize, and one distinguishes between different types of synchronization depending on the nature of the attractor [19–21]. For example, when f is a chaotic map $\mathcal{M}$ may contain an attractor in the weak Milnor sense [22], that is, for some set of initial conditions with positive Lebesgue measure, and the corresponding behavior is then termed weak synchronization. Systems similar to (8) arise also in continuous time in different applications. For a general introduction the reader is referred to [23].

The effect of the network topology on synchronization is determined by the properties of the coupling operator. For (8), the relevant operator is the (normalized) Laplacian with matrix representation

$$L = D^{-1}A - I \tag{9}$$

where $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ is the diagonal matrix of vertex degrees and $A = [a_{ij}]$ is the adjacency matrix of the graph, defined by $a_{ij} = 1$ if $i \sim j$ and zero otherwise. The eigenvalues of L are real and nonpositive, which we denote by $-\lambda_i$. Zero is always an eigenvalue, and has multiplicity 1 for a connected graph. The smallest nontrivial eigenvalue, denoted λ_1, is called the spectral gap of the Laplacian, and is the important quantity for the synchronization of coupled chaotic systems. Larger values of λ_1 enable chaotic synchronization for a larger set of parameter values. This result holds in both continuous and discrete time, for the Laplacian (9) that we consider here, as well as the combinatorial Laplacian $A - D$ [11–13]. In the following, we shall show how to construct a graph having a prescribed degree sequence and an arbitrarily small spectral gap, that is, a poor synchronizer.

Let $G = (E, V)$ be a connected graph, with edge set E and vertex set V. We denote the cardinality of a set S by $|S|$. Thus $|E(G)|$ is the number of edges of G. For a subset $S \subset V$, we define

$$h_G(S) = \frac{|E(S, V - S)|}{\min(\sum_{v\in S} d_v, \sum_{u\in V-S} d_u)}, \tag{10}$$

where $|E(S, V - S)|$ denotes the number of edges with one endpoint in S and one endpoint in $V - S$. The *Cheeger constant* h_G is defined as [24, Section 2.2]

$$h_G = \min_{S\subset V} h(S).$$

This quantity provides an upper bound for the smallest nontrivial eigenvalue.

Lemma 5 *$\lambda_1 \le 2h_G$.*

This result is basically Lemma 2.1 in [24]. There it is proved for the smallest nontrivial eigenvalue of the matrix $\tilde{L} = [l_{ij}]$ defined by

$$l_{ij} = \begin{cases} a_{ij}/\sqrt{d_i d_j} & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$$

Since $\tilde{L} = D^{-1/2}AD^{-1/2} - I = D^{1/2}LD^{-1/2}$ is similar to the matrix L in (9), its eigenvalues coincide with those of L. Hence, the above lemma also holds for L.

Based on the estimate given in Lemma 5, we construct, from a given degree distribution, a realization whose spectral gap λ_1 is small. We first consider regular graphs.

Theorem 1 *Suppose $\pi = (k, \ldots, k)$ is graphic with n elements, with $2 \leq k < n/2$. Then π has a connected realization G such that*

$$\lambda_1(G) \leq \frac{4}{|E(G)| - k}.$$

Proof. We split π into two graphic sequences π_1 and π_2, which are either equal (for n equal 0 modulo 4 or n equal 2 modulo 4 and k is even), or π_1 has one element less than π_2 (for n equal 1 or 3 modulo 4), or π_1 has two elements less than π_2 (for n equal 2 modulo 4 and k is odd). (By splitting we mean that the union of the lists π_1 and π_2 is equal to π.) By Lemmas 2 and 3, π_1 and π_2 have connected realizations G_1 and G_2, respectively. Now we construct a connected realization G of π as follows (see Figure 1). Let uv and xy be edges of G_1 and G_2 that are in a cycle. We delete uv and xy, and add new edges ux and vy. This new graph G is connected, and it is a realization of π since vertex degrees remain unchanged after this operation. Furthermore, if S denotes the smaller of the vertex sets $V(G_1)$ and $V(G_2)$, then $(n-2)/2 \leq |S|$ (see above), and we have

$$h_G(S) = \frac{2}{\sum_{v \in S} d_v} \leq \frac{2}{\frac{1}{2}nk - k} = \frac{2}{|E(G)| - k} \tag{11}$$

Thus by Lemma 5, $\lambda_1(G) \leq 4/(|E(G)| - k)$. ■

Next we consider the small-world variants of regular graphs.

