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Dirac-Wave Maps

by

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Abstract. We introduce a functional that couples the nonlinear sigma model with a spinor field: $L = \int_{R^{1+1}}[|d\phi|^2 + \langle\psi, \not{D}\psi\rangle]$. In two dimensions, it is conformally invariant. The critical points of this functional are called Dirac-wave maps. We prove that there exists global solution for the Cauchy data.

1. Introduction

Let $\{R^{1+1}, \{h_{\alpha\beta}\}\{t,x\}\}$ be two dimensional Minkowski and $\{M^n, \{g_{ij}\}, \{y^i\}\}$ be a compact Riemannian manifold. $P_{SO(1,1)} \to R^{1+1}$ its oriented orthonormal frame bundle. A *Spin*-structure is a lift of the structure group $SO(1,1)$ to $Spin(1,1)$, *i.e.* there exists a principal *Spin*-bundle $P_{Spin(1,1)} \to R^{1+1}$ such that there is a bundle map

$$\begin{array}{ccc} P_{Spin(1,1)} & \longrightarrow & P_{SO(1,1)} \\ \downarrow & & \downarrow \\ R^{1+1} & \longrightarrow & R^{1+1} \end{array}$$

Let $\Sigma^+ R^{1+1} := P_{Spin(1,1)} \times_\rho \mathcal{C}$ be a complex line bundle over R^{1+1} associated to $P_{Spin(1,1)}$. This is the bundle of positive half-spinors. Its complex conjugate $\Sigma^- R^{1+1} := \overline{\Sigma^+ R^{1+1}}$ is called the bundle of negative half-spinors. The spinor bundle is $\Sigma R^{1+1} := \Sigma^+ R^{1+1} \oplus \Sigma^- R^{1+1}$.

There exists a Clifford multiplication

$$\begin{array}{rcl} TR^{1+1} \times_{\mathbb{C}} \Sigma^+ R^{1+1} & \to & \Sigma^- R^{1+1} \\ TR^{1+1} \times_{\mathbb{C}} \Sigma^- R^{1+1} & \to & \Sigma^+ R^{1+1} \end{array}$$

denoted by $v \otimes \psi \to v \cdot \psi$, which satisfies the Clifford relations

$$v \cdot w \cdot \psi + w \cdot v \cdot \psi = -2h(v,w)\psi,$$

for all $v, w \in TR^{1+1}$ and $\psi \in \Sigma R^{1+1}$.

On the spinor bundle ΣR^{1+1} there is a hermitian metric $\langle \cdot, \cdot \rangle$ and a connection ∇ compatible with the hermitian metric. Since ΣR^{1+1} is trivial, so ∇ is trivial. Let ϕ be a map from R^{1+1} to M. Denote $\phi^{-1}TM$ the pull-back bundle of TM by ϕ and consider the twisted bundle $\Sigma R^{1+1} \otimes \phi^{-1}TM$. Let D be the Levi-Civita connection on $\phi^{-1}TM$. On twisted bundle $\Sigma \otimes \phi^{-1}TM$ there is a metric and connection $\widetilde{\nabla}$ induced from the metrics and the connections on ΣR^{1+1} and $\phi^{-1}TM$.

In local coordinates, the section ψ of $\Sigma R^{1+1} \otimes \phi^{-1}TM$ can be expressed by

$$\psi(t,x) = \sum_{j=1}^n \psi^j(t,x) \frac{\partial}{\partial y^j}(\phi(t,x)),$$

where ψ^i is a spinor and $\{\frac{\partial}{\partial y^j}\}$ is the natural local basis. $\widetilde{\nabla}$ can be expressed by

$$\widetilde{\nabla}\psi = \sum_{i=1}^{n} \nabla\psi^i(t,x)\frac{\partial}{\partial y^j}(\phi(t,x)) + \sum_{i,j,k=1}^{n} \Gamma^i_{jk}\partial\phi^j(t,x)\psi^k(t,x)\frac{\partial}{\partial y^i}(\phi(t,x)).$$

If we write ψ^j as column vector with two components $\psi^j = (\psi^j_1, \psi^j_2)^T$ and $\overline{\psi}^j = (\overline{\psi}^j_1, \overline{\psi}^j_2)^T$, then

$$\begin{aligned}\psi(t,x) &= (\sum_{j=1}^{n} \psi^j_1(t,x)\frac{\partial}{\partial y^j}(\phi(t,x)), \sum_{j=1}^{n} \psi^j_2(t,x)\frac{\partial}{\partial y^j}(\phi(t,x)))^T \\ &=: (\psi_1, \psi_2)^T\end{aligned}$$

