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Chern Simons and String Theory

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Chern-Simons and String Theory

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List of Topics

1. Some Historical Remarks
2. Knot Invariants via Chern-Simons TQFT
3. WRT-Invariants of 3-Manifolds
4. The Conifold Transition
5. Chern-Simons and String Theory

Recently several of us (lucky mathematical physicists) attended the Bayrischzell workshop 2004 on noncommutativity and physics. One of the speakers told us that gauge theory, open string theory, closed string theory and noncommutative QFT are the same. Perhaps the heady alpine air had someting to do with this euphoric statement. Nevertheless several bits and pieces of evidence supporting the existence of links between these theories has accumulated over the last few years. However, stealing a line from Cliff Taubes, I would like to say that at least from a mathematical point of view we would be lucky if in a few years we know what are the right questions to ask. There is a saying "If you steal from one you are a thief. If you steal from many you are a business (or a researcher)". As you will see this is a research talk.
This year we are celebrating the 25th anniversary of the marriage between Gauge Theory and the Geometry of Fiber Bundles from the sometime warring tribes of Physics and Mathematics. Marriage brokers were none other than Chern and Simons. The 1978 paper by Wu and Yang can be regarded as the announcement of this union. It has led to many wonderful offspring. The theories of Donaldson, Chern-Simons, Floer-Fukaya, Seiberg-Witten, and TQFT are just some of the more famous members of their extended family. Quantum Groups, CFT, Supersymmetry, String Theory and Gravity also have close ties with this family. In this talk we will discuss one particular relationship that has recently come to light. The qualitative aspects of Chern-Simons theory as string theory were investigated by Witten almost ten years ago. Before recounting the main idea of this work we review the Feynman path integral method of quantization which is particulary suited for studying topologial quantum field theories.

A **quantum field theory** may be considered as an assignment of the **quantum expectation** $< \Phi >_\mu$ to each gauge invariant function $\Phi : \mathcal{A}(M) \to \mathbf{C}$, where $\mathcal{A}(M)$ is the space of gauge potentials for a given gauge group G. Φ is called an **observable** in quantum field theory. In the Feynman path integral approach to quantization the quantum expectation $< \Phi >_\mu$ of an observable is given by the following expression.

$$< \Phi >_\mu = \frac{\int_{\mathcal{A}(M)} e^{-S_\mu(\omega)} \Phi(\omega) \mathcal{D}\mathcal{A}}{\int_{\mathcal{A}(M)} e^{-S_\mu(\omega)} \mathcal{D}\mathcal{A}}, \tag{1}$$

where $\mathcal{DA}$ is a suitably defined measure on $\mathcal{A}(M)$. It is customary to express the quantum expectation $< \Phi >_\mu$ in terms of the **partition function** Z_μ

defined by

$$Z_\mu(\Phi) := \int_{\mathcal{A}(M)} e^{-S_\mu(\omega)}\Phi(\omega)\mathcal{D}\mathcal{A}. \qquad (2)$$

Thus we can write

$$< \Phi >_\mu = \frac{Z_\mu(\Phi)}{Z_\mu(1)}. \qquad (3)$$

In the above equations we have written the quantum expectation as $< \Phi >_\mu$ to indicate explicitly that, in fact, we have a one-parameter family of quantum expectations indexed by the coupling constant μ in the action. There are several examples of gauge invariant functions. For example, primary characteristic classes evaluated on suitable homology cycles give an important family of gauge invariant functions. The instanton number k of $P(M, G)$ belongs to this family, as it corresponds to the second Chern class evaluated on the fundamental cycle of M representing the fundamental class $[M]$. The pointwise norm $|F_\omega|_x$ of the gauge field at $x \in M$, the absolute value $|k|$ of the instanton number k and the Yang-Mills action S_μ are also gauge invariant functions. Another important example is the Wilson loop functional well known in the physics literature.

