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Tau-functions on spaces of Abelian and quadratic differentials and determinants of Laplacians in Strebel metrics of finite volume

by

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Tau-functions on spaces of Abelian and quadratic differentials and determinants of Laplacians in Strebel metrics of finite volume

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1 Introduction

Various tau-functions play the central role in the theory of integrable partial and ordinary differential equations and numerous areas of their application. In particular, the zeros of Jimbo-Miwa isomonodromic tau-function [12] govern the solvability of matrix Riemann-Hilbert problem. For a class of Riemann-Hilbert problems with special monodromy groups [7, 21] the Jimbo-Miwa tau-function gives rise to a horizontal section of a line bundle over Hurwitz spaces (spaces of ramified coverings of the Riemann sphere) [16, 17]; this horizontal section was called in [16] the Bergman tau-function, due to its close link with Bergman projective connection. The Bergman tau-function on Hurwitz spaces was computed in [18] in terms of theta-functions and prime-forms on branched coverings; this allowed to find explicit expressions for the G-function of Hurwitz Frobenius manifolds and get a new formula for the determinant of Laplace operator in Poincaré metric over Riemann surfaces.

In this paper we define and compute the tau-function on different strata of the spaces $\mathcal{H}_g$ and $\mathcal{Q}_g$, where $\mathcal{H}_g$ is the space of Abelian differentials over Riemann surfaces i.e. the space of pairs $(\mathcal{L},w)$, where $\mathcal{L}$ is a compact Riemann surface of genus $g\geq 1$ and w is an Abelian differential (i. e. a holomorphic 1-differential) on $\mathcal{L}$; $\mathcal{Q}_g$ is the space of pairs $(\mathcal{L},W)$, where W is a meromorphic quadratic differential on $\mathcal{L}$ with at most simple poles and which is not the square of an Abelian differential. The space $\mathcal{H}_g$ has dimension $4g-3$; it is stratified according to multiplicities of zeros of Abelian differentials w. The space $\mathcal{Q}_g$ is infinite-dimensional, since the quadratic differential W is allowed to have an arbitrary number of poles; however, this space can be stratified according to the number of poles and multiplicities of zeros of W; each stratum then is finite-dimensional. The corresponding strata generally have several connected components. The classification of these connected components is given in [19, 22]. In particular, the stratum of the space $\mathcal{H}_g$ having the highest dimension (on this stratum all the zeros of w are simple) is connected; the maximal number of connected components of a given stratum of the space $\mathcal{H}_g$ equals three ([19]). (See [22] for discussion of existing results for the space $\mathcal{Q}_g$.)

We apply the developed formalism to computation of the determinants of Laplacians in flat metrics with conical singularities over Riemann surfaces given by the modulus square of an Abelian differential $|w|^2$ or by the modulus of a meromorphic quadratic differential $|W|$ with at most simple poles (Strebel metrics of finite volume).

To illustrate our results we consider here the case of holomorphic quadratic differentials with $4g-4$ simple zeros. Let $\mathcal{Q}_g(1,\dots,1)$ be the stratum of $\mathcal{Q}_g$, consisting of pairs $(\mathcal{L}, W)$, where W is such a quadratic differential. There exists a two-fold covering (which is called canonical) $\pi : \tilde{\mathcal{L}} \to \mathcal{L}$ such that $\pi^* W = w^2$, where w is an Abelian differential on $\tilde{\mathcal{L}}$.

The covering $\tilde{\mathcal{L}}$ is ramified at the zeros $R_1, \dots, R_{4g-4}$ of the quadratic differential W.

Denote by $*$ the holomorphic involution on $\tilde{\mathcal{L}}$ interchanging the sheets of canonical covering. Denote the basis of cycles on $\mathcal{L}$ by (a_α, b_α). The Abelian differential w is anti-invariant under the involution: $w(P^*) = -w(P)$. The canonical basis of cycles on $\tilde{\mathcal{L}}$ will be denoted as follows [8]:

$$\{a_\alpha, b_\alpha, a_{\alpha'}, b_{\alpha'}, a_m, b_m\} \tag{1.1}$$

where $\alpha, \alpha' = 1, \dots g$; $m = 1, \dots 2g-3$; this basis has the following invariance properties under the involution $*$:

$$a_\alpha^* + a_{\alpha'} = b_\alpha^* + b_{\alpha'} = 0 \tag{1.2}$$

and

$$a_m^* + a_m = b_m^* + b_m = 0 \tag{1.3}$$

The full set of independent coordinates on the space $\mathcal{Q}_g(1,\dots,1)$ is obtained by integrating the differential $w(P)$ over basic cycles on $\tilde{\mathcal{L}}$ as follows [24]:

$$A_\alpha := \oint_{a_\alpha} w \qquad B_\alpha := \oint_{b_\alpha} w \qquad A_m := \oint_{a_m} w \qquad B_m := \oint_{b_m} w \tag{1.4}$$

for $\alpha = 1, \dots, g$, $m = 1, \dots, 2g-3$; the total number of these coordinates equals $6g-6$ i.e. coincides with dimension of the space $\mathcal{Q}_g(1,\dots,1)$. According to (1.2), and due to anti-invariance of w under the involution $*$, the integration of w over cycles $(a_{\alpha'}, b_{\alpha'})$ does not give any new independent coordinates.

Let us introduce on the covering $\tilde{\mathcal{L}}$ the coordinate $z(P) = \int_{R_1}^P w$; the coordinate $z(P)$ can be chosen as a local parameter everywhere on $\tilde{\mathcal{L}}$ outside of the ramification points R_s.

Dependence of the holomorphic differentials and the matrix of b-periods on the moduli of Riemann surfaces is given by the Rauch variational formulas [28]. Here we prove analogs of the Rauch variational formulas on the spaces $\mathcal{Q}(1,\dots,1)$. For example, for the matrix of b-periods $\mathbf{B}_{\alpha\beta}$ of the surface $\mathcal{L}$ as a function of coordinates (1.4) we have:

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial A_\gamma} = -\oint_{b_\gamma} \frac{\pi^* w_\alpha \pi^* w_\beta}{w}\,, \qquad \frac{\partial \mathbf{B}_{\alpha\beta}}{\partial B_\gamma} = \oint_{a_\gamma} \frac{\pi^* w_\alpha \pi^* w_\beta}{w} \tag{1.5}$$

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial A_m} = -\frac{1}{2}\oint_{b_m} \frac{\pi^* w_\alpha \pi^* w_\beta}{w}\,, \qquad \frac{\partial \mathbf{B}_{\alpha\beta}}{\partial B_m} = \frac{1}{2}\oint_{a_m} \frac{\pi^* w_\alpha \pi^* w_\beta}{w} \tag{1.6}$$

where $\alpha, \beta, \gamma = 1, \dots, g$, $m = 1, \dots, 2g-3$, $\{w_\alpha\}_{\alpha=1}^g$ is the canonical basis of Abelian differentials on $\mathcal{L}$. (In what follows in similar formulas we shall often omit the operator π^*).

Consider on the Riemann surface $\mathcal{L}$ the canonical meromorphic bidifferential

$$\mathbf{w}(P,Q) = d_P d_Q \log E(P,Q) \tag{1.7}$$

where $E(P,Q)$ is the prime-form. The bidifferential $\mathbf{w}(P,Q)$ has the following local behavior near the diagonal $P \to Q$:

$$\mathbf{w}(P,Q) = \left(\frac{1}{(x(P)-x(Q))^2} + \frac{1}{6}S_B(x(P)) + o(1)\right) dx(P)dx(Q), \tag{1.8}$$

where $x(P)$ is a local parameter. The term $S_B(x(P))$ is a projective connection which is called *the Bergman projective connection*. We recall that a projective connection S is a quantity transforming as follows under the coordinate change $z = z(t)$:

$$S(t) = S(z)\left(\frac{dz}{dt}\right)^2 + \{z,t\},$$

where

$$\{z,t\} = \frac{z'''(t)z'(t) - \frac{3}{2}(z''(t))^2}{(z'(t))^2}$$

is the Schwarzian derivative (see, e. g., ([31])). The difference of two arbitrary projective connections is a quadratic differential.

The tau-function $\tau(\mathcal{L},W)$ on the stratum $\mathcal{Q}_g(1,\dots,1)$ is defined locally by the following system of compatible equations:

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial A_\alpha} = \frac{1}{12\pi i}\oint_{b_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1,\dots,g; \tag{1.9}$$

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial B_\alpha} = -\frac{1}{12\pi i}\oint_{a_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1,\dots,g; \tag{1.10}$$

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial A_m} = \frac{1}{24\pi i}\oint_{b_m} \frac{S_B - S_w}{w}, \quad m = 1,\dots,2g-3; \tag{1.11}$$

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial B_m} = -\frac{1}{24\pi i}\oint_{a_m} \frac{S_B - S_w}{w}, \quad m = 1,\dots,2g-3, \tag{1.12}$$

where S_B is the Bergman projective connection on $\mathcal{L}$; $S_w(\zeta) := \left\{\int^P w, \zeta\right\}$, the difference between two projective connections S_B and S_w is a meromorphic quadratic differential on $\mathcal{L}$ with poles at the zeros of W.

In global terms, $\tau(\mathcal{L},W)$ is a horizontal holomorphic section of certain line bundle over a given connected component of the covering $\widehat{\mathcal{Q}_g}(1,\dots,1)$ of the space $\mathcal{Q}_g(1,\dots,1)$. The space $\widehat{\mathcal{Q}_g}(1,\dots,1)$ consists of triples $(\mathcal{L},W,\{a_\alpha,\, b_\alpha, a_{\alpha'}, b_{\alpha'},\, a_m, b_m\})$, where $(\mathcal{L},W) \in \mathcal{Q}_g(1,\dots,1)$ and $\{a_\alpha,\, b_\alpha, a_{\alpha'},\, b_{\alpha'}, a_m, b_m\}$ is a basis of homologies on surface $\tilde{\mathcal{L}}$ subject to conditions (1.2), (1.3).

The name "tau-function" for the function $\tau(\mathcal{L},W)$ defined by (1.9 – 1.12) is inherited from the Bergman tau-function on Hurwitz space ([16]) (or, equivalently, the tau-function of Hurwitz Frobenius manifolds [7, 17]). Although at the moment we don't know whether the function $\tau(\mathcal{L},W)$ itself can be interpreted as the Jimbo-Miwa tau-function of some isomonodromy problem (perhaps on a Riemann surface of genus higher than zero), we still call it tau-function because its definition and role in computation of determinants of Laplacians are similar to those of the Bergman tau-function on Hurwitz space.

The system (1.9) – 1.12) can be solved in terms of the multi-valued holomorphic $g(1-g)/2$-differential $\mathcal{C}(P)$ on $\mathcal{L}$ given by the formula

$$\mathcal{C}(P) := \frac{1}{\mathcal{W}[w_1,\dots,w_g](P)} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \theta(K^P)}{\partial_{z_{\alpha_1}}\dots\partial_{z_{\alpha_g}}} w_{\alpha_1}\dots w_{\alpha_g}(P), \tag{1.13}$$

where

$$\mathcal{W}(P) := \det_{1\le\alpha,\beta\le g}||w_\beta^{(\alpha-1)}(P)||$$

is the Wronskian of holomorphic differentials at the point P; K^P is the vector of Riemann constants corresponding to the basepoint P. Assume that the fundamental domain $\hat{\mathcal{L}}$ is chosen in such a way that $\mathcal{A}\big((W)\big) + 4K^P = 0$, where $\mathcal{A}$ is the Abel map with the basepoint P (this choice of fundamental domain is always possible). Then, it is easy to verify that the expression

$$\mathcal{F} := W^{(g-1)/4}(P)\mathcal{C}(P)\prod_{s=1}^{4g-4} E^{(1-g)/4}(R_s, P)$$

does not depend on $P \in \mathcal{L}$ (being a holomorphic section of the trivial line bundle over $\mathcal{L}$ with respect to P); the prime-forms are evaluated at the ramification points R_s in the so-called *distinguished* [30] local coordinates

$$\lambda_s(P) := \left(\int_{R_s}^{P} w\right)^{2/3}. \tag{1.14}$$

The tau-function on $\mathcal{Q}_g(1,\dots,1)$ i.e. the holomorphic solution of the system (1.9– 1.12) is given by the following expression:

$$\tau(\mathcal{L}, W) = \mathcal{F}^{2/3} \prod_{s,r=1 \ s<r}^{4g-4} [E(R_s, R_r)]^{1/24}, \tag{1.15}$$

where again the prime-forms are evaluated at P_m in the local parameters (1.14).

Introduce the Laplacian $\Delta = 4\rho^2(z,\bar{z})\partial^2_{z\bar{z}}$ corresponding to the flat singular metric $\rho^{-2}(z,\bar{z})\widehat{dz} = \rho^{-2}(z,\bar{z})dx\,dy = |w|^2$ or $\rho^{-2}(z,\bar{z})\widehat{dz} = |W|$ and acting in the trivial line bundle over the surface $\mathcal{L}$. (Since the metrics have conical singularities, the Laplacian is not essentially self-adjoint and one has to choose a proper self-adjoint extension: here we deal with the Friedrichs extension.) The main goal of this paper is to apply the formalism of tau-function on spaces of Abelian and quadratic differentials to computation of the determinants of such Laplacians. There are several difficulties in the definition and computation of these determinants implied by the conical singularities of the metrics $|w|^2$ and $|W|$ at the zeros of w and zeros and poles of W.

First, these singularities cause a potential problem with the standard definition of the determinant of the corresponding Laplacian through the operator zeta-function,

$$\det\Delta = \exp\{-\zeta'_\Delta(0)\}\ ;$$

to ensure that $\zeta_\Delta(z)$ has no pole at $z = 0$ one needs to apply some additional information about the asymptotics of the heat kernel on conical manifolds.

The next potential difficulty appears in the study of dependence of detΔ on the point $(\mathcal{L}, w)$ (or $(\mathcal{L}, W)$). For smooth metrics dependence of the determinant of Laplacian on the moduli and the metric is well-known (see [9]). The Polyakov formula ([26], see also [9], p. 62),

$$\frac{\det \Delta^{\mathbf{g}_1}}{\det \Delta^{\mathbf{g}_0}} = \frac{\int_{\mathcal{L}} \rho_1^{-2}(z,\bar z)\widehat{dz}}{\int_{\mathcal{L}} \rho_0^{-2}(z,\bar z)\widehat{dz}} \exp\left\{\frac{1}{3\pi}\int_{\mathcal{L}} \log\frac{\rho_1}{\rho_0}\, \partial^2_{z\bar z} \log \rho_1\rho_0 \widehat{dz}\right\}, \tag{1.16}$$

which shows how the determinant of Laplacian depends on the metric within a given conformal class can be considered as a very special case of the general variational formulas from ([9]). Here $\mathbf{g}_1 := \rho_1^{-2}(z,\bar z)\widehat{dz}$ and $\mathbf{g}_0 := \rho_0^{-2}(z,\bar z)\widehat{dz}$ are two *smooth* ($\rho_{0,1}$ and $\rho_{0,1}^{-1} \in C^\infty$) metrics on $\mathcal{L}$; $\Delta^{\mathbf{g}_{0,1}} = 4\rho_{0,1}^2\partial\bar\partial$, the determinants of Laplacians are defined via standard ζ-function regularization (we adopt the notation from [9]).

For metrics with conical singularities, to the best of our knowledge, the variational formulas were absent. If one of the metrics in (1.16) has conical singularity, the right-hand side of Polyakov formula does not make sense. In principal, one may choose some kind of regularization of the arising divergent integral (this idea was widely used in string theory literature). It leads to an alternative definition of the determinant of Laplacian in conical metrics: one may simply take for $\mathbf{g}_0$ some smooth metric and define the determinant of Laplacian in conical metric $\mathbf{g}_1$ through formula (1.16) with properly regularized right-hand side. Such a way was chosen in the work of Sonoda [29] (see also [5]) for metrics given by the modulus square of an Abelian differential, which constitutes a partial case of our considerations, or metrics given by the modulus square of a meromorphic 1-differential, when the Laplacians have continuous spectrum and their determinants have no rigorous definition. In [29] the smooth reference metric $\mathbf{g}_0$ is chosen to be the Arakelov metric. Since the determinant of Laplacian in Arakelov metric is known (it was calculated in [6] in functional integral approach; the expression found in [6] was then rigorously proved in [9]), such an approach leads to a heuristic formula for $\det \Delta^{\mathbf{g}_1}$. This result heavily depends on the choice of the regularization procedure. The naive choice of the regularization leads to dependence of $\det \Delta^{\mathbf{g}_1}$ on the reference metric $\mathbf{g}_0$ which is obviously unsatisfactory. More sophisticated (and used in [29] and [5]) procedure of regularization eliminates the dependence on $\mathbf{g}_0$ but provides an expression which behaves as a tensor with respect to local coordinates at the zeros of the differential w and, therefore, also can not be considered as completely satisfactory. In any case it is unclear whether this heuristic formula for $\det \Delta^{\mathbf{g}_1}$ for singular $\mathbf{g}_1$ has something to do with the determinant of Laplacian det$\Delta^{\mathbf{g}_1}$ defined via the spectrum of the operator $\Delta^{\mathbf{g}_1}$.