Therefore we can consider ψ_1, ψ_2 as vectors on $\phi^{-1}TM$, so $\widetilde{\nabla}$ can be written as

$$\widetilde{\nabla}\psi = (D\psi_1, D\psi_2)^T$$

Now we define the norm of ψ and $\widetilde{\nabla}\psi$ by

$$\begin{aligned}\|\psi\|^2 &=: g_{ij}(\psi^i, \psi^j) = g_{ij}Re((\overline{\psi}^i)^T\psi^j) \\ &= g_{ij}Re(\overline{\psi}^i_1\psi^j_1) + g_{ij}Re(\overline{\psi}^i_2\psi^j_2) \\ &= \|\psi_1\|^2 + \|\psi_2\|^2 \\ \|\widetilde{\nabla}\psi\|^2 &=: \|D\psi_1\|^2 + \|D\psi_2\|^2\end{aligned}$$

Define the *Dirac operator along the map* ϕ by
(1.1)

$$D\!\!\!\!/\,\psi = \sum_i \partial\!\!\!/\psi^i(t,x)\frac{\partial}{\partial y^i}(\phi(t,x)) + \sum_{i,j,k=1}^{n} \Gamma^i_{jk}\partial_{e_\alpha}\phi^j(t,x)e_\alpha\cdot\psi^k(\phi(t,x))\frac{\partial}{\partial y^i}(\phi(t,x)),$$

where e_1, e_2 is the local orthonormal basis of R^{1+1} and $\partial\!\!\!/ := \sum_{\alpha=1}^{2} e_\alpha\cdot\nabla_{e_\alpha}$ is the usual Dirac operator. The Dirac operator $D\!\!\!\!/$ is formally self-adjoint, i.e.,

$$\int_{R^2}\langle\psi, D\!\!\!\!/\,\xi\rangle = \int_{R^2}\langle D\!\!\!\!/\,\psi, \xi\rangle, \tag{1.2}$$

for all $\psi, \xi \in \Gamma(\Sigma R^{1+1}\otimes\phi^{-1}TM)$, the space of smooth section of $\Sigma R^{1+1}\otimes\phi^{-1}TM$ and ψ or ξ has compact support. Set

$$\mathcal{X} := \{(\phi,\psi)\,|\,\phi\in C^\infty(R^{1+1}, M) \text{ and } \psi\in\Gamma(\Sigma R^{1+1}\otimes\phi^{-1}TM)\}.$$

On $\mathcal{X}$, we consider the following functional

$$L(\phi,\psi) = \int_{R^2}[g_{ij}(\phi)(\frac{\partial\phi^i}{\partial t}\frac{\partial\phi^j}{\partial t} - \frac{\partial\phi^i}{\partial x}\frac{\partial\phi^j}{\partial x}) + g_{ij}(\phi)\langle\psi^i, D\!\!\!\!/\,\psi^j\rangle]dtdx, \tag{1.3}$$

The Euler-Lagrange equations of L are:

$$\Box(\phi) = \mathcal{R}(\phi,\psi), \tag{1.4}$$

$$D\!\!\!\!/\,\psi = 0, \tag{1.5}$$

where $\Box(\phi)$ is the tension field of the map ϕ and $\mathcal{R}(\phi,\psi)\in\Gamma(\phi^{-1}TM)$ defined by

$$\mathcal{R}(\phi,\psi)(x) = \frac{1}{2}\sum R^m_{lij}(\phi(x))\langle\psi^i, d\phi^l\cdot\psi^j\rangle\frac{\partial}{\partial y^m}(\phi(x)). \tag{1.6}$$

Here R^m_{lij} are components of the Riemannian curvature tensor of g. Solutions (ϕ, ψ) to (1.4) and (1.5) are called *Dirac-harmonic maps*.