Wilson loop functional: Let ρ denote a representation of G on V. Let $\alpha \in \Omega(M, x_0)$ denote a loop at $x_0 \in M$. Let $\pi : P(M, G) \to M$ be the canonical projection and let $p \in \pi^{-1}(x_0)$. If ω is a connection on P, then the parallel translation along α maps the fiber $\pi^{-1}(x_0)$ into itself. Let $\hat{\alpha}_\omega : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$ denote this map. Since G acts transitively on the fibers, $\exists g_\omega \in G$ such that $\hat{\alpha}_\omega(p) = pg_\omega$. Now define

$$\mathcal{W}_{\rho,\alpha}(\omega) := Tr[\rho(g_\omega)] \; \forall \omega \in \mathcal{A}. \qquad (4)$$

We note that g_ω and hence $\rho(g_\omega)$, change by conjugation if, instead of p, we choose another point in the fiber $\pi^{-1}(x_0)$, but the trace remains unchanged. We call these $\mathcal{W}_{\rho,\alpha}$ the Wilson loop functionals associated to the representation ρ and the loop α. In the particular case when $\rho = Ad$ the adjoint representation of G on $\mathbf{g}$, our constructions reduce to those considered in physics.

In the 1980s, Jones discovered his polynomial invariant $V_\kappa(q)$, called the **Jones polynomial**, while studying Von Neumann algebras and gave its interpretation in terms of statistical mechanics. These new polynomial invariants have led to the proofs of most of the Tait conjectures. As with the earlier invariants, Jones' definition of his poynomial invariants is algebraic and combinatorial in nature and was based on representations of the braid groups and related Hecke algebras. The Jones polynomial $V_\kappa(t)$ of κ is a Laurent polynomial in t (polynomial in t and t^{-1}) which is uniquely determined by a simple set of properties similar to the well known axioms for the Alexander-Conway polynomial. More generally, the Jones polynomial can be defined for any oriented link L as a Laurent polynomial in $t^{1/2}$.

A geometrical interpretation of the Jones' polynomial invariant of links was provided by Witten by applying ideas from QFT to the Chern-Simons Lagrangian constructed from the Chern-Simons action

$$\mathcal{A}_{CS} = \frac{k}{4\pi} \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

where A is the gauge potential of the $SU(n)$ connection ω. Chern-Simons action is not guage invariant. Under a gauge transformation g the action transforms as follows:

$$\mathcal{A}_{CS}(A^g) = \mathcal{A}_{CS}(A) + 2\pi k \mathcal{A}_{WZ}, \tag{5}$$

where $\mathcal{A}_{WZ}$ is the **Wess-Zumino action functional**. It can be shown that the Wess-Zumino functional is integer valued and hence, if the Chern-Simons coupling constant k is taken to be an integer, then the partition function Z defined by

$$Z(\Phi) := \int_{\mathcal{A}(M)} e^{-i\mathcal{A}_{CS}(\omega)} \Phi(\omega) \mathcal{D}\mathcal{A}$$

is gauge invariant.

We denote the Jones polynomial of L simply by V. Recall that there are 3 standard ways to change a link diagram at a crossing point. The Jones polynomials of the corresponding links are denoted by V_+, V_- and V_0 respectively. To verify the defining relations for the Jones' polynomial of a link L in S^3, Witten starts by considering the Wilson loop functionals for the associated links L_+, L_-, L_0. Witten obtains the following skein relation for the polynomial invariant V of the link

$$t^{n/2} V_+ - t^{-n/2} V_- = (t^{1/2} - t^{-1/2}) V_0 \tag{6}$$

where we have put

$$< \Phi > = \ V(t), \text{ and } t = e^{2\pi i/(k+n)}.$$

We note that the result makes essential use of the Verlinde fusion rules in $2d$ conformal field theory.

For $SU(2)$ Chern-Simons theory, equation (6) is the skein relation that defines a variant of the original Jones' polynomial. This variant also occurs in the work of Kirby and Melvin where the invariants are studied by using representation theory of certain Hopf algebras and the topology of framed links. It is not equivalent to the Jones polynomial. In an earlier work I had observed that under the transformation $\sqrt{t} \to -1/\sqrt{t}$, it goes over into the equation which is the skein relation characterizing the Jones polynomial. The Jones polynomial belongs to a different family that corresponds to the negative values of the level. Note that the coefficients in the skein relation (6) are defined for positive valuse of the level k. To extend them to negative values of the level we must also note that the shift in k by the dual Coxeter number would now change the level $-k$ to $-k-n$. If in equation (6) we now allow negative values of n and take t to be a formal variable, then the extended family includes both positive and negative levels.