To derive rigorous formulas for the determinants of Laplacians in the metrics $|W|$ and $|w|^2$ we find analogs of the variational formulas for $\det \Delta$ in smooth metrics for the case of flat metrics with conical singularities. As an essential intermediate step we extend the machinery of analytic surgery due to Burghelea, Friedlander and Kappeler ([3]), proving an analog of the analytic surgery formula for flat surfaces with conical singularities. Then it turns out that variations of $\log \det \Delta^{|W|}$ and $\log |\tau(\mathcal{L}, w)|^2$ with respect to local coordinates in $\mathcal{Q}_g(1, \dots, 1)$ coincide up to a simple additional term; this leads to the following explicit formula:

$$\frac{\det \Delta^{|W|}}{\mathrm{Vol}(\mathcal{L}, |W|) \det \Im \mathbf{B}} = C|\tau(\mathcal{L}, W)|^2 , \tag{1.17}$$

where $\mathrm{Vol}(\mathcal{L}, |W|) := \int_{\mathcal{L}} |W|$ is the area of $\mathcal{L}$ in the metric $|W|$; $\mathbf{B}$ is the matrix of b-periods of $\mathcal{L}$; $\tau(\mathcal{L}, W)$ is the tau-function on the space $\mathcal{Q}_g(1, \dots, 1)$; C is a constant (which in general case can be different for different connected components of a given startum of $\mathcal{Q}_g$).

The formula (1.17) can be considered as a natural higher genus generalization of two previously known results in genera 1 and 0. The first one is the famous Kronecker formula for the determinant of Laplacian on elliptic surface in flat nonsingular metric $|dz|^2$:

$$\det\Delta^{|dz|^2} = C|\Im\sigma|^2|\eta(\sigma)|^4 ,$$

where σ is the period of elliptic surface and η is the Dedekind eta-function (see [27]). The second is the following expression for the determinant of the Laplacian $\Delta^{|W|}$ on the Riemann sphere corresponding to the flat singular metric $|W| = \left(\prod_{j=1}^4 |z-a_j|^{-1}\right)|dz|^2$:

$$\det\Delta^{|W|} = C\prod_{i<j}|a_i-a_j|^{1/3}\int\int_{\mathbb{C}}\frac{|dz|^2}{|z-a_1||z-a_2||z-a_3||z-a_4|} , \tag{1.18}$$

which appeared in the recent preprint ([1]).

From the explicit expression (1.17) one can derive a nice analog of Polyakov's formula (1.16) for flat metrics. Consider two holomorphic quadratic differentials with simple zeros W and $\tilde W$; denote the zeros of W by R_s, and the zeros of $\tilde W$ by $\tilde R_s$. Then

$$\frac{\det\Delta^{|W|}}{\det\Delta^{|\tilde W|}} = C\frac{\mathrm{Vol}(\mathcal{L},|W|)}{\mathrm{Vol}(\mathcal{L},|\tilde W|)}\prod_{s=1}^{4g-4}\Big|\frac{W(\tilde R_s)}{\tilde W(R_s)}\Big|^{1/24} \tag{1.19}$$

where C is a constant which might be not equal to 1 if the points $(\mathcal{L},W)$ and $(\mathcal{L},\tilde W)$ belong to different connected components of the space $\mathcal{Q}(1,\dots,1)$. Here $\tilde W(R_s) = \frac{\tilde W}{(d\lambda_s)^2}\Big|_{\lambda_s=0}$, where $\lambda_s(P) = \left(\int_{R_s}^P\sqrt{W}\right)^{2/3}$ is the distinguished local parameter near the zero R_s of the quadratic differential W. Analogously, $W(\tilde R_s) = \frac{W}{(d\tilde\lambda_s)^2}\Big|_{\tilde\lambda_s=0}$, where $\tilde\lambda_s(P) = \left(\int_{\tilde R_s}^P\sqrt{\tilde W}\right)^{2/3}$ is the distinguished local parameter near the zero $\tilde R_s$ of $\tilde W$.

The paper is organized as follows. In Section 2 we introduce and compute tau-functions on the space of Abelian differentials over Riemann surfaces. In particular, here we derive variational formulas of Rauch type on the spaces of Abelian differentials. In section 3 similar results are presented for spaces of quadratic differentials with at most simple poles. In Section 4 we derive variational formulas for determinants of Laplace operators in flat metrics with conical singularities using the technique of analytical surgery. Comparison of variational formulas for the tau-functions with variational formulas for the determinant of Laplacian, together with explicit computation of the tau-functions, leads to the explicit formulas for the determinants. (The formulas (1.17) and (1.15) are their specifications for the case of the stratum $\mathcal{Q}_g(1,\dots,1)$)). This is done in Section 5. Finally, in the same Section 5 we prove an analog of the Polyakov formula for the determinants of Laplacians in Strebel metrics of finite volume (choosing the generic situation of the differentials with simple zeros).

2 Tau-function on spaces of Abelian differentials over Riemann surfaces

2.1 Spaces of Abelian differentials

Following ([19]), define the space $\mathcal{H}_g$ as the moduli space of pairs $(\mathcal{L},w)$, where $\mathcal{L}$ is a compact surface of genus g, and w is a holomorphic 1-differential on $\mathcal{L}$. This space is stratified via the multiplicities of zeros of w. Denote by $\mathcal{H}_g(k_1,\dots,k_M)$ the stratum of $\mathcal{H}_g$, consisting of differentials w which have M zeros on $\mathcal{L}$ of multiplicities $(k_1,\dots,k_M)$. Denote the zeros of w by $P_1,\dots,P_M$, so the divisor of differential w is given by $(w)=\sum_{m=1}^M k_m P_m$. Let us choose a canonical basic of cycles (a_α,b_α) on the Riemann surface $\mathcal{L}$ (the basis of "absolute" homologies) and the homology classes (on $\mathcal{L}\setminus(w)$) of paths l_m connecting the zero P_1 with other zeros P_m of w (the "relative" homologies), $m=2,\dots,M$. Then the local coordinates on $\mathcal{H}_g(k_1,\dots,k_M)$ can be chosen as follows ([20]):

$$A_\alpha := \oint_{a_\alpha} w\;, \qquad B_\alpha := \oint_{b_\alpha} w\;, \qquad z_m := \int_{l_m} w\;, \qquad \alpha=1,\dots,g;\; m=2,\dots,M\;. \tag{2.1}$$

2.2 Variational formulas on $\mathcal{H}_g(k_1,\dots,k_M)$

Here we shall prove variational formulas, which describe dependence of the normalized holomorphic differentials, the matrix of b-periods, the canonical meromorphic bidifferential and the Bergman projective connection on $\mathcal{L}$ on coordinates (2.1) on the spaces $\mathcal{H}_g(k_1,\dots,k_M)$.

Denote by $w_a(P)$ the basis of holomorphic 1-forms on $\mathcal{L}$ normalized by $\int_{a_\alpha} w_\beta = \delta_{\alpha\beta}$.

Theorem 1 *The holomorphic normalized differentials w_α depend as follows on A_α, B_α and z_k:*

$$\frac{\partial w_\alpha(P)}{\partial A_\beta}\Big|_{z(P)} = -\frac{1}{2\pi i}\oint_{b_\beta}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)} \tag{2.2}$$

$$\frac{\partial w_\alpha(P)}{\partial B_\beta}\Big|_{z(P)} = \frac{1}{2\pi i}\oint_{a_\beta}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)} \tag{2.3}$$

$$\frac{\partial w_\alpha(P)}{\partial z_m}\Big|_{z(P)} = \mathrm{res}|_{Q=P_m}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)} \tag{2.4}$$

where we assume that $z(P)=\int_{P_1}^P$ is kept constant under differentiation with respect to A_α, B_α and z_m.

Proof. Let us write down the differential w_α (outside of the points P_m) with respect to the local parameter $z(P)$: $w_\alpha(P)=f_\alpha(z)dz$. Then the variational formula (2.4) is a complete analog of the formula for the derivatives of the normalized holomorphic differentials on a branched covering with respect to a branch point of order k_m (see formula (2.19) of [16]). Indeed, in a neighborhood of P_m, $z(P)$ is just a function on $\mathcal{L}$ with critical value z_m (z_m is the analog of branch point λ_m in [16]).

For example, let us prove (2.3). Consider some point $P_0\in\mathcal{L}$ such that $z_0:=z(P_0)$ is independent of the moduli $\{A_\beta,B_\beta,z_m\}$. Let us dissect the surface $\mathcal{L}$ along the basic cycles started at P_0 to get the fundamental polygon $\hat{\mathcal{L}}$. Denote the images of the different shores of the basic cycles in z-plane by a_β^-, a_β^+, b_β^- and b_β^+. The endpoints of these contours coincide with the points z_0, z_0+A_β, z_0+B_β and $z_0+A_\beta+B_\beta$. Let us write down the differential w_α in terms of the local parameter z as follows: $w_\alpha(P)=f_\alpha(z)dz$, where $z=z(P)$.

The function $f_\alpha(z)$ is the same on the different shores of the cuts a_β^- and a_β^+ i.e. $f_\alpha(z+B_\beta)=f_\alpha(z)$ for $z\in a_\beta^-$. Differentiating this relation with respect to B_β, we get

$$\frac{\partial f_\alpha}{\partial B_\beta}(z+B_\beta)=\frac{\partial f_\alpha}{\partial B_\beta}(z)-\frac{\partial f_\alpha}{\partial z}(z)\;;$$

obviously, this is the only discontinuity of the differential $\frac{\partial w_\alpha(P)}{\partial B_\beta}|_{z(P)}$ on $\mathcal{L}$. Therefore, the differential $\frac{\partial w_\alpha(P)}{\partial B_\beta}|_{z(P)}$ has all vanishing a-periods and the jump $-\frac{\partial f_\alpha}{\partial z}(z(P))dz(P)$ on the contour a_β^-; outside of the cycle a_β this differential is holomorphic (in other words, it solves the scalar Riemann-Hilbert problem on the contour a_β). Such a differential can be easily written (see, e.g., [35]) in terms of the canonical bidifferential $\mathbf{w}(P,Q)$ as a contour integral over a_β in the form (2.3) (in terms of coordinate $z(P)$ we have $w(Q) = dz(Q)$).

The formula (2.2) can be proved in the same way; the only difference is the change of sign which appears due to the asymmetry between the cycles a_β and b_β imposed by their intersection index $a_\beta \circ b_\beta = -b_\beta \circ a_\beta = 1$.

□

Denote by $\mathbf{B}$ the matrix of b-periods of the surface $\mathcal{L}$: $\mathbf{B}_{\alpha\beta} := \oint_{b_\alpha} w_\beta$. Integrating the variational formulas (2.2), (2.3),(2.4) along the basic b-cycles we get the following

Corollary 1 *The matrix of b-periods depends as follows on the coordinates on a given stratum of the space of holomorphic 1-differentials:*

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial A_\gamma} = -\oint_{b_\gamma} \frac{w_a w_b}{w} \qquad \frac{\partial \mathbf{B}_{\alpha\beta}}{\partial B_\gamma} = \oint_{a_\gamma} \frac{w_a w_b}{w} \tag{2.5}$$

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial z_m} = 2\pi i \,\mathrm{res}|_{P_m} \frac{w_\alpha w_\beta}{w} \tag{2.6}$$

for $\alpha, \beta, \gamma = 1, \dots, g$, $m = 2, \dots, M$.

We shall also use the variational formula for the canonical bidifferential $\mathbf{w}(P,Q)$ given by the following

Theorem 2 *The bidifferential $\mathbf{w}(P,Q)$ depends on A_α, B_α and z_m as follows:*

$$\frac{\partial \mathbf{w}(P,Q)}{\partial A_\beta} = -\frac{1}{2\pi i}\oint_{b_\beta} \frac{\mathbf{w}(P,R)\mathbf{w}(Q,R)}{w(R)} \tag{2.7}$$

$$\frac{\partial \mathbf{w}(P,Q)}{\partial B_\beta} = \frac{1}{2\pi i}\oint_{a_\beta} \frac{\mathbf{w}(P,R)\mathbf{w}(Q,R)}{w(R)} \tag{2.8}$$

$$\frac{\partial \mathbf{w}(P,Q)}{\partial z_m} = -\mathrm{res}|_{R=P_m} \frac{\mathbf{w}(P,R)\mathbf{w}(Q,R)}{w(R)} \tag{2.9}$$

for $\alpha, \beta = 1, \dots, g$, $m = 2, \dots, M$, where we assume that $z(P)$ and $z(Q)$ are kept constant under differentiation.

The proof of this theorem is completely parallel to the proof of Theorem 1.

Denote by $S_w(\zeta(P))$ the Schwarzian derivative $\left\{\int^P w, \zeta(P)\right\}$. Then the difference of two projective connections $S_B - S_w$ is a (meromorphic) quadratic differential on $\mathcal{L}$; its dependence on the moduli is given by the following

Corollary 2 *The variational formulas take place:*

$$\frac{\partial}{\partial A_\alpha}(S_B(P) - S_w(P))\Big|_{z(P)} = \frac{3}{\pi i}\oint_{b_\beta} \frac{\mathbf{w}^2(P,R)}{w(R)} \tag{2.10}$$

$$\frac{\partial}{\partial B_\alpha}(S_B(P) - S_w(P))\Big|_{z(P)} = -\frac{3}{\pi i}\oint_{a_\beta}\frac{\mathbf{w}^2(P,R)}{w(R)} \tag{2.11}$$

$$\frac{\partial}{\partial z_m}(S_B(P) - S_w(P))\Big|_{z(P)} = -\frac{1}{6}\mathrm{res}|_{R=P_m}\frac{\mathbf{w}^2(P,R)}{w(R)} \tag{2.12}$$

Proof. The formulas (2.10), (2.11), (2.12) follow from the variational formulas for the bidifferential $\mathbf{w}(P,Q)$ (2.7), (2.8) and (2.9) in the limit $P \to Q$ if we write down these formulas with respect to the local coordinate $z(P)$ (in this local coordinate the projective connection S_w vanishes) and take into account the definition (1.8) of the Bergman projective connection.

2.3 Basic Beltrami differentials for $\mathcal{H}_g(k_1, \dots, k_M)$

Here we construct the Beltrami differentials which correspond to the variations of the conformal structure of the surface $\mathcal{L}$ under variations of the coordinates on the spaces of Abelian differentials.

Let a pair $(\mathcal{L}, w)$ belong to the space $\mathcal{H}_g(k_1, \dots, k_M)$. Consider a thin strip Π_α on the surface $\mathcal{L}$ around basic cycle a_α (one considers smooth smooth curve representing the homology class); one part of the oriented boundary of Π_α is homologically equivalent to a_α; another part is homologically equivalent to $-a_\alpha$. Assume that this strip is thin enough not to contain the zeros of w. Let χ be a function from $C^\infty(\mathcal{L})$ which is equal to 1 in a neighborhood of the cycle a_α and vanishes in a neighborhood of $-a_\alpha$. Consider a $(0,1)$-form σ which coincides with $\bar\partial\chi$ in the strip Π_α and vanishes outside Π_α.

Introduce the Beltrami differential μ_{B_α} from C^∞ by

$$\mu_{B_\alpha} = \frac{\sigma}{w}. \tag{2.13}$$

Using the Stokes theorem, it is easy to see that

$$\int_{\mathcal{L}} \mu_{B_\alpha} q = \oint_{a_\alpha}\frac{q}{w} \tag{2.14}$$

for any meromorphic quadratic differential q which is holomorphic outside of the zeroes of w.

Therefore, due to Corollary 1, one has the relation

$$\frac{\partial F}{\partial B_\alpha} = \delta_{\mu_{B_\alpha}} F$$

for any differentiable function F on the Teichmüller space T_g. In the same manner we construct Beltrami differentials μ_{A_α} and μ_{z_m} from C^∞ responsible for the deformation of the complex structure on the surface $\mathcal{L}$ under infinitesimal shifts of coordinates A_α and z_m. Note that the Beltrami differential μ_{z_m} can be chosen to be supported in an annulus centered at the point P_m:

$$\mathbf{A}_m = \{r_1 \le |x_m| \le r_2\} \tag{2.15}$$

with some $r_1, r_2 > 0$. One has

$$\int_{\mathcal{L}} \mu_{z_m} W = 2\pi i \mathrm{res}|_{P_m}\frac{W}{w} \tag{2.16}$$

for any meromorphic quadratic differential W with $(W) \ge -2P_1 - \dots - 2P_M$. We call the Beltrami differentials μ_{A_α}, μ_{B_α} and μ_{z_m} *basic.*

2.4 Definition of the Bergman tau-function

Definition 1 *The Bergman tau-function $\tau(\mathcal{L}, w)$ on the stratum $\mathcal{H}_g(k_1, \dots, k_M)$ of the space of Abelian differentials is locally defined by the following system of equations:*

$$\frac{\partial \log \tau(\mathcal{L}, w)}{\partial A_\alpha} = \frac{1}{12\pi i} \oint_{b_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1, \dots, g; \tag{2.17}$$

$$\frac{\partial \log \tau(\mathcal{L}, w)}{\partial B_\alpha} = -\frac{1}{12\pi i} \oint_{a_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1, \dots, g; \tag{2.18}$$

$$\frac{\partial \log \tau(\mathcal{L}, w)}{\partial z_m} = -\frac{1}{6} \text{res}\Big|_{P_m} \frac{S_B - S_w}{w}, \quad m = 2, \dots M\,. \tag{2.19}$$

where S_B is the Bergman projective connection; $S_w(\zeta) := \left\{ \int^P w, \zeta \right\}$; the difference between two projective connections S_B and S_w is a meromorphic quadratic differential with poles at the zeros of w.

To justify this definition one needs to prove the following proposition.