Theorem 2 *Let π be a k-regular degree sequence with n elements, with $k < n/2$, and let π' be a small-world degree sequence obtained from π by adding or replacing c edges. Then π' has a connected realization G such that*

$$\lambda_1(G) \leq \frac{2(c+2)}{|E(G)| - (k+c)}.$$

Proof. First we split the k-regular degree sequence π in two parts, as in the previous proof, obtaining the situation shown in Figure 1, and obtain the estimate (11) for $h_G(S)$, where S is the smaller of the vertex sets $V(G_1)$ and $V(G_2)$. Now we add or replace c edges in G. The numerator in (11) can then increase by at most c, in case the c added adges are

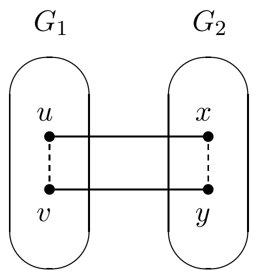

Figure 1: Partitioning the degree sequence π in two parts π_1 and π_2, where G_1 and G_2 are realizations of π_1 and π_2, respectively.

all between G_1 and G_2. Furthermore, the denominator can decrease by at most c, in case all the removed edges, if any, are all in G_1. Hence,

$$h_G(S) \leq \frac{2+c}{|E(G)| - k - c}$$

and the result follows by Lemma 5 as before. ■

Remark By the same argument, a connected graph G can be constructed from a k-regular graph by *removing* c edges, whose spectral gap satisfies

$$\lambda_1(G) \leq \frac{4}{|E(G)| - (k+c)}.$$

The method of proof for the above results is actually applicable to a large class of degree sequences, including the full sequences defined in Section 2.

Theorem 3 *Let $\pi = (d_{\max}, \ldots, d_{\max}, d_{\max} - 1, \ldots, d_{\max} - 1, \ldots, 1, \ldots, 1)$ be a full graphic sequence with n elements, $d_{\max} \leq n/4$, for which the sum of the elements is greater or equal than $2n + d_{\max}$, and each k, $1 \leq k \leq d_{\max}$, appears more than once. Then π has a connected realization G such that*

$$\lambda_1(G) \leq \frac{4}{|E(G)| - d_{\max}/2}.$$

Proof. For $k = 1, \ldots, d_{\max}$, let n_k be the number of times the element k appears in the sequence π. We construct two sequences π_1 and π_2 from π as follows. The sequences π_1 and π_2 get $\lfloor n_k/2 \rfloor$ elements from each k for $k = 1, \ldots, d_{\max}$, where the notation $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Let $d_1, \ldots, d_s$ be the remaining elements of π, where the d_i are distinct. For the set $N = \{1, 2, \ldots, s\}$, we define

$$r_\pi = \min_{J \subseteq N} \left| \sum_{j \in J} d_j - \sum_{j \notin J} d_j \right|.$$

By induction on the number of elements, it is easy to see that $r_\pi \leq d_{\max}$. Now we spread the remaining elements $d_1, \ldots, d_s$ to the sequences π_1 and π_2 such that r_π is minimum. By Lemmas 4 and 2, π_1 and π_2 have connected realizations G_1 and G_2, respectively. Let uv and xy be edges of G_1 and G_2 that are in a cycle or a vertex of the edges has degree one. (Such edges exist since G_1 and G_2 are connected.) We rewire these as before (Fig. 1), and the rest of the proof follows analogously to the proofs of Theorems 1 and 2. ■

We finally generalize to include degree sequences where some degrees between 1 and $d_{\max}$ may be missing.