Let $V^{(n)}$ denote the Jones-Witten polynomial corresponding to the skein relation (6), (with $n \in \mathbf{Z}$) then the family of polynomials $\{V^{(n)}\}$ can be shown to be equivalent to the two variable HOMFLY polynomial $P(\alpha, z)$ which satisfies the following skein relation

$$\alpha P_+ - \alpha^{-1} P_- = z P_0. \tag{7}$$

If we put $\alpha = t^{-1}$ and $z = (t^{1/2} - t^{-1/2})$ in equation (7) we get the skein relation for the original Jones polynomial V. If we put $\alpha = 1$ we get the skein relation for the Alexander-Conway polynomial.

To compare our results with those of Kirby and Melvin we note that they use q to denote our t and t to denote its fourth root. They construct a modular Hopf algebra U_t as a quotient of the Hopf algebra $U_q(sl(2, \mathbf{C}))$ which is the well known q-deformation of the universal envelopping algebra of the Lie algebra $sl(2, \mathbf{C})$. Jones polynomial and its extensions are obtained by studying the representaions of the algebras U_t and U_q.

If $Z_k(1)$ exists, it provides a numerical invariant of M. For example, for $M = S^3$ and $G = SU(2)$, using the Chern-Simons action Witten obtains the following expression for this partition function as a function of the level k

$$Z_k(1) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right). \tag{8}$$

This partition function provides a new family of invariants of S^3. Such a partition function can be defined for a more general class of 3-manifolds and gauge groups. More precisely, let G be a compact, simply connected, simple Lie group and let $k \in \mathbf{Z}$. Let M be a 2-framed closed, oriented 3-manifold. We define the **Witten invariant** $\mathcal{T}_{G,k}(M)$ of the triple (M, G, k) by

$$\mathcal{T}_{G,k}(M) := Z(1) := \int_{\mathcal{A}(M)} e^{-i\mathcal{A}_{CS}} \mathcal{D}\mathcal{A}, \tag{9}$$

where $\mathcal{D}\mathcal{A}$ is a suitable measure on $\mathcal{A}(M)$.

We note that no precise definition of such a measure is available at this time and the definition is to be regarded as a formal expression. Indeed, one of the aims of TQFT is to make sense of such formal expressions. We define the **normalized Witten invariant** $\mathcal{W}_{G,k}(M)$ of a 2-framed, closed, oriented 3-manifold M by

$$\mathcal{W}_{G,k}(M) := \frac{\mathcal{T}_{G,k}(M)}{\mathcal{T}_{G,k}(S^3)}. \tag{10}$$

Then we have the following theorem.
Theorem (Witten, Reshetikhin, Turaev):
Let G be a compact, simply connected, simple Lie group. Let M, N be two 2-framed, closed, oriented 3-manifolds. Then we have the following results:

$$\mathcal{T}_{G,k}(S^2 \times S^1) = 1 \tag{11}$$

$$\mathcal{T}_{SU(2),k}(S^3) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) \tag{12}$$

$$\mathcal{W}_{G,k}(M \# N) = \mathcal{W}_{G,k}(M)\mathcal{W}_{G,k}(N) \tag{13}$$

In his work Kohno defined a family of invariants $\Phi_k(M)$ of a 3-manifold M by its Heegaard decomposition along a Riemann surface Σ_g and representations of its mapping class group in the space of conformal blocks. The agreement of his results (up to normalization) with those of Witten may be regarded as strong evidence for the usefulness of the ideas from TQFT and CFT in low dimensional geometric topology. We remark that a mathematically precise definition of the Witten invariants via solutions of the Yang-Baxter equations and representations of the corresponding quantum groups was given by Reshetikhin and Turaeev. For this reason, we now refer to them as Witten-Reshetikhin-Turaeev or WRT invariants. The invariant is well defined only at roots of unity and perhaps near roots of unity if a perturbative expansion is possible. This situation occurs in the study of classical modular functions and Ramanujan's mock theta functions.