Proposition 1 *The system of equations (2.17), (2.18), (2.19) is compatible.*

Proof. Denote the right-hand sides of equations (2.17-2.19) by H^{A_α}, H^{B_α} and H^{z_m} respectively (these are analogs of isomonodromic Hamiltonians from [7], [17]).

We have to show that $\frac{\partial H^{A_\alpha}}{\partial B_\beta} = \frac{\partial H^{B_\beta}}{\partial A_\alpha}$, $\frac{\partial H^{z_m}}{\partial A_\alpha} = \frac{\partial H^{A_\alpha}}{\partial z_m}$, etc. Most of these equations immediately follow from the variational formulas (2.7), (2.8), (2.9) and the symmetry of the bidifferential $\mathbf{w}(P, Q)$.

For example, to prove that

$$\frac{\partial H^{A_\alpha}}{\partial A_\beta} = \frac{\partial H^{A_\beta}}{\partial A_\alpha} \tag{2.20}$$

for $\alpha \neq \beta$ we write down the left-hand side as

$$\frac{\partial H^{A_\alpha}}{\partial A_\beta} = -\frac{1}{4\pi^2} \oint_{a_\alpha} \oint_{a_\beta} \frac{\mathbf{w}^2(P, Q)}{w(P) w(Q)} \tag{2.21}$$

which is obviously symmetric with respect to interchange of A_α and A_β since the cycles a_α and a_β always can be chosen non-intersecting. Similarly, one can prove all other symmetry relations where the integration contours don't intersect (interpreting the residue at P_m in terms of the integral over a small contour encircling P_m).

The only equations which require interchange of the order of integration over intersecting cycles are

$$\frac{\partial H^{A_\alpha}}{\partial B_\alpha} = \frac{\partial H^{B_\alpha}}{\partial A_\alpha}\,. \tag{2.22}$$

To prove (2.22) we denote the intersection point of a_α and b_α by Q_α; then we have:

$$\frac{\partial H^{A_\alpha}}{\partial B_\alpha} \equiv \frac{1}{12\pi i} \frac{\partial}{\partial B_\alpha} \left\{ \oint_{b_\alpha} \frac{S_B - S_w}{w} \right\} = \frac{1}{12\pi i} \frac{S_B - S_w}{w}(Q_a) - \frac{1}{4\pi^2} \oint_{b_\alpha} \oint_{a_\alpha} \frac{\mathbf{w}^2(P, Q)}{w(P) w(Q)} \tag{2.23}$$

where the value of 1-form $\frac{1}{w}(S_B - S_w)$ at the point Q_α is computed in coordinate the $z(P)$. The additional term in (2.23) arises from dependence of the cycle b_α in the z-plane on B_α (the difference between the initial and endpoints of the cycle b_α in z-plane is exactly B_α), which has to be taken into account in the process of differentiation.

In the same way we find that

$$\frac{\partial H^{B_\alpha}}{\partial A_\alpha} \equiv -\frac{1}{12\pi i}\frac{\partial}{\partial A_\alpha}\left\{\oint_{a_\alpha}\frac{S_B - S_w}{w}\right\} = -\frac{1}{12\pi i}\frac{S_B - S_w}{w}(Q_a) - \frac{1}{4\pi^2}\oint_{a_\alpha}\oint_{b_\alpha}\frac{\mathbf{w}^2(P,Q)}{w(P)w(Q)} \tag{2.24}$$

(note the change of the sign in front of the term $\frac{1}{w}(S_B - S_w)(Q_\alpha)$ in (2.24) comparing with (2.23)). Interchanging the order of integration in, say, (2.23) we come to (2.24) after elementary analysis of the behavior of the integrals in a neighborhood of the singular point Q_α (one should carefully account the interplay between different branches of logarithm).

□

Remark 1 The right-hand side of formulas (2.17), (2.18) and (2.19) depends not only on the choice of the canonical basis of absolute homologies on the surface $\mathcal{L}$, but also on mutual positions of the basic cycles and the points of the divisor (w), i.e. it depends on the choice of both absolute and relative homology basis on the punctured Riemann surface $\mathcal{L}\setminus(w)$. This means that the proper global definition of the tau-function should be as a horizontal section of some (flat) line bundle over the covering $\widehat{H}_g(k_1,\dots,k_M)$ of the space $H_g(k_1,\dots,k_M)$. Here $\widehat{\mathcal{H}}_g(k_1,\dots,k_M)$ is the space of triples $(\mathcal{L}, w, \{a_\alpha, b_\alpha, l_m\}_{\alpha=1}^g)$, where $(\mathcal{L}, w)\in H_g(k_1,\dots,k_M)$; $\{a_\alpha,b_\alpha\}_{\alpha=1}^g$ is a canonical basis of cycles on $\mathcal{L}$; $\{l_m\}_{m=2}^M$ form a basis of relative homologies on $\mathcal{L}\setminus(w)$.

2.5 Differential $\mathcal{C}$ and its variation

Dissecting the surface $\mathcal{L}$ along the basic cycles, we get the fundamental polygon $\widehat{\mathcal{L}}$. Define the following $g(1-g)/2$-differential on $\widehat{\mathcal{L}}$ which has multipliers 1 and $\exp[-\pi i(g-1)^2\mathbf{B}_{\alpha\alpha} - 2\pi i(g-1)K_\alpha^P]$ along the cycles a_α and b_α respectively:

$$\mathcal{C}(P) := \frac{1}{\mathcal{W}[w_1,\dots,w_g](P)}\sum_{\alpha_1,\dots,\alpha_g=1}^{g}\frac{\partial^g\theta(K^P)}{\partial_{z_{\alpha_1}}\dots\partial_{z_{\alpha_g}}}w_{\alpha_1}\dots w_{\alpha_g}(P), \tag{2.25}$$

where

$$\mathcal{W}(P) = \det_{1\le\alpha,\beta\le g}||w_\beta^{(\alpha-1)}(P)||$$

is the Wronskian of holomorphic differentials at the point P; K^P is the vector of Riemann constants corresponding to the basepoint P.

We shall also make use of the following multi-valued differential of two variables $\mathbf{s}(P,Q)$ $(P,Q\in\widehat{\mathcal{L}})$ built from $\mathcal{C}(P)$:

$$\mathbf{s}(P,Q) := \left(\frac{\mathcal{C}(P)}{\mathcal{C}(Q)}\right)^{1/(1-g)}. \tag{2.26}$$

which can be also written as follows:

$$\mathbf{s}(P,Q) = \exp\left\{-\sum_{\alpha=1}^{g}\oint_{a_\alpha}w_\alpha(R)\log\frac{E(R,P)}{E(R,Q)}\right\}, \tag{2.27}$$

where $E(R,P)$ is the prime-form (see [9]). The right-hand side of (2.27) is a non-vanishing holomorphic $g/2$-differential on $\widehat{\mathcal{L}}$ with respect to P and a non-vanishing holomorphic $(-g/2)$-differential with respect to Q. Being lifted on the universal covering of $\mathcal{L}$ it has the automorphic factors $\exp[(g-1)\pi i \mathbf{B}_{\alpha\alpha} + 2\pi i K_\alpha^P]$ with respect to P and the multiplier $\exp[(1-g)\pi i \mathbf{B}_{\alpha\alpha} - 2\pi i K_\alpha^Q]$ with respect to Q along the cycle b_α.

For arbitrary two points $P_0, Q_0 \in \mathcal{L}$ we introduce the following multi-valued 1-differential

$$\Omega^{P_0}(P) = \mathbf{s}^2(P, Q_0) E(P, P_0)^{2g-2} (w(Q_0))^g (w(P_0))^{g-1} \tag{2.28}$$

(the Q_0-dependence of the right-hand side of (2.28) plays no important role and is not indicated).

The differential $\Omega^{P_0}(P)$ has automorphy factors 1 and $\exp(4\pi i K_\alpha^{P_0})$ along the basic cycles a_α and b_α respectively. The only zero of the 1-form Ω^{P_0} on $\widehat{\mathcal{L}}$ is P_0; its multiplicity equals $2g-2$.

Definition 2 *The projective connection S_{Fay}^P on $\mathcal{L}$ given by the Schwarzian derivative*

$$S_{Fay}^P(x(Q)) = \left\{ \int^Q \Omega^P, x(Q) \right\}, \tag{2.29}$$

where $x(Q)$ is a local coordinate on $\mathcal{L}$, is called the Fay projective connection (more precisely, we have here a family of projective connections parametrized by point $P \in \mathcal{L}$).

The following variational formula for $\mathcal{C}$ was proved (in a slightly different form) in ([9], p.58, formula (3.25)).

Theorem 3 *The variation of the differential $\mathcal{C}$ under the variation of the conformal structure of the Riemann surface $\mathcal{L}$ defined by a smooth Beltrami differential μ is given by the following expression:*

$$\delta_\mu \log\{\mathcal{C} w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = -\frac{1}{8\pi i} \text{v.p.} \int_{\mathcal{L}} \mu \{S_B - S_{Fay}^P\}, \tag{2.30}$$

where S_B is the Bergman projective connection; S_{Fay}^P is the projective connection (2.29); $S_B - S_{Fay}^P$ is a meromorphic quadratic differential on $\mathcal{L}$. The coordinate $z(P) = \int_{P_1}^P w$ is kept fixed under differentiation in (2.30).

The product of C by a power of w in the left-hand side of (2.30) is a scalar function (i.e. it has zero tensor weight) on $\hat{\mathcal{L}}$, as well as the right-hand side.

Substituting into (2.30) the basic Beltrami differentials μ_{A_α}, μ_{A_α} and μ_{C_m} introduced in section 2.3, we get the following expressions for the derivatives of $\mathcal{C}(P)$ with respect to the coordinates on the space of holomorphic differentials:

Corollary 3 *The following variational formulas for differential $\mathcal{C}(P)$ on the space $\mathcal{H}(k_1, \dots, k_M)$ take place:*

$$\frac{\partial}{\partial A_\alpha} \log\{\mathcal{C} w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = \frac{1}{8\pi i} \oint_{b_\alpha} \frac{1}{w} \left(S_B - S_{Fay}^P\right) \tag{2.31}$$

$$\frac{\partial}{\partial B_\alpha} \log\{\mathcal{C} w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = -\frac{1}{8\pi i} \oint_{a_\alpha} \frac{1}{w} \left(S_B - S_{Fay}^P\right) \tag{2.32}$$

$$\frac{\partial}{\partial C_m} \log\{\mathcal{C} w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = -\frac{1}{4} \text{res}|_{P_m} \left\{ \frac{1}{w} \left(S_B - S_{Fay}^P\right) \right\}, \tag{2.33}$$

where the local parameter $z(P)$ is kept fixed under differentiation.

2.6 Dirichlet integral: variational formulas and holomorphic factorization

Let a pair $(\mathcal{L}, w)$ belong to the space $\mathcal{H}_g(k_1, \dots, k_M)$. Recall that the surface $\mathcal{L}$ can be provided with the system of local parameters defined by the holomorphic differential w. Outside the zeroes $P_1, \dots, P_M$ of w we use the local parameter z, defined by the relation $z(P) = \int_{P_1}^{P} w$. We set $z_m = z(P_m)$ as before. In a small neighborhood of the zero P_m of multiplicity k_m we use the local parameter x_m such that $w = (k_m + 1)x_m^{k_m} dx_m$ or, equivalently, $x_m(P) = (z(P) - z_m)^{1/(k_m+1)}$, which can be also written as $z(P) = z_m + x_m^{k_m+1}$.

Introduce the real-valued function ϕ by the equation

$$\phi(z, \bar{z}) = \log \left| \frac{\Omega^{P_0}}{w} \right|^2 . \tag{2.34}$$

for a fixed $P_0 \in \mathcal{L}$. The function ϕ is defined outside the zeros P_m of the differential w. Analogously, in neighborhoods of the zeros P_m define M functions $\phi^{int}(x_m, \bar{x}_m)$ by

$$e^{\phi^{int}(x_m, \bar{x}_m)} |dx_m|^2 = |\Omega^{P_0}|^2 .$$

Near the zeroes P_m we have the following asymptotics

$$|\phi_z(z, \bar{z})|^2 = \left(\frac{k_m}{k_m + 1} \right)^2 |z - z_m|^{-2} + O\left(|z - z_m|^{-2+1/(k_m+1)} \right), \quad \text{as} \quad P \to P_m . \tag{2.35}$$

Near the point P_0 there is the asymptotics

$$|\phi_z(z, \bar{z})|^2 = 4(g-1)^2 |z(P) - z(P_0)|^{-2} + O\left(|z(P) - z(P_0)|^{-1} \right) , \quad \text{as} \quad P \to P_0 . \tag{2.36}$$

Let $\rho > 0$, set

$$\mathcal{L}_\rho = \mathcal{L} \setminus \left[\cup_{m=1}^{M} \{ P \in \mathcal{L} : |z(P) - z_m| \le \rho \} \cup \{ z \in \mathcal{L} : |z(P) - z(P_0)| \le \rho \} \right] .$$

Define the regularized Dirichlet integral $\mathbf{D}$ by the equality

$$\mathbf{D} = \frac{1}{\pi} \lim_{\rho \to 0} \left\{ \int_{\mathcal{L}_\rho} |\phi_z|^2 |dz|^2 + 2\pi \left(\sum_{m=1}^{M} \frac{k_m^2}{k_m + 1} + 4(g-1)^2 \right) \log \rho \right\} . \tag{2.37}$$

Due to asymptotics (2.35) and (2.36) the limit in the right-hand side of (2.37) is finite.

2.6.1 Holomorphic factorization of the Dirichlet integral

First we recall that in order to make the z-coordinate $z(P) = \int_{P_1}^{P} w$ single-valued one should fix a fundamental cell of the surface $\mathcal{L}$, cutting $\mathcal{L}$ along some cuts homotopic to the basic cycles $\{a_\alpha, b_\alpha\}$; we shall denote these cuts by the same letters $\{a_\alpha, b_\alpha\}$.

An alternative system of cuts on $\mathcal{L}$ (we call them $\{\tilde{a}_\alpha, \tilde{b}_\alpha\}$) can be introduced to make the multiplicative differential Ω^{P_0} and Fay's projective connection single-valued. We shall assume that in homologies $a_\alpha = \tilde{a}_\alpha$ and $b_\alpha = \tilde{b}_\alpha$.

For technical reasons it is convenient to keep these two systems of cuts different (although defining the same basis of the homology group). In further calculations some auxiliar functions will have jumps which arise either due to the jumps of the z-coordinate or due to the nonsinglevaluedness of the differential Ω^{P_0}. To trace these jumps it is easier to analyse them separately.

Introduce also the following notation:

$$\Omega^{P_0}(P_m) = \frac{\Omega^{P_0}(x_m)}{dx_m}\Big|_{x_m=0}, \quad m = 1, \dots, M,$$

where x_m is the local parameter near the zero P_m. Analogously, let

$$\sigma(P, Q) = \mathbf{s}(P, Q) w(P)^{-g/2} w(Q)^{g/2}. \tag{2.38}$$

Theorem 4 *The regularized Dirichlet integral* $\mathbf{D}$ *admits the following representation:*

$$\mathbf{D} = \ln \left| \sigma^{4-4g}(P_0, Q_0) \prod_{m=1}^{M} \left\{ \Omega^{P_o}(P_m) \right\}^{k_m} \exp\{4\pi i \langle \mathbf{r}, K^{P_0} \rangle\} \right|^2 + \sum_{m=1}^{M} k_m \log(k_m + 1), \tag{2.39}$$

where vector $\mathbf{r}$ *has integer components given by*

$$2\pi r_\alpha = \mathrm{Var}\Big|_{\tilde{a}_\alpha} \left\{ \mathrm{Arg}\, \frac{\Omega^{P_0}(P)}{w(P)} \right\}.$$

Proof. By the Stokes theorem

$$\int_{\mathcal{L}_\rho} |\phi_z|^2 \widehat{dz} = \frac{1}{2i} \left\{ \sum_{m=1}^{M} \oint_{P_m} + \oint_{P_0} + \sum_{\alpha=1}^{g} \int_{\tilde{a}_\alpha^+ \cup \tilde{a}_\alpha^-} \right\} \phi_z \phi \, dz,$$

where $\oint_{P_m}$ and $\oint_{P_0}$ are integrals over circles of radius ρ centred at P_m and P_0, $\tilde{a}_\alpha^+$ and $\tilde{a}_\alpha^-$ are different shores of the cut $\tilde{a}_\alpha$ having opposite orientation. We have

$$\frac{1}{2i} \int_{\tilde{a}_\alpha^+ \cup \tilde{a}_\alpha^-} \phi_z \phi dz = \pi r_\alpha \log \left| \exp 4\pi i K_\alpha^{P_0} \right|^2. \tag{2.40}$$

One gets also

$$\frac{1}{2i} \oint_{P_m} \phi_z \phi dz = \frac{1}{2i} \oint_{|x_m| = \rho^{1/(k_m+1)}} \left(\frac{1}{k_m + 1} \phi_{x_m}^{int} x_m^{-k_m} + \left(\frac{1}{k_m + 1} - 1 \right) x_m^{-k_m - 1} \right)$$

$$(\phi^{int} - 2k_m \log|x_m| - 2\log(k_m + 1))(k_m + 1) x_m^{k_m} dx_m =$$

$$= \pi k_m \phi^{int}(x_m)|_{x_m=0} - \frac{k_m^2}{k_m + 1} 2\pi \log\rho + 2\pi k_m \log(k_m + 1) + o(1)$$

and

$$\frac{1}{2i} \oint_{P_0} \phi_z \phi dz = \frac{1}{2i} \oint_{|z - z(P_0)| = \rho} \log \left| \sigma^2(P_0, Q_0)(z - z(P_0))^{2g-2} \left\{ 1 + O(z - z(P_0)) \right\} \right|^2$$

-

$$\left(\frac{2g - 2}{z - z(P_0)} + O(1) \right) dz = -\pi \log \left| \sigma^{4g-4}(P_0, Q_0) \right|^2 - 8\pi (g - 1)^2 \log\rho + o(1)$$

as $\rho \to 0$, which implies (2.39). □

2.6.2 Variation of the Dirichlet integral

The following proposition together with Theorem 4 provide an explicit expression for the Bergman tau-function on the space of Abelian differentials.

