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by

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COMPACTNESS OF A_r-SPIN EQUATIONS

HUIJUN FAN, TYLER JARVIS AND YONGBIN RUAN

Abstract. We introduce the W-spin structures on a Riemann surface Σ and give a precise definition to the corresponding W-spin equations for W being a quasi-homogeneous polynomial. When W is the A_r-potential, then they correspond to the r-spin structures and the r-spin equations considered by E. Witten [W2]. If the number of the Ramond marked points on Σ is at least 1, then Witten's lemma does not hold any more and the W-spin equations may have nontrivial solutions. An nontrivial solution of r-spin equation is given in this case. We demonstrate the "inner compactness" of the W-spin equations when W is one of the superpotentials: A_r, D_r, E_r or pure Neveu-Schwarz. Especially if W is A_r-potential, then the solution space of the r-spin equation is compact in suitable topology.

1. Introduction

Let $\mathcal{M}_{g,s}$ be the moduli space of complex Riemann surfaces of genus g and s marked points. Let $\overline{\mathcal{M}}_{g,s}$ be its Deligne-Mumford compactification. $\overline{\mathcal{M}}_{g,s}$ consists of stable curves of genus g and s marked points. It can be shown that $\overline{\mathcal{M}}_{g,s}$ is an smooth orbifold with complex dimension $3g-3+s$. Mumford [Mum] also introduced tantological cohomology classes asssociated to the universal curve $\mathbf{C}_{g,s} \to \overline{\mathcal{M}}_{g,s}$. For instance, let x_i be a marked point on Σ, then x_i has a cotangent bundle $T^*_x\Sigma$. When Σ varies in $\overline{\mathcal{M}}_{g,s}$, one gets a complex line bundle $L_i \to \overline{\mathcal{M}}_{g,s}$. Consider the following intersection numbers:

$$\langle \prod_{i=1}^{s} C_1(L_i)^{n_i}, \overline{\mathcal{M}}_{g,s}\rangle, \forall n_i \geq 0.$$

Witten [W1] conjectured these intersection numbers can be assembled to a potential function which satisfies the KdV hierarchy (i.e. the semiclassical limit of the KdV equation). Later this conjecture was proved by Kontsevich [K]. This provides an unexpected link between the algebraic geometry of these moduli spaces and integrable systems.

Based on the theory of $\overline{\mathcal{M}}_{g,s}$, people can construct many moduli theories and propose similar problems discussed as above. There are two ways to generalize the above moduli theory on $\overline{\mathcal{M}}_{g,s}$. Let V be a smooth projective variety or a symplectic manifold. The first generalization is to consider the moduli space $\overline{\mathcal{M}}_{g,s}(V)$, which consists of the stable maps from a Rimann surface Σ to V. Using the evaluation maps from $\overline{\mathcal{M}}_{g,s}(V)$ to V, one can define the Gromov-Witten invariants, which satisfy Manin and Konstevich's axiom system [KM]. This motivates another well-known conjecture by Eguchi-Hori-Xiong-Katz in quantum cohomology that the generating function of Gromov-Witten invariants satisfies a set of equations which form a virasoro algebra. This conjecture is commonly known as the *Virasoro Conjecture*.

The second less known generalization was proposed by Witten [W2] to study the moduli space $\overline{\mathcal{M}}_{g,s}^{\frac{1}{r}}$ of r-spin curves. Roughly speaking, an element $[\Sigma, L] \in \mathcal{M}_{g,s}^{\frac{1}{r}}$ is an automorphism class of tuple (Σ, L), where Σ is a smooth curve of s marked points, and L is the r-th root of the twisted canonical bundle $K_\Sigma(-\Sigma_{i=1}^s m_i \cdot p_i)$. If the degree $2g-2-\Sigma_1^s m_i$ is divisible by r, then L exists and there exactly r^{2g} isomorphism classes of L on Σ.

If we want to discuss the compactification $\overline{\mathcal{M}}_{g,s}^{\frac{1}{r},\mathbf{m}}$ of $\mathcal{M}_{g,s}^{\frac{1}{r},\mathbf{m}}$ $(\mathbf{m} = (m_1, \cdots, m_s))$, we have to consider the degeneracy of the line bundle L along a circle in Σ. This forces us to consider the sheaf $\mathcal{L}$ on Σ. There are exactly r possibilities for the degeneracy. According to the degeneracy behavior, all nodal points of Σ can be divided into two types:

- (1) if near the nodal point Q, $\mathcal{L}$ is a locally free sheaf, then Q is called a Ramond nodal point;
- (2) Otherwise the nodal point Q is called a Neveu-Schwarz point.

If Q is a Ramond nodal point, then the sheaf $\mathcal{L}$ can be obtained from the sheaf $\tilde{\mathcal{L}}$ over its normalization $\pi : \tilde{\Sigma} \to \Sigma$. Take a local coordinate $B_i = \{z_i \in \mathbb{C} | |z_i| < 1, z_i(Q_i) = 0\}, \forall i = 1, 2$, around the lifting points Q_i, then $\tilde{\mathcal{L}_i}$ is generated by $(\frac{dz_i}{z_i})^{\frac{1}{r}}$. Now L is obtained by a gluing $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\phi((\frac{dz_1}{z_1})^{\frac{1}{r}})^{\otimes r} = -\frac{dz_2}{z_2}$. If Q is NS nodal point, then there exist integers $(m_1(Q), m_2(Q))$ satisfying $0 \le m_1(Q), m_2(Q) \le r-2, m_1(Q) + m_2(Q) = r-2$ such that $\mathcal{L} = \pi_*(\oplus \mathcal{O}_{B_i}(z_i^{m_i} dz_i)^{\frac{1}{r}})$ around Q. $(m_1(Q), m_2(Q))$ is called the index of Q. If Q is of Ramond type, then we take $(m_1, m_2) = (-1, -1)$.

If $m \equiv m'(\mod r)$, we can construct a canonical isomorphism from $\overline{\mathcal{M}}_{g,s}^{\frac{1}{r},m}$ to $\overline{\mathcal{M}}_{g,s}^{\frac{1}{r},m'}$. So we can restrict m_i such that $-1 \le m_i \le r-2$.

Similarly we call the marked point P as a Ramond marked point if the twisted coefficient m of the divisor at P is -1. The other marked points are called the Neveu-Schwarz marked points.

After introducing an extra assistant notion called "coherent net", the second author [J1, J2] can prove

Theorem 1.1. $\overline{\mathcal{M}}_{g,s}^{\frac{1}{r},m}$ *is a smooth orbifold.*

In [JKV], Jarvis-Kimura-Vaitrob considered a stratum $\overline{\mathcal{M}}_{\Gamma}^{\frac{1}{r}}$ of $\overline{\mathcal{M}}_{g,s}^{\frac{1}{r},m}$ which consists of all the elements in $\overline{\mathcal{M}}_{g,s}^{\frac{1}{r},m}$ having the stable decorated graph Γ. Motivated by the construction of Gromov-Witten invariants from the moduli space of stable maps, they introduce axioms which must be satisfied by a cohomology class $C_{\Gamma}^{\frac{1}{r}}$ (called the *virtual cycle*) on the moduli space of r-spin curves $\overline{\mathcal{M}}_{\Gamma}^{\frac{1}{r}}$. These axioms will ensure to obtain a cohomological field theory (CohFT) of rank $r-1$ in the sense of Kontsevich and Manin [KM].

To construct the virtual cycle, they introduce the following condition:

JKV Condition*: Assume there is at most one Ramond marked point, i.e., all* $m_i \ge 0$ *except possible one* $m_j = -1$.

Under the JKV condition, they [JKV] proved the existence of the virtual cycle $C_{\Gamma}^{\frac{1}{r}}$ in cases $r = 2$ or genus $g = 0$. Further with the JKV condition, Mochizuki [Mo], Polishchuk and Vaintrob [PV] proved the existence in general case.

Let us describe roughly the construction in [Mo]. This method constructing $C_{\Gamma}^{\frac{1}{r}}$ was originally proposed by Witten [W2] even before one knew how to construct the moduli space $\overline{\mathcal{M}}_{g,s}^{\frac{1}{r}}$.

In each complex surface Σ, there is a canonical operator $\bar{\partial}_{\Sigma}$. This operator depends smoothly on the complex structure j_{Σ} on Σ. Now if we let $\bar{\partial}_{\Sigma}$ move on the moduli space $\mathcal{M}_{g,s}$, we then get a family of $\bar{\partial}$-operator parametrized by $\mathcal{M}_{g,s}$. Seeley-Singer [SS] can extend $\bar{\partial}_{\Sigma}$ continuously to the whole compactification space $\overline{\mathcal{M}}_{g,s}$ in the sense of graph norm. Then we can consider the Banach bundles and bundle maps as shown in the following figure:

Set $D = \frac{(r-2)(g-1)}{r} + \Sigma_{i=1}^{s} m_i$ and T to be an positive integer. Let F_{Σ}^{D+T} be a finite-dimensional space containing coker$(\bar{\partial}_{\Sigma})$, and let $E_{\Sigma}^{T} = \bar{\partial}_{\Sigma}^{-1}(F_{\Sigma})$. Now we have a pull-back bundle $\pi^*F \to E$. The "top-Chern class" called by Witten (i.e. $C_{g,s}^{\frac{1}{r}}$) $C_D(F,E) = \pi_*(C_{D+T}(\pi^*F))$. However this can't be done directly, since E is not a compact manifold. So Witten introduced a section $\omega : E \to \pi^*F$ such that the zero locus lies only on the zero section of $\pi : E \to \overline{\mathcal{M}}_{g,s}$. Such a section permits us to define a cohomology class $C_{D+T}(\pi^*F;\omega)$ in $H^*_{\text{cpt}}(E)$. Then we can define

$$(C_{g,s}^{\frac{1}{r}} :=) C_D(F,E;\omega) = \pi_*(C_{D+T}(\pi^*F;\omega)) \tag{1}$$

one should check the definition of this cohomology class is independent of the choices of E, F.

The section Witten chosen is

$$\omega = \bar{\partial}s + \bar{s}^{r-1} : \Omega^0(L) \to \Omega^{0,1}(L). \tag{2}$$

This section is not chosen arbitrarily, it has physical meaning as mentioned by Witten [W2]. To make the above definition meaningful, one should have the following identifications for which Witten only gave a very rough formulation:

$$\bar{L}^{\otimes r-1} \cong \bar{L}^{\otimes r} \otimes L \cong \bar{\omega}_{\Sigma} \otimes L \otimes \overline{\mathcal{O}(-\Sigma m_i \cdots p_i)} \hookrightarrow \bar{\omega}_{\Sigma} \otimes L. \tag{3}$$

The following Witten's lemma is essential to define the "top Chern class".

Lemma 1.2. *If there is no Ramond marked point in a nodal curve, then $\omega(s) = 0$ iff $s = 0$.*

So a natural problem arises: if the JKV condition does not hold, can one construct the virtual cycle $C_{\Gamma}^{\frac{1}{r}}$? On the other hand, Witten [W2] also asked if one can construct the analogous theories for the superpotentials E_r, D_r not only A_r superpotential.

In this paper, we attempt to generalize Witten's theory about r-spin curves to more generalized settings. In section 2, we will define the W-spin structures on an orbicurve, where $W \in \mathbb{C}[x_1, \cdots, x_t]$ is a quasi-homogeneous polynomial. If $W = x^r$ and forget the orbifold structure on the orbicurve, we recover the r-spin structure on a Riemann surface. We will give a precise definition to the following W-spin equation:

$$\bar{\partial} u_j + \frac{\bar{\partial} W(u_1, \cdots, u_t)}{\partial u_j} = 0, \forall j = 1, \cdots, t. \tag{4}$$

Note that if $W = x^r$, then the above W-spin equation is just the r-spin equation. If we let W be the D_r, E_r or pure Neveu-Schwarz potential, then we can get the corresponding spin equations. We also discuss some local properties of the W-spin equations.

An important thing is that the Witten's lemma does not hold when the number of Ramond marked points is at least 1. In this case, Witten's formulation of constructing the virtual cycle is ineffective, because the corresponding r-spin equation may really have solutions. This will be shown by an example in section 5. Hence to obtain the virtual cycle, it is natural to consider the moduli space which should includes the information of the solutions of r-spin equation. Hence we need to consider the compactness of the solution space of the r-spin equation.