Ramanujan had introduced his mock theta functions in a letter to Hardy in 1920 (the famous last letter) to describe some power series in variable $q = e^{2\pi iz}, z \in \mathbf{C}$. He also wrote down (without proof, as was usual in his work) a number of identities involving these series which were completely verified only in 1988 . Recently, Lawrence and Zagier have obtained several different formulas for the Witten invariant $\mathcal{W}_{SU(2),k}(M)$ of the Poincaré homology sphere $M = \Sigma(2, 3, 5)$. They show how the Witten invariant can be extended from integral k to rational k and give its relation to the mock theta function. In particular, they obtain the following fantastic formula, a la Ramanujan, for the Witten invariant of the Poincaré homology sphere

$$\mathcal{W} = 1 + \sum_{n=1}^{\infty} x^{-n^2}(1 + x)(1 + x^2)\dots(1 + x^{n-1})$$

where $x = e^{\pi i/(k+2)}$. We note that the series on the right hand side of this formula terminates after $k + 2$ terms.

As we remarked at the begining of this talk the question "what is the relationship between gauge theory and string theory?" is not meaningful at this time. However, interesting special cases where such relationship can be established are emerging. For example, Witten has argued that Chern-Simons gauge theory on a 3-manifold M can be viewed as a string theory constructed by using a topological sigma model with target space T^*M. The perturbation theory of this string will coincide with Chern-Simons perturbation theory, in the form discussed by Axelrod and Singer. The coefficient of

k^{-r} in the perturbative expansion of $SU(n)$ theory in powers of $1/k$ comes from Feynman digrams with r loops. Witten shows how each diagram can be replaced by a Riemann surface Σ of genus g with h holes (boundary components) with $g = (r - h + 1)/2$. Gauge theory would then give an invariant $\Gamma_{g,h}(M)$ for every topological type of Σ. Witten shows that this invariant would equal the corresponding string partition function $Z_{g,h}(M)$.
We now give an example of gauge theory to string theory correspondence relating the non-perturbative WRT invariants in Chern-Simons theory with gauge group $SU(n)$ and topological string amplitudes which generalize the GW (Gromov-Witten) invariants of Calabi-Yau 3-folds. The passage from real 3 dimensional Chern-Simons theory to the 10 dimensional string theory and further onto the 11 dimensional M-theory can be schematically represented by the following:

$$\begin{aligned} 3+3 &= 6 \text{ (real symplectic 6-manifold)} \\ &= 6 \text{ (conifold in } \mathbf{C}^4 \text{)} \\ &= 6 \text{ (Calabi-Yau manifold)} \\ &= 10-4 \text{ (string compactification)} \\ &= (11-1)-4 \text{ (M-theory)} \end{aligned}$$

We now discuss the significance of the various terms of the above equation array.

The first line suggests that we consider open topological strings on the cotangent bundle T^*S^3 with Dirichlet boundary conditions on the zero section S^3. We can compute the open topological string amplitudes from the $SU(n)$ Chern-Simons theory. Conifold transition has the effect of closing up the holes in open strings to give closed strings on the Calabi-Yau manifold obtained by the usual string compactification from 10 dimensions. Thus we recover a topological gravity result starting from gauge theory. In fact, as we discussed earlier, Witten had anticipated such a gauge theory string theory correspondance almost ten years ago. Significance of the last line is based on the conjectured equivalence of M-theory compactified on S^1 to type IIA strings compactified on a Calabi-Yau threefold.

To understand the relation of the WRT invariant of S^3 for $SU(n)$ Chern-Simons theory with open and closed topological string amplitudes on "Calabi-Yau" manifolds we need to discuss the concept of conifold transition. From the geometrical point of view this corresponds to symplectic surgery in six dimensions. It replaces a vanishing Lagrangian 3-sphere by a symplectic S^2. The starting point of the construction is the observation that T^*S^3 minus its zero section is symplectomorphic to the cone $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ minus the origin in $\mathbf{C}^4$, where each manifold is taken with its standard symplectic structure. The complex singularity at the origin can be smoothed out by the manifold M_τ defined by $z_1^2 + z_2^2 + z_3^2 + z_4^2 = \tau$ producing a Lagrangian S^3 vanishing cycle. There are also two so called small resolutions $M^\pm$ of the singularity with exceptional set $\mathbf{CP}^1$.