It is obvious that there are two types of trivial solutions. One is $(\phi, 0)$, where ϕ is a wave map, and another is (y, ψ), where y is a point in M viewed as a constant map from $R^{1+1} \to M$ and ψ is a wave spinor, *i.e*, $\not{D}\psi = 0$. The main purpose of this paper is to prove that there exists nontrival global solution of equation (1.4) and (1.5). We stated it as following:

Theorem 1. *Suppose (M, g) is compact Riemmannian manifold, then the equation (1.4) and (1.5) have unique global smooth solutions with given initial smooth conditions,*

$$\phi(0, x) = \phi_0,\ \ \phi_t(0, x) = \phi_1,\ \ \psi(0, x) = \psi_0.$$

2. Dirac-Wave Maps

In this section, we establish some basic facts for the functional L and equations (1.4)–(1.5).

Proposition 2.1. *The Euler-Lagrange equations for L are*

$$\Box(\phi) = \mathcal{R}(\phi, \psi) \tag{2.1}$$

$$\not{D}\psi = 0, \tag{2.2}$$

where $\Box(\phi)$ is the tension field of the map ϕ and $\mathcal{R}$ is defined by (1.6).

Proof. Equation (2.2) is easy to derive. Consider a family of ψ_s with $d\psi_s/ds = \eta$ at $s = 0$ and η has compact support, fix ϕ. Since $\not{D}$ is formally self-adjoint for such η, we have

$$\begin{aligned} \frac{dL}{ds}|_{s=0} &= \int_{R^2} \langle \eta, \not{D}\psi\rangle + \langle \psi, \not{D}\eta\rangle \\ &= 2\int_{R^2} \langle \eta, \not{D}\psi\rangle. \end{aligned}$$

Hence, we get (2.2).

Next, we consider a variation $\{\phi_s\}$ of ϕ such that $d\phi_s/ds = \xi$ at $s = 0$ and ξ has compact support, fix ψ. We choose $\{e_\alpha\}$ as a local orthonormal basis on R^{1+1} such that $[e_\alpha, \partial_s] = 0$, $\nabla_{e_\alpha} e_\beta = 0$ at a considered point.

(2.3)

$$\frac{dL(\phi_s)}{ds}|_{s=0} = \int_{R^2} \frac{\partial}{\partial s}[g_{ij}(\phi_s)(\frac{\partial \phi^i_s}{\partial t}\frac{\partial \phi^j_s}{\partial t} - \frac{\partial \phi^i_s}{\partial x}\frac{\partial \phi^j_s}{\partial x})]|_{s=0} + \int_{R^2} \frac{\partial}{\partial s}\langle \psi, \not{D}\psi\rangle|_{s=0} := I + II.$$

It is easy to check that

$$I = -2\int_{R^2} \Box^i(\phi) g_{im}\xi^m. \tag{2.4}$$

Now we compute II. First we compute the variation of $\not{D}\psi$. We have

$$\begin{aligned} \frac{d}{ds}\not{D}\psi &= e_\alpha \cdot \nabla_{\partial_s}\nabla_{e_\alpha}\psi \\ &= e_\alpha \cdot \nabla_{e_\alpha}\psi^i \otimes \nabla_{\partial_s}\partial_{y_i} + e_\alpha \cdot \psi^i \otimes \nabla_{\partial_s}\nabla_{e_\alpha}\partial_{y_i} \\ &= e_\alpha \cdot \nabla_{e_\alpha}\psi^i \otimes \nabla_{\partial_s}\partial_{y_i} + e_\alpha \cdot \psi^i \otimes [\nabla_{e_\alpha}\nabla_{\partial_s}\partial_{y_i} + R(\partial_s, e_\alpha)\partial_{y_i}] \\ &= e_\alpha \cdot \nabla_{e_\alpha}(\psi^i \otimes \nabla_{\partial_s}\partial_{y_i}) + e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_s), d\phi(e_\alpha))\partial_{y_i}. \end{aligned}$$

Hence, we have

$$\begin{aligned}
II &= \int_{R^2} \langle \psi, \frac{d}{ds} \not{D}\psi \rangle |_{s=0} \\
&= \int_{R^2} \langle \psi, \not{D}(\psi^i \otimes \nabla_{\partial_s}\partial_{y_i})\rangle |_{s=0} + \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_s), d\phi(e_\alpha))\partial_{y_i}\rangle |_{s=0} \\
&= \int_{R^2} \langle \not{D}\psi, \psi^i \otimes \nabla_{\partial_s}\partial_{y_i}\rangle |_{s=0} + \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_s), d\phi(e_\alpha))\partial_{y_i}\rangle |_{s=0} \\
&= \int_{R^2} \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(d\phi(\partial_s), d\phi(e_\alpha))\partial_{y_i}\rangle |_{s=0} \\
&= \int_{R^2} \langle \psi, e_\alpha \cdot \psi^i \otimes R^N(\xi^m \partial_{y_m}, \phi^l_\alpha \partial_{y_l})\partial_{y_i}\rangle \\
&= \int_{R^2} \langle \psi, e_\alpha \cdot \psi^i \otimes \xi^m \phi^l_\alpha R^j_{iml}\partial_{y_j}\rangle \\
&= \int_M \langle \psi^i, d\phi^l \cdot \psi^j \rangle R_{mlij}\xi^m,
\end{aligned}$$