The working spaces are the weighted sobolev spaces. In Section 3, we shall give the L^p estimate of the $\bar{\partial}$ operator in weighted sobolev spaces and prove it is a Fredholm operator under some mild restraints.

In Section 4, we will demonstrate the "inner compactness" of the solution spaces of the A_r, D_r, E_r-spin equations and pure Neveu-Schwarz spin equations. Let R be the sum of the residue of $W(u_1, \cdots, u_t)$ at each Ramond marked points. If $u_1, \cdots, u_t$ are the nontrivial solutions of the W-spin equation for W being A_r, D_r, E_r or pure Neveu-Schwarz potential, then R is a non-zero real number. The main conclusion is that the space of solution spaces satisfying $R \leq C < \infty$ is compact in L^p_1 topology for some $p \geq 2$.

In section 5 we will concentrate on the easiest equation, A_r-spin equation. We will prove if adding the singular solutions to the solution space of regular solutions, the whole solution space is compact in suitable topology.

2. Spin structures on orbicurves and spin equations

In this section we will introduce the W-spin structures on orbicurves, where $W \in \mathbb{C}[x_1, \cdots, x_t]$ is a non-degenerate quasi-homogeneous polynomial. By means of W-spin structures one can define the W-spin equations on orbicurves.

Let $W \in \mathbb{C}[x_1, \cdots, x_t]$ be a quasi-homogeneous polynomial, i,e, there exist degrees $d, k_1, \cdots, k_t \in \mathbf{Z}^{>0}$ such that for any $\lambda \in \mathbb{C}^*$

$$W(\lambda^{k_1} x_1, \cdots, \lambda^{k_t} x_t) = \lambda^d W(x_1, \cdots, x_t).$$

Definition 2.1. W is called nondegenerate if

(1) the fractional degrees $q_i = \frac{k_i}{d}$ are uniquely determined by W; and
(2) the hypersurface defined by W in weighted projective space is non-singular, or equivalently, the affine hypersurface defined by W has an isolated singularity at the origin.

From now on, we always assume the quasi-homogeneous polynomial W is non-degenerate and the corresponding degrees $d, k_1, \cdots, k_t$ of W are the *least* positive integer degrees.

Definition 2.2. We say W is pure Neveu-Schwarz (or pure NS) if for every $i \in \{1, \cdots, t\}$ there is a positive integer n_i such that W contains a monomial of the form x_i^n.

So pure NS implies that $q_i = \frac{1}{n_i}$. Actually every non-degenerate pure NS, quasi-homogeneous polynomial W is a deformation of a Brieskorn singularity:

$$W = x_1^{n_1} + \cdots + x_t^{n_t} + V,$$

where $V \in \mathbb{C}[x_1, \cdots, x_t]$ is itself a suitably quasi-homogeneous polynomial with $\deg(x_i) = \frac{1}{n_i}$ for every i.

Example 2.3. If $W(x) = x^r; W(x,y) = x^r + xy^2$, then the two polynomials are called A_r and D_r potentials respectively.

Lemma 2.4. *If W is non-degenerate, then the group*

$$H := \{(\alpha_1, \dots, \alpha_t) \in (\mathbb{C}^*)^t | W(\alpha_1 x_1, \dots, \alpha_t x_t) = W(x_1, \dots, x_t)\}$$

of diagonal symmetries of W is finite. In particular, we have

$$H \subseteq \mu_{d/k_1} \times \cdots \times \mu_{d/k_t} \cong k_1\mathbb{Z}/d \times \dots, \times k_t\mathbb{Z}/d$$

where μ_l is the group of lth roots of unity.

Proof. First write $W = \sum_{j=1}^s W_j$ with $W_j = c_j \prod x_l^{b_{l,j}}$ and with $c_j \neq 0$. The uniqueness of the fractional degrees is equivalent to saying that the matrix $B = (b_{l,j})$ has rank t. We may as well assume that B is invertible. Now write $h = (h_1, \dots, h_t) \in H$, as $h_j = exp(u_j + v_j i)$ for $u_j \in \mathbf{R}$ uniquely determined and $v_j \in \mathbf{R}$ determined up to integral multiple of $2\pi i$. The equations $W(h_1 x_1, \dots, h_t x_t) = W(x_1, \dots, x_t)$ can now be written as $B(u + vi) \equiv 0 (mod 2\pi i \mathbf{Z})$. Invertibility of B shows that $u_l = 0$ for all l–thus H lives in $U(1)^t$, and a straightforward argument shows that the number of solutions $(mod 2\pi i \mathbf{Z})$ to the equation $B(vi) \equiv 0 (mod 2\pi i \mathbf{Z})$ is also finite. □

W-spin structures on smooth orbicurves. Let $(\tilde{\Sigma}, \mathbf{z}, \mathbf{m})$ be a smooth orbicurve (or orbifold Riemann surface) as defined in [CR2], i.e., $(\tilde{\Sigma}, \mathbf{z}, \mathbf{m})$ is a Rimann surface Σ with marked points $\mathbf{z} = \{z_i\}$ having orbifold structure near each marked point z_i given by a faithful action of $\mathbf{Z}/m_i$. In another word, a neighborhood of each marked point is uniformized by the branched covering map $z \to z^{m_i}$. Let $\varrho : \tilde{\Sigma} \to \Sigma$ the natural projection to the coarse Riemann surface Σ.

A line bundle L on Σ can be uniquely lifted to an orbifold line bundle on $\tilde{\Sigma}$. We denote by the same L the lifting.

Definition 2.5. Let K be the canonical bundle of Σ, and let

$$K_{log} := K \otimes \mathcal{O}(z_1) \otimes \cdots \otimes \mathcal{O}(z_k)$$

be the *log-canonical bundle*. K_{log} can be thought as canonical bundle of the punctured Riemann surface $\Sigma - \{z_1, \cdots, z_k\}$. Suppose that $L_1, \dots, L_t$ are orbifold line bundles on $\tilde{\Sigma}$ with isomorphisms $\varphi_j : W_j(L_1, \dots, L_t) \xrightarrow{\sim} K_{\log}$ where by $W_j(L_1, \dots, L_t)$ we mean the jth monomial of W in L_i

$$W_j(L_1, \dots, L_t) = L_1^{\otimes b_{1j}} \otimes \dots, \otimes L_t^{\otimes b_{tj}},$$

and where K_{log} is identified with its pull-back to $\tilde{\Sigma}$. The tuple $(L_1, \ldots, L_t, \varphi_1, \ldots, \varphi_s)$ is called a *W-spin structure*.

Definition 2.6. Suppose that the chart of $\tilde{\Sigma}$ at an orbifold point z_i is $D/(\mathbf{Z}/m)$ with action $e^{\frac{2\pi i}{m}}(z) = e^{\frac{2\pi i}{m}} z$. Suppose that the local trivialization of an orbifold line bundle L is $(D \times \mathbf{C})/(\mathbf{Z}/m)$ with the action

$$e^{\frac{2\pi i}{m}}(z, w) = (e^{\frac{2\pi i}{m}} z, e^{\frac{2\pi i v}{m}} w). \tag{5}$$

When $v = 0$, we say that L is *Ramond* at z_i. When $v > 0$, we say L is *Neveu-Schwarz (NS)* at z_i.

A W-spin structure $(L_1, \ldots, L_t, \varphi_1, \ldots, \varphi_s)$ is called *Ramond* at the point z_i if the group element $h = (\exp(2\pi i v_1/m), \ldots, \exp(2\pi i v_t/m))$ defined by the orbifold action on the line bundles L_j at z_i is Ramond.

Remark 2.7. If L is an orbifold line bundle on a smooth orbifold Riemann surface $\tilde{\Sigma}$, then the sheaf of local invariant sections of L is locally free of rank one, and hence dual to a unique orbifold line bundle $|L|$ on Σ. We also denote $|L|$ by $\varrho_* L$, and it corresponds to the desingularization of L [CR1](Prop 4.1.2). It can be constructed as follows.

We keep the local trivialization at other places and change it at the orbifold point z_i by a $\mathbf{Z}/m$-equivariant map $\Psi : (D - \{0\}) \times \mathbf{C} \to (D - \{0\}) \times \mathbf{C}$ by

$$(z, w) \to (z^m, z^{-v} w) \tag{6}$$

where $\mathbf{Z}/m$ acts trivially on the second $(D - \{0\}) \times \mathbf{C}$. Then, we extend $L|_{((D-\{0\})\times \mathbf{C})}$ to a smooth holomorphic line bundle over Σ by the second trivialization. Since $\mathbf{Z}/m$ acts trivially, this gives a line bundle over Σ, which is $|L|$. Note that if L is Ramond at z_i, then $|L| = L$ locally. When L is Neveu-Schwarz at z_i, then $|L|$ will differ from L.

Example 2.8. A smooth orbifold Riemann surface $\tilde{\Sigma} = (\Sigma, \mathbf{z}, \mathbf{m})$ has a natural orbifold canonical bundle $K_{\tilde{\Sigma}}$, defined as the top wedge product of its (orbifold) cotangent bundle. The desingularization is related to the canonical bundle of Σ by

$$|K_{\tilde{\Sigma}}| = K_\Sigma \otimes_i \mathcal{O}(-(m_i - 1) z_i).$$

On the other hand, the desingularization of the log-canonical bundle of $\tilde{\Sigma}$ is again the log-canonical bundle of Σ, since K_{log} is Ramond at every marked point.

Next we study the sections. Suppose that s is a section of $|L|$ having local representative $g(u)$. Then, $(z, z^v g(z^m))$ is a local section of L. Therefore, we obtain a section $\varrho^*(s) \in \Omega^0(L)$ which equals s away from orbifold points under the identification given by Equation 6. It is clear that if s is holomorphic, so is $\varrho^*(s)$. If we start from an analytic section of L, we can reverse the above process to obtain a section of $|L|$. In particular, L and $|L|$ have isomorphic spaces of holomorphic sections. In the same way, there is a map $\varrho^* : \Omega^{0,1}(|L|) \to \Omega^{0,1}(L)$, where $\Omega^{0,1}(L)$ is the space of orbifold $(0, 1)$-forms with values in L. Suppose that $g(u) d\bar{u}$ is a local representative of a section of $t \in \Omega^{0,1}(|L|)$. Then $\varrho^*(t)$ has a local representative $z^v g(z^m) m \bar{z}^{m-1} d\bar{z}$. Moreover, ϱ induces an isomorphism from $H^1(|L|) \to H^1(L)$.

Suppose now that $L^r \cong K_{log}$ with action of $\mathbf{Z}/m$ on L as in Equation 5. Since K_{log} is Ramond at every marked point, we must have $rv = lm$ for some l. The integer l is non-zero precisely when v is, and thus L is Neveu-Schwarz at z_i if and only if $l > 0$. Moreover, we have $v < m$, so $l < r$, and of course $\frac{v}{m} = \frac{l}{r}$. Suppose that $s \in \Omega^0(|L|)$ with local representative $g(u)$. Then, s^r has local representative $z^{rv} g^r(z^m) = z^{ml} g^r(z^m) = u^l g^r(u)$. Hence, $s^r \in \Omega^0(K_{log} \otimes \mathcal{O}((-l_i)z_i)$, and thus when $l_i > 0$, or equivalently, when L is Neveu-Schwarz at every z_i, we have $s^r \in \Omega^0(K)$.

Remark 2.9. More generally, if $L^r \cong K_{log}$ on a smooth orbicurve with action of the local group on L defined by l_i (as above) at each marked point, then we have

$$(\varrho_* L)^r = |L|^r = K_{log} \otimes \mathcal{O}((-l_i)z_i),$$

locally, near z_i.