They are defined by

$$M^\pm := \left\{ z \in \mathbf{C}^4 \mid \frac{z_1 + iz_2}{z_3 \pm iz_4} = \frac{-z_3 \pm iz_4}{z_1 - iz_2} \right\} .$$

Note that $M_0 \backslash \{0\}$ is symplectomorphic to each of $M^\pm \backslash \mathbf{CP}^1$. Blowing up the exceptional set $\mathbf{CP}^1 \subset M^\pm$ gives a resolution of the singularity which can be expressed as a fiber bundle F over $\mathbf{CP}^1$. Going from the fiber bundle T^*S^3 over S^3 to the fiber bundle F over $\mathbf{CP}^1$is referred to in the physics literature as the conifold transition. We note that holomorphic automorphism of $\mathbf{C}^4$ given by $z_4 \mapsto -z_4$ switches the two small resolutions $M^\pm$ and changes the orientation of S^3. Conifold transition can also be viewed as an application of mirror symmetry to Calabi-Yau manifolds with singularities. Such an interpretation requires the notion of symplectic Calabi-Yau manifolds and the corresponding enumerative geometry.

To find the relation between the large n limit of $SU(n)$ Chern-Simons theory on S^3 to a special topological string amplitude on a Calabi-yau manifold we begin by recalling the formula for the partition function (vacuum amplitude) of the theory $\mathcal{T}_{SU(n),k}(S^3)$ or simply $\mathcal{T}$. Upto a phase, it is given by

$$\mathcal{T} = \frac{1}{\sqrt{n(k+n)^{(n-1)}}} \prod_{j=1}^{n-1} \left[2 \sin \left(\frac{j\pi}{k+n} \right) \right]^{n-j} . \tag{14}$$

Let us denote by $F_{(g,h)}$ the amplitude of an open topological string theory on T^*S^3 of a Riemann surface of genus g with h holes. Then the generating function for the free energy can be expressed as

$$-\sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \lambda^{2g-2+h} n^h F_{(g,h)} \tag{15}$$

This can be compared directly with the result from Chern-Simons theory by expanding the $\log \mathcal{T}$ as a double power series in λ and n.

Instead of that we use the conifold transition to get the topological amplitude for a closed string on a Calabi-Yau manifold. We want to obtain the large n expansion of this amplitude in terms of parameters λ and τ which are defined in terms of the Chern-Simons parameters by

$$\lambda = \frac{2\pi}{k+n}, \tau = n\lambda = \frac{2\pi n}{k+n}. \tag{16}$$

The parameter λ is the string coupling constant and τ is the 't Hooft coupling $n\lambda$ of the Chern-Simons theory. The parameter τ has the geometric interpretation as the Kähler modulus of a blown up S^2 in the string amplitude expansion. If $F_g(\tau)$ denotes the amplitude for a closed string at genus g then we have

$$F_g(\tau) = \sum_{h=1}^{\infty} \tau^h F_{(g,h)} \tag{17}$$

So summing over the holes amounts to filling them up to give the closed string amplitude.

The large n expansion of $\mathcal{T}$ in terms of parameters λ and τ is given by

$$\mathcal{T} = \exp\left[-\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(\tau) \right] , \tag{18}$$

where F_g defined in (17) can be interpreted on the string side as the contribution of closed genus g Riemann surfaces. For $g > 1$ the F_g can be expressed in terms of the Euler characteristic χ_g and the Chern class c_{g-1} of the Hodge bundle of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g as follows

$$F_g = \int_{\mathcal{M}_g} c_{g-1}^3 - \frac{\chi_g}{(2g-3)!} \sum_{n=1}^{\infty} n^{2g-3} e^{-n(\tau)} . \tag{19}$$

The integral appearing in the formula for F_g can be evaluated explicitly to give

$$\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{(-1)^{(g-1)}}{(2\pi)^{(2g-2)}} 2\zeta(2g-2)\chi_g. \tag{20}$$