where we have used (2.2).Consequently, we have

$$\frac{dL(\phi_s)}{ds}|_{s=0} = \int_{R^2} [-2g_{mi}\Box^i(\phi) + R_{mlij}\langle \psi^i, d\phi^l \cdot \psi^j\rangle]\xi^m,$$

and hence (2.1). □

3. Global Existence

In this section we will prove the main theorem. Before we prove the theorem, let us note the following facts. Consider $\mathbb{R}^{1+1}$ with the Euclidean metric $dt^2 - dx^2$. Let $e_1 = \frac{\partial}{\partial t}$ and $e_2 = \frac{\partial}{\partial x}$ be the standard orthonormal frame. A spinor field is simply a map $\Psi : \mathbb{R}^{1+1} \to \Delta_2 = \mathbb{C}^2$, and e_1 and e_2 acting on spinor fields can be identified by multiplication with matrices

$$e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

If $\Psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \mathbb{R}^{1+1} \to \mathbb{C}^2$ is a spinor field, then the Dirac operator is

$$\partial\!\!\!/ \Psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_1}{\partial t} \\ \frac{\partial \psi_2}{\partial t} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_1}{\partial x} \\ \frac{\partial \psi_2}{\partial x} \end{pmatrix} = 2 \begin{pmatrix} -\frac{\partial \psi_2}{\partial \xi} \\ \frac{\partial \psi_1}{\partial \eta} \end{pmatrix},$$

where $\xi = \frac{t+x}{2}, \eta = \frac{t-x}{2}$ is characteristic coordinates.

Using this fact we can write

$$\not{D}\psi^i = 2 \begin{pmatrix} -\frac{\partial \psi_2^i}{\partial \xi} - \Gamma^i_{jk}(\phi)\frac{\partial \phi^j}{\partial \xi}\psi_2^k \\ \frac{\partial \psi_1^i}{\partial \eta} + \Gamma^i_{jk}(\phi)\frac{\partial \phi^j}{\partial \eta}\psi_1^k \end{pmatrix} = 2 \begin{pmatrix} -D_\xi \psi_2^i \\ D_\eta \psi_1^i \end{pmatrix}$$

Therefore the equation (2.2) is equivalent the following systems of equations of first order

$$D_\xi \psi_2^i = 0 \tag{3.1}$$

$$D_\eta \psi_1^i = 0 \tag{3.2}$$

We also write the equation (2.1) in the simple form

$$D_\eta \phi_\xi = 0 \tag{3.3}$$

or, equivalently,

$$D_\xi \phi_\eta = 0 \tag{3.4}$$

where $\phi_\xi = \frac{\partial \phi}{\partial \xi}$ are the tangent vectors of the ξ-curve which are the image of the characteristics η =const. in the R^{1+1}, and D_η is the symbol for covariant derivatives of the $\eta-$curves. ϕ_η and $\not{D}_\xi$ are defined similarly.

So we transform the original problem to the following systems:

$$\frac{\partial u^i}{\partial \eta} + \Gamma^i_{jk}(z) u^j v^k = \frac{1}{2} R^i_{lkj}(z) \langle \psi^k, u^l e_\xi \cdot \psi^j \rangle + \frac{1}{2} R^i_{lkj}(z) \langle \psi^k, v^l e_\eta \cdot \psi^j \rangle \tag{3.5}$$

$$\frac{\partial v^i}{\partial \xi} + \Gamma^i_{jk}(y) v^j u^k = \frac{1}{2} R^i_{lkj}(y) \langle \psi^k, u^l e_\xi \cdot \psi^j \rangle + \frac{1}{2} R^i_{lkj}(y) \langle \psi^k, v^l e_\eta \cdot \psi^j \rangle \tag{3.6}$$