Theorem 4 *Let $\pi = (d_{\max}, \ldots, d_{\min})$ be a graphic sequence with n elements, where $d_{\max} \leq n/6$, $d_{\min} \geq 2$, and each degree appears more than once. Let c be the sum of distinct integers between 1 and $d_{\max}$ which do not appear in π. Then π has a connected realization G such that*

$$\lambda_1(G) \leq \frac{2c}{|E(G)| - d_{\max}/2}. \tag{12}$$

Proof. First, we follow the proof of Theorem 3 and construct two sequences π_1 and π_2 from π as before. The sequences π_1 and π_2 get $\lfloor n_k/2 \rfloor$ elements from each present degree. We also handle the remaining elements of π as before. So, the sum of the elements of the sequences differ by at most $d_{\max}$. Notice that by construction, the sequences π_1 and π_2 have at least $n/3$ elements. It is possible that π_1 and π_2 are not graphic. By the following process, we make these sequences graphic by adding new vertices and edges. For a nonnegative integer sequence $s = (d_1, \ldots, d_n)$ with at least one positive element, let d be the largest element of s and p be the number of positive elements of s. If $d > p - 1$, then we add $d - p + 1$ new elements to s that are equal to one, otherwise the sequence s does not change. Now we obtain the sequence s' from s by deleting its largest element d and subtracting 1 from its d next largest elements. We repeat this for s' and so forth until the considered sequence has no positive elements. Let z be the total number of new elements that are added in each step. By Lemma 1, the sequence s is graphic by adding at most z edges. We call this process a *lay-on process* and say that the sequence s *is graphic with at most z new edges*. In other words, if we add to the sequence $s = (d_1, \ldots, d_n)$ z new elements that are equal one, then the new sequence $s^* = (d_1, \ldots, d_n, 1 \ldots, 1)$ is a degree sequence. We call s^* as *extension* of s. Since we can add all the missing elements of $d_{\max}, \ldots, 1$ to π_1 and π_2 and by Lemma 4 these new sequences are graphic, it follows that the sequences π_1 and π_2 are graphic with at most c new edges.

Claim 1: If the sum of the elements of s is even (odd), then z is even (odd).

The proof of the claim follows by induction on the number of elements of s and the lay-on process.

The sums of the elements of π_1 and π_2 are either both even or both odd, because the sum of the elements of π is even. Then the total number of new elements of π_1 and π_2 have the same parity, by the above Claim. We say a sequence $s = (d_1, \ldots, d_n)$ has a connected *extension*, if its extension s^* has a connected realization.

Claim 2: The sequence $s = (d_1, \ldots, d_n)$, $d_1 \geq \cdots \geq d_n \geq 2$, has a connected extension.

We prove the claim by induction on the number of elements of s. If s has one element, it is trivial. We assume that the claim is true, if s has less than n elements. Let s have n elements. If $d_n \geq 3$, then s has a connected extension, by induction and the lay-on process. Suppose now that $d_n = 2$. If $d_1 \geq d_2 \geq 3$, then we delete d_n and subtract one from d_1 and d_2, obtaining a sequence with $n-1$ elements that are greater than one. By induction this sequence has a connected extension. It follows that s has a connected extension. It remains to consider the case when $d_1 > 2$ and all other elements are equal to two. For $d_1 \leq n-4$, there is a vertex v_1 with d_1 neighbors and the rest of the vertices can build up a cycle. We switch an edge of the cycle and an edge of v_1, obtaining a connected extension of s. If $d_1 \geq n-1$, then by construction s has connected extension. If $d_1 = n-3$, then there are two remaining vertices with degree two. By construction, there is an edge between them and both of them have an edge with neighbors of v_1. It follows that s has a connected extension. The case $d_1 = n-2$ is similar.
The sequences π_1 and π_2 are graphic with at most c new edges, they have the same parity, and, by Claim 2, π_1 and π_2 have connected extension. It remains to construct a connected realization of the degree sequence π. Let G_1 and G_2 be the connected realizations of the extensions π_1^* and π_2^*, respectively. For the vertices v_1 and v_2 of G_1 and G_2 that are adjacent to the vertices with degree one, respectively. We delete these edges and add an edge between v_1 and v_2, therefore we get a connected realization G of π. Assume that there remain in G_1 two edges ux_1 and vx_2, where x_i have degree one (because of the same parity this case is the only possibility). Then we delete these edges, as well as an edge ab of G_2, and add the edges au and bv. Notice that after deleting and adding edges, the new graph is also connected. G can be split in two parts with at most c edges between them, and the sum of the degrees of these two parts differ by at most $d_{\max}$. The estimate (12) follows as before from (10) and Lemma 5. ■

Remark The upper bound for λ_1 given in Theorem 4 is proportional to the sum of the missing degrees c; hence, it is smaller if the degree sequence is closer to being a full sequence.