Proposition 2 *Let the z-coordinate, $z(P_0) = \int_{P_1}^{P_0} w$, of the point P_0 entering in the definition of the multiplicative differential Ω (and, hence, in the definition of Dirichlet integral (2.37)) be kept constant when the moduli $A_1, \dots, A_g$, $B_1, \dots, B_g$, $z_2, \dots, z_M$ vary. Then Dirichlet integral (2.37) satisfies the following system of equations:*

$$\frac{\partial \mathbf{D}}{\partial A_\alpha} = -\frac{1}{\pi i}\oint_{b_\alpha} \frac{S_{Fay}^{P_0} - S_w}{w}, \quad \alpha = 1, \dots, g; \tag{2.41}$$

$$\frac{\partial \mathbf{D}}{\partial B_\alpha} = \frac{1}{\pi i}\oint_{a_\alpha} \frac{S_{Fay}^{P_0} - S_w}{w}, \quad \alpha = 1, \dots, g; \tag{2.42}$$

$$\frac{\partial \mathbf{D}}{\partial z_m} = 2\mathrm{res}\Big|_{P_m} \frac{S_{Fay}^{P_0} - S_w}{w}, \quad m = 2, \dots M\ . \tag{2.43}$$

Proof. Similar statements (about the derivatives of some Dirichlet integral on a branched covering with respect to branch points of this covering) were proved in ([16]) and ([18]). As in ([16], [18]) we start with the following standard lemma. A similar lemma was exploited in [33] in order to evaluate the derivatives of the Liouville integral with respect to the moduli of (noncompact) Riemann surfaces of genus zero. The lemma from ([33]) is equivalent to the Ahlfors lemma from Teichmüller theory, whereas our lemma is of more elementary nature, being essentially a simple consequence of "thermodynamic" identity (2.58).

Lemma 1 *Let $\widehat{\mathcal{L}}$ be a fundamental polygon defined by the system of cuts $\{\tilde a_\alpha, \tilde b_\alpha\}$. Define the map $\mathcal{U}\,:\,\widehat{\mathcal{L}} \ni P \mapsto \xi \in \mathbf{C}$ by*

$$\xi = \mathcal{U}(P) = \int_{P_1}^{P} \Omega\ .$$

Then

- *The following relations hold true:*

$$\frac{\partial \phi}{\partial z_m} + F_m^Z \frac{\partial \phi}{\partial z} + \frac{\partial F_m^Z}{\partial z} = 0\ , \qquad m = 2, \dots, M\ ; \tag{2.44}$$

$$\frac{\partial \phi}{\partial A_\alpha} + F_\alpha^A \frac{\partial \phi}{\partial z} + \frac{\partial F_\alpha^A}{\partial z} = 0\ , \qquad \alpha = 1, \dots, g\ ; \tag{2.45}$$

$$\frac{\partial \phi}{\partial B_\alpha} + F_\alpha^B \frac{\partial \phi}{\partial z} + \frac{\partial F_\alpha^B}{\partial z} = 0\ , \qquad \alpha = 1, \dots, g\ , \tag{2.46}$$

where

$$F_m^Z = -\frac{\mathcal{U}_{z_m}}{\mathcal{U}_z}\ , \qquad F_\alpha^A = -\frac{\mathcal{U}_{A_\alpha}}{\mathcal{U}_z}\ , \qquad F_\alpha^B = -\frac{\mathcal{U}_{B_\alpha}}{\mathcal{U}_z}\ . \tag{2.47}$$

- *Fix some number m, $2 \le m \le M$. Near the point P_n denote $\Phi^Z_n(x_n) := F^Z_m(z_n + x_n^{d_n})$; then*

$$\Phi^Z_n(0) = \delta^n_m \; ; \qquad \left(\frac{d}{dx_n}\right)^k \Phi^Z_n(0) = 0, \qquad k = 1, \dots, k_n - 1 \,, \tag{2.48}$$

where δ^n_m is the Kronecker symbol. Denote $\Phi^A_\alpha(x_n) := F^A_\alpha(z_n + x_n^{k_n+1})$, $\Phi^B_\alpha(x_n) := F^B_\alpha(z_n + x_n^{k_n+1})$ near P_n for $\alpha = 1, \dots, g$.. Then

$$\left(\frac{d}{dx_n}\right)^k \Phi^A_\alpha(0) = \left(\frac{d}{dx_n}\right)^k \Phi^B_\alpha(0) = 0, \quad k = 0, \dots, k_n - 1 \,. \tag{2.49}$$

- *Let P^+, P^- be points on the different shores of the cut a_α which are glued to a single point of the surface $\mathcal{L}$. Then*

$$F^B_\alpha(P^+) - F^B_\alpha(P^-) = -1, \tag{2.50}$$

$$F^B_\beta(P^+) - F^B_\beta(P^-) = 0, \quad \text{for} \quad \beta \neq \alpha, \quad \beta = 1, \dots, g \;, \tag{2.51}$$

and

$$F^A_\beta(P^+) - F^A_\beta(P^-) = 0, \quad \beta = 1, \dots, g \;, \tag{2.52}$$

$$F^Z_n(P^+) - F^Z_n(P^-) = 0, \quad n = 2, \dots, M \;. \tag{2.53}$$

- *Let Q^+, Q^- be points on different shores of the cut b_α which are glued together to a single point of the surface $\mathcal{L}$. Then*

$$F^A_\alpha(Q^+) - F^A_\alpha(Q^-) = 1, \tag{2.54}$$

$$F^A_\beta(Q^+) - F^A_\beta(Q^-) = 0, \quad \text{for} \quad \beta \neq \alpha, \quad \beta = 1, \dots, g \;, \tag{2.55}$$

and

$$F^B_\beta(Q^+) - F^B_\beta(Q^-) = 0, \quad \beta = 1, \dots, g \;, \tag{2.56}$$

$$F^Z_n(Q^+) - F^Z_n(Q^-) = 0, \quad n = 2, \dots, M \;. \tag{2.57}$$

Proof. To get equations (2.44–2.47) one has to differentiate the relation $\phi(z, \bar{z}) = \log |\mathcal{U}'(z)|^2$ with respect to moduli $\{z_m, A_\alpha, B_\alpha\}$. Due to holomorphic dependence of the map $\mathcal{U}$ on the moduli, the result follows immediately. To get other statements of the lemma set $\mathcal{R} = z \circ \mathcal{U}^{-1}$; $z = \mathcal{R}(\xi)$. Writing the moduli dependence explicitly, we have

$$\mathcal{U}\left(\{z_m, A_\alpha, B_\alpha\}; \mathcal{R}(\{z_m, A_\alpha, B_\alpha\}; \xi)\right) = \xi. \tag{2.58}$$

Differentiating this identity with respect to moduli and using (2.47), we get the relations

$$F^Z_m = \frac{\partial \mathcal{R}}{\partial z_m}, \quad F^A_\alpha = \frac{\partial \mathcal{R}}{\partial A_\alpha} \;, \quad F^B_\alpha = \frac{\partial \mathcal{R}}{\partial B_\alpha} \;. \tag{2.59}$$

To prove (2.48) note that $\mathcal{R}(\xi) = z_m + (\xi - \xi(P_m))^{k_m+1} f(\xi; \{z_m, A_\alpha, B_\alpha\})$, where function f is holomorphic with respect to ξ. Now (2.48) follows from the first relation (2.59). The remaining statements of the lemma also follow from (2.59).

Corollary 4 *The following six* 1*-forms are exact:*

$$(\phi_z\phi)_{z_m}dz - \{F_m^Z|\phi_z|^2d\bar{z} + (F_m^Z)_z\phi_{\bar{z}}d\bar{z}\},$$

$$(\phi_z\phi)_{A_\alpha}dz - \{F_\alpha^A|\phi_z|^2d\bar{z} + (F_\alpha^A)_z\phi_{\bar{z}}d\bar{z}\},$$

$$(\phi_z\phi)_{B_\alpha}dz - \{F_\alpha^B|\phi_z|^2d\bar{z} + (F_\alpha^B)_z\phi_{\bar{z}}d\bar{z}\},$$

and

$$\{F_m^Z|\phi_z|^2d\bar{z} - (F_m^Z)_z\phi_z dz + (F_m^Z)_z\phi_{\bar{z}}d\bar{z}\} - \{F_m^Z(2\phi_{zz} - \phi_z^2)dz + \phi\phi_{zz_m}dz\}\ ,$$

$$\{F_\alpha^A|\phi_z|^2d\bar{z} - (F_\alpha^A)_z\phi_z dz + (F_\alpha^A)_z\phi_{\bar{z}}d\bar{z}\} - \{F_\alpha^A(2\phi_{zz} - \phi_z^2)dz + \phi\phi_{zA_\alpha}dz\}\ ,$$

$$\{F_\alpha^B|\phi_z|^2d\bar{z} - (F_\alpha^B)_z\phi_z dz + (F_\alpha^B)_z\phi_{\bar{z}}d\bar{z}\} - \{F_\alpha^B(2\phi_{zz} - \phi_z^2)dz + \phi\phi_{zB_\alpha}dz\}\ .$$

The proof can be obtained by a straightforward calculation.

Now we are able to proceed with the proof of proposition 2. Consider equation (2.41). Recall that we assume that the coordinate $z(P_0) := \int_{P_1}^{P_0} w$ of the point P_0 does not change under the variation of the moduli $A_1, \dots, A_g, B_1, \dots, B_g, z_2, \dots, z_M$. Setting $I_\rho := \int_{\mathcal{L}_\rho} |\phi_z|^2|dz|^2$, we get

$$\frac{\partial I_\rho}{\partial A_\alpha} = \frac{1}{2i}\left\{\oint_{P_0} + \sum_{m=1}^{M}\oint_{P_m}\right\}(\phi_z\phi)_{A_\alpha}dz + \frac{1}{2i}\sum_{\beta=1}^{g}\int_{\tilde{a}_\beta^+\cup\tilde{a}_\beta^-}(\phi_z\phi)_{A_\alpha}dz. \tag{2.60}$$

Using Lemma 1 and its corollary, the holomorphy of $(F_\alpha^A)_z\phi_z$ and the relation $(\phi_z\phi)_z dz = d(\phi_z\phi) - \phi_z\phi_{\bar{z}}d\bar{z}$, we rewrite the right-hand side of (2.60) as

$$\frac{1}{2i}\Bigg(\left\{\oint_{P_0} + \sum_{m=1}^{M}\oint_{P_m}\right\}\{F_\alpha^A|\phi_z|^2d\bar{z} - (F_\alpha^A)_z\phi_z dz + (F_\alpha^A)_z\phi_{\bar{z}}d\bar{z}\} -$$

$$- \sum_{\beta=1}^{g}\left[\int_{\tilde{a}_\beta^+\cup\tilde{a}_\beta^-} + \int_{\tilde{b}_\beta^+\cup\tilde{b}_\beta^-}\right](F_\alpha^A)_z\phi_z dz + \sum_{\beta=1}^{g}\int_{\tilde{a}_\beta^+\cup\tilde{a}_\beta^-}(\phi_z\phi)_{A_\alpha}dz\Bigg)\ , \tag{2.61}$$

where we have used the fact that $(F_\alpha^A)_z$ has no jumps on the cuts a_β and b_β, $\beta = 1, \dots, g$. By means of asymptotical expansions of the integrands at the zeroes P_m of w (Lemma 1 plays here a central role; cf. the proof of Theorem 4 in [16]) one gets the relation

$$\frac{1}{2i}\left\{\sum_{m=1}^{M}\oint_{P_m}\right\}\{F_\alpha^A|\phi_z|^2d\bar{z} - (F_\alpha^A)_z\phi_z dz + (F_\alpha^A)_z\phi_{\bar{z}}d\bar{z}\} \tag{2.62}$$

$$= -\pi\sum_{m=1}^{M}\frac{1}{k_m!}\left(1 - \frac{1}{(k_m+1)^2}\right)\left(\frac{d}{dx_m}\right)^{k_m+1}\Phi_\alpha^A(x_m)\Big|_{x_m=0} + o(1)$$

as $\rho \to 0$. By Corollary 4 we have

$$\frac{1}{2i}\oint_{P_0} F_\alpha^A|\phi_z|^2d\bar{z} - (F_\alpha^A)_z\phi_z dz + (F_\alpha^A)_z\phi_{\bar{z}}d\bar{z} = \frac{1}{2i}\oint_{P_0} F_\alpha^A(2\phi_{zz} - (\phi_z)^2)dz + o(1)\ . \tag{2.63}$$

The Cauchy theorem and asymptotical expansions at P_m imply that

$$0=\frac{1}{2i}\left\{\sum_{m=1}^{M}\oint_{P_m}+\oint_{P_0}+\sum_{\beta=1}^{g}\left[\int_{\tilde a_\beta^+\cup\tilde a_\beta^-}+\int_{\tilde b_\beta^+\cup\tilde b_\beta^-}\right]+\sum_{\beta=1}^{g}\left[\int_{a_\beta^+\cup a_\beta^-}+\int_{b_\beta^+\cup b_\beta^-}\right]\right\}F_\alpha^A(2\phi_{zz}-(\phi_z)^2)dz$$

$$=-\pi\sum_{m=1}^{M}\frac{1}{k_m!}\left(1-\frac{1}{(k_m+1)^2}\right)\left(\frac{d}{dx_m}\right)^{k_m+1}\Phi_\alpha^A(x_m)\Big|_{x_m=0} \tag{2.64}$$

$$+\frac{1}{2i}\left\{\oint_{P_0}+\sum_{\beta=1}^{g}\left[\int_{\tilde a_\beta^+\cup\tilde a_\beta^-}+\int_{\tilde b_\beta^+\cup\tilde b_\beta^-}\right]+\sum_{\beta=1}^{g}\left[\int_{a_\beta^+\cup a_\beta^-}+\int_{b_\beta^+\cup b_\beta^-}\right]\right\}F_\alpha^A(2\phi_{zz}-(\phi_z)^2)dz+o(1)$$

(cf. [16], Lemma 6; we emphasize that generically F_α^A has jumps on the a and b cuts as well on $\tilde a$ and $\tilde b$-cuts). Using (2.62) and the information about the jumps of F_α^A on a- and b-cuts given in Lemma 1, we get the equality

$$\frac{1}{2i}\sum_{m=1}^{M}\oint_{P_m}\{F_\alpha^A|\phi_z|^2d\bar z-(F_\alpha^A)_z\phi_z dz+(F_\alpha^A)_z\phi_{\bar z}d\bar z\}=$$

$$=-\frac{1}{2i}\oint_{P_0}F_\alpha^A(2\phi_{zz}-(\phi_z)^2)dz-\frac{1}{i}\oint_{b_\alpha}\{\phi_{zz}-\frac{1}{2}(\phi_z)^2\}dz- \tag{2.65}$$

$$-\frac{1}{2i}\sum_{\beta=1}^{g}\left[\int_{\tilde a_\beta^+\cup\tilde a_\beta^-}+\int_{\tilde b_\beta^+\cup\tilde b_\beta^-}\right]F_\alpha^A(2\phi_{zz}-(\phi_z)^2)dz+o(1).$$

It remains to note that due to Corollary (4),

$$\int_{\tilde a_\beta^+\cup\tilde a_\beta^-}(\phi_z\phi)_{A_\alpha}dz=\int_{\tilde a_\beta^+\cup\tilde a_\beta^-}(\phi_z\phi)_{A_\alpha}dz=$$

$$=\int_{\tilde a_\beta^+\cup\tilde a_\beta^-}[(2\phi_{zz}-(\phi_z)^2)F_\alpha^A+(F_\alpha^A)_z\phi_z]dz \tag{2.66}$$

and the corresponding integrals over pairs of different shores of $\tilde b$-cuts vanish. Now substituting (2.65),(2.63) and (2.66) into (2.61), we get the relation

$$\frac{\partial I_\rho}{\partial A_\alpha}=-\frac{1}{i}\oint_{b_\alpha}\{\phi_{zz}-\frac{1}{2}(\phi_z)^2\}dz+o(1). \tag{2.67}$$

To get (2.41) from (2.67) one needs to use the classical relation

$$\phi_{zz}-\frac{1}{2}(\phi_z)^2=\left\{\int^P\Omega^{P_0},z\right\}=S_{Fay}^{P_0}(z)$$

and take into account that all $o(1)$ above are uniform with respect to the moduli belonging to a compact neighbourhood of the initial point in moduli space.