Proposition 2.10. *Let $(L_1, \ldots, L_t)$ be a W-spin structure on a smooth orbicurve. And suppose that the local group G_z of z acts on L_j by $exp(2\pi i/m)(z, w_j) = (exp(2\pi i/m)z, exp(2\pi i v_j/m)w_j)$. There is a unique element $h \in H$ such that $exp(2\pi i v_j/m) = h_j = ext(2\pi i a_j(h)) = exp(2\pi i c_j(h)/d)$ for every j. Moreover, for every monomial W_i we have*

$$W_i(|L_1|, \ldots, |L_t|) \cong K_{\log} \otimes \mathcal{O}(-\sum_{j=1}^{t} b_{ij} a_j(h) z)$$

near the point z. Letting h_l define the action of the local group G_{z_l} near z_l we have the global isomorphism

$$\begin{aligned} W_i(|L_1|), \ldots, |L_t|) &\cong K_{log} \otimes \mathcal{O}(-\sum_{l=1}^{k}\sum_{j=1}^{t} b_{ij} a_j(h_l) z_l) \\ &\cong K_\Sigma \otimes \mathcal{O}(-\sum_{l=1}^{k}\sum_{j=1}^{t} b_{ij}(a_j(h_l) - q_j) z_l). \end{aligned}$$

Proof. The existence and uniqueness of $h \in H$ is a straightforward generalization of the argument for $W = W_{A_{r-1}}$, given above. The rest is an immediate consequence of the description of h as $h = (exp(2\pi i a_1(h), \ldots, exp(2\pi i a_t(h))$ and the description of $|L_j|$ in terms of the action of the local group G_z given above. $\square$

W-spin equations. Let $D = -\sum_{l=1}^{k}\sum_{j=1}^{t} b_{ij}(a_j(h_l) - q_j)z_l$ be the divisor, then there is a canonical section $s_0 \in H^0(\Sigma, \mathcal{O}[D])$ with the divisor D. This section provides the identification

$$K_\Sigma \otimes \mathcal{O}(D) \overset{s_0^{-1}}{\cong} K_\Sigma(D).$$

Take a coordinate chart $\{U_\alpha\}$ of Σ, and let e_j^α be a holomorphic base of the line bundle $|L_j|$ in the chart U_α. For simplicity, we will omit the upper index α if the discussed chart is known. Now near the marked points z_l the metric defined on $K_\Sigma(D)$ induces the singular metric in each line bundle $|L_j|$ such that the singular metric near z_l is

$$|e_j|_s = |z|^{a_j(h_l) - q_j} \ q_j = \frac{k_j}{d}, \ 0 \le a_j(h_l) \le 1 - q_j.$$

As before, we assume that $W = \Sigma W_i = \Sigma_i(c_i \prod_l x_l^{b_{il}})$. Let $u_j = \tilde{u}_j e_j$, then it is easy to see that

$$\overline{\frac{\partial W}{\partial u_j}} \in \bar{K}_\Sigma \otimes \overline{|L_j|}^{-1}.$$

Define an isomorphism $I_1 : \Omega(\Sigma, \overline{|L_j|}^{-1} \otimes \Lambda^{0,1}) \to \Omega(\Sigma, |L_j| \otimes \Lambda^{0,1})$ such that for a section $v = \tilde{v} e'_j$ there is

$$I_1(\tilde{v}\bar{e}'_j \otimes d\bar{z}) = \tilde{v}|e'_j|^2 e_j \otimes d\bar{z},$$

where e'_j is the holomorphic base of $|L_j|^{-1}$ such that $e'_j \cdot e_j = dz$.

It is obvious that I_1 is the unique metric-preserving isomorphism between the corresponding two spaces and it is independent of the choice of the local charts.

Since $I_1(\overline{\frac{\partial W}{\partial u_j}}) \in \overline{K}_\Sigma \otimes |L_j|$, the W-spin equation is defined below as the first order system of the sections $u_1, \cdots, u_t$:

$$\bar{\partial} u_j + I_1(\overline{\frac{\partial W}{\partial u_j}}) = 0, \forall j = 1, \cdots, t.$$

Remark 2.11. The de-singularization of the orbifold line bundle L_j induces isomorphisms $\varrho_j : \Omega^0(|L_j|) \to \Omega^0(L_j)$ and $\varrho^*_j : \Omega^{0,1}(|L_j|) \to \Omega^{0,1}(L_j)$. It is easy to see that ϱ^* commute with $\bar{\partial}$ and $\frac{\bar{W}}{\partial \bar{u}_j}$, hence the above W-spin equation can be regarded also as equations defined on orbifold. However we study the W-spin equations in the resolution line bundel $|L_j|$.

We can define the weighted sobolev spaces $L^p_1(\Sigma, |L_j|)$ and $L^p(\Sigma, |L_j| \otimes \Lambda^{0,1})$. Their norms are defined as:

- if $u \in L^p_1(\Sigma, |L_i|)$, then
$$||u||^p_{1,p} := \int |u|^p_s + |\partial u|^p_s + |\bar{\partial} u|^p_s;$$
- if $u \in L^p(\Sigma, |L_i| \otimes \Lambda^{0,1})$, then
$$||u||^p_p = \int |u|^p_s.$$

Definition 2.12. The sections $(u_1, \cdots, u_t)$ are said to be the *regular* solutions of the W-spin equations

$$\bar{\partial} u_j + I_1(\overline{\frac{\partial W}{\partial \bar{u}_j}}) = 0, \tag{7}$$

if for each j, $u_j \in L^2_1(\Sigma, |L_j|)$, $I_1(\overline{\frac{\partial W}{\partial u_j}}) \in L^2(\Sigma, |L_j| \otimes \Lambda^{0,1})$ and $(u_1, \cdots, u_t)$ satisfy the W-spin equations almost everywhere.

The spin equation $\bar{\partial} u_j + I_1(\overline{\frac{\partial W}{\partial \bar{u}_j}}) = 0$ has different properties near the Ramond marked points and Neveu-Schwarz marked points. We discuss them respectively.

(1) Near Ramond marked points

Let $u_j = \tilde{u}_j e_j$ in a local coordinate near a Ramond marked point z_l, then this can be expressed by

$$\frac{\bar{\partial}\tilde{u}}{\partial \bar{z}} + \overline{\frac{\partial W(\tilde{u}_1, \cdots, \tilde{u}_t)}{\partial \tilde{u}_j} \frac{1}{z}} |e'_j|^2 = 0 \tag{8}$$

In polar coordinate, this equation can be rewritten as

$$\frac{1}{2} r (\frac{\partial}{\partial r} + \sqrt{-1} \frac{1}{r} \frac{\partial}{\partial \theta}) \tilde{u}_j + \overline{\frac{\partial W(\tilde{u}_1, \cdots, \tilde{u}_t)}{\partial \tilde{u}_j}} r^{2q_j} = 0. \tag{9}$$

Example 2.13. (**A local solution of r-spin equation near the Ramond marked points**)

Near a Ramond marked point, the equation (9) becomes

$$\rho(\frac{\partial}{\partial\rho}+\sqrt{-1}\frac{1}{\rho}\frac{\partial}{\partial\theta})\tilde{u}+2r\bar{\tilde{u}}^{r-1}\rho^{\frac{2}{r}}=0.$$

If we assume further that u is a real function and depends only on the radius ρ, then we have

$$\frac{d\tilde{u}}{d\rho}=-2r\tilde{u}^{r-1}\rho^{\frac{2}{r}-1}.$$

Now a special local solution is given by

$$\tilde{u}=(r^2(r-2)\rho^{\frac{2}{r}}+C)^{-\frac{1}{r-2}},$$

where C is a positive constant.

An easy computation shows that $\tilde{u}\in L^p_1$ if and only if $p<\frac{2}{1-\frac{1}{r}}$.

If we set $\phi_j=\tilde{u}_j z^{-q_j}$, then an easy computation shows that the equation (9) of $\tilde{u}_1,\cdots,\tilde{u}_t$ turns to the simple equation of $\phi_1,\cdots,\phi_t$ below:

$$\bar{\partial}\phi_j+\overline{\frac{\partial W}{\partial\phi_j}}=0,\forall j=1,\cdots,t \tag{10}$$

(2) Near the Neveu-Schwarz points

Let $u_j=\tilde{u}_j e_j$ near a Neveu-Schwarz point z_l. Then the W-spin equation becomes

$$\frac{\bar{\partial}\tilde{u}_i}{\partial\bar{z}}+\Sigma_j\overline{\frac{\partial W_j(\tilde{u}_1,\cdots,\tilde{u}_t)}{\partial\tilde{u}_i}z^{\Sigma_{s=1}^t b_{js}(a_s(h_l)-q_s)}}|e_i'|^2=0 \tag{11}$$

3. $\bar{\partial}$ operator in Weighted Sobolev Spaces

In this section, we will give the L^p estimate of the $\bar{\partial}$ operator and show that it is a Fredholm operator under some mild assumption.

Firstly for the convenience of the reader, we list the standard representations of the $\bar{\partial}$ operator in different coordinate systems.

(1) in (x,y)-coordinate

$$\begin{aligned}
&z=x+iy\\
&\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-\sqrt{-1}\frac{\partial}{\partial y}),\ \frac{\bar{\partial}}{\partial\bar{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+\sqrt{-1}\frac{\partial}{\partial y})\\
&dz=dx+\sqrt{-1}dy,\ d\bar{z}=dx-\sqrt{-1}dy
\end{aligned}$$

(2) in (r,θ)-coordinate

$$\begin{aligned}
&z=re^{i\theta}\\
&\frac{\partial}{\partial z}=\frac{1}{2}e^{-i\theta}(\frac{\partial}{\partial r}-\frac{\sqrt{-1}}{r}\frac{\partial}{\partial\theta}),\ \frac{\bar{\partial}}{\partial\bar{z}}=\frac{1}{2}e^{i\theta}(\frac{\partial}{\partial r}+\frac{\sqrt{-1}}{r}\frac{\partial}{\partial\theta})\\
&dz=e^{i\theta}(dr+\sqrt{-1}rd\theta),\ d\bar{z}=e^{-i\theta}(dr-\sqrt{-1}rd\theta)
\end{aligned}$$

(3) in cylindrical coordinate (t,θ), where $r=e^{-t}$:

$$\frac{\partial}{\partial z}=-\frac{1}{2}e^{t-i\theta}(\frac{\partial}{\partial t}+\sqrt{-1}\frac{\partial}{\partial\theta}),\ \frac{\bar\partial}{\partial\bar z}=-\frac{1}{2}e^{t+i\theta}(\frac{\partial}{\partial t}-\sqrt{-1}\frac{\partial}{\partial\theta})$$
$$dz=-e^{i\theta-t}(dt-\sqrt{-1}d\theta),\ d\bar z=-e^{-i\theta-t}(dt+\sqrt{-1}d\theta)$$
$$\partial=\frac{1}{2}(\frac{\partial}{\partial t}+\sqrt{-1}\frac{\partial}{\partial\theta})(dt-\sqrt{-1}d\theta)$$
$$\bar\partial=\frac{1}{2}(\frac{\partial}{\partial t}-\sqrt{-1}\frac{\partial}{\partial\theta})(dt+\sqrt{-1}d\theta)$$

Theorem 3.1. *If* $\frac{2}{1+2q_j}<p<\frac{2}{q_j},p\neq\frac{2}{1+q_j}$, *then* $\bar\partial: L^p_1(\Sigma,|L_j|)\to L^p(\Sigma,|L_j|\otimes\Lambda^{0,1})$ *is a Fredholm operator*

Proof. **Step 1 L^p Estimate near a marked point z_l**

Firstly we assume that $a_j(h_l)-q_j\neq 0$, since the L^p estimate under the $a_j(h_l)-q_j=0$ case corresponds to the classical L^p estimate without weight.