We note that many of the assumptions in Theorems 1-4 can be relaxed. Furthermore, the estimates obtained for λ_1 are certainly not tight. Our aim here is not to obtain estimates in full generality, but rather show that the spectral gap cannot be determined from the degree distribution. Hence, we content ourselves to constructing connected graphs for which λ_1 is inversely proportional to the number of edges, i.e. is arbitrarily small for large graph sizes. Thus, based on Theorems 1-4, we conclude that the degree distribution of a network in general is not sufficient to determine its synchronizability.

The construction used in the above proofs has a quite general nature. In essence, the idea is to split a given sequence π into two sequences π_1, π_2 more or less equal in size, which have the connected realizations G_1, G_2, respectively. Then, as schematically shown in Figure 1, switching the edges between two pairs of vertices yields a connected graph G as a realization of π. The spectral gap λ_1 is estimated from the Cheeger constant of the sets G_1 or G_2, and is small since the numerator in (10) is 2 and the denominator can be rather large. This gives an algorithm for constructing networks having a prescribed degree distribution and a small spectral gap. In certain cases, it may require some effort to obtain connected realizations for the partitions π_1,π_2. Nevertheless, by arguments similar to the above, it is not hard to show that if π_1 has a realization with m components, then it is possible to construct a realization G of π such that λ_1 is essentially given by $\lambda_1 \sim 2m/|E(G)|$. So, the basic construction is indeed applicable to even more general cases and distributions than we have considered here. On the other hand, it should be kept in mind that certain degree sequences are not amenable to such manipulation, in particular those that have a unique representation. For instance, the n-element sequences $(n-1,\ldots,n-1)$ and $(1,1,\ldots,1,n-1)$ correspond uniquely to the complete graph K_n and the star S_n, respectively; hence the value of λ_1 is uniquely determined for them.

We conclude this section with an example illustrating the main ideas.

Example. Consider a constant degree sequence $\pi = (n-1,\ldots,n-1)$ of $2n$ elements, where $n \geq 3$. We split it into two constant sequences $\pi_1 = \pi_2 = (n-1,\ldots,n-1)$ each with n elements. Since either n or $n-1$ is even, both subsequences are graphic by Lemma 3 and have connected realizations G_1, G_2 by Lemma 2. In fact, in this special case the realization is unique: $G_1 = G_2 = K_n$, the complete graph on n vertices where each vertex is connected to the remaining $n-1$. We now remove one edge each from G_1 and G_2, and rewire to construct the structure shown in Figure 1, i.e., a connected graph G having the original degree sequence π. The subgraphs G_1, G_2 are complete graphs each missing an edge. To show that G has a small spectral gap, Theorem 1 can be invoked to obtain the bound

$$\lambda_1 \leq \frac{4}{(n-1)^2} \tag{13}$$

which is plotted in Figure 2 against n, together with the true values of λ_1. The figure shows that λ_1 decreases rapidly with increasing graph size, and is in fact estimated very well by the upper bound (13) for large n. In this special case the estimate (13) can actually be somewhat improved, but the point is that the value of λ_1 is on the order of n^{-2} for large n. It is worth noting that the average distance in the constructed graph G is small and approaches 2 as $n \to \infty$, and the maximum distance between any two vertices is equal to 3. Despite these small distances, G is a poor synchronizer even at moderate sizes n.

4 Discussion and conclusion

We have shown that the degree sequence is generally not sufficient to characterize the synchronizability of a network. The method of proof is based on the construction of a graph which has a specified degree distribution and a small spectral gap for the graph Laplacian.

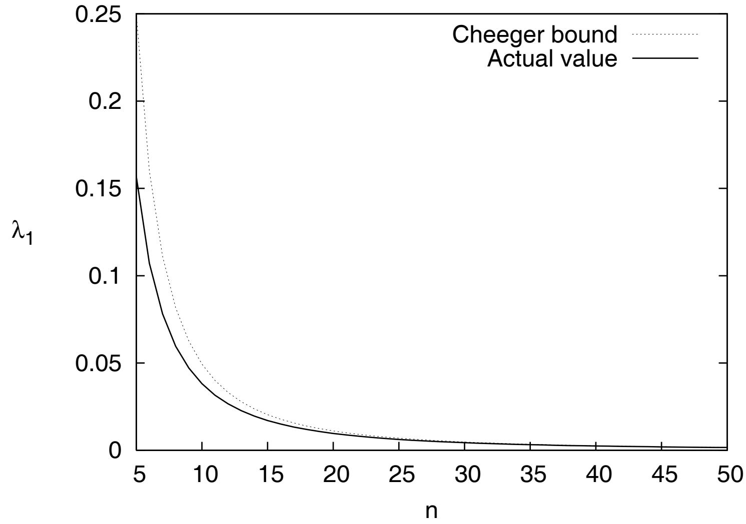

Figure 2: The estimated and actual values of λ_1 for the graph constructed in Example.