$$\frac{\partial y^i}{\partial \xi} = u^i, \qquad \frac{\partial z^i}{\partial \eta} = v^i \tag{3.7}$$

$$\frac{\partial \psi_2^i}{\partial \xi} + \Gamma^i_{jk}(y) u^j \psi_2^k = 0 \tag{3.8}$$

$$\frac{\partial \psi_1^i}{\partial \eta} + \Gamma^i_{jk}(z) v^j \psi_1^k = 0 \tag{3.9}$$

together with the initial conditions

$$\begin{gathered} y^i(0,x) = z^i(0,x) = \phi_0^i(x), \quad \psi^i(0,x) = \psi_0^i \\ u^i(0,x) = \frac{\partial \phi_0^i(x)}{\partial x} + \phi_1^i(x), \quad v^i(0,x) = -\frac{\partial \phi_0^i(x)}{\partial x} + \phi_1^i(x) \end{gathered} \tag{3.10}$$

In order to prove the theorem it is sufficient to prove that the equations (3.5)-(3.9) have global solutions, provided that the initial data satisfy (3.10). Now we turn to the proof of the theorem.

Proof. Define

$$M = \sup_{|x| \le L} \{\|u\|_0, \|v\|_0\}, \quad M_0 = \sup_{|x| \le L} \{\|\psi_1\|_0, \|\psi_2\|_0\}$$

$$M_1 = \sup_{|x| \le L} \{\|D_\xi u\|_0, \|D_\xi v\|_0, \|D_\eta u\|_0, \|D_\eta v\|_0\}$$

$$M_2 = \sup_{|x| \le L} \{\|D_\xi \psi_1\|_0, \|D_\xi \psi_2\|_0, \|D_\eta \psi_1\|_0, \|D_\eta \psi_2\|_0\}$$

where $\|\|_0$ denote the value of a vector at $t = 0$. We shall use C below for uniform bound of R_{ijkl}, Γ^i_{jk} and all their derivatives.

First we will prove the existence of the solutions on Λ_k using the method of iteration, where $\Lambda_k = \{-k \leq -\eta \leq \xi \leq k\}$, $k = \min\{L, \frac{1}{4CM_0^2}\}$, L is a big number.

It is easily seen that $y^i(t,x) = z^i(t,x)$ and (2.1) are satisfied by them. Let u_0^i, v_0^i be any smooth functions satisfying the initial conditions (3.10) and subjected to the following restriction: the function y_0^i, z_0^i defined by

$$\frac{\partial y_0^i}{\partial \xi} = u_0^i, \ \frac{\partial z_0^i}{\partial \eta} = v_0^i$$

$$y_0^i(0,x) = z_0^i(0,x) = \phi_0^i(x) \tag{3.11}$$

Suppose that we have constructed y_{m-1}^i, z_{m-1}^i, u_{m-1}^i, v_{m-1}^i which satisfy the initial conditions (3.10). Define y_m^i, z_m^i, u_m^i, v_m^i , ψ_{1m}, ψ_{2m} by the equations

(3.12)
$$\frac{\partial u_m^i}{\partial \eta} + \Gamma_{jk}^i(z_{m-1})u_m^j v_{m-1}^k = \frac{1}{2}R_{lkj}^i(z_{m-1})\langle \psi_m^k, u_m^l e_\xi \cdot \psi_m^j\rangle + \frac{1}{2}R_{lkj}^i(z_{m-1})\langle \psi_m^k, v_m^l e_\eta \cdot \psi_m^j\rangle$$

(3.13)
$$\frac{\partial v_m^i}{\partial \eta} + \Gamma_{jk}^i(y_{m-1})v_m^j u_{m-1}^k = \frac{1}{2}R_{lkj}^i(y_{m-1})\langle \psi_m^k, u_m^l e_\xi \cdot \psi_m^j\rangle + \frac{1}{2}R_{lkj}^i(y_{m-1})\langle \psi_m^k, v_m^l e_\eta \cdot \psi_m^j\rangle$$

$$\frac{\partial y_m^i}{\partial \xi} = u_m^i, \qquad \frac{\partial z_m^i}{\partial \eta} = v_m^i \tag{3.14}$$

$$\frac{\partial \psi_{2m}^i}{\partial \xi} + \Gamma_{jk}^i(y_{m-1})u_{m-1}^j \psi_{2m}^k = 0 \tag{3.15}$$