Let $\bar\partial u=f$, where $f\in C^\infty(B_1(z_l),|L_j|\otimes\Lambda^{0,1})$. Choose cylindrical coordinate $(r=e^{-t},\theta)$. Let

$$u=\tilde u e^{-t}e_j$$

$$f=\tilde f e_j\otimes d\bar z=-\tilde f e^{-i\theta-t}e_j\otimes(dt+\sqrt{-1}d\theta)$$

The equation becomes

$$\bar\partial(\tilde u e^{-t})=\frac{1}{2}(\frac{\partial(\tilde u e^{-t})}{\partial t}-\sqrt{-1}\frac{\partial(\tilde u e^{-t})}{\partial\theta})(dt+\sqrt{-1}d\theta)=-\tilde f e^{-i\theta-t},$$

i.e.,

$$\frac{\partial\tilde u}{\partial t}-\tilde u-\sqrt{-1}\frac{\partial\tilde u}{\partial\theta}=-2\tilde f e^{-i\theta} \tag{12}$$

Since $\int|f|_s^p<\infty$,this is equivalent to

$$\int_0^\infty|\tilde f|^p e^{-p(a_j(h_l)-q_j)t-2t}dtd\theta<\infty$$

Let $a_{j,l}=a_j(h_l)-q_j+\frac{2}{p}$, then the integral becomes

$$\int|\tilde f e^{-a_{j,l}t}|^p<\infty.$$

Let $\hat u=\tilde u e^{-a_{j,l}t},\hat f=\tilde f e^{-a_{j,l}t-i\theta}$. Then by (12) the equation of $\hat u$ is

$$\frac{\partial\hat u}{\partial t}=\frac{\partial\tilde u}{\partial t}e^{-a_{j,l}t}-a_{j,l}e^{-a_{j,l}t}\tilde u=(\tilde u+\sqrt{-1}\frac{\partial\tilde u}{\partial\theta}-2\tilde f e^{-i\theta})e^{-a_{j,l}t}-a_{j,l}\hat u=(1-a_{j,l})\hat u+\sqrt{-1}\frac{\partial\hat u}{\partial\theta}-2\hat f,$$

i.e.,

$$\frac{\partial\hat u}{\partial t}+L_\theta\hat u=-2\hat f,\ t\in[0,\infty) \tag{13}$$

where $L_\theta=-\sqrt{-1}\frac{\partial}{\partial\theta}-(1-a_{j,l})$.

A special solution u_s

We seek for a special solution u_s satisfying the inhomogeneous $\bar\partial$ equation.

Extend $\hat{f}$ symmetrically to $(-\infty,\infty)$ and we get an equation of (13) defined in the real line.

Since the spectrum of $-\sqrt{-1}\frac{\partial}{\partial\theta}$ is the set $\mathbb{Z}$, and the eigenspace is $\{\oplus_n e^{in\theta}, n\in\mathbb{Z}\}$, the spectrum of L_θ is $\{\lambda_n = n + a_{j,l}, n\in\mathbb{Z}\}$. Note that

$$-q_j + \frac{2}{p} \le a_{j,l} \le 1 - q_j + \frac{2}{p}$$

and $1 < p < \infty$, we have the restriction

$$-q_j \le a_{j,l} \le 3 - q_j.$$

The possible integer that $a_{j,l}$ can achieve is $0,1,2$. Since $a_j(h_l) - q_j = sq_j$ for some integer s satisfying $-1 \le s \le [\frac{1}{q_j}] - 1$, further analysis shows if

$$p \ne \frac{2}{q_j}, \frac{2}{1-sq_j}, \frac{2}{2-sq_j} \tag{14}$$

for integer s in the interval $[-1, [\frac{1}{q_j}] - 1]$, then 0 is not a spectrum point (since we exclude $a_{j,l} = 0$ case at the beginning of the Step 1) and L_θ is an invertible operator on S^1. Therefore under the condition (14), (13) has a unique bounded solution $\hat{u}$. If $-2\hat{f} = \Sigma_n \rho_{\lambda_n}(t) e^{i(n+1)\theta}$, then $\hat{u}$ has the following formula

$$\hat{u}(t,\theta) = -\Sigma_{\lambda_n<0} e^{\lambda_n t}\int_t^\infty e^{-\lambda_n\tau}\rho_{\lambda_n}(\tau)d\tau\cdot e^{i(n+1)\theta} + \Sigma_{\lambda_n>0} e^{-\lambda_n t}\int_{-\infty}^t e^{\lambda_n\tau}\rho_{\lambda_n}(\tau)d\tau\cdot e^{i(n+1)\theta}.$$

We denote the corresponding solution of the $\bar{\partial}$ equation as $u_s := Q_s \circ f$.

Now we can get the estimate from [D], Lemma 3.22,

$$||\hat{u}||_p \le C||\hat{f}||_p, \tag{15}$$

where C is a constant depending on p, q_j, L_j, z_l.

From the inequality (15), we have

$$\int_{-\infty}^\infty |\tilde{u}e^{-a_{j,l}t}|^p dt d\theta \le C\int_{-\infty}^\infty |\tilde{f}e^{-a_{j,l}t - i\theta}|^p.$$

So

$$\int_{-\infty}^\infty |\tilde{u}e^{-t}|^p e^{tp} e^{-p(a_j(h_l)-q_j)t-2t} dt d\theta \le C\int_0^\infty |\tilde{f}|^p e^{-p(a_j(h_l)-q_j)t-2t} dt d\theta,$$

which induces

$$\int_{B_1(0)} |\frac{u_s}{z}|_s^p |dz d\bar{z}| \le C\int_{B_1(0)} |f|_s^p |dz d\bar{z}|. \tag{16}$$

Let $D = \frac{\partial}{\partial t} + L_\theta$. Let $B_n = S^1 \times [n, n+1]$, for $n = 0,1,2\cdots$, and let B_n^+ is a band slightly bigger than B_n. Since D is a first order elliptic operator, we have the classical L^p estimate:

$$(\int_{B_n} |\partial_t\hat{u}|^p + |\partial_\theta\hat{u}|^p)^{\frac{1}{p}} \le C\left((\int_{B_n^+} |D\hat{u}|^p)^{\frac{1}{p}} + (\int_{B_n^+} |\hat{u}|^p)^{\frac{1}{p}}\right) \le C\left((\int_{B_n^+} |\hat{f}|^p)^{\frac{1}{p}} + (\int_{B_n^+} |\hat{u}|^p)^{\frac{1}{p}}\right)$$

Summing over n, we get

$$(\int_{S^1\times(0,\infty)} |\partial_t\hat{u}|^p + |\partial_\theta\hat{u}|^p)^{\frac{1}{p}} \le 2C(||\hat{f}||_{L^p(S^1\times(-1,\infty))} + ||\hat{u}||_{L^p(S^1\times(-1,\infty))}) \le C||\hat{f}||_{L^p(S^1\times(-1,\infty))}.$$

This plus (15) gives

$$(\int_{B_1} |\partial u_s|_s^p)^{\frac{1}{p}} \leq C(\int_{B_1^+} |f|_s^p)^{\frac{1}{p}}, \tag{17}$$

where B_1^+ is a neighborhood of z_l slightly bigger than B_1.

For the estimate of L^p norm of u_s, we have

$$\int |u_s|_s^p = \int_{S^1\times(0,\infty)} |\tilde{u}e^{-t}|^p e^{-pa_{j,l}t} \leq \int_{S^1\times(0,\infty)} |\tilde{u}|^p e^{-pa_{j,l}t} = \int |\hat{u}|^p \leq C\int |f|_s^p. \tag{18}$$

Combining (17) and (18), we obtain

$$||u_s||_{L_1^p(B_1)} \leq C||f||_{L^p(B_1^+)} \tag{19}$$

Now we apply the ordinary sobolev embedding theorem to the function $u_s r^a$, where $a = (a_j(h_l) - q_j)$ to get the weighted sobolev embedding inequalities. We use $||\cdot||_{o,k,p}$ to represent the ordinary $W^{k,p}$ norm.

If $p \leq 2$, then $\forall 1 < q < \frac{2p}{2-p}$,

$$\begin{aligned}
&||u_s r^a||_{o,q} \leq C||u_s r^a||_{o,1,p}\\
&\leq C(\int |\partial_r(u_s r^a)|^p + \frac{1}{r^p}|\frac{\partial}{\partial\theta}(u_s r^a)|^p + |u_s r^a|^p)^{\frac{1}{p}}\\
&= C(\int (|\partial_r u_s|^p + \frac{1}{r^p}|\partial_\theta u_s|^p + |\frac{u_s}{r}|^p + |u_s|^p) r^{ap})^{\frac{1}{p}}\\
&\leq C(\int (|\partial_r u_s|^p + \frac{1}{r^p}|\partial_\theta u_s|^p) r^{ap} + |u_s|^p r^{ap})^{\frac{1}{p}} = C||u_s||_{1,p},
\end{aligned}$$

where the second inequality comes from the relation (16). Especially, when $p = 2$, we have for any $1 < q < \infty$,

$$||u_s||_q \leq C||u_s||_{1,2}.$$

If $p > 2$, by similar argument we have

$$|||u_s|_s||_{C^\alpha} \leq C||u_s||_{1,p},$$

where $0 < \alpha < 1 - \frac{2}{p}$.

In summary, we have

Lemma 3.2. *If $f \in L^p(B_1(0), |L_i| \otimes \Lambda^{0,1})$ for p satisfying the choice (14), then the special solution $u_s = Q_s \circ f$ satisfies the following estimate:*

(1) *if $1 < p < \infty$,*

$$||\frac{u_s}{z}||_{p;B_1(0)} \leq C||f||_{p;B_1(0)} \tag{20}$$

(2) *if $1 < p \leq 2$, and $1 < q < \frac{2p}{2-p}$, then*

$$||u_s||_{q;B_1(0)} \leq C||u_s||_{1,p;B_1(0)} \leq C||f||_{p;B_1(0)}; \tag{21}$$

(3) *if $p > 2$, and $0 < \alpha < 1 - \frac{2}{p}$, then*

$$||u_s r^a||_{C^\alpha(B_1(0))} \leq C||u_s||_{1,p;B_1(0)} \leq C||f||_{p;B_1(0)}, \tag{22}$$

where $a = a_{j,l} - q_j$.

Estimate of the homogeneous solution

Let $\bar\partial u = 0$ in $B_1(0)$. We have the inner estimate

$$||u||_{o,k,s;B_1(0)} \le C||u||_{o,s:B_1^+(0)},$$

where $B_1^+(0)$ is a ball containing virtually $B_1(0)$.

Let $p > \frac{2}{1+2q_j}$. Since the function $\frac{2s}{2-s(1-2q_j)}$ of s is monotone increasing near $s=1$, there exits $s>1$ such that

$$s < \frac{2s}{2-(1-2q_j)s} < p. \tag{23}$$

Let $a = a_j(h_l) - q_j$, then $-q_j \le a \le 1-2q_j$. This and the above inequality induce

$$\frac{2s}{2+q_js} \le \frac{2s}{2-as} \le \frac{2s}{2-(1-2q_j)s} < p, \tag{24}$$

i.e.,

$$-\frac{pas}{p-s} > -2. \tag{25}$$

Now

$$\begin{aligned} ||u||^s_{o,s;B_1^+(0)} &= \int_{B_1^+(0)} |u|^s|z|^{sa}|z|^{-sa} \\ &\le (\int |u|^p|z|^{ap})^{\frac{s}{p}}(\int |z|^{\frac{-sap}{p-s}})^{\frac{p-s}{p}} \end{aligned}$$

$$\le C(\int |u|^p|z|^{ap})^{\frac{s}{p}},$$

Where the inequality (25) ensures the integrability of

$$\int |z|^{\frac{-sap}{p-s}}.$$

Therefore for any $k \ge 0$, we have

$$||u||_{o,k,s;B_1(0)} \le C||u||_{p;B_1^+(0)} \tag{26}$$

Actually we can have a refined inequality

$$||u||_{o,k,s;B_1(0)} \le C||u||_{p;B_1^+(0)\setminus B_{\frac{1}{2}}(0)}, \tag{27}$$

Since $|u|$ is a subharmonic function and we can use the maximum principle.