Relations (2.42) and (2.43) can be proved in the same way. The only slight modification will be in the proof of (2.43):

- After differentiation of I_ρ with respect to the coordinate z_m the analog of equation (2.64) will contain the additional term

$$-\frac{2\pi}{(k_m-1)!(k_m+1)}\left(\frac{d}{dx_m}\right)^{k_m-1} S_{Fay}^{P_0}(x_m)\Big|_{x_m=0} = -2\pi\,\mathrm{res}|_{P_m}\left\{\frac{S_{Fay}^{P_0}-S_w}{w}\right\}$$

at the right-hand side;

- The right-hand side of the analogs of equations (2.60) and (2.62) will contain extra terms $-\frac{1}{2i}\oint_{P_m}|\phi_z|^2 d\bar{z}$ and $\frac{1}{2i}\oint_{P_m}|\phi_z|^2 d\bar{z}$ respectively. After summation they will cancel out.

(cf. [18], Theorem 3).

2.7 Explicit formula for the Bergman tau-function

Theorem 5 *The Bergman tau-function on the space $\mathcal{H}(k_1,\dots,k_M)$ is given by the following formula:*

$$\tau(\mathcal{L},w) = e^{-\frac{\pi i}{3}\langle \mathbf{r},K^{P_0}\rangle}\mathcal{C}(P_0)^{2/3}\left\{\sigma^{4-4g}(P_0,Q_0)\prod_{m=1}^{M}\left\{\Omega^{P_0}(P_m)\right\}^{k_m}\right\}^{-1/12}, \tag{2.68}$$

where the integer vector $\mathbf{r}$ *is defined by the equality*

$$\mathcal{A}\big((w)\big) + 2K^{P_0} + \mathbf{B}\mathbf{r} + \mathbf{s} = 0\;; \tag{2.69}$$

$\mathbf{s}$ *is another integer vector, (w) is the divisor of the differential w, the initial point of the Abel map $\mathcal{A}$ coincides with P_0 and all the paths are chosen inside the same fundamental polygon $\widehat{\mathcal{L}}$.*

Proof. Due to (2.30), one has the relation

$$\partial_\mu \log|\tau(\mathcal{L},w)|^2 = \partial_\mu \log\mathcal{C}(P_0)^{2/3} - \frac{1}{12\pi i}\mathrm{v.p.}\int_{\mathcal{L}}\mu(S_{Fay}^{P_0}-S_w)$$

for any basic Beltrami differential μ. Therefore, taking into account the variational formulas (2.41), (2.42), (2.43) for Dirichlet integral (2.39), we get the formula

$$|\tau|^2 = |\mathcal{C}(P_0)|^{4/3}\exp\{-\mathbf{D}/12\}. \tag{2.70}$$

Using the theorem 4 which gives the holomorphic factorization of the Dirichlet integral $\mathbf{D}$, we get the formula (2.68) for the solution of system (2.17), (2.18), (2.19).

Expression (2.68) is independent of the choice of point Q_0, this immediately follows from (2.26), (2.28) and the relation $\sum k_m = 2g-2$. In addition, it must be independent of the choice of the point P_0. Analysing the monodromy of (2.68) with respect to P_0 along a-cycles we get (2.69).

□

To simplify the expression for the tau-function we need the following lemma.

Lemma 2 *The expression*

$$\mathcal{F} := [w(P)]^{\frac{g-1}{2}} e^{-\frac{\pi i}{2}\langle \mathbf{r},K^P\rangle}\left\{\prod_{m=1}^{M}[E(P,P_m)]^{\frac{(1-g)k_m}{2}}\right\}\mathcal{C}(P) \tag{2.71}$$

is independent of P.

To prove the lemma it is enough to observe that expression (2.71) has tensor weight 0 with respect to P, is nonsingular and has trivial monodromies when P encircles a- and b-cycles.

Now taking in (2.68) $Q_0 = P_0$ and using the equality $\mathcal{C}(P)/\mathcal{C}(Q) = (\mathbf{s}(P,Q))^{1-g}$ and Lemma 2, we get the following expression for the Bergman tau-function.

Theorem 6 *The solution of equations (2.17), (2.18), (2.19) defining the Bergman tau-function is given by*

$$\tau(\mathcal{L}, w) = \mathcal{F}^{1/3} \left\{ \prod_{m=1}^{M} \mathcal{C}^{k_m}(P_m) \right\}^{\frac{1}{6(g-1)}} , \tag{2.72}$$

where $\mathcal{F}$ is given by the formula (2.71).

The expression (2.72), (2.71) for the Bergman tau-function can be slightly simplified for the case of the highest stratum $\mathcal{H}(1,\dots,1)$.

Lemma 3 *Let all the zeros of the Abelian differential w be simple. Then the fundamental cell $\hat{\mathcal{L}}$ can always be chosen such that $\mathcal{A}((w)) + 2K^P = 0$.*

Proof. For an arbitrary choice of the fundamental cell we can claim that the vector $\mathcal{A}((w)) + 2K^P$ coincides with 0 on the Jacobian of the surface $\mathcal{L}$, i.e. there exist two integer vectors $\mathbf{r}$ and $\mathbf{r}$ such that

$$\mathcal{A}((w)) + 2K^P + \mathbf{B}\mathbf{r} + \mathbf{s} = 0 \, . \tag{2.73}$$

Fix some zero P_1 of w; according to our assumption this zero is simple. By a smooth deformation of a cycle a_α within a given homological class we can stretch this cycle such that the point P_1 crosses this cycle; two possible directions of the crossing correspond to the jump of component $\mathbf{r}_\alpha$ of the vector $\mathbf{r}$ to $+1$ or -1. Similarly, if we deform a cycle b_α such that it crosses the point R_1, the component $\mathbf{s}_\alpha$ of the vector $\mathbf{s}$ also jumps by ± 1. Repeating such procedure, we come to fundamental domain where $\mathbf{r} = \mathbf{s} = 0$.

□

We notice that the choice of the fundamental domain such that $\mathcal{A}((w)) + 2K^P = 0$ is obviously possible if differential w has at least one simple zero.

Corollary 5 *Consider the highest stratum $\mathcal{H}(1,\dots,1)$ of the space $\mathcal{H}_g$ containing Abelian differentials w with simple zeros. Let us choose the fundamental cell $\hat{\mathcal{L}}$ such that $\mathcal{A}((w)) + 2K^P = 0$. Then the Bergman tau-function on $\mathcal{H}(1,\dots,1)$ can be written as follows:*

$$\tau(\mathcal{L}, w) = \mathcal{F}^{2/3} \prod_{m,l=1 \;\; m<l}^{2g-2} [E(P_m, P_l)]^{1/6} \tag{2.74}$$

where expression

$$\mathcal{F} := [w(P)]^{\frac{g-1}{2}} \mathcal{C}(P) \prod_{m=1}^{2g-2} [E(P, P_m)]^{\frac{(1-g)}{2}} \tag{2.75}$$

does not depend on P; all prime-forms are evaluated at the points P_m in the distinguished local parameters $x_m(P) = \left(\int_{P_m}^{P} w \right)^{1/2}$.

3 Tau-function on the spaces of quadratic differentials

3.1 Spaces of quadratic differentials with simple poles

The space $\mathcal{Q}_g$ of quadratic differentials on the Riemann surfaces of genus g is the moduli space of pairs $(\mathcal{L}, W)$, where $\mathcal{L}$ is the Riemann surface of genus g and W is a meromorphic quadratic differential on $\mathcal{L}$ having at most simple poles. The space $\mathcal{Q}_g$ is infinite-dimensional, since the number of poles can be arbitrary. This space is stratified in a family of finite-dimensional strata accordiang to the multiplicities of zeros and the number of poles of quadratic differentials. Denote by $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ the stratum of the space $\mathcal{Q}_g$ which consists of quadratic differentials which have M_1 zeros of odd multiplicities $k_1, \dots, k_{M_1}$, M_2 zeros of even multiplicities $l_1, \dots, l_{M_2}$ and L simple poles, and which are not the squares of Abelian differentials (the last condition makes sense if $M_{-1} = L = 0$). Since the degree of divisor (W) equals $4g-4$, the multiplicities of the zeros and the number of poles are connected by the equality $k_1 + \dots + k_{M_1} + l_1 + \dots + l_{M_2} - L = 4g - 4$. In particular, the number $L + M_1$ is always even. For any pair $(\mathcal{L}, W)$ from $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ one can construct the so-called canonical two-fold covering

$$\pi : \tilde{\mathcal{L}} \to \mathcal{L} \tag{3.1}$$

such that $\pi^* W = w^2$, where w is a holomorphic 1-differential on $\tilde{\mathcal{L}}$. This covering is ramified over the poles and the zeros of odd multiplicity of W.

Counting the zeros of the holomorphic Abelian differential w on $\tilde{\mathcal{L}}$, we can compute the genus $\tilde{g}$ of the surface $\tilde{\mathcal{L}}$. Each zero of even multiplicity l_s of W gives rise to two distinct zeros of w of multiplicity $l_s/2$, whereas each zero of W of odd multiplicity k_s corresponds to a single zero of w of multiplicity $k_s + 1$. Thus, one has the relation

$$2\tilde{g} - 2 = l_1 + \dots + l_{M_2} + k_1 + \dots + k_{M_1} + M_1 = 4g - 4 + L + M_1 \;, \tag{3.2}$$

therefore, $\tilde{g} = 2g + (L + M_1)/2 - 1$.

3.2 Coordinates on stratum $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$

The coordinates on the space $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ can be constructed as follows ([23], [20], [24]). Let $R_1, \dots, R_{M_1}$ and $P_1, \dots, P_{M_2}$ be the zeros of a quadratic differential W of (respectively) odd and even multiplicities and let $S_1, \dots, S_L$ be its poles.

Denote by $*$ the holomorphic involution on $\tilde{\mathcal{L}}$ interchanging the sheets of covering (3.1).

The differential $w(P)$ is anti-invariant with respect to involution $*$:

$$w(P^*) = -w(P) \tag{3.3}$$

Denote the canonical basis of cycles on $\mathcal{L}$ by (a_α, b_α). The canonical basis of cycles on $\tilde{\mathcal{L}}$ will be denoted as follows [8]:

$$\{a_\alpha, b_\alpha, a_{\alpha'}, b_{\alpha'}, a_m, b_m\} \tag{3.4}$$

where $\alpha, \alpha' = 1, \dots g$; $m = 1, \dots (L + M_1)/2 - 1$; this basis can always be chosen to have the following invariance properties under the involution $*$:

$$a_\alpha^* + a_{\alpha'} = b_\alpha^* + b_{\alpha'} = 0 \tag{3.5}$$

and

$$a_m^* + a_m = b_m^* + b_m = 0 \;. \tag{3.6}$$

For corresponding canonical basis of normalized holomorphic differentials u_α, $u_{\alpha'}$, u_m on $\tilde{\mathcal{L}}$ we have as a corollary of (3.5, 3.6):

$$u_\alpha(P^*) = -u_{\alpha'}(P) \;, \qquad u_m(P^*) = -u_m(P) \;. \tag{3.7}$$

The canonical basis of normalized holomorphic differentials on $\mathcal{L}$ is then given by

$$w_\alpha(P) = u_\alpha(P) - u_{\alpha'}(P) \;, \qquad \alpha = 1, \dots, g \tag{3.8}$$

The canonical meromorphic differential $\tilde{\mathbf{w}}(P,Q)$ on $\tilde{\mathcal{L}}$ satisfies the following relation:

$$\tilde{\mathbf{w}}(P^*, Q^*) = \tilde{\mathbf{w}}(P,Q) \tag{3.9}$$

for any $P, Q \in \tilde{\mathcal{L}}$; it is related to the meromorphic canonical differential $\mathbf{w}(P,Q)$ on $\mathcal{L}$ as follows:

$$\mathbf{w}(P,Q) = \tilde{\mathbf{w}}(P,Q) + \tilde{\mathbf{w}}(P,Q^*) \;, \qquad P, Q \in \mathcal{L} \;. \tag{3.10}$$

Now we are to introduce the local coordinates on the stratum $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$. First consider the case when $L + M_1 > 0$ (quadratic differentials have a zero of odd multiplicity or a pole). The complex dimension of the stratum $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ is $2g + (L + M_1 - 2) + M_2$. The first $2g + (L + M_1 - 2)$ coordinates on the space $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ can be chosen by integrating the differential $w(P)$ over the basic cycles on $\tilde{\mathcal{L}}$ as follows ([20], [24]):

$$A_\alpha := \oint_{a_\alpha} w \qquad B_\alpha := \oint_{b_\alpha} w \qquad A_m := \oint_{a_m} w \qquad B_m := \oint_{b_m} w \tag{3.11}$$

for $\alpha = 1, \dots, g$, $m = 1, \dots, (L + M_1)/2 - 1$.

To introduce remaining M_2 coordinates we denote by P_k^+ and P_k^- the zeros of differential w such that $\pi(P_k^+) = \pi(P_k^-) = P_k$; $k = 1, \dots, M_2$. The points $R_1, \dots, R_{M_1}, S_1, \dots, S_L$ have unique pre-images under the covering map π (which are the ramification points of the covering). In the sequel we shall denote these points and their pre-images by the same letters. Dissect the surface $\tilde{\mathcal{L}}$ along the basic cycles obtaining a fundamental polygon. Choose as a basic point one of the points $R_1, \dots, R_{M_1}, S_1, \dots, S_L$, say, R_1. Then the last M_2 coordinates are given by integrals of w over the paths connecting the basic point R_1 and the points $P_1^+, \dots, P_{M_2}^+$:

$$z_k = \int_{R_1}^{P_k^+} w; \quad k = 1, \dots, M_2; \tag{3.12}$$

all the paths of integration lie inside the fundamental polygon.

Now consider the case $L + M_1 = 0$. In other words we deal here with holomorphic quadratic differentials which have only zeros of even multiplicity and are not the squares of Abelian differentials. (An example of such a differential can be found in [23]). In this case the canonical covering is unramified. The dimension of the stratum $\mathcal{Q}_g(l_1, \dots, l_{M_2})$ is $2g + M_2 - 1$. The local coordinates are given by integrals

$$A_\alpha := \oint_{a_\alpha} w \qquad B_\alpha := \oint_{b_\alpha} w \qquad z_k := \int_{P_1^+}^{P_k^+} w, \tag{3.13}$$

where $\alpha = 1, \dots g$; $k = 2, \dots, M_2$.

In the sequel we shall restrict ourselves to the case $M_1 + L > 0$. Treatment of the case $M_1 = L = 0$ is completely parallel to the treatment of the ramified covering case $M_1 + L > 0$.

If a quadratic differential W is the square of a holomorphic 1-differential w (in particular, all zeros of W have even multiplicity), then the canonical covering becomes a disjoint union of two copies $\mathcal{L}_+$ and $\mathcal{L}_-$ of the surface $\mathcal{L}$ and $\sqrt{\pi^* W} = \pm w$ on $\mathcal{L}_\pm$. Thus, the moduli space of such quadratic differentials coincides with the moduli space of Abelian differentials.

3.3 Variational formulas on $\mathcal{Q}_g(k_1,\dots,k_{M_1},l_1,\dots,l_{M_2},[-1]^L)$

Let a pair $(\mathcal{L},W)$ belong to the stratum $\mathcal{Q}_g(k_1,\dots,k_{M_1},l_1,\dots,l_{M_2},[-1]^L)$; recall that $\pi:\tilde{\mathcal{L}}\to\mathcal{L}$ is the canonical covering and $\pi^*W=w^2$, where w is an Abelian differential on $\tilde{\mathcal{L}}$. Let us introduce on $\tilde{\mathcal{L}}$ the coordinate $z(P)=\int_{R_1}^P w$; $z(P)$ can be chosen as local parameter everywhere on $\tilde{\mathcal{L}}$ outside of the divisor (w).

Theorem 7 *The basic differentials $w_\alpha(P)$ on $\mathcal{L}$ depend as follows on coordinates (3.11) if $z(P)$ is kept fixed under differentiation:*

$$\frac{\partial w_\alpha(P)}{\partial A_\beta}\Big|_{z(P)}=-\frac{1}{2\pi i}\oint_{b_\beta}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)}\,,\qquad \frac{\partial w_\alpha(P)}{\partial B_\beta}\Big|_{z(P)}=\frac{1}{2\pi i}\oint_{a_\beta}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)} \tag{3.14}$$

$$\frac{\partial w_\alpha(P)}{\partial A_m}\Big|_{z(P)}=-\frac{1}{4\pi i}\oint_{b_m}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)}\,,\qquad \frac{\partial w_\alpha(P)}{\partial B_m}\Big|_{z(P)}=\frac{1}{4\pi i}\oint_{a_m}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)} \tag{3.15}$$

$$\frac{\partial w_\alpha(P)}{\partial z_k}\Big|_{z(P)}=\mathrm{res}|_{Q=P_k^+}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)}, \tag{3.16}$$

where the integrals in the right hand side are computed on $\tilde{\mathcal{L}}$ (since the 1-form $w(P)$ is well-defined on $\tilde{\mathcal{L}}$ only). In the formulas (3.15) we understand $w_\alpha(Q)$ and $\mathbf{w}(P,Q)$ as the natural lift of these differentials from $\mathcal{L}$ to $\tilde{\mathcal{L}}$.