The construction works for a wide class of degree distributions, including Poisson, exponential, and power-law distributions, regular and small-world networks, and many others. Thus, sychronizability is not an intrinsic property of a degree distribution. Furthermore, small diameters or average distances in a graph do not necessarily imply synchronization, since the poorly-synchronizing graphs we construct can have small diameters, as the above Example shows.

The method of proof presented here is essentially an algorithm to construct a graph with a given degree distribution and a small spectral gap. Such a poor-synchronizing realization of a degree distribution is certainly not unique. An interesting query is then finding how many poor-synchronizers there are, as compared to all realizations of a given distribution? This turns out to be a difficult question. In general it is not easy to estimate of the number of realizations of a given degree sequence. McKay and Wormald [25] give an estimate that depends on the maximum degree. To state their result, let $\pi = (d_1, \ldots, d_n)$ be a degree sequence without zero elements, n_1 entries of value 1, and n_2 entries of value 2. If $n_1 = O(n^{1/3})$, $n_2 = O(n^{2/3})$, and $d_{\max} \le \frac{1}{3} \log n / \log \log n$, then the number of the realizations of π is asymptotically

$$\frac{M!}{(M/2)! 2^{M/2} d_1! \cdots d_n! n!} \exp\left(-O(d_{\max}^6) - O\left(\frac{d_{\max}^{10}}{12n}\right) + O\left(\frac{d_{\max}^3}{M}\right)\right),$$

where $M = \sum d_i = 2|E(G)|$ [25].

As an attempt to apply this result to the case considered here, let $\pi = (d_1, \ldots, d_{2n})$ be a full degree sequence with $2n$ elements and $\pi_1 = (d_1, \ldots, d_n)$ be one of the partitions with n

elements such that $\sum_{i=1}^{2n} d_i = 2\sum_{i=1}^{n} d_i$. Then the ratio of the number of realizations π_1 to π is approximately (by using $n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$)

$$d_{n+1}! \cdots d_{2n}! n^n 2^{-M+2n} M^{-M/2} \exp\left(M/2 - n - O\left(\frac{d_{\max}^{10}}{12n}\right)\right),$$

where $M = \sum_{i=1}^{n} d_i = O(n \log n / \log\log n)$. For a certain partition π_1 the ratio is very small. On the other hand it is hard to give an estimate on the number of suitable partitions like π_1 that make λ_1 very small. Hence, at this point there seems to be no general estimates for the relative size of poor-synchronizers belonging to the same degree sequence, and the question remains open for future research.

At any rate, it is clear that care is needed when making general statements about the synchronizability of, say, scale-free networks, even in a statistical or asymptotical sense. In this context it is also worth noting that most results about the scale-free architecture are based on the algorithm of Barabàsi and Albert [26], which, by way of construction, possibly introduces more structure to affect the dynamics than is reflected in its power-law degree distribution. In fact, another preferential attachment type algorithm introduced in [27] yields also graphs with power-law degree distributions, but significantly smaller first eigenvalues than those constructed by the algorithm of Barabàsi and Albert or random graphs. While the algorithm of [26] lets nodes receive new connections with a probability proportional to the number of connections they already possess, the one of [27] introduces new nodes by first connecting to an arbitrary node and then adding further connections that complete triangles, that is, to nearest neighbors of nodes already connected to. Since the probability to be a nearest node of another node is also proportional to the number of links a nodes possesses, that algorithm also implements a preferential attachment scheme and therefore produces a power-law graph. The small first eigenvalues reported in [27] agree with our more general results in showing that it is possible to construct power-law graphs with arbitrarily small spectral gap. Furthermore, this observation holds regardless of the exponent in the power law.

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