By sobolev embedding theorem, we have

$$||u||_{C^k(B_1(0))} \le C||u||_{p;B_1^+(0)\setminus B_{\frac{1}{2}}(0)}.$$

Lemma 3.3. *Let $\bar\partial u = 0$ and $u \in L^p(B_1^+(0))$. We have the estimate:*

(1) *for any $k \ge 0, 1 < q < \infty, \frac{2}{1+2q_j} < p < \infty$, there exists C such that*

$$||u||_{o,k,q;B_1(0)} \le C||u||_{p;B_1^+(0)\setminus B_{\frac{1}{2}}(0)}.$$

(2) *if $a \ge 0$, then for any $k \ge 0, 1 < q < \infty, \frac{2}{1+2q_j} < p < \infty$, there exists C such that*

$$||u||_{k,q;B_1(0)} \le C||u||_{p;B_1^+(0)\setminus B_{\frac{1}{2}}(0)}.$$

(3) *if $a=-q_j$, then for any $k\geq 0, 1<q<\frac{2}{q_j}$ and $\frac{2}{1+2q_j}<p<\infty$, there exists C such that the above inequality holds.*

Combining the estimate about the inhomogeneous solution and the homogeneous solutions, one has

Lemma 3.4. *Let $\bar\partial u = f$ in $B_1^+(0)$, where $u,f\in L^p$. Then $u\in L_1^p$ and the inequality*

$$||u||_{1,p;B_1(0)} \leq C(||u||_{p,B_1^+(0)\backslash B_{\frac12}(0)} + ||f||_{p,B_1^+(0)}) \tag{28}$$

holds if the following two conditions are satisfied:

- *if $p\neq \frac{2}{q_j}, \frac{2}{1-sq_j}, \frac{2}{2-sq_j}$ for integer s in the interval $[-1,[\frac{1}{q_j}]-1]$*
- *either $a\geq 0$, $\frac{2}{1+2q_j}<p<\infty$ or $a=-q_j$, $\frac{2}{1+2q_j}<p<\frac{2}{q_j}$.*

Proof. Under the assumptions of the parameter a,p, one has

$$||u||_{1,p;B_1(0)} \leq ||u-u_s||_{1,p} + ||u_s||_{1,p} \leq C(||u-u_s||_{p,B_1^+(0)\backslash B_{\frac12}(0)} + ||f||_p) \tag{29}$$

$$\leq C(||u||_{p,B_1^+(0)\backslash B_{\frac12}(0)} + ||u_s||_p + ||f||_p) \leq C(||u||_{p,B_1^+(0)\backslash B_{\frac12}(0)} + ||f||_{p,B_1^+(0)}) \tag{30}$$

□

Step 2 Global L^p estimate

Let $z_1,\cdots,z_m$ be the m marked points. Take a C^∞ function β such that $\beta\equiv 1$ on $\Sigma\setminus\cup_{l=1}^m B_1(z_l)$, and $\beta\equiv 0$ on $\cup_{l=1}^m B_{\frac12}(z_l)$. Then

$$\begin{aligned}&||u||_{1,p}\leq ||\beta u||_{1,p} + ||(1-\beta)u||_{1,p}\\ &\leq C(||\bar\partial(\beta u)||_p + ||\beta u||_p + ||\bar\partial(1-\beta)u||_p + ||(1-\beta)u||_{p,\Sigma\backslash\cup_{l=1}^m B_{\frac12}(z_l)})\\ &\leq C(||u||_{L^p(\Sigma\backslash\cup_{l=1}^m B_{\frac12}(z_l))} + ||\bar\partial u||_p).\end{aligned}$$

Lemma 3.5. *Let $\bar\partial u = f$ on Σ, where $u,f\in L^p$. Then $u\in L_1^p$ and the inequality*

$$||u||_{1,p}\leq C(||u||_{L^p(\Sigma\backslash\cup_{l=1}^m B_{\frac12}(z_l))} + ||\bar\partial u||_p). \tag{31}$$

holds if the following two conditions are satisfied:

- *if $p\neq \frac{2}{q_j}, \frac{2}{1-sq_j}, \frac{2}{2-sq_j}$ for integer s in the interval $[-1,[\frac{1}{q_j}]-1]$*
- *either $\frac{2}{1+2q_j}<p<\infty$ in the case that there is no Ramond marked points, or $\frac{2}{1+2q_j}<p<\frac{2}{q_j}$ if there exists Ramond marked points.*

Corollary 3.6. *Let $\bar\partial u = f$ on Σ, where $u,f\in L^p$. If there exists Ramond marked points on Σ, then for $\frac{2}{1+2q_j}<p<\frac{2}{q_j}, p\neq\frac{2}{1+q_j}$ there is*

$$||u||_{1,p}\leq C(||u||_{L^p(\Sigma\backslash\cup_{l=1}^m B_{\frac12}(z_l))} + ||\bar\partial u||_p). \tag{32}$$

Because $L_1^p(\Sigma)\hookrightarrow L^p(\Sigma\setminus\cup_{l=1}^m B_{\frac12}(z_l))$ is a compact inclusion, the above inequality shows that $\bar\partial: L_1^p\to L^p$ has a finite dimensional kernel and a closed image.

Step 3 the cokernel of $\bar\partial$ is a finite dimensional space

Note that the inner product $(\cdot,\cdot)$ is defined as

$$(\eta,\phi)=\int \eta\wedge *\phi, \forall \eta,\phi\in L^p(\Sigma,|L_j|\otimes\Lambda^{0,1}).$$

If $u\in L^p_1(\Sigma,|L_j|), v\in L^q(\Sigma,|L_j|^{-1}\otimes\Lambda^{0,1})$, we have

$$(\bar\partial u,v)=(u,\bar\partial^* v).$$

Thus $v\in\operatorname{coker}\bar\partial$ iff $\bar\partial * v=0$. Hence $*v\in L^q(\Sigma,\Lambda^{1,0}\otimes|L_j|)$ satisfies the equation $\bar\partial(*v)=0$. Because of the restriction of q (which is the dual index of p), the solutions $*v$ are global meromorphic sections of $\Lambda^{1,0}\otimes|L_j|$ with the possible singularities at the marked points. The order of those poles are lower bounded by a constant depending only on q_j. Therefore $\ker\bar\partial^*=\operatorname{coker}\bar\partial$ is a finite dimensional space.

Therefore we have proved that $\bar\partial$ is a Fredholm operator. □

Sobolev embedding theorem for weighted sobolev spaces

Let $||\cdot||_{o,k,p}$ denote the ordinary sobolev norm and $a\in\mathbb{R}, 1<p<\infty$. We introduce two spaces:

$$C^m_s(\mathbb{R}^2)=\{u|u|z|^a\in C^m_0(\mathbb{R}^2)\},$$

and

$$C^\infty_{s,0}(\mathbb{R}^2)=\{g\in C^\infty_0(\mathbb{R}^2)|,\operatorname{supp} g\cap\{0\}=\emptyset\}.$$

We denote by $L^p_{k,s}(\mathbb{R}^2)$ the closure of $C^\infty_{s,0}(\mathbb{R}^2)$ with respect to the weighted sobolev norm $||\cdot||_{k,p}$ with weight function $|z|^a$.

Theorem 3.7. *Let $p>1, p\neq\frac{2}{1-a}$. We have the embedding*

$$L^p_{k,0}(\mathbb{R}^2)\cap L^p_{k-1,s}(\mathbb{R}^2)\hookrightarrow\begin{cases} L^{\frac{np}{n-kp}}(\mathbb{R}^2) & \text{if } n>kp\\ C^m_s(\mathbb{R}^2) & \text{if } 0\le m<k-\frac{n}{p}\end{cases}$$

Proof. We only prove $k=1$ case. The other case can be treated similarly. For $f\in C^\infty_0(0,\infty), \lim_{t\to 0}$, we have Hardy inequality [HLP]

$$\int_0^\infty |\frac{f(t)}{t}|^p t^\epsilon dt\le\left[\frac{p}{|\epsilon-p+1|}\right]^p\int_0^\infty |f'(t)|^p t^\epsilon dt,$$

which holds for $\epsilon\neq p-1$.

Now by means of the Hardy inequality, we can apply the ordinary sobolev embedding theorem to the function $u|z|^a$ to get the weighted sobolev embedding theorem. For example, consider the $p<n$ case. We have

$$\begin{aligned}&||ur^a||_{o,q}\le C||ur^a||_{o,1,p}\\ &\le C(\int|\partial_r(ur^a)|^p+\frac{1}{r^p}|\frac{\partial}{\partial\theta}(ur^a)|^p+|ur^a|^p)^{\frac1p}\\ &=C(\int(|\partial_r u|^p+\frac{1}{r^p}|\partial_\theta u|^p+|\frac{u}{r}|^p+|u|^p)r^{ap})^{\frac1p}\\ &\le C(\int(|\partial_r u|^p+\frac{1}{r^p}|\partial_\theta u|^p)r^{ap}+|u|^p r^{ap})^{\frac1p}=C||u||_{1,p},\end{aligned}$$

where in the second inequality we used the Hardy inequality.

□

4. Inner compactness of the solution spaces of A_r, D_r, and E_r spin equations

In this section, we will discuss the compactness problem for the A_r, D_r, E_r spin equations. We will prove if R, the sum of the residue of $W(u_1, \cdots, u_t)$ at each Ramond marked points is finite, then the corresponding solution space is compact, hence the so called "inner compactness" holds. However as shown by an example in the next section, if R is infinite, then the space of the regular solutions is not compact. The singular solutions of the W spin equations should be added to compactify the solution space.

Above all we shall prove that the regular solutions of the W-spin equations lie in L_1^p space for some $p > 2$.

Denote by $P_i(u)$ the nonlinear term of the W-spin equations (7). Then $u_i = u_{i,s} + (u_i - u_{i,s})$, where $u_{i,s} = Q_s \circ P_i(u)$ is the special solution we constructed before. We have the estimate

$$||u_{i,s}||_q \leq C||u_{i,s}||_{1,2} \leq C||P_i(u)||_2, \tag{33}$$

for any $1 < q < \infty$. On the other other hand $u_i - u_{i,s}$ is a meromorphic section with the possible singularity at the marked points. Since $u_i - u_{i,s} \in L_1^2$, by the restriction of integrability $u_i - u_{i,s}$ should be holomorphic sections. There are two cases:

(1) if $a_i(h_l) - q_i \geq 0$ (i.e, $|L_i|$ is Neveu-Schwarz at z_l), then $u_i - u_{i,s}$ is L^q integrable for any $q, 1 < q < \infty$.
(2) if $a_i(h_l) = 0$ (i,e, $|L_i|$ is Ramond at z_l), then $u_i - u_{i,s}$ is L^q integrable for $1 < q < \frac{2}{q_i}$.

So at least, u_i is L^q integrable for $1 < q < \frac{2}{q_i}, i = 1, \cdots, t$. Moreover, by Lemma 3.3, we have

$$||u_i - u_{i,s}||_q \leq C||u_i - u_{i,s}||_2 \leq C(||u_i||_2 + ||u_{i,s}||_2) \tag{34}$$

for $1 < q < \frac{2}{q_i}, i = 1, \cdots, t$. The inequalities (34) and (33) induce

$$||u_i||_q \leq C(||u_i||_2 + ||P_i(u)||_2) \tag{35}$$

for $1 < q < \frac{2}{q_i}, i = 1, \cdots, t$.

We estimate the norm $||\frac{\partial W}{\partial u_i}||_p^p$ for some $p > 2$. For simplicity, we take a monomial W_l. Since $\Sigma_j b_{lj} q_j = 1$, we have $b_{lj} q_j < 1$ for each j. Choose p, ϵ such that $0 < \epsilon < q_i, 2 < p$ and $p(1-\epsilon) < 2$. We have Hölder index group

$$\left(\frac{1-\epsilon}{b_{l1}q_1}, \cdots, \frac{1-\epsilon}{(b_{li}-1)q_i}, \cdots, \frac{1-\epsilon}{b_{lt}q_t}, \frac{1-\epsilon}{q_i - \epsilon}\right).$$

To make each entry greater than 1, we let ϵ sufficiently small. Then by Hölder inequality, we have

$$\begin{aligned} &||\frac{\partial W}{\partial u_i}||_p^p = \int |u_1|_s^{pb_{l1}} \cdots |u_i|_s^{(b_i-1)p} \cdots |u_t|_s^{pb_{lt}} \\ &\leq (\int |u_1|_s^{\frac{p(1-\epsilon)}{q_1}})^{\frac{q_1 b_{l1}}{1-\epsilon}} \cdots (\int |u_i|_s^{\frac{p(1-\epsilon)}{q_i}})^{\frac{q_i(b_{li}-1)}{1-\epsilon}} \cdots (\int |u_t|_s^{\frac{p(1-\epsilon)}{q_t}})^{\frac{q_t b_{lt}}{1-\epsilon}} |\Sigma|^{\frac{q_i - \epsilon}{1-\epsilon}} \\ &\leq C(||u_i||_2, ||P_i(u)||_2) \leq \infty \end{aligned} \tag{36}$$

Thus if we let $\delta = \min\{q_1, \cdots, q_t\}$ and choose $2 \leq p < \frac{2}{1-\delta}$, then $\frac{\partial W}{\partial u_i}$ is L^p integrable for any i.