Proof. Since $w(P)$ is holomorphic 1-differential on $\tilde{\mathcal{L}}$, we can write down the analogs of formulas (2.2), (2.3), (2.4) for this differential. However, due to existence of involution $*$ on the surface $\tilde{\mathcal{L}}$ the differential $\frac{\partial w_a(P)}{\partial A_\beta}\Big|_{z(P)}$ is a differential with vanishing a-periods and jumps on contours b_β and $b_{\beta'}$; this allows to write it down in terms of the canonical meromorphic bidifferential $\tilde{\mathbf{w}}(P,Q)$ on $\tilde{\mathcal{L}}$ as follows:

$$\frac{\partial w_a(P)}{\partial A_\beta}\Big|_{z(P)}=-\frac{1}{2\pi i}\oint_{b_\beta}\frac{w_\alpha(Q)\tilde{\mathbf{w}}(P,Q)}{w(Q)}+\frac{1}{2\pi i}\oint_{b_{\beta'}}\frac{w_\alpha(Q)\tilde{\mathbf{w}}(P,Q)}{w(Q)} \tag{3.17}$$

Since $b_{\beta'}=-b_\beta^*$, $w(Q^*)=-w(Q)$ and $w_\alpha(Q^*)=w_\alpha(Q)$, the second term at the right hand side is equal to

$$-\frac{1}{2\pi i}\oint_{b_\beta}\frac{w_\alpha(Q)\tilde{\mathbf{w}}(P,Q^*)}{w(Q)}$$

which leads to the first of equations (3.14). Up to the sign, the proof of the second formula in (3.14) is completely parallel.

Let us prove the first formula of (3.15). The differential $\frac{\partial w_\alpha(P)}{\partial A_m}\Big|_{z(P)}$ has jump on $\tilde{\mathcal{L}}$ only on the cycle b_m; all the a-periods of this differential vanish. Therefore, we can write it in terms of the meromorphic differential $\tilde{\mathbf{w}}(P,Q)$ as follows:

$$\frac{\partial w_\alpha(P)}{\partial A_m}\Big|_{z(P)}=-\frac{1}{2\pi i}\oint_{b_m}\frac{w_\alpha(Q)\tilde{\mathbf{w}}(P,Q)}{w(Q)} \tag{3.18}$$

Taking into account that $b_m=-b_m^*$, $w(Q^*)=-w(Q)$ and $w_\alpha(Q^*)=w_\alpha(Q)$, we get

$$\oint_{b_m}\frac{w_\alpha(Q)\tilde{\mathbf{w}}(P,Q)}{w(Q)}=\frac{1}{2}\left\{\oint_{b_m}\frac{w_\alpha(Q)\tilde{\mathbf{w}}(P,Q)}{w(Q)}+\oint_{b_m}\frac{w_\alpha(Q)\tilde{\mathbf{w}}(P,Q^*)}{w(Q)}\right\}=\frac{1}{2}\oint_{b_m}\frac{w_\alpha(Q)\mathbf{w}(P,Q)}{w(P)} \tag{3.19}$$

which gives the first formula in (3.15); the remaining formulas can be proved in the same way.
□

Integration of the formulas (3.14), (3.15) over the b-cycles of $\mathcal{L}$ leads to the following

Corollary 6 *The matrix of b-periods of the Riemann surface $\mathcal{L}$ depends as follows on coordinates $A_\alpha, B_\alpha, A_m, B_m, z_k$:*

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial A_\gamma} = -\oint_{b_\gamma} \frac{w_\alpha w_\beta}{w}\,, \qquad \frac{\partial \mathbf{B}_{\alpha\beta}}{\partial B_\gamma} = \oint_{a_\gamma} \frac{w_\alpha w_\beta}{w} \tag{3.20}$$

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial A_m} = -\frac{1}{2}\oint_{b_m} \frac{w_\alpha w_\beta}{w}\,, \qquad \frac{\partial \mathbf{B}_{\alpha\beta}}{\partial B_m} = \frac{1}{2}\oint_{a_m} \frac{w_\alpha w_\beta}{w} \tag{3.21}$$

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial z_k} = 2\pi i \mathrm{res}|_{P_k^+} \frac{w_\alpha w_\beta}{w} \tag{3.22}$$

where $\alpha, \beta, \gamma = 1, \dots, g$, $m = 1, \dots (L + M_1)/2 - 1$, $k = 1, \dots, M_2$

Dependence of the meromorphic bidifferential $\mathbf{w}(P, Q)$ on the moduli is described by the following theorem whose proof is parallel to the proofs of theorems 1 and 7:

Theorem 8 *The following variational formulas take place:*

$$\frac{\partial \mathbf{w}(P,Q)}{\partial A_\beta} = -\frac{1}{2\pi i}\oint_{b_\beta} \frac{\mathbf{w}(Q,R)\mathbf{w}(P,R)}{w(R)}\,, \qquad \frac{\partial \mathbf{w}(P,Q)}{\partial B_\beta} = -\frac{1}{2\pi i}\oint_{a_\beta} \frac{\mathbf{w}(Q,R)\mathbf{w}(P,R)}{w(R)} \tag{3.23}$$

$$\frac{\partial \mathbf{w}(P,Q)}{\partial A_m} = -\frac{1}{4\pi i}\oint_{b_m} \frac{\mathbf{w}(Q,R)\mathbf{w}(P,R)}{w(R)}\,, \qquad \frac{\partial \mathbf{w}(P,Q)}{\partial B_m} = \frac{1}{4\pi i}\oint_{a_m} \frac{\mathbf{w}(Q,R)\mathbf{w}(P,R)}{w(R)} \tag{3.24}$$

$$\frac{\partial \mathbf{w}(P,Q)}{\partial z_k} = -\mathrm{res}|_{P=P_k^+} \frac{\mathbf{w}(Q,R)\mathbf{w}(P,R)}{w(R)}, \tag{3.25}$$

where $z(P)$ and $z(Q)$ are kept fixed under differentiation; $\alpha, \beta, \gamma = 1, \dots, g$, $m = 1, \dots (L+M_1)/2-1$, $k = 1, \dots, M_2$.

Finally, we shall need the analogs of formulas (2.10), (2.11), (2.12) for the Bergman projective connection:

Corollary 7 *The following variational formulas take place:*

$$\frac{\partial}{\partial A_\beta}(S_B(P) - S_w(P)) = -\frac{3}{\pi i}\oint_{b_\beta} \frac{\mathbf{w}^2(P,R)}{w(R)}\,, \qquad \frac{\partial}{\partial B_\beta}(S_B(P) - S_w(P)) = \frac{3}{\pi i}\oint_{a_\beta} \frac{\mathbf{w}^2(P,R)}{w(R)}\,, \tag{3.26}$$

$$\frac{\partial}{\partial A_m}(S_B(P) - S_w(P)) = -\frac{3}{2\pi i}\oint_{b_m} \frac{\mathbf{w}^2(P,R)}{w(R)}\,, \qquad \frac{\partial}{\partial B_m}(S_B(P) - S_w(P)) = \frac{3}{2\pi i}\oint_{a_m} \frac{\mathbf{w}^2(P,R)}{w(R)} \tag{3.27}$$

$$\frac{\partial}{\partial z_k}(S_B(P) - S_w(P)) = -\frac{1}{6}\mathrm{res}|_{P=P_k^+} \frac{\mathbf{w}^2(P,R)}{w(R)}, \tag{3.28}$$

where the local parameter $z(P)$ is kept fixed under differentiation; $\alpha, \beta, \gamma = 1, \dots, g$, $m = 1, \dots (L + M_1)/2 - 1$, $k = 1, \dots, M_2$.

3.4 Basic Beltrami differentials for $\mathcal{Q}_g(k_1,\dots,k_{M_1},l_1,\dots,l_{M_2},[-1]^L)$

Basic Beltrami differentials corresponding to variation of the point of the space $\mathcal{Q}_g(k_1,\dots,k_{M_1},l_1,\dots,l_{M_2},[-1]^L)$ can be constructed similarly to the basic Beltrami differentials from section 2.3. As in the section 2.3, we denote the smooth Beltrami differentials responsible for the change of complex structure on the surface $\mathcal{L}$ under infinitesimal shifts of the coordinates $A_\alpha, B_\beta, A_m, B_m, z_k$ by $\mu_{A_\alpha}, \mu_{B_\beta}, \mu_{A_m}, \mu_{B_m}, \mu_{z_k}$ respectively. The Beltrami differentials μ_{A_α} and μ_{B_β} have supports in thin strips along the cycles a_α and b_α of the surface $\mathcal{L}$, the differentials μ_{A_m} and μ_{B_m} have supports along the projections on the surface $\mathcal{L}$ of the thin strips along the cycles a_m and b_m on the covering $\tilde{\mathcal{L}}$. The supports of the differentials μ_{z_k} belong to small annuli centered at the points P_k. The construction of these Beltrami differentials, which we again call *basic*, is completely parallel to the case of spaces $\mathcal{H}_g(k_1,\dots,k_M)$. The only modification is that instead of Corollary 1 one should make use of its analog for the spaces of quadratic differentials, Corollary 6.

3.5 Definition of Bergman tau-function

The definition of the Bergman tau-function on the spaces of quadratic differentials is similar to the spaces of Abelian differentials.

Definition 3 *The Bergman tau-function $\tau(\mathcal{L},W)$ on the stratum $\mathcal{Q}_g(k_1,\dots,k_{M_1},l_1,\dots,l_{M_2},[-1]^L)$ of the space of quadratic differentials over Riemann surface $\mathcal{L}$ is locally defined by the following system of equations:*

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial A_\alpha} = \frac{1}{12\pi i}\oint_{b_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1,\dots,g; \tag{3.29}$$

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial B_\alpha} = -\frac{1}{12\pi i}\oint_{a_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1,\dots,g; \tag{3.30}$$

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial A_m} = \frac{1}{24\pi i}\oint_{b_m} \frac{S_B - S_w}{w}, \quad m = 1,\dots,(L+M_1)/2-1; \tag{3.31}$$

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial B_m} = -\frac{1}{24\pi i}\oint_{a_m} \frac{S_B - S_w}{w}, \quad m = 1,\dots,(L+M_1)/2-1, \tag{3.32}$$

$$\frac{\partial \log \tau(\mathcal{L},W)}{\partial z_k} = -\frac{1}{6}\mathrm{res}|_{P=P_k^+} \frac{S_B - S_w}{w}, \quad k = 1,\dots,M_2, \tag{3.33}$$

where S_B is the Bergman projective connection on $\mathcal{L}$, $S_w(\zeta) := \left\{\int^P w, \zeta\right\}$; the difference between two projective connections S_B and S_w is a meromorphic quadratic differential on $\mathcal{L}$ with poles at the zeros of W.

The compatibility of this system follows from Corollary 7 and the symmetry of the canonical meromorphic bidifferential in complete analogy to Proposition 1.

3.6 Variation of differential $\mathcal{C}$

Here is the analog of the Corollary (3) for the spaces of quadratic differentials.

Corollary 8 *The following variational formulas for differential $\mathcal{C}(P)$ on the space $\mathcal{Q}_g(k_1,\dots,k_{M_1},l_1,\dots,l_{M_2},[-1]^L)$ take place:*

$$\frac{\partial}{\partial A_\alpha}\log\{\mathcal{C}w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = \frac{1}{8\pi i}\oint_{b_\alpha}\frac{1}{w}\left(S_B - S^P_{Fay}\right), \tag{3.34}$$

$$\frac{\partial}{\partial B_\alpha}\log\{\mathcal{C}w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = -\frac{1}{8\pi i}\oint_{a_\alpha}\frac{1}{w}\left(S_B - S^P_{Fay}\right), \tag{3.35}$$

$$\frac{\partial}{\partial A_m}\log\{\mathcal{C}w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = \frac{1}{16\pi i}\oint_{b_m}\frac{1}{w}\left(S_B - S^P_{Fay}\right), \tag{3.36}$$

$$\frac{\partial}{\partial B_m}\log\{\mathcal{C}w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = -\frac{1}{16\pi i}\oint_{a_m}\frac{1}{w}\left(S_B - S^P_{Fay}\right), \tag{3.37}$$

$$\frac{\partial}{\partial z_k}\log\{\mathcal{C}w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = -\frac{1}{4}\mathrm{res}|_{P=P_k^+}\frac{1}{w}\left(S_B - S^P_{Fay}\right), \tag{3.38}$$

where the coordinate $z(P)=\int_{R_1}^P w$ is kept constant under differentiation.

3.7 Dirichlet integral: variational formulas and holomorphic factorization

Let a pair $(\mathcal{L},W)$ belong to the space $\mathcal{Q}_g(k_1,\dots,k_{M_1},l_1,\dots,l_{M_2},[-1]^L)$. Let $\pi:\tilde{\mathcal{L}}\to\mathcal{L}$ be the canonical covering and let Ω^{P_0} be the multi-valued 1-differential defined by (2.28). We denote by $\tilde{\Omega}^{P_0}=\pi^*(\Omega^{P_0})$ the lift of the differential Ω^{P_0} to $\tilde{\mathcal{L}}$. The multi-valued 1-differential $\tilde{\Omega}^{P_0}$ has on $\tilde{\mathcal{L}}$ simple zeros at the ramification points $R_1,\dots,R_{M_1}$, $S_1,\dots,S_L$ of the canonical covering and zeros of multiplicity $2g-2$ at the preimages P_0^+ and P_0^- of the point P_0. It is single-valued along the cycles $a_\alpha, a_{\alpha'}, a_m, b_m$ and gains the multipliers $\exp(4\pi i K_\alpha^{P_0})$ and $\exp(-4\pi i K_\alpha^{P_0})$ along the cycles b_α and $b_{\alpha'}$. The divisor of the differential w on the covering $\tilde{\mathcal{L}}$ can be written as follows

$$(w)=\sum_{m=1}^{M_2}\left(\frac{l_m}{2}P_k^+ + \frac{l_m}{2}P_k^-\right)+\sum_{s=1}^{M_1}(k_s+1)R_s\,. \tag{3.39}$$

(Abelian differential w is non-singular and non-vanishing at the poles S_l of W). Assume for convenience that $M_1\neq 0$. Then the system of local parameters on the covering $\tilde{\mathcal{L}}$ is given by

$$z(P)=\int_{R_1}^P w \tag{3.40}$$

outside the divisor (w);

$$x_m(P)=\left(\int_{P_m^\pm}^P w\right)^{\frac{2}{l_m+2}} \tag{3.41}$$

near the points $P_m^\pm$ and

$$\zeta_s(P)=\left(\int_{R_s}^P w\right)^{\frac{1}{k_s+2}} \tag{3.42}$$

near the points R_s.

Setting $\phi(z,\bar z)=\log|\frac{\tilde\Omega^{P_0}}{w}|^2$ we define the regularized Dirichlet integral

$$\mathbb{D}=\frac{1}{2\pi}\lim_{\rho\to 0}\left\{\int_{\tilde{\mathcal{L}}\setminus U_\rho}|\phi_z|^2|dz|^2+\left(2\sum_{m=1}^{M_2}\frac{l_m^2}{l_m+2}+\sum_{s=1}^{M_1}\frac{k_s^2}{k_s+2}+2L+16(g-1)^2\right)\pi\log\rho\right\},\qquad(3.43)$$

where U_ρ is the union of the disks of radius ρ centered at $P_0, P_0^*,\ P_1^\pm,\dots,P_{M_2}^\pm,\ R_1,\dots,R_{M_1}$ and $S_1,\dots,S_L$. In order to follow the proof of theorem 4 and factorize this integral we have to find the asymptotics of the line integrals $\oint_Q\phi_z\phi\,dz$, where Q is one of the just listed points, as $\rho\to 0$. Obviously, we have the following asymptotics

$$\phi_z=-\frac{l_m}{l_m+2}x_m^{-\frac{l_m}{2}-1}+O(x_m^{-\frac{l_m}{2}}),\qquad(3.44)$$

near the points $P_m^\pm$. To get the asymptotics near the point R_s we note that

$$\phi_z=-\frac{k_s+1}{k_s+2}\zeta_s^{-k_s-2}+\frac{1}{k_s+2}\zeta^{-k_s-1}\frac{\tilde\Omega^{P_0}_{\zeta_s}(\zeta_s)}{\tilde\Omega^{P_0}(\zeta_s)}$$

and the function $\tilde\Omega^{P_0}(\zeta_s)$ has the first order zero at $\zeta_s=0$. Thus,

$$\phi_z=-\frac{k_s}{k_s+2}\zeta_s^{-k_s-2}+O(\zeta_s^{-k_s-1})\qquad(3.45)$$

near the point R_s. Near the point S_r one has the asymptotics

$$\phi_z=\frac{1}{\xi_r}+O(1),\qquad(3.46)$$

where we put $\xi_r(P)=z(P)-z(S_r)$. The asymptotics of ϕ_z near the points $P_0^\pm$ is the same as in (2.36). Thus, we have

$$\frac{1}{2i}\int_{P_m^\pm}\phi_z\phi\,dz=\frac{\pi l_m}{2}\log\left|\tilde\Omega^{P_0}(x_m)|_{x_m=0}\right|^2-\pi\frac{l_m^2}{l_m+2}\log\rho-\pi l_m\log(\frac{l_m}{2}+1)+o(1),\qquad(3.47)$$

as $\rho\to 0$. The asymptotics of integrals around the points R_s looks as follows:

$$\frac{1}{2i}\int_{R_s}\phi_z\phi\,dz=\frac{1}{2i}\int_{|\zeta_s|=\rho^{\frac{1}{k_s+2}}}\left[-\frac{k_s}{k_s+2}\zeta_s^{-k_s-2}+O(\zeta_s^{-k_s-1})\right]$$

$$\left[\log\left|\frac{\tilde\Omega^{P_0}(\zeta_s}{\zeta_s}\right|^2-2k_s\log|\zeta_s|-2\log(k_s+2)\right](k_s+2)\zeta_s^{k_s+1}d\zeta_s=$$

$$=\pi k_s\log\left|\frac{\tilde\Omega^{P_0}(\zeta_s)}{\zeta_s}|_{\zeta_s=0}\right|^2-\pi\frac{k_s}{k_s+2}\log\rho-2\pi k_s\log(k_s+2)+o(1).\qquad(3.48)$$