The Euler characteristic is given by the Harer-Zagier formula

$$\chi_g = \frac{(-1)^{(g-1)}}{(2g)(2g-2)} B_{2g} \ , \tag{21}$$

where B_{2g} is the $(2g)$-th Bernoulli number. We omit the special formulas for the genus 0 and genus 1 cases. The formulas for F_g for $g \geq 0$ coincide with those of the g-loop topological string amplitude on a suitable Calabi-Yau manifold. This result can be extended to show that the expectation value of the quantum observable defined by the Wilson loop in the Chern-Simons theory also has a similar interpretation in terms of a topological string amplitude. The change in geometry that leads to this calculation can be thought of as the result of coupling to gravity. Such a situation occurs in the quantization of Chern-Simons theory. Here the classical Lagrangian does not depend on the metric, however, coupling to the gravitational Chern-Simons term is necessary to make it TQFT.
It is well known that every closed connected oriented 3-manifold can be obtained by surgery on a framed link in S^3. Moreover, such a 3-manifold is the boundary of a 4-manifold which is a 4-ball with finitely many 2-handles attached. It would be interesting to relate invariants of 4-manifolds to the quantum invariants of 3-manifolds. The use of Wilson loops reminds us of Ashtekar's formalism for gravity. Chern-Simons theory is also closely related to the spin network formulation of 3-dimensional quantum gravity. Recently Kontsevich has proposed an extension of TQFT by using some old ideas of Grothendieck. Perhaps such a theory may be viewed as a time evolution of 3-dimensional Chern-Simons states. Following a propossal of Ooguri, Crane and Yetter have found a new invariant of a compact oriented 4-manifold M by applying ideas from modular tensor category to the 3-skeleton of M.
This invariant has an analytic expression in terms of the level k WRT invariant of the $SU(2)$ Chern-Simons theory and the corresponding string coupling constant and two classical topological invariants of M, namely, the Hirzebruch signature $\sigma(M)$ and the Euler characteristic $\chi(M)$. We note that Temperley-Lieb algebras associated to link diagrams and the Kirillov-Reshetikhin approach to link invariants via representations of the quantum deformation of the universal enveloping algebra of $su(2)$, play a crucial role in obtaining the above formula. It can be thought of as a local combinatorial

description of $\sigma(M)$, in the spirit of Gelfand and Macpherson's interpretation of Pontryagin classes.

We now interrupt this regular lecture to bring you the following

BREAKING NEWS

Over the years, I have been incredibaly lucky to get the breaking news about exciting developments in mathematics. From small gems like Apery's proof of the irrationality of $\zeta(3)$ (from Dieudonne) to Kohinoors like Andrew Wiles proof of FLT (from George Booth via Bill Messing), I remember them all vividly. Here is the news about some interesting conjectures which will soon be renamed as theorems.

Kepler's conjecture: Kepler was an extraordinary observer of nature. His observations of snowflackes, honeycombs and the packing of seeds in various fruit led him to his lesser known study of the sphere packing problem. For dimensions 1, 2 and 3 he found the answers to be 2, 6 and 12 respectively. The lattice structures on these spaces played a crucial role in Kepler's "proof". The three dimensional problem came to be known as Kepler's conjecture.

The slow progress in the solution of this problem led John Milnor to remark that here is a problem that nobody can solve but its answer is known to every schoolboy. It was only solved recently by Tom Hales and the problem in higher dimensions is still wide open.

It was the study of this problem that led John Conway to the discovery of his sporadic simple group. Soon thereafter the last holdouts in the complete list of the 26 finite sporadic simple groups were found. All the infinite families of finite simple groups (such as the groups $\mathbf{Z}_p$, for p a prime number and alternating groups $A_n, n > 4$ that we study in the first course in algebra) were already known. So the classification of finite simple groups was complete. It ranks as the greatest achievement of twentieth century mathematics. Hundreds of mathematicians contributed to it. The various parts of the classification together fill more than ten thousand pages.

Conway's group and other sporadic simple groups are closely related to the symmetries of lattices. The study of representations of the largest of these groups (called the Friendly Giant or Fisher-Griess Monster) has led to the creation of a new field of mathematics called Vertex algebras. They turn out to be closely related to the chiral algebras in conformal field theory. These and other ideas inspired by string theory have led to a proof of Conway and Norton's Moonshine conjectures (Borcherds, Frenkel, Lepowski, Meurman). The monster Lie algebra is the simplest example of a Lie algebra of physical states of a chiral string on a 26-dimensional orbifold. This algebra can be defined by using the infinite dimensional graded representation of the monster simple group. Its quantum dimension is related to Jacobi's $SL(2, \mathbf{Z})$ hauptmodule (elliptic modular function of genus 0) $j(q)$, where $q = e^{2\pi iz}, z \in \mathbf{H}$ by