Lemma 4.1. *Suppose* $(u_1, \cdots, u_t)$ *are solutions of the* W*-spin equations (7), then* u_i *is* L_1^p *integrable and* $\frac{\partial W}{\partial u_i}$ *is* L^p *integrable for* $2 \le p < \frac{2}{1-q_i}$*, and there is the estimate*

$$||u_i||_{1,p} \le C(||u_i||_p + ||\frac{\partial W}{\partial u_i}||_p) \le C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2),$$

where $C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2)$ *is a constant depending on the norms* $||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2$.

Corollary 4.2. *Suppose* u *is the solution of* r*-spin equation, then* u *is smooth away from the Ramond marked points and is* L_1^p *integrable for* $2 \le p < \frac{2}{1-\frac{1}{r}}$

Further Estimate of the W-spin equations

Consider the following integral

$$\Sigma_i(\bar{\partial} u_i, I_1(\overline{\frac{\partial W}{\partial \bar{u}_i}}))_{L^2}$$

over Σ.

We will show the Neveu-Schwarz marked points and the Ramond marked points have different contribution to the integral. For simplicity, we assume there is only one marked point on a smooth curve Σ.

1. Assume this marked point $z_l = 0$ is a Ramond marked point, then

$$\begin{aligned}
&\Sigma_i(\bar{\partial} u_i, I_1(\overline{\frac{\partial W}{\partial \bar{u}_i}}))_{L^2} = \Sigma_i \int (\frac{\bar{\partial}\tilde{u}_i}{\partial \bar{z}} d\bar{z} \otimes e_i, \overline{\frac{\partial W(\tilde{u}_1, \cdots, \tilde{u}_t)}{\partial \tilde{u}_i} \frac{1}{z}} |e_i'|^2 e_i \otimes d\bar{z}) \\
&= \Sigma_i \int \frac{\bar{\partial}\tilde{u}_i}{\partial \bar{z}} \frac{\partial W(\tilde{u}_1, \cdots, \tilde{u}_t)}{\partial \tilde{u}_i} \frac{1}{z} dz \wedge d\bar{z} \frac{\sqrt{-1}}{2} \quad (\text{since } *(|e_i'|^2 e_i = e_i')) \\
&= \frac{\sqrt{-1}}{2} \int \frac{\bar{\partial}}{\partial \bar{z}} (W(\tilde{u}_1, \cdots, \tilde{u}_t)) \frac{1}{z} dz \wedge d\bar{z} \\
&= \lim_{\epsilon \to 0} \frac{\sqrt{-1}}{2} \int_{\partial B_\epsilon(0)} \frac{W(\tilde{u}_1, \cdots, \tilde{u}_t)}{z} dz = -\pi W(\tilde{u}_1(0), \cdots, \tilde{u}_t(0)).
\end{aligned}$$

2. Assume this marked point is Neveu-Schwarz, then

$$\begin{aligned}
&\Sigma_i(\bar{\partial} u_i, I_1(\overline{\frac{\partial W}{\partial \bar{u}_i}}))_{L^2} = \Sigma_i \int (\frac{\bar{\partial}\tilde{u}_i}{\partial \bar{z}} d\bar{z} \otimes e_i, \Sigma_j \overline{\frac{\partial W_j(\tilde{u}_1, \cdots, \tilde{u}_t)}{\partial \tilde{u}_i} z^{\Sigma_{s=1}^t b_{js}(a_s(h_0) - q_s)}} |e_i'|^2 e_i \otimes d\bar{z}) \\
&= \lim_{z \to 0} -\pi \Sigma_j W_j(\tilde{u}_1(z), \cdots, \tilde{u}_t(z)) z^{\Sigma_{s=1}^t b_{js}(a_s(h_0))}.
\end{aligned}$$

Denote by $P_i(\tilde{u})$ the nonlinear term in equation (11). Let $\tilde{u}_{i,s} = -Q_s \circ P_i(\tilde{u})$. By Lemma 3.2 and Lemma 4.1, there is

$$\int_{B_1(0)} |\frac{\tilde{u}_{i,s}}{z}|_s^p \le \int_{B_1(0)} |P_i(\tilde{u})|_s^p = ||\frac{\partial W}{\partial u_i}(u_1, \cdots, u_t)||_{p,B_1(0)}^p \le C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2), \tag{37}$$

if $2 \le p < \frac{2}{1-q_i}$. Since $\bar{\partial}(\tilde{u}_i - \tilde{u}_{i,s}) = 0$, by Lemma 3.3 we have

$$||\tilde{u}_i - \tilde{u}_{i,s}||_{o,k,q;B_1(0)} \le C(||u_i - u_{i,s}||_{p,B_1^+(0)}) \le C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2), \tag{38}$$

if $2 \le p < \frac{2}{1-q_i}$. Now let $c_{i0} = a_i(h_0) - q_i$. Since $|L_i|$ is Neveu-Schwarz at $z_l = 0$, $c_{i0} > 0$. So if $2 \le p < \frac{2}{1-q_i}$, then

$$\int |\frac{\tilde{u}_i - \tilde{u}_{i,s}}{z}|^p |z|^{pc_{i0}} \le C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2) \int |z|^{-p+pc_{i0}} \le C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2). \quad (39)$$

Combining (37), (39) and Lemma 4.1, we obtain

$$||u_i||_{1,p;B_1(0)} + ||\frac{u_i}{z}||_{p,B_1(0)} \le C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2),$$

for $2 \le p < \frac{2}{1-q_i}$. This is equivalent to

$$||\tilde{u}_i r^{c_{i0}}||_{o,1,p;B_1(0)} \le C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2). \quad (40)$$

By sobolev embedding inequality, we have

$$|\tilde{u}_i(z) r^{c_{i0}}|_{C^0(B_1(0))} \le C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2),$$

Therefore

$$|\Sigma_j W_j(\tilde{u}_1(z), \cdots, \tilde{u}_t(z)) z^{\Sigma_{s=1}^t b_{js}(a_s(h_0))}| \le \Sigma_j |W_j(\tilde{u}_1(z), \cdots, \tilde{u}_t(z)) z^{\Sigma_{s=1}^t b_{js}(a_s(h_0))}|$$
$$\le \Sigma_j |W_j(\tilde{u}_1(z) r^{c_{10}}, \cdots, \tilde{u}_t(z) r^{c_{t0}})| r$$

So

$$\Sigma_i (\bar{\partial} u_i, I_1(\overline{\frac{\partial W}{\partial \bar{u}_i}}))_{L^2} = 0.$$

If Σ is a nodal curve, then by the similar argument, we can prove that the nodal points have no contribution to the integral.

In general, one has

$$\Sigma_i (\bar{\partial} u_i, I_1(\overline{\frac{\partial W}{\partial \bar{u}_i}}))_{L^2} = -\pi \Sigma_{z_l:Ramond} W(\tilde{u}_1(z_l), \cdots, \tilde{u}_t(z_l)) \quad (41)$$

Hence, we have

$$0 = \Sigma_i (\bar{\partial} u_i, \bar{\partial} u_i + I_1(\overline{\frac{\partial W}{\partial \bar{u}_i}}))_{L^2} = ||\bar{\partial} u||_2^2 - \pi \Sigma_{l=1}^m W(\tilde{u}_1(z_l), \cdots, \tilde{u}_t(z_l)).$$

Here $||\bar{\partial} u||_2^2 = \Sigma_i ||\bar{\partial} u_i||_2^2$.

Let $R := \Sigma_{l=1}^m W(\tilde{u}_1(z_l), \cdots, \tilde{u}_t(z_l)) = \Sigma_l \mathrm{Res} W(u_1, \cdots, u_t)|_{z_l}$, then the above equality is

$$||\bar{\partial} u||_2^2 = \pi R \quad (42)$$

From (42), we have

$$||\partial W||_2^2 := \Sigma_i ||\frac{\partial W}{\partial u_i}||_2^2 = \pi R \quad (43)$$

Control norms of u_i by R

Our aim is to control the suitable norms (sobolev norms or Hölder continuous norms) of the solutions u_i by R, the sum of residues of W at Ramond marked points. In general, since the W-spin equation is an elliptic system, it is hard (sometimes even impossible) to control the maximum norm of each section u_i. Here we hope to use the special structure of W to get the control. Up to now, we only know the following cases:

(1) A_r case
(2) D_n case
(3) pure Neveu-Schwarz case.

We will treat them respectively.

Notice that by Lemma 4.1 for any solutions u_i, we have

$$||u_i||_{1,p} \leq C(||u_i||_2, ||\frac{\partial W}{\partial u_i}||_2), \tag{44}$$

for $2 \leq p < \frac{2}{q_i}$.

Therefore we hope to get the L^2 integrable information of u_i from the estimate of the nonlinear term $||\partial W||_2$.

A_r **case**

The r-spin equation is

$$\bar{\partial} u + I_1(r\bar{u}^{r-1}) = 0 \tag{45}$$

The fractional degree $q = \frac{1}{r}$.

If u is a solution of the r-spin equation, then (43) gives

$$\pi R = ||I_1(r\bar{u}^{r-1})||_2 = ||ru^{r-1}||_2 = r||u||^{r-1}_{2(r-1)} \geq ||u||_2^{r-1}|\Sigma|^{-\frac{r-2}{2}},$$

i.e.,

$$||u||_2 \leq CR^{\frac{1}{r-1}}. \tag{46}$$

D_n**-case**

In this case, $W = x^n + xy^2, \partial_x W = nx^{n-1} + y^2,\ \partial_y W = 2xy$. The W spin equation becomes

$$\bar{\partial} u + I_1(\overline{nu^{n-1} + v^2}) = 0 \tag{47}$$

$$\bar{\partial} v + I_1(\overline{2uv}) = 0 \tag{48}$$

The fractional degree $(q_1, q_2) = (\frac{1}{n}, \frac{n-1}{2n})$.

By (43), we have

$$||nu^{n-1} + v^2||_2^2 + ||2uv||_2^2 = \pi R \tag{49}$$

Let C_R be the constant depending on $R, |\Sigma|$, but not on u, v.

We have

$$\int_{|u|\leq|v|} |u|^4 \leq \int_{|u|\leq|v|} |u|^2|v|^2 \leq C_R \tag{50}$$

$$\int_{|u|\geq|v|} |v|^4 \leq \int_{|u|\geq|v|} |u|^2|v|^2 \leq C_R. \tag{51}$$

On the other hand, we have

$$\int_{|u|\geq|v|} n^2|u|^{2(n-1)} \leq 2\int_{|u|\geq|v|} |nu^{n-1} + v^2|^2 + 2\int_{|u|\geq|v|} |v|^4. \tag{52}$$

$$\leq C_R + 2\int_{|u|\geq|v|} |v|^2|u|^2 \leq C_R \tag{53}$$

Since $n \geq 2$, from (53) and (50) we get

$$\int |u|^2 \leq C_R \tag{54}$$

By equations, we get from (35) the following equality:

$$||u||_q \leq C(||nu^{n-1} + v^2||_2 + ||u||_2) \leq C_R,$$

for $1 < q < \frac{2}{q_1} = 2n$. Especially, we have

$$||u||_{2(n-1)} \leq C_R \tag{55}$$

So

$$\int |v|^4 \leq 2\int |nu^{n-1} + v^2|^2 + 2n^2 \int |u|^{2(n-1)} \leq C_R, \tag{56}$$

hence

$$||v||_2 \leq C_R.$$

Pure Neveu-Schwarz case

In this case, $W(u_1, \cdots, u_n) = u_1^n + \cdots + u_n^n + u_1 \cdots u_n$, and

$$L_i := \frac{\partial W}{\partial u_i} = nu_i^{n-1} + u_1 \cdots \hat{u}_i \cdots u_n.$$

We have $\Sigma_i ||L_i||_2^2 \leq C_R$. Now using the fundamental inequalities, we have

$$\begin{aligned}
&\int n(\Sigma |L_i|^2) \geq \int |\Sigma_i L_i|^2 \\
&= \int |n(u_1^{n-1} + \cdots + u_n^{n-1}) + \Sigma_i (u_1 \cdots \hat{u}_i \cdots u_n)|^2 \\
&\geq \int |n(|u_1|^{n-1} + \cdots + |u_n|^{n-1}) - \Sigma_i(|u_1| \cdots |\hat{u}_i| \cdots |u_n|)|^2 \\
&\geq \int |n(|u_1|^{n-1} + \cdots + |u_n|^{n-1}) \\
&- \Sigma_i \frac{|u_1|^{n-1} + \cdots + |\hat{u}_i|^{n-1} + \cdots + |u_n|^{n-1}}{n-1}|^2 \\
&\geq \int |\Sigma_i (n-1)|u_i|^{n-1}|^2.
\end{aligned}$$

The above inequality shows that the L^2 norms of any sections u_i can be controlled by C_R. Hence using the L^p estimate and the weighted sobolev embedding theorem, we can control any norms of u_i.