$$\frac{\partial \psi_{1m}^i}{\partial \eta} + \Gamma_{jk}^i(z_{m-1})v_{m-1}^j \psi_{1m}^k = 0 \tag{3.16}$$

and the initial conditions (3.10). The equations are linear; thus their solutions are well defined on Λ_k. The geometric meaning of (3.15) and (3.16) is that the vector field ψ_{2m}^i, ψ_{1m}^i are parallel along the curves $y_{m-1}^i(\xi,\eta_0), y_{m-1}^i(\xi_0,\eta)$ respectively. Since parallel translation keeps the length of the vector unchanged , we have $\|\psi_{1m}\| = \|\psi_{1m}\|_0$, $\|\psi_{2m}\| = \|\psi_{2m}\|_0$. Now we estimate $\|u_m\|, \|v_m\|$.

$$\begin{aligned}
\|u_m(\xi,\eta)\|^2 - \|u_m(\xi,\eta_0)\|^2 &= \int_{\eta_0}^{\eta} \frac{d}{d\eta}\|u_m\|^2 d\eta \\
&= 2\int_{\eta_0}^{\eta} \langle D_\eta u_m, u_m\rangle
\end{aligned}$$

So we have

$$\begin{aligned}
\|u_m(\xi,\eta)\|^2 &\leq 2\int_{\eta_0}^{\eta} |\langle D_\eta u_m, u_m\rangle| + \|u_m\|_0 \\
&\leq (\eta-\eta_0)\sum(|R_{lkj}^i\langle\psi_m^k, u_m^l e_\xi \cdot \psi^j\rangle u_m^i| + |R_{lkj}^i\langle\psi_m^k, v_m^l e_\eta \cdot \psi^j\rangle u_m^i|) + \|u_m\|_0 \\
&\leq C(\eta-\eta_0)(\|\psi_m\|^2\|u_m\|^2 + \|\psi_m\|^2\|u_m\|\|v_m\|) + M^2 \\
&\leq Ck(\|\psi_m\|^2\|u_m\|^2 + \|\psi_m\|^2\|u_m\|\|v_m\|) + M^2 \\
&\leq CkM_0^2(\|u_m\|^2 + \|u_m\|\|v_m\|) + M^2 \\
&\leq \frac{1}{4}(\|u_m\|^2 + \|u_m\|\|v_m\|) + M^2
\end{aligned}$$

where we have used $k \leq \frac{1}{4CM_0^2}$. Similarly we can get

$$\|v_m(\xi,\eta)\|^2 \leq \frac{1}{4}(\|v_m\|^2 + \|u_m\|\|v_m\|) + M^2$$

Combining these two equations we can

$$\begin{aligned}\|u_m\|^2 + \|v_m\|^2 &\leq \frac{1}{4}(\|u_m\|^2 + \|v_m\|^2 + 2\|u_m\|\|v_m\|) + 2M^2 \\ &\leq \frac{1}{2}(\|u_m\|^2 + \|v_m\|^2) + 2M^2\end{aligned}$$

So we have

$$\|u_m\|^2 + \|v_m\|^2 \leq 4M^2. \tag{3.17}$$

We claim that the equation (3.17) implies more regularity of u_m, v_m, ψ_m. In fact we differentiate the equation (3.15) covariantly, then we can get

$$D_{\phi_{m-1}(\eta)} D_{\phi_{m-1}(\eta)} \psi_{2m} = 0 \tag{3.18}$$

$$0 = D_{\phi_{m-1}(\xi)} D_{\phi_{m-1}(\eta)} \psi_{2m} = D_{\phi_{m-1}(\eta)} D_{\phi_{m-1}(\xi)} \psi_{2m} + R(\partial_\xi \phi_{m-1}, \partial_\eta \phi_{m-1})\psi_{2m}. \tag{3.19}$$

From the equation (3.18) we can know

$$\|D_{\phi_{m-1}(\eta)}\psi_{2m}\| = \|D_{\phi_{m-1}(\eta)}\psi_{2m}\|_0 \leq M_2$$