In the same manner we get

$$\frac{1}{2i}\int_{S_r}=-\pi\log\left|\frac{\tilde\Omega^{P_0}(\xi_r)}{\xi_r}|_{\xi_r=0}\right|^2-2\pi\log\rho+o(1).\qquad(3.49)$$

The integrals around $P_0^{\pm}$ have exactly the same asymptotics as in Section 2.6.1. Altogether, these asymptotics yield the following factorization formula for the Dirichlet integral:

$$\mathbb{D} = \log\Big|\sigma^{4-4g}(P_0,Q_0)\prod_{m=1}^{M_2}\left(\Omega^{P_0}(x_m)|_{x_m=0}\right)^{\frac{l_m}{2}}\prod_{s=1}^{M_1}\left(\frac{\tilde\Omega^{P_0}(\zeta_s)}{\zeta_s}|_{\zeta_s=0}\right)^{\frac{k_s}{2}}\times \tag{3.50}$$

$$\times\prod_{r=1}^{L}\left(\frac{\tilde\Omega^{P_0}(\xi_r)}{\xi_r}\right)^{-\frac{1}{2}}\exp\{4\pi i\langle\mathbf{r},K^{P_0}\rangle\}\Big|^2 + \text{const}$$

where

$$2\pi r_\alpha = \text{Var}|_{a_\alpha}\left\{\text{Arg}\frac{\tilde\Omega^{P_0}(P)}{w(P)}\right\} = \text{Var}|_{a_{\alpha'}}\left\{\text{Arg}\frac{\tilde\Omega^{P_0}(P)}{w(P)}\right\}$$

and σ is defined in (2.38). (One should make use of the obvious relation

$$|\sigma(P_0,Q_0)|^2 = |\tilde\sigma(P_0^+,Q_0^+)\tilde\sigma(P_0^-,Q_0^-)|,$$

where $\tilde\sigma = \pi^*\sigma$.) To rewrite (3.50) using only objects defined on the surface $\mathcal{L}$ and not on the covering $\tilde{\mathcal{L}}$ introduce the local parameters according to the following definition [30]:

Definition 4 *The local parameters on $\mathcal{L}$*

$$\lambda_s := \zeta_s^2 = \left(\int_{R_s}^P w\right)^{\frac{2}{k_s+2}}, \quad \theta_r := \xi_r^2 = \left(\int_{S_r}^P w\right)^2, \quad x_m(P) := \left(\int_{P_m}^P w\right)^{\frac{2}{l_m+2}} \tag{3.51}$$

near the points R_s, S_r and P_m respectively, are called distinguished.

Setting $\Omega^{P_0}(R_s) = \Omega^{P_0}(\lambda_s)|_{\lambda_s=0}$ and $\Omega^{P_0}(S_r) = \Omega^{P_0}(\theta_r)|_{\theta_r=0}$ we eventually get the final form of the factorization formula.

Theorem 9 *The following representation of the Dirichlet integral (3.43) holds:*

$$\mathbb{D} = \log\Big|\sigma^{4-4g}(P_0,Q_0)\prod_{m=1}^{M_2}\left(\Omega^{P_0}(P_m)\right)^{\frac{l_m}{2}}\prod_{s=1}^{M_1}\left(\Omega^{P_0}(R_s)\right)^{\frac{k_s}{2}}\times \tag{3.52}$$

$$\times\prod_{r=1}^{L}\left(\Omega^{P_0}(S_r)\right)^{-\frac{1}{2}}\exp\{4\pi i\langle\mathbf{r},K^{P_0}\rangle\}\Big|^2 + \text{const}.$$

The proof of the following proposition is completely parallel to the proof of proposition 2. The only difference is due to the fact that one should work with the covering $\tilde{\mathcal{L}}$ instead of $\mathcal{L}$.

Proposition 3 *Dirichlet integral (3.43) satisfies the following system of equations:*

$$\frac{\partial\mathbb{D}}{\partial A_\alpha} = -\frac{1}{\pi i}\oint_{b_\alpha}\frac{S_{Fay}^{P_0} - S_w}{w}, \quad \alpha = 1,\dots,g; \tag{3.53}$$

$$\frac{\partial\mathbb{D}}{\partial B_\alpha} = \frac{1}{\pi i}\oint_{a_\alpha}\frac{S_{Fay}^{P_0} - S_w}{w}, \quad \alpha = 1,\dots,g; \tag{3.54}$$

$$\frac{\partial \mathbb{D}}{\partial A_m} = -\frac{1}{2\pi i}\oint_{b_m} \frac{S_{Fay}^{P_0} - S_w}{w}, \quad m = 1, \dots, (L+M_1)/2 - 1; \tag{3.55}$$

$$\frac{\partial \mathbb{D}}{\partial B_m} = \frac{1}{2\pi i}\oint_{a_m} \frac{S_{Fay}^{P_0} - S_w}{w}, \quad m = 1, \dots, (L+M_1)/2 - 1; \tag{3.56}$$

$$\frac{\partial \mathbb{D}}{\partial z_k} = -\frac{1}{6}\mathrm{res}|_{P=P_k^+} \frac{S_{Fay}^{P_0} - S_w}{w}, \quad m = 1, \dots, (L+M_1)/2 - 1. \tag{3.57}$$

where $w^2 = W$; the projective connection S_w is given by $S_w(\zeta(P) := \{\int^P w, \zeta(P)\}$ in a local parameter $\zeta(P)$.

3.8 Explicit formula for the tau-function

The explicit formula for the tau-function on the space $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ follows from formula (2.30), Theorem 9 and Proposition 3. To write down this formula in a compact form we introduce the following notation for the divisor (W):

$$(W) = -\sum_{r=1}^{L} S_r + \sum_{s=1}^{M_1} k_s R_s + \sum_{m=1}^{M_2} l_m P_m := \sum_{k=1}^{L+M_1+M_2} n_k Q_k\ ; \tag{3.58}$$

the distinguished local parameters near the points Q_k are given in Definition 4.

Proposition 4 *The tau-function on the space $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ is given by the following formula:*

$$\tau(\mathcal{L}, W) = e^{-\frac{\pi i}{3}\langle \mathbf{r}, K^{P_0}\rangle} \mathcal{C}(P_0)^{2/3} \sigma^{(1-g)/3}(P_0, Q_0) \prod_{k=1}^{L+M_1+M_2} \left\{\Omega^{P_0}(Q_k)\right\}^{-\frac{n_k}{24}} . \tag{3.59}$$

The integer vector $\mathbf{r}$ in (3.59) is subject to the condition

$$\mathcal{A}\big((W)\big) + 4K^{P_0} + \mathbf{B}\mathbf{r} + \mathbf{s} = 0, \tag{3.60}$$

where $\mathbf{s}$ is another integer vector, the initial point of the Abel map $\mathcal{A}$ coincides with P_0 and all the paths are chosen inside the same fundamental polygon $\widehat{\mathcal{L}}$.

An alternative expression for the same tau-function is given by the following theorem:

Theorem 10 *The Bergman tau-function on the space $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ is given by the following expression:*

$$\tau(\mathcal{L}, W) = \mathcal{F}^{\frac{1}{3}} \prod_{k=1}^{L+M_1+M_2} \mathcal{C}^{\frac{n_k}{12(g-1)}}(Q_k)\ , \tag{3.61}$$

where the expression

$$\mathcal{F} = W^{(g-1)/4}(P)\mathcal{C}(P) e^{-\pi i \langle \mathbf{r}, K^P\rangle} \prod_{k=1}^{L+M_1+M_2} E^{(1-g)n_k/4}(Q_k, P)$$

does not depend on $P \in \mathcal{L}$; the prime-forms and differential $\mathcal{C}$ at the points of divisor (W) are evaluated in the distinguished local parameters given in Definition 4.

The proof is parallel to the proof of the theorem 6 for the spaces of Abelian differentials.

Next we show how to simplify this expression if the quadratic differential W is holomorphic and all the zeros of W are simple (then $L = M_2 = 0$).

In analogy to Lemma 3, one can show that for an arbitrary quadratic differential W with simple zeros the fundamental cell can always be chosen such that

$$\mathcal{A}((W)) + 4K^P = 0 \tag{3.62}$$

(for an arbitrary choice of the fundamental cell this vector vanishes only up to an integer combination of lattice vectors defining the Jacobian of the Riemann surface $\mathcal{L}$).

Corollary 9 *The tau-function on the stratum $\mathcal{Q}_g(1, \dots, 1)$ of the space $\mathcal{Q}_g$, which consists of holomorphic quadratic differentials with simple zeros, is given by the following formula:*

$$\tau(\mathcal{L}, W) = \mathcal{F}^{2/3} \prod_{r,s=1 \ r<s}^{4g-4} [E(R_r, R_s)]^{1/24}, \tag{3.63}$$

where expression

$$\mathcal{F} = W^{(g-1)/4}(P)\mathcal{C}(P) \prod_{s=1}^{4g-4} E^{(1-g)/4}(R_s, P)$$

does not depend on P. The prime-forms and differential $\mathcal{C}(P)$ are evaluated at the points R_s with respect to distinguished local parameters

$$\lambda_s(P) = \left(\int_{R_s}^P w \right)^{2/3} ; \tag{3.64}$$

the fundamental cell $\hat{\mathcal{L}}$ is chosen such that $\mathcal{A}((W)) + 4K^P = 0$.

4 Variational formulas for determinants of Laplacians in Strebel metrics of finite volume

4.1 Preliminaries: determinants of Laplacians in smooth metrics

Introduce the smooth metric

$$\mathbf{g} := \rho^{-2}(z, \bar{z})dxdy = \rho^{-2}(z, \bar{z})\widehat{dz} \tag{4.1}$$

on the Riemann surface $\mathcal{L}$. Here $z = x + iy$ is a local parameter on $\mathcal{L}$, ρ is a smooth positive function of local parameter (we adopt the notation from [9]). The Laplacian $\Delta^{\mathbf{g}} = 4\rho^2(z, \bar{z})\partial^2_{z\bar{z}}$ with domain given by C^∞-functions on $\mathcal{L}$ is an essentially self-adjoint operator with discrete spectrum. The determinant $\det\Delta^{\mathbf{g}}$ is defined via standard ζ-regularization

$$\det\Delta^{\mathbf{g}} = \exp\{-\zeta'(0)\},$$

where $\zeta(s)$ is the ζ-function of the operator $\Delta^{\mathbf{g}}$ (see, e.g., [9]). The formula describing dependence of $\det\Delta^{\mathbf{g}}$ on the smooth metric $\mathbf{g}$ within a given conformal class was first derived by Polyakov [26]. We shall need original Polyakov's version of this formula which is related to the formula (1.16) (a proof of which can be found in [9]) via integration by parts.

Recall that the Gaussian curvature of the metric (4.1) is given by

$$K = 4\rho^2 \partial^2_{z\bar z} \log \rho\,. \tag{4.2}$$

Now, for two arbitrary smooth metrics $\mathbf{g}_0 := \rho_0^{-2}(z,\bar z)dxdy$ and $\mathbf{g}_1 := \rho_1^{-2}(z,\bar z)dxdy$ the Polyakov's formula tells that

$$\log \frac{\det\Delta^{\mathbf{g}_1}}{\mathrm{Vol}(\mathcal{L},\mathbf{g}_1)} - \log \frac{\det\Delta^{\mathbf{g}_0}}{\mathrm{Vol}(\mathcal{L},\mathbf{g}_0)} = -\frac{1}{3\pi}\left\{\int_{\mathcal{L}} \left|\partial_z \log\frac{\rho_0}{\rho_1}\right|^2 \widehat{dz} + \frac{1}{2}\int_{\mathcal{L}} \left(\log\frac{\rho_0}{\rho_1}\right) K_0 \rho_0^{-2}\widehat{dz}\right\}\,, \tag{4.3}$$

where K_0 is the Gaussian curvature of the metric $\mathbf{g}_0$. To derive (4.3) from (1.16) we rewrite the right-hand side of (1.16) as

$$\frac{1}{3\pi}\left(\int_{\mathcal{L}} \log\frac{\rho_1}{\rho_0}\partial^2_{z\bar z}\log\frac{\rho_1}{\rho_0}\widehat{dz} + \int_{\mathcal{L}}\left(\log\frac{\rho_1}{\rho_0}\right)\rho_0^2\{\partial^2_{z\bar z}\log\rho_0^2\}\rho_0^{-2}\widehat{dz}\right)\,. \tag{4.4}$$

To get (4.3) one needs to integrate the first term by parts.

Remark 2 The notation from ([9]) which we adopt here is very convenient though seemingly noninvariant. A routine reasoning shows that all the objects in (4.3, 4.4) actually have the invariant sense: for instance, $\log\frac{\rho_1}{\rho_0}$ is a scalar as well as $\rho_0^2\{\partial^2_{z\bar z}\log\rho_0^2\}$, $\bar\partial\partial\log\frac{\rho_1}{\rho_0}$ is a $(1,1)$-form and can be integrated over $\mathcal{L}$, etc.

In each conformal class there exists the unique metric of constant curvature: the Poincaré metric. Variational formula describing dependence of $\det\Delta$ in the Poincaré metric on the moduli of a Riemann surface was given in ([34]). To describe this result we recall that the surface $\mathcal{L}$ is biholomorphically equivalent to the quotient space $\mathbb{H}/\Gamma$, where $\mathbb{H} = \{u \in \mathbb{C} : \Im u > 0\}$; Γ is a strictly hyperbolic Fuchsian group. Denote by $\pi_F : \mathbb{H} \to \mathcal{L}$ the natural projection. The Fuchsian projective connection S_F is given by the Schwarzian derivative $S_F(x) = \{u, x\}$, where x is a local coordinate of a point $P \in \mathcal{L}$, $w \in \mathbb{H}$, $\pi_F(u) = P$. The Poincaré metric $\mathbf{g}^P$ on $\mathcal{L}$ is given by the projection onto $\mathcal{L}$ of the metric $|\Im u|^{-2}|du|^2$ on $\mathbb{H}$. The Poincaré metrics has constant curvature -1 and, due to the Gauss-Bonnet theorem, the volume $\mathrm{Vol}(\mathcal{L},\mathbf{g}^P)$ of the surface $\mathcal{L}$ in this metric is independent of moduli and equals $2\pi(2g-2)$, where g is the genus of $\mathcal{L}$. Denote by $\Delta^{\mathbf{g}^P}$ the Laplacian in the Poincaré metrics. Then there is the following variational formula for its ζ-regularized determinant:

$$\delta_\mu \log\left(\frac{\det\Delta^{\mathbf{g}^P}}{\det\Im\mathbf{B}\mathrm{Vol}(\mathcal{L},\mathbf{g}^P)}\right) = -\frac{1}{12\pi i}\int_L (S_B - S_F)\mu\,, \tag{4.5}$$

where μ is a Beltrami differential defining the deformation of the complex structure on $\mathcal{L}$, $S_B - S_F$ (the difference of the Bergman and the Fuchsian projective connections) is a quadratic differential on $\mathcal{L}$.

The constant factor $\mathrm{Vol}(\mathcal{L},\mathbf{g}^P)$ in the left-hand side of (4.5) is of no importance and may be omitted. It should be noted that formula (4.5) can be considered as a partial case of the general Fay variational formula for analytic torsion (see [9], in particular, p. 97 for derivation of (4.5)).

4.2 Definition of $\det\Delta$ for Strebel metrics of finite volume

Any meromorphic quadratic differential W with only simple poles defines a natural flat metric on the Riemann surface $\mathcal{L}$ given by $|W|$. This metric has conical singularities at the zeroes and poles of

W. The cone angles of the metric $|W|$ equal π at the simple poles of W and $(k+2)\pi$ at the zeros of W of multiplicity k. The unbounded symmetric operator $4|W|^{-1}\partial\bar{\partial}$ in $L_2(\mathcal{L}, |w|^2)$ with domain $C_0^\infty(\mathcal{L} \setminus (W))$ admits the closure and has the self-adjoint Friedrichs extension which we denote by $\Delta^{|W|}$. It is known ([4, 25]) that the spectrum $\sigma(\Delta^{|W|})$ of $\Delta^{|W|}$ is discrete and

$$N(\lambda) = O(\lambda), \tag{4.6}$$

as $\lambda \to +\infty$, where $N(\lambda)$ is the number of eigenvalues of the positive operator $-\Delta^{|W|}$ not exceeding λ (counting multiplicity). The ζ-function of the operator $\Delta^{|W|}$ defined by the sum over positive eigenvalues:

$$\zeta(s) = \sum_{\lambda_j > 0} \lambda_j^{-s}$$

for $\Re s > 1$ (in this domain the series converges due to asymptotics (4.6)) admits analytic continuation to a meromorphic function in $\mathbf{C}$ which is regular at $s = 0$ ([14]). The regularized determinant of the operator $\Delta^{|W|}$ is defined by the equality

$$\det \Delta^{|W|} = \exp\{-\zeta'(0)\} \ .$$

Remark 3 For smooth metrics the analogs of the above statements (the spectral properties of the Laplacian, the regularity of the ζ-function at $s = 0$, etc) are more or less standard and are nicely summarized in [9] in the general context of holomorphic bundles of arbitrary rank. (Here we deal only with trivial line bundles.) In our present case (when the metric has conical singularities) the proofs are less known. All the appropriate references could be found in [14].