Compactness of the space of solutions

Assume that $\{u_1^n, \cdots, u_t^n\}$ are a sequence of solutions of the W-spin equation

$$\bar{\partial} u_i^n + I_1(\frac{\partial W}{\partial u_i}(u_1^n, \cdots, u_t^n)) = 0.$$

Let $u_i^n = \tilde{u}_i^n e_i$. Because of the different types of singularity, we will discuss the compactness of the solutions in three domains.

(1). Compactness in the inner domain away from the marked points.

In this case, the W-spin equations have the following form:

$$\bar\partial\tilde u_i^n+\overline{\frac{\partial W}{\partial\tilde u_i}(\tilde u_1^n,\cdots,\tilde u_t^n)}\phi=0,$$

where ϕ is a C^∞ function. By Lemma 4.1, we have

$$||\tilde u_i^n||_{o,1,p;inn}\le C_R.$$

Here "inn" means the inner domain which has a positive distance to those marked points. Therefore by the standard argument of compactness, there exist a C^∞ function $\tilde u_i$ and a subsequence (still denoted by $\tilde u_i^n$)$\tilde u_i^n$ such that

$$\tilde u_i^n\to\tilde u_i \text{ in } C^k \text{ and ordinary } L_k^p \text{ norms},$$

for any integer $k\ge 0$. $\tilde u_i$ are certainly the solutions of the W-spin equations in the interior part.

(2). Compactness in the neighborhood of Neveu-Schwarz points.

Denote by $P_i(\tilde u)$ the nonlinear term in equation (11). Let $\tilde u_{i,s}^n=-Q_s\circ P_i(\tilde u)$. By Lemma 3.2, there is

$$\int_{B_1(0)}|\frac{\tilde u_{i,s}^n}{z}|_s^p\le\int_{B_1(0)}|P_i(\tilde u)|_s^p=||\frac{\partial W}{\partial u_i}(u_1^n,\cdots,u_t^n)||_{p,B_1(0)}^p\le C(||u_i^n||_2,||\frac{\partial W}{\partial u_i}||_2)\le C_R, \tag{57}$$

if $2\le p<\frac{2}{1-q_i}$. Here the second inequality is due to Lemma 4.1. Since $\bar\partial(\tilde u_i^n-\tilde u_{i,s}^n)=0$, by Lemma 3.3 we have

$$||\tilde u_i^n-\tilde u_{i,s}^n||_{o,k,q;B_1(0)}\le C(||u_i^n-u_{i,s}^n||_{p,B_1^+(0)})\le C(||u_i^n||_2,||\frac{\partial W}{\partial u_i}||_2)\le C_R, \tag{58}$$

if $2\le p<\frac{2}{1-q_i}$. Now let $c_{il}=a_i(h_l)-q_i$. Since $|L_i|$ is Neveu-Schwarz at z_l, $c_{il}>0$. So if $2\le p<\frac{2}{1-q_i}$, then

$$\int|\frac{\tilde u_i^n-\tilde u_{i,s}^n}{z}|^p|z|^{pc_{il}}\le C_R\int|z|^{-p+pc_{il}}\le C_R. \tag{59}$$

Combining (57), (59) and Lemma 4.1, we obtain

$$||u_i^n||_{1,p;B_1(0)}+||\frac{u_i^n}{z}||_{p,B_1(0)}\le C_R,$$

for $2\le p<\frac{2}{1-q_i}$. This is equivalent to

$$||\tilde u_i^n r^{c_{il}}||_{o,1,p;B_1(0)}\le C_R. \tag{60}$$

Using the ordinary sobolev compact embedding theorem, there exists $\tilde u_i$ such that $\tilde u_i r^{c_{il}}\in C^{\alpha_i}\cap L^q$(ordinary q norm), for $0<\alpha_i<q_i, 1<q<\infty$, and

$$\tilde u_i^n r^{c_{il}}\to\tilde u_i r^{c_{il}} \text{ in } C^{\alpha_i'}, \tag{61}$$

where $0<\alpha_i'<\alpha_i$. By Lemma 3.4 for any $2\le p<\frac{2}{1-q_i}$, one has

$$||u_i^n-u_i^m||_{1,p;B_1(0)}\le C(||u_i^n-u_i^m||_{p;B_1^+(0)}+||\frac{\partial W}{\partial u_i}(u_1^n,\cdots,u_t^n)-\frac{\partial W}{\partial u_i}(u_1^m,\cdots,u_t^m)||_{p;B_1^+(0)})$$
$$=C(||(\tilde u_i^n-\tilde u_i^m)r^{c_{il}}||_{o,p;B_1^+(0)}+||\frac{\partial W}{\partial u_i}(\tilde u_1^n r^{c_{1l}},\cdots,\tilde u_t^n r^{c_{tl}})-\frac{\partial W}{\partial u_i}(\tilde u_1^m r^{c_{1l}},\cdots,\tilde u_t^m r^{c_{tl}})||_{o,p;B_1^+(0)}).$$

This shows that $\{u_i^n\}$ is a cauchy sequence in $L_1^p(B_1(0))$ and

$$\begin{aligned}
&u_i^n \to u_i \text{ in } L_1^p \\
&\frac{\partial W}{\partial u_i}(u_1^n,\cdots,u_t^n) \to \frac{\partial W}{\partial u_i}(u_1,\cdots,u_t) \text{ in } L^p,
\end{aligned}$$

for $2 \le p < \frac{2}{1-q_i}$. Therefore $(u_1,\cdots,u_t)$ are the solutions of the W-spin equations in $B_1(0)$.

(3). In the neighborhood of Ramond marked points.

In the case, the inequality (58) is not true for $2 \le p < \frac{2}{1-q_i}$. We can't use the same argument in case (2). But $\tilde{u}_{i,s}^n(0) = 0$, we have the decomposition

$$\tilde{u}_i^n = \tilde{u}_i^n - \tilde{u}_i^n(0) + (\tilde{u}_i^n - \tilde{u}_{i,s}^n)(0).$$

By Lemma 3.3,

$$\begin{aligned}
&|\tilde{u}_i^n(0)| = |(\tilde{u}_i^n - \tilde{u}_{i,s}^n)(0)| \le C||\tilde{u}_i^n - \tilde{u}_{i,s}^n||_{2,B_1(0)} \\
&\le C(||u_i^n||_{2,B_1(0)} + ||u_{i,s}^n||_{2,B_1(0)}) \le C_R.
\end{aligned} \tag{62}$$

So there exists a constant A_i such that $\tilde{u}_i^n(0) \to A_i$ (of course,we take the subsequence as usual).

On the other hand, if $2 \le p < \frac{2}{1-q_i}$, we have

$$\begin{aligned}
&||u_i^n - u_i^n(0)||_{1,p;B_1(0)} \le ||u_i^n||_{1,p;B_1(0)} + ||u_i^n(0)||_{p,B_1(0)} \\
&\le C_R + (\int |z|^{-pq_i})^{\frac{1}{p}} |\tilde{u}_i^n(0)| \le C_R.
\end{aligned}$$

By the weighted sobolev embedding theorem, there exists a subsequence and a function $\tilde{v}_i$ such that

$$(\tilde{u}_i^n(z) - \tilde{u}_i^n(0))r^{-q_i} \to \tilde{v}_i r^{-q_i} \text{ in } C^\alpha, \forall 0 < \alpha < q_i.$$

Especially, $\forall \varepsilon > 0$, there exists N such that $\forall n > N$,

$$|\tilde{u}_i^n - \tilde{u}_i^n(0) - \tilde{v}_i| \le \varepsilon r^{q_i}.$$

Therefore $\forall z \in B_1(0)$,

$$|\tilde{u}_i^n(z) - A_i - \tilde{v}_i(z)| \le \varepsilon(1 + r^{q_i}) \le C\varepsilon,$$

i.e., $\tilde{u}_i^n \to A_i + \tilde{v}_i := \tilde{u}_i$ in $C^0(B_1(0))$. Hence

$$\frac{\partial W}{\partial u_i}(\tilde{u}_1^n,\cdots,\tilde{u}_t^n) \to \frac{\partial W}{\partial u_i}(\tilde{u}_1,\cdots,\tilde{u}_t) \text{ in } C^0(B_1(0)).$$

Similarly if $2 \le p < \frac{2}{1-q_i}$, we have

$$||u_i^n - u_i^m||_{1,p;B_1(0)} \le C(||u_i^n - u_i^m||_{p;B_1^+(0)}$$
$$+ ||(\frac{\partial W}{\partial u_i}(\tilde{u}_1^n, \cdots, \tilde{u}_t^n) - \frac{\partial W}{\partial u_i}(\tilde{u}_1^m, \cdots, \tilde{u}_t^m))r^{-1+2q_i}||_{p;B_1^+(0)})$$
$$\le C \max\left\{|\tilde{u}_i^n - \tilde{u}_i^m|_{C^0}, |\frac{\partial W}{\partial u_i}(\tilde{u}_1^n, \cdots, \tilde{u}_t^n) - \frac{\partial W}{\partial u_i}(\tilde{u}_1^m, \cdots, \tilde{u}_t^m)|_{C^0}\right\}$$
$$((\int |z|^{-pq_i})^{\frac{1}{p}} + (\int |z|^{(-1+q_i)p})^{\frac{1}{p}})$$
$$\le C \max\left\{|\tilde{u}_i^n - \tilde{u}_i^m|_{C^0}, |\frac{\partial W}{\partial u_i}(\tilde{u}_1^n, \cdots, \tilde{u}_t^n) - \frac{\partial W}{\partial u_i}(\tilde{u}_1^m, \cdots, \tilde{u}_t^m)|_{C^0}\right\}.$$

The above inequality shows $\{u_i^n\}$ is a cauchy sequence in L_1^p and

$$u_i^n \to u_i \text{ in } L_1^p$$
$$\frac{\partial W}{\partial u_i}(u_1^n, \cdots, u_t^n) \to \frac{\partial W}{\partial u_i}(u_1, \cdots, u_t) \text{ in } L^p,$$

for $2 \le p < \frac{2}{1-q_i}$. Therefore $(u_1, \cdots, u_t)$ is the solutions of the W-spin equations in $B_1(0)$.

Theorem 4.3. *Let M_s be the space of the solutions $u = (u_1, \cdot, u_t)$ satisfying the W-spin equations. Let $E(u) = \Sigma_{z_l:Ramond} Res_{z_l}(W(u_1, \cdots, u_t))$. Then*

1. *for any $a \in \mathbb{C} - (0, \infty), E^{-1}(a) = \{0\}$.*
2. *for any $a \in (0, \infty)$, $E^{-1}((0, a))$ is a compact space in L_1^p topology for $2 \le p < \frac{2}{1-\delta}$, where $\delta = \min\{q_1, \cdots, q_t\}$.*

5. Compactifying the solution space of A_r-spin equation

The following example shows that the space of the regular solutions is not compact and a sequence of regular solutions of the A_r-spin equation will converge to some singular solutions away from the Ramond marked points.