From the equation (3.19) we can get

$$\begin{aligned}\|D_{\phi_{m-1}(\xi)}\psi_{2m}\|^2 &= \int_{\eta_0}^{\eta} \frac{d}{d\eta}\|D_{\phi_{m-1}(\xi)}\psi_{2m}\|^2 + \|D_{\phi_{m-1}(\xi)}\psi_{2m}\|_0^2 \\ &\leq 2\int_{\eta_0}^{\eta} \langle D_{\phi_{m-1}(\eta)} D_{\phi_{m-1}(\xi)}\psi_{2m}, D_{\phi_{m-1}(\xi)}\psi_{2m}\rangle + M_2^2 \\ &\leq 2kC\|u_{m-1}\|\|v_{m-1}\|\|\psi_{2m}\|\|D_{\phi_{m-1}(\xi)}\psi_{2m}\| + M_2^2 \\ &\leq kC(\|u_{m-1}\|^2 + \|v_{m-1}\|^2)\|\psi_{2m}\|\|D_{\phi_{m-1}(\xi)}\psi_{2m}\| + M_2^2 \\ &\leq 4kCM^2M_0\|D_{\phi_{m-1}(\xi)}\psi_{2m}\| + M_2^2\end{aligned}$$

So,

$$\|D_{\phi_{m-1}(\xi)}\psi_{2m}\| \leq \frac{M^2}{M_0} + M_2.$$

By the same argument we can obtain

$$\|D_{\phi_{m-1}(\xi)}\psi_{1m}\|_0 \leq M_2$$

and

$$\|D_{\phi_{m-1}(\eta)}\psi_{1m}\| \leq \frac{M^2}{M_0} + M_2.$$

We differentiate the equation (3.13), then

$$\begin{aligned}D_\eta D_\eta u_m &= \frac{1}{2}R^i_{lkj,m}(z_{m-1})v^m_{m-1}\langle\psi^k_m, u^l_m e_\xi \cdot \psi^j_m\rangle\frac{\partial}{\partial y^i} + \frac{1}{2}R^i_{lkj}(z_{m-1})\langle\widetilde{\nabla}_\eta\psi^k_m, u^l_m e_\xi \cdot \psi^j_m\rangle\frac{\partial}{\partial y^i} \\ &+\frac{1}{2}R^i_{lkj}(z_{m-1})\langle\psi^k_m, D_\eta u^l_m e_\xi \cdot \psi^j_m\rangle\frac{\partial}{\partial y^i} + \frac{1}{2}R^i_{lkj}(z_{m-1})\langle\psi^k_m, u^l_m e_\xi \cdot \widetilde{\nabla}_\eta\psi^j_m\rangle\frac{\partial}{\partial y^i} \\ &+\frac{1}{2}R^i_{lkj}(z_{m-1})\langle\psi^k_m, u^l_m e_\xi \cdot \psi^j_m\rangle v^m_{m-1}\Gamma^n_{mi}(z_{m-1})\frac{\partial}{\partial y^n}\end{aligned}$$

plus a similar term with u^l_m replaced by v^l_m, e_ξ replaced by e_η in the $\langle\cdot,\cdot\rangle$. Using the method as above we can estimate

$$\begin{aligned}\|D_\eta u_m\|^2 &\leq 2kC(4M^2M_0^2+4MM_1M_0(\frac{M^2}{M_0}+M_1^2)+4M_0^2M^2C)\|D_\eta u_m\| \\ &\quad +kCM_0^2\|D_\eta u_m\|^2+kCM_0^2\|D_\eta u_m\|\|D_\eta v_m\|+M_1^2 \\ &\leq C_1\|D_\eta u_m\|+\frac{1}{4}(\|D_\eta u_m\|^2+\|D_\eta u_m\|\|D_\eta v_m\|)+M_1^2\end{aligned}$$

Similarly,

$$\|D_\eta v_m\|^2 \leq C_1\|D_\eta v_m\|+\frac{1}{4}(\|D_\eta u_m\|^2+\|D_\eta u_m\|\|D_\eta v_m\|)+M_1^2$$

where $C_1 = 2(M^2+4\frac{MM_1}{M_0}(\frac{M^2}{M_0}+M_1^2)+M^2C)$ Therefore we have

$$\|D_\eta u_m\|+\|D_\eta v_m\| \leq 4C_1+4M_1.$$

By the same argument we can get more regularity. So we can proof the sequences $\{y_m\}, \{z_m\}$ and $u_m, v_m, \psi_{1m}, \psi_{2m}$ and the sequences of their partial derivatives converge uniformly on Λ_k and the limit of $u_m(t,x), v_m(t,x), \psi_{1m}(t,x), \psi_{2m}(t,x)$ is a smooth solution of the equation (3.5)-(3.9) and the limit of $y_m(t,x)$ is a smooth solution of the equation (2.1). From the proof we can see that the constant k only depends on M_0 and C. Because at any time $\|\psi_1\|, \|\psi_2\| \leq M_0$, provided that $|x| \leq L$, then using this procedure successively we can conclude that there exists global solutions of the equation (3.5)-(3.9) This completes the proof of the existence part of the theorem. The uniqueness is easy.