$$j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots$$

Witten's SW = D conjecture: The classification problem for closed 4-manifolds has made rapid progress since the introduction of two new sets of invariants. Donaldson's polynomial invariants were defined by using instanton solutions of Yang-Mills equations in gauge theory with gauge group $SU(2)$ in the mid 1980s. In less than 10 years Seiberg-Witten equations were obtained as a by product of the study of super Yang-Mills equations. These equations are defined using a $U(1)$ monopole gauge theory and the Dirac operator obtained by coupling to a $Spin^c$ structure. Kronheimer and Mrowka obtained a structure theorem for the Donaldson invariants in terms of their basic classes and introduced a technical property called simple type for a closed, simply connected 4-manifold M. There is also a simple type condition and basic classes in SW theory. Witten used the idea of taking ultraviolate and infrared limits of $N = 2$ supersymmetric quantum Yang-Mills theory and metric independence of correlation functions to relate D and SW invariants.

The precise form of Witten's conjecture can be expressed as follows:
A closed, simply connected 4-manifold M has KM-simple type if and only if it has SW-simple type. If M has simple type and if $D(\alpha)$ (resp. $SW(\alpha)$) denote the generating function series for the Donaldson (resp. Seiberg-Witten) invariants with $\alpha \in H_2(M; \mathbf{R})$, then we have

$$D(\alpha) = 2^c e^{\iota_M(\alpha)/2} SW(\alpha), \forall \alpha \in H_2(M; \mathbf{R}).$$

In the above formula ι_M is the intersection form of M and c is a constant given by

$$c = 2 + \frac{7\chi(M) + 11\sigma(M)}{4}.$$

A mathematical approach to a proof of Witten's conjecture was proposed by Pidstrigatch and Tyurin. Feehan and Lenses have used similar ideas but employ an $SO(3)$ monopole gauge theory which generalizes both the instanton and $U(1)$ monopole theories. The problem of relating this proof to Witten's TQFT argument remains open.

Poincaré conjecture is one of the most celebrated conjectures in geometric topology. Its generalizations for spheres $S^n, n > 3$ have been proved. The most recent of these is the case $n = 4$ which is subsumed under the classification of topological 4-manifolds by Freedman. The original case $n = 3$ has resisted all attempts till now and forms a part of the Thurston geometrization conjecture for closed 3-manifolds.

Over the last quarter century Hamilton has studied the topology of a smooth manifold M by considering the Ricci flow equation for the evolution of its metric. This evolution implies the evolution equation for the full Riemann curvature as well as for Ricci and scalar curvatures. Hamilton obtains topological information on M by studying these flows. The full implemntation of Hamilton's program by Perelman is expected to prove the Thurston geometrization conjecture for closed 3-manifolds and in particular, the Poincaré conjecture.

In Perelman's work, the Ricci flow is perturbed by a scalar field which corrosponds in String theory to the dilaton. It is suppossed to determine the overall strength of all interactions. The value of the dilaton field can be thought of as the size of an extra dimension of space. This would give the space 11 dimensions as required in the M-theory. The low energy effective action of the dilaton field is given by the functional $\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2) e^{-f}$. The corresponding variational equations lead to the evolution equations

$$(g_{ij})_t = -2(R_{ij} + \nabla_i \nabla_j f), f_t = -R - \Delta f.$$

After applying a suitable diffeomorphism these equations lead to the gradient flow equations. This modified Ricci flow can be pushed through the singularities by surgery and rescaling. A detailed case by case analysis is then used to prove the geometrization conjecture.

Property P conjecture: In early 1960s Bing tried to find a counterexample to the Poincaré conjecture by constructing 3-manifolds by surgery on knots. Bing and Martin later formalized this search by defining property P of a knot as follows: A knot K has property P if every 3-manifold Y obtained by non-trivial Dehn surgery on K has non-trivial fundamental group.

Conjecture: Every non-trivial knot K has property P. In particular, Y is not a homotopy 3-sphere.

Recently, Kronheimer and Mrowka proved the Property P conjecture by showing that $\pi_1(Y)$ admits a non-trivial homomorphism to the group $SO(3)$. The proof uses several recent results in gauge theory, symplectic and contact geometry and proof of Witten's conjecture relating the Seiberg-Witten and Donaldson invariants.

Verse of the Day

Die meisten
Mathematiker
glauben.
Aber alle
Physiker
wissen.

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