Example 5.1. Let $(\mathbb{C}P^1, 3, \mathbf{z})$ be a marked sphere with two Ramond marked points and one non-orbifold marked point. We shall construct a sequence of global regular solutions of the A_r-spin equations.

Let $\mathbb{C}P^1 = U_0 \cup U_1$, where $U_0 = \{[Z_0, Z_1] | Z_0 \ne 0\}$. Let $z = \frac{Z_1}{Z_0}$ be the affine coordinate in U_0. on U_0 the Fubini-Study metric is given by

$$\omega = \frac{\sqrt{-1}}{2\pi} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

So the induced metric on the canonical bundle is given by

$$|dz| = 1 + |z|^2.$$

In U_0 the r-spin equation is given by

$$\frac{\bar{\partial}\tilde{u}_0}{\partial \bar{z}} + \frac{r}{\bar{z}}(1+|z|^2)^{-\frac{2}{r}}|z|^{\frac{2}{r}}\bar{\tilde{u}}_0^{r-1} = 0.$$

If we only consider the real-valued solution, then the above equation becomes

$$\frac{d\tilde{u}_0}{d\rho} = -2r\tilde{u}_0^{r-1}\rho^{\frac{2}{r}-1}(1+\rho^2)^{-\frac{2}{r}}.$$

Therefore

$$\tilde{u}_0(\rho) = [2r(r-2)\int_0^{\rho} \tau^{\frac{2}{r}-1}(1+\tau^2)^{-\frac{2}{r}}d\tau + u_0^{-(r-2)}(0)]^{-\frac{1}{r-2}}.$$

It is easy to check that $u = \tilde{u}_0(\frac{dz}{z})^{\frac{1}{z}}$ is really a global solution of the r-spin equation. Namely if we represent u by $\tilde{u}_1$ in the other local chart U_1, then it also satisfies the r-spin equation.

one can easily obtain the relation:

$$R = \tilde{u}_1^r(0) + \tilde{u}_0^r(0) = u_0^r(0) - (\int_0^{\infty} (\frac{\tau}{1+\tau^2})^{\frac{2}{r}} \frac{1}{\tau} d\tau + u_0(0)^{-(r-2)})^{-\frac{r}{r-2}} > 0.$$

Thus if $R \to \infty$(or $u_0(0) \to \infty$), then

$$\tilde{u}_0(\rho) \to [2r(r-2)\int_0^{\rho} \tau^{\frac{2}{r}-1}(1+\tau^2)^{-\frac{2}{r}}d\tau]^{-\frac{1}{r-2}},$$

which is not a regular solution of the r-spin equation.

Definition 5.2. The sections $(u_1, \cdots, u_t)$ are called the singular solutions of the W-spin equations if they satisfy the W-spin equations pointwise away from the Ramond marked points and are not the regular solutions of the W-spin equations.

In the following part, we only consider the compactification of the solution space of the r-spin equation.

To compactifying the solution space in suitable topology, we have to consider the asymptotic behavior of the singular solutions near the Ramond marked points.

We assume that 0 is the unique Ramond marked point in $B_2(0)$ and that u is the singular solution in $B_2(0) - \{0\}$ of the r-spin equation:

$$u_{\bar{z}} + r\bar{u}^{r-1}\frac{1}{\bar{z}}|z|^{\frac{2}{r}} = 0.$$

Let $u = z^{\frac{1}{r}}\varphi$, then we can get the equation of φ:

$$\varphi_{\bar{z}} + r\bar{\varphi}^{r-1} = 0, \tag{63}$$

where φ is locally smooth function. Note that φ^r is a well-defined function in $B_2(0) - \{0\}$.

By equation (63), we have

$$\partial_z\partial_{\bar{z}}\varphi = r^2(r-1)|\varphi|^{2(r-2)}\varphi.$$

We have the computation,

$$\begin{aligned}\partial_z\partial_{\bar{z}}(\varphi\cdot\bar{\varphi}) &= \partial_z\partial_{\bar{z}}\varphi\cdot\bar{\varphi} + \varphi\cdot\partial_z\partial_{\bar{z}}\bar{\varphi} + \partial_{\bar{z}}\varphi\cdot\partial_z\bar{\varphi} + \partial_z\varphi\cdot\partial_{\bar{z}}\bar{\varphi}\\ &= 2r^2(r-1)|\varphi|^{2(r-2)}\varphi\cdot\bar{\varphi} + |\partial_{\bar{z}}\varphi|^2 + |\partial_z\varphi|^2\end{aligned}$$

So we have the equation of $|\varphi|^2$:

$$\Delta|\varphi|^2 = 8r^2(r-1)|\varphi|^{2(r-1)} + 4|\partial_{\bar{z}}\varphi|^2 + 4|\partial_z\varphi|^2. \tag{64}$$

This implies the maximum principle (see[CW])

Lemma 5.3. *For any $p > 0, 1 < \theta < 1$ and any $R > 0$ such that $B_R(z) \in B_2(0) - \{0\}$, $z \in B_2(0) - \{0\}$, we have*

$$\sup_{B_{\theta R}(z)} |\varphi| \leq C \left(\frac{1}{|B_R(z)|} \int_{B_R(z)} |\varphi|^p \right)^{\frac{1}{p}} \tag{65}$$

Lemma 5.4. *Let φ be a solution of (63) in $B_2(0)-\{0\}$, then there exists a constant C_r only depending on r such that for any $z \in B_2(0) - \{0\}$,*

$$|\varphi(z)| \leq C_r |z|^{-\frac{1}{r-2}}. \tag{66}$$

Proof. Multiplying the two sides of (63) by $r\varphi^{r-1}$, we have

$$(\varphi^r)_{\bar{z}} + r^2 |\varphi|^{2(r-1)} = 0. \tag{67}$$

Let $\psi^\beta, \beta > 0$, be a cut-off function with support away from the origin, we have

$$\int (\varphi^r)_{\bar{z}} \psi^\beta + r^2 |\varphi|^{2(r-1)} \psi^\beta = 0.$$

Integrating by parts and using Hölder inequality, we have

$$\begin{aligned} &\int r^2 |\varphi|^{2(r-1)} \psi^\beta = \int \varphi^r \beta \psi^{\beta-1} \psi_{\bar{z}} \\ &\leq \int |\varphi|^r \beta \psi^{\beta-1} |\psi_{\bar{z}}| \\ &\leq \beta \left(\int |\varphi|^{2(r-1)} \psi^\beta \right)^{\frac{r}{2(r-1)}} \left(\int (\psi^{\beta \frac{r-2}{2(r-1)} - 1} |\psi_{\bar{z}}|)^{\frac{2(r-1)}{(r-2)}} \right)^{\frac{r-2}{2(r-1)}} \end{aligned}$$

Thus we have

$$\int |\varphi|^{2(r-1)} \psi^\beta \leq C_r \int \psi^{\beta - \frac{2(r-1)}{(r-2)}} |\psi_{\bar{z}}|^{\frac{2(r-1)}{r-2}}. \tag{68}$$

Now we take $\beta = \frac{2(r-1)}{r-2}$, and choose ψ satisfying the requirement that $\psi = 1$ in $B_{\frac{|z|}{4}}(z)$, vanishing outside $B_{\frac{|z|}{2}}(z)$ and $|\nabla \psi| \leq \frac{4}{|z|}$. Here $z \neq 0$. Thus we obtain from (68) the following estimate,

$$\int_{B_{\frac{|z|}{4}(z)}(z)} |\varphi|^{2(r-1)} \leq C_r \int_{B_{\frac{|z|}{2}}(z)} |z|^{-\frac{2(r-1)}{(r-2)}} = C_r |z|^{-\frac{2}{r-2}} \tag{69}$$

Now the estimate (69) and the lemma of maximum principle induces the required pointwise estimate. □

By the pointwise estimate, we can get a uniform L^p-estimate:

Corollary 5.5. *Let $r \geq 3$ and $1 < p < 2(r-2)$. If φ is the solution of (63) in $B_2(0) - \{0\}$, then φ is an integrable function in $B_2(0)$ and furthermore*

$$||\varphi||_{p, B_2(0)} \leq C,$$

where C depends only on r, p.

We can also obtain the Harnack inequality for $|\varphi|$.

Lemma 5.6. *Let $0 \leq \theta < 1$ be a fixed number, $0 < \epsilon < 1$. Assume that φ be a solution of the equation (63) in $B_2(0) - \{0\}$, then*

$$\sup_{z \in T(\epsilon(1-\theta),\epsilon)} |\varphi(z)| \leq C(r,\theta) \inf_{z \in T(\epsilon(1-\theta),\epsilon)} |\varphi(z)|,$$

where $T(\epsilon(1-\theta),\epsilon)$ is the annulus with radius between $\epsilon(1-\theta)$ and ϵ and $C(r,\theta)$ is a constant only depending on r, θ.

Proof. By equation (63), we have (since $\varphi \neq 0$)

$$(\log \varphi)_{\bar{z}} = -r\bar{\varphi}^{r-1}\varphi^{-1}.$$

Let

$$g(z) = -\frac{1}{\pi}\int_{T(1-\theta,1)} \frac{-r\bar{\varphi}^{r-1}\varphi^{-1}(\zeta)dv}{\zeta - z},$$

then $g_{\bar{z}} = -r\bar{\varphi}^{r-1}\varphi^{-1}$, for $z \in T(1-\theta,1)$. Since $|\varphi(z)| \leq C_r(1-\theta)^{-\frac{1}{r-2}}$ for $z \in T(1-\theta,1)$, then $|-r\bar{\varphi}^{r-1}\varphi^{-1}|(z) \leq C_r(1-\theta)^{-1}$. Hence $|g(z)| \leq C(r,\theta)$, and g is a Hölder continuous function in $T(1-\theta,1)$. Let $\hat{\Psi} = \log\varphi - g$, then

$$\hat{\Psi}_{\bar{z}} = 0.$$

Since $\hat{\Psi}$ is continuous, $\hat{\Psi}$ is an analytic function. We have $\varphi = e^g e^{\hat{\psi}}$. Let $\Psi = e^{\hat{\Psi}}$, then $\varphi = e^g\Psi$, where g is Hölder continuous and Ψ is analytic and also the following estimate holds

$$|\Psi(z)| \leq |e^{-g}\varphi(z)| \leq C(r,\theta) =: e^L \tag{70}$$

Since g is bounded, so to prove the Harnack inequality of φ, we need only to prove the Harnack inequality of Ψ. Now $L - \log|\Psi(z)|$ is a nonnegative and harmonic function, so we have the gradient estimate:

$$|\nabla(L - \log|\Psi(z)|)| \leq C(r,\theta)(L - \log|\Psi(z)|) \leq C(r,\theta)L,$$

i,e, $|\nabla \log|\Psi(z)|| \leq C(r,\theta)$, which implies the Harnack inequality in $T(1-\theta,1)$,

$$\sup_{T(1-\theta,1)} |\Psi(z)| \leq C \inf_{T(1-\theta,1)} |\Psi(z)|. \tag{71}$$

To prove the Harnack inequality in annulus $T(\epsilon(1-\theta),\epsilon)$, we use the scaling invariance of the equation (63). Namely if φ is the solution of (63) in $T(\epsilon(1-\theta),\epsilon)$, then $\varphi_\epsilon(z) := \epsilon^{\frac{1}{r-2}}\varphi(\epsilon z)$ is the solution of (63) in the annulus $T(1-\theta,1)$. Thus one can easily get the same conclusion in the annulus $T(\epsilon(1-\theta),\epsilon)$. □

Now by maximum principle and the Harnack inequality, one can easily get a convergence corollary:

Corollary 5.7. *Let φ be a solution of the equation (63) in $B_2(0) - \{0\}$, then either $\varphi(z)$ is bounded near the origin or $\lim_{z\to 0}|\varphi(z)| = \infty$.*

Now it is easy to get the following compactness theorem:

Theorem 5.8. *Let $\mathcal{M}(\Sigma, L)$ be the solution space of the r-spin equation, which contains the regular solutions and the singular solutions on the Riemann surface Σ with the spin structure L, then $\mathcal{M}(\Sigma, L)$ is compact with respect to the topology in $L^P \cap C^\infty_{loc}(\Sigma \setminus \{$Ramond marked points$\})$.*

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