Let (ϕ^1,ψ^1) and (ϕ^2,ψ^2) are two solutions with the same data, then $\phi^1-\phi^2$ and $\psi^1-\psi^2$ have zero Cauchy data at $t=0$. Let $\phi=\phi^1-\phi^2$ and $\psi=\psi^1-\psi^2$, then they satisfy the following equations,

$$\begin{aligned}\frac{\partial u}{\partial\eta}+\Gamma(\phi_1)uv^1 &= (\Gamma(\phi^2)-\Gamma(\phi^1))u^2v^1+\Gamma(\phi^2)u^2(v^2-v^1) \\ &\quad +\frac{1}{2}(R(\phi^1)-R(\phi^2))\langle\psi^1,u^1e_\xi\cdot\psi^1\rangle+\frac{1}{2}R(\phi^2)\langle(\psi^1-\psi^2),u^1e_\xi\cdot\psi^1\rangle \\ &\quad +\frac{1}{2}R(\phi^2)\langle\psi^2,u^1e_\xi\cdot(\psi^1-\psi^2)\rangle+\frac{1}{2}R(\phi^2)\langle\psi^2,(u^1-u^2)e_\xi\cdot\psi^2\rangle \\ &\quad +\frac{1}{2}(R(\phi^1)-R(\phi^2))\langle\psi^1,v^1e_\eta\cdot\psi^1\rangle+\frac{1}{2}R(\phi^2)\langle(\psi^1-\psi^2),v^1e_\eta\cdot\psi^1\rangle \\ &\quad +\frac{1}{2}R(\phi^2)\langle\psi^2,v^1e_\eta\cdot(\psi^1-\psi^2)\rangle+\frac{1}{2}R(\phi^2)\langle\psi^2,(v^1-v^2)e_\eta\cdot\psi^1\rangle\end{aligned}$$

$$\frac{\partial v}{\partial\xi}+\Gamma(\phi_1)vu^1 = (\Gamma(\phi^2)-\Gamma(\phi^1))v^2u^1+\Gamma(\phi^2)v^2(u^2-u^1)+f$$

f denote the right terms of the last equation excluding the first two terms.

$$\frac{\partial\psi_2}{\partial\eta}+\Gamma(\phi_1)u^1\psi_2 = (\Gamma(\phi_2)-\Gamma(\phi_1))u^1\phi_2^2+\Gamma(\phi_2)(u^2-u^1)\psi_2^2$$

$$\frac{\partial\psi_1}{\partial\xi}+\Gamma(\phi_1)v^1\psi_1 = (\Gamma(\phi_2)-\Gamma(\phi_1))v^1\phi_1^2+\Gamma(\phi_2)(v^2-v^1)\psi_1^2$$

Frow these equations we can estimate $\|u\|, \|v\|$ and $\|\psi_1\|, \|\psi_2\|$ as above, then we can get

$$\|\psi_2\| \leq 2kC(\|\phi\|\|u^1\|\|\psi_2^2\|+\|u\|\|\psi_2^2\|) \qquad (3.20)$$

$$\|\psi_1\| \leq 2kC(\|\phi\|\|uv_1\|\|\psi_1^2\| + \|v\|\|\psi_1^2\|) \tag{3.21}$$

and

(3.22)

$$\|u\|+\|v\| \leq 2kC(\|u\|+\|v\|)(\|u^2\|+\|v^2\|)+4kC\|\psi^2\|^2(\|u\|+\|v\|)+2kC(\|\phi\|+\|\psi\|)$$

If k is sufficiently small, we have

$$\|u\| + \|v\| \leq C_1(\|\phi\| + \|\psi\|) \tag{3.23}$$

Put the equation (3.20) and (3.21) into the above equation, we can get

$$\|u\| + \|v\| \leq C_2\|\phi\|$$

From Gronwall's inequality, we know that $\phi \equiv 0$. Then the equation (3.20) and (3.21) imply $\psi \equiv 0$ immediately and we finish the proof. □