4.3 Variational formulas for $\det \Delta$ in Strebel metrics

The well-known Fay's variational formula (see [9], pp. 59-60) describes the variation of analytic torsion under (generally nonconformal) variations of smooth metric on the surface. Here we state an analog of this result for metrics with conical singularities defined by quadratic (or Abelian) differentials. Namely we describe the behavior of the determinants of the Laplacians in these metrics under the variations of coordinates on the spaces $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ and $\mathcal{H}(k_1, \dots, k_M)$.

Let pairs $(\mathcal{L}, W)$ and $(\mathcal{L}, w)$ belong to the spaces $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ and $\mathcal{H}(k_1, \dots, k_M)$ respectively. For the pair $(\mathcal{L}, W)$ we define

$$T(\mathcal{L}, |W|) = \log\left\{\frac{\det \Delta^{|W|}}{\mathrm{Vol}(\mathcal{L}, |W|)\det\Im\mathbf{B}}\right\} ,$$

where for an arbitrary metric $\mathbf{g}$ on $\mathcal{L}$ we denote by $\mathrm{Vol}(\mathcal{L}, \mathbf{g})$ the area of the Riemann surface $\mathcal{L}$ in metric $\mathbf{g}$.

Analogously, we denote

$$T(\mathcal{L}, |w|^2) = \log\left\{\frac{\det \Delta^{|w|^2}}{\mathrm{Vol}(\mathcal{L}, |w|^2)\det\Im\mathbf{B}}\right\}$$

for the pair $(\mathcal{L}, w)$.

The following key theorem describes the variation of $T(\mathcal{L}, |W|)$ and $T(\mathcal{L}, |w|^2)$ on the spaces $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ and $\mathcal{H}(k_1, \dots, k_M)$.

Theorem 11 *The following formulas for the derivatives of $T(\mathcal{L}, W)$ with respect to standard coordinates in $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$ hold true*

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial A_\alpha} = \frac{1}{12\pi i} \int_{b_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1, \dots, g; \tag{4.7}$$

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial B_\alpha} = -\frac{1}{12\pi i} \oint_{a_\alpha} \frac{S_B - S_w}{w}, \qquad \alpha = 1, \dots, g; \tag{4.8}$$

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial A_m} = \frac{1}{24\pi i} \oint_{b_\alpha} \frac{S_B - S_w}{w}, \qquad m = 1, \dots, (L + M_1)/2 - 1; \tag{4.9}$$

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial B_m} = -\frac{1}{24\pi i} \oint_{a_\alpha} \frac{S_B - S_w}{w}, \qquad m = 1, \dots, (L + M_1)/2 - 1; \tag{4.10}$$

$$\frac{\partial T(\mathcal{L}, |W|)}{\partial z_k} = -\frac{1}{6} \mathrm{res}\Big|_{P_k^+} \frac{S_B - S_w}{w}, \qquad k = 1, \dots M_2 \ , \tag{4.11}$$

where S_B is the Bergman projective connection, S_w is the projective connection given by the Schwarzian derivative

$$\Big\{ \int^P \sqrt{W}, \zeta \Big\},$$

$S_B - S_w$ is a meromorphic quadratic differential with poles of the second order at the poles and zeroes of W.

The following formulas for the derivatives of $T(\mathcal{L}, |w|^2)$ with respect to standard coordinates in $\mathcal{H}(k_1, \dots, k_M)$ hold true

$$\frac{\partial T(\mathcal{L}, |w|^2)}{\partial A_\alpha} = \frac{1}{12\pi i} \oint_{b_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1, \dots, g; \tag{4.12}$$

$$\frac{\partial T(\mathcal{L}, |w|^2)}{\partial B_\alpha} = -\frac{1}{12\pi i} \oint_{a_\alpha} \frac{S_B - S_w}{w}, \quad \alpha = 1, \dots, g; \tag{4.13}$$

$$\frac{\partial T(\mathcal{L}, |w|^2)}{\partial z_m} = -\frac{1}{6} \mathrm{res}\Big|_{P_m} \frac{S_B - S_w}{w}, \quad m = 2, \dots M \ , \tag{4.14}$$

where S_B is the Bergman projective connection, S_w is the projective connection given by the Schwarzian derivative

$$\Big\{ \int^P w, \zeta \Big\},$$

$S_B - S_w$ is a meromorphic quadratic differential with poles of the second order at the zeroes P_m of w.

The proof is given in the next section.

4.4 Proof of Theorem 11

The proof of Theorem 11 uses the following tools:

- The Polyakov formula and the formula for the variation of the determinant of the Laplacian in the Poincaré metric (see section 4.1).

- The machinery of the Dirichlet integrals, providing generating functions for the values of Fuchsian projective connection in distinguished local parameters at the zeros of quadratic (Abelian) differentials (see sections 4.4.1 and 4.4.2).
- A version of Burghelea-Friedlander-Kappeler analytic surgery for metrics with conical singularities (see section 4.4.4).

4.4.1 Fuchsian Dirichlet integral for metrics $|w|^2$

Introduce the functions $\phi^{ext}(z,\bar z)$ of local parameter $z(P)=\int_{P_1}^P w$ on the surface $\mathcal{L}$ outside the zeros P_m of the differential w:

$$e^{\phi^{ext}(z,\bar z)}|dz|^2=\frac{|du|^2}{|\Im u|^2}\,, \tag{4.15}$$

where u is the Fuchsian coordinate on the universal covering $\mathbb{H}$ of $\mathcal{L}$. Analogously, define the functions $\phi^{int}(x_m)$ of the distinguished local parameter x_m near P_m by

$$\exp\phi^{int}(x_m,\bar x_m)|dx_m|^2=|du|^2/|\Im u|^2. \tag{4.16}$$

Define the regularized Dirichlet integral $\operatorname{reg}\mathbb{D}_F$ as

$$\mathbb{D}_F^{reg}=\lim_{\epsilon\to 0}\left(\int_{\Lambda_\epsilon}|\phi_z^{ext}|^2\widehat{dz}+2\pi\sum_{m=1}^M\frac{k_m^2}{k_m+1}\log\epsilon\right), \tag{4.17}$$

where $\Lambda_\epsilon=\mathcal{L}\setminus\cup_{m=1}^M\{|x_m|\le\epsilon\}$.

Let

$$\mathcal{D}=\left\{\mathbb{D}_F^{reg}-2\pi\sum_{m=1}^M k_m\phi^{int}(x_m,\bar x_m)\Big|_{x_m=0}\right\}. \tag{4.18}$$

Lemma 4 *The equations hold true*

$$\frac{\partial\mathcal{D}}{\partial A_\alpha}=\frac{1}{i}\oint_{b_\alpha}\left(\frac{S_F-S_w}{w}\right), \tag{4.19}$$

$$\frac{\partial\mathcal{D}}{\partial B_\alpha}=-\frac{1}{i}\oint_{a_\alpha}\left(\frac{S_F-S_w}{w}\right), \tag{4.20}$$

$$\frac{\partial\mathcal{D}}{\partial z_m}=-2\pi\operatorname{Res}|_{P_m}\left(\frac{S_F-S_w}{w}\right), \tag{4.21}$$

where $\alpha=1,\dots,g$, $m=2,\dots,M$.

The proof of this Lemma (as well as the proof of Lemma 5 below) is a routine, though somewhat cumbersome, reasoning similar to the proof Proposition 2. Moreover, in the part concerning the coordinates z_m it coincides with the proof of Theorem 8 from [16]. Recall that in [16] we considered the Hurwitz spaces of meromorphic functions on Riemann surfaces. The coordinates on the Hurwitz spaces (the critical values λ_m of meromorphic function λ or, what is the same, the integrals $\int^{\lambda_m}d\lambda$) are complete analogs of our coordinates z_m on the spaces of Abelian differentials. So, the proof of (4.21) coincides with the proof of Theorem 8 from [16] verbatim. To prove relations (4.19) and (4.20) one has to play with jumps of the derivatives of the potential ϕ^{ext} on the a- and b-cycles similarly to the proof of Proposition 2.

4.4.2 Fuchsian Dirichlet integral for metrics $|W|$

Let us formulate an analog of Lemma 4 for the spaces of quadratic differentials.

Let $z = z(P)$ be a local parameter on the canonical covering $\tilde{\mathcal{L}}$, $z(P) = \int_{R_1}^P w$. Define the function $\phi^{ext}(z, \bar{z})$ by (4.15) where u is the Fuchsian coordinate corresponding to the point P of the surface $\mathcal{L}$. Let λ_s, θ_r, x_m be the distinguished parameters (see Definition 4) near the points R_s, S_r, P_m of the surface $\mathcal{L}$. Define the functions $\phi^{int}(\lambda_s, \bar{\lambda}_s)$, $\phi^{int}(\theta_r, \bar{\theta}_r)$ and $\phi^{int}(x_m, \bar{x}_m)$ analogously to (4.16). Set $\phi^{int}(R_s) = \phi^{int}(\lambda_s, \bar{\lambda}_s)\Big|_{\lambda_s=0}$ and define $\phi^{int}(S_r)$ and $\phi^{int}(P_m)$ in the same way.

Lemma 5 *Let*

$$\mathcal{D}_Q = \frac{1}{2}\left\{\text{reg}\int_{\tilde{\mathcal{L}}} |\phi_z^{ext}|^2 \widehat{dz} - 2\pi\left(\sum_{s=1}^{M_1} k_s \phi^{int}(R_s) + \sum_{s=1}^{M_2} l_s \phi^{int}(P_s) - \sum_{s=1}^{L} \phi^{int}(L_s)\right)\right\} \tag{4.22}$$

Then the following equations hold true.

$$\frac{\partial \mathcal{D}_Q}{\partial A_\alpha} = \frac{1}{i}\oint_{b_\alpha}\left(\frac{S_F - S_w}{w}\right), \tag{4.23}$$

$$\frac{\partial \mathcal{D}_Q}{\partial B_\alpha} = -\frac{1}{i}\oint_{a_\alpha}\left(\frac{S_F - S_w}{w}\right), \tag{4.24}$$

$$\frac{\partial \mathcal{D}_Q}{\partial A_m} = \frac{1}{2i}\oint_{b_m}\left(\frac{S_F - S_w}{w}\right), \tag{4.25}$$

$$\frac{\partial \mathcal{D}_Q}{\partial B_m} = -\frac{1}{2i}\oint_{a_m}\left(\frac{S_F - S_w}{w}\right), \tag{4.26}$$

$$\frac{\partial \mathcal{D}_Q}{\partial z_k} = -2\pi \text{Res}|_{P_k}\left(\frac{S_F - S_w}{w}\right), \tag{4.27}$$

where $\alpha = 1, \dots, g$, $m = 1, \dots, (L + M_1)/2 - 1$, $k = 1, \dots, M_2$.

4.4.3 Smoothing of conical metric

The proof of theorem 11 is based on the following fact. In the distinguished local parameters near the conical points the metrics $|W|$ and $|w|^2$ all have the standard form $|\zeta|^n |dz|^2$, where ζ is the corresponding distinguished local parameter, n is an integer. Smoothing these metrics in standard neighborhoods $|\zeta| < \epsilon$ of the conical points one gets a new metrics without singularities. These smooth metrics can be chosen to have the same volume as the old (conical) ones. It turns out that variations w. r. t. moduli of the logarithms of determinants of Laplacians in old (singular) metrics and new (smooth) metrics coincide. (Here moduli mean the coordinates on the space of corresponding Abelian or quadratic differentials.) To prove this coincidence we apply the machinery of analytic surgery. For smooth metrics the surgery formula (BFK formula in what follows) belongs to Burghelea, Friedlander and Kappeler ([3]). Using their ideas together with some known facts about the heat kernel on manifolds with conical singularities, we establish here a version of the BFK formula for conical metrics. These two versions of BFK formula imply the required coincidence of the variations.

We restrict ourselves to the case of the space of Abelian differentials, the proofs in the case of quadratic differentials differ only in notation. Let x_m be a distinguished local parameter $x_m(P) =$

$\left(\int_{P_m}^P w\right)^{1/(k_m+1)}$ near the zero P_m of the differential w. Take some $\epsilon > 0$. In the disks $\{|x_m| \leq \epsilon\}$ all the metrics w has the form $|x_m|^{2k_m}|dx_m|^2$. Let $f(x_m)$ be a strictly positive C^∞-function such that $f^{-2}(x_m) = |x_m|^{2k_m}$ for $\epsilon/2 \leq |x_m| \leq \epsilon$, $\int_{D_\epsilon} f^{-2}(x_m)|dx_m|^2 = \int_{D_\epsilon} |x_m|^{2k_m}|dx_m|^2$. Introduce the metric $\mathbf{g} := \rho_\epsilon^{-2}|dz|^2$ which coincides with $|w|^2$ outside the discs $D_\epsilon = \{|x_m| \leq \epsilon\}$ and equals $f^{-2}(x_m)|dx_m|^2$ inside these disks. This metric is everywhere nonsingular, belongs to the conformal class of the metric $|w|^2$ and has the same volume as $|w|^2$.

Proposition 5 *Let ∂_t be the differentiation w. r. t. one of the coordinates on the space $\mathcal{H}_g(k_1, \dots, k_M)$ and let $\det\Delta^{\mathbf{g}}$ be the standard ζ-regularized determinant of the Laplacian in the metric $\mathbf{g}$. Then one has the relation*

$$\partial_t \log \det\Delta^{|w|^2} = \partial_t \log \det\Delta^{\mathbf{g}}. \tag{4.28}$$

Proof. For simplicity suppose first that $M = 1$. Let $D = D_\epsilon = \{|x_1| \leq \epsilon\}$, $\Sigma = \mathcal{L} \setminus D$. Let $\Delta^{\mathbf{g}}$ be the Laplacian related to the metric $\mathbf{g}$. Let $(\Delta^{|w|^2}|D)$ and $(\Delta^{|w|^2}|\Sigma)$ be the operators of the Dirichlet boundary problem for $\Delta^{|w|^2}$ in domains D and Σ respectively. Define the Neumann jump operator (a pseudodifferential operator on ∂D of order 1) $R : C^\infty(\partial D) \to C^\infty(\partial D)$ by

$$R(f) = \partial_\nu(V^- - V^+),$$

where ν is the outward normal to ∂D, the functions V^- and V^+ are the solutions of the boundary value problems $\Delta^\rho V^- = 0$ in D, $V^-|_{\partial D} = f$ and $\Delta^\rho V^+ = 0$ in Σ, $V^+|_{\partial D} = f$. (For brevity we are omitting the index t in ρ_t.) In what follows it is crucial that the Neumann jump operator does not change if we vary the metric within the same conformal class (operators $4|w|^{-2}\partial\bar\partial$ and $4\rho^2\partial\bar\partial$ give rise to the same Neumann jump operator). Due to Theorem B^* from [3], we have

$$\log \det\Delta^{\mathbf{g}} = \log \det(\Delta^{\mathbf{g}}|D) + \log\ \det(\Delta^{\mathbf{g}}|\Sigma) + \log\ \det R + \log \mathrm{Vol}(\mathcal{L}, \mathbf{g}) - \log l(\partial D), \tag{4.29}$$

where $l(\partial D)$ is the length of the contour ∂D in the metric $\mathbf{g}$. Analogous statement holds if the metric defining the Laplacian has a conical singularity inside D. The only change which should be made is one in the definition of the solution of the boundary value problem for the Laplacian in D. Namely, the asymptotical condition $U(P) = O(1)$ near the conical point should be imposed on such a solution (this corresponds to the choice of the Friedrichs extension of the Laplacian). Under this condition we have the surgery formula for the operator $\Delta^{|w|^2}$:

$$\log \det\Delta^{|w|^2} = \log \det(\Delta^{|w|^2}|D) + \log\ \det\,(\Delta^{|w|^2}|\Sigma) + \log \det R + \log \mathrm{Vol}(\mathcal{L}, |w|^2) - \log l(\partial D). \tag{4.30}$$

The proof of formula (4.30) will be given in the next section. Note that the variations of the first terms in right hand sides of (4.29) and (4.30) vanish (these terms are independent of t) whereas the variations of all the remaining terms coincide. This leads to (4.28). To consider the general case ($M > 1$) one should apply an obvious generalization of the surgery formula to the case when the domain D consists of union of several non-overlapping discs; similar result can be found in ([32], Remark on page 326). □

4.4.4 Analytic surgery for flat metrics with conical singularities.

Let $\mathbf{g}^c = \rho^{-2}|dz|^2$ be a flat metric with conical singularity on the surface $\mathcal{L}$. Let D be a disk around the conical point (for simplicity we assume that there is only one such) and let $\Sigma = \mathcal{L} \setminus D$. Let the Neumann jump operator R and the operators of boundary value problems $(\Delta^{\mathbf{g}^c}|D)$ and $(\Delta^{\mathbf{g}^c}|\Sigma)$ for the Laplacian $\Delta^{\mathbf{g}^c} = 4\rho^2\bar\partial\partial$ be as in the previous section.

Theorem 12 *We have the surgery formula for the operator* $\Delta^{\mathbf{g}^c}$*:*

$$\log\det\Delta^{\mathbf{g}^c} = \log\det(\Delta^{\mathbf{g}^c}|D) + \log\,\det\,(\Delta^{\mathbf{g}^c}|\Sigma) + \log\det R + \log \mathrm{Vol}(\mathcal{L}, \mathbf{g}^c) - \log l(\partial D)\;. \tag{4.31}$$

Proof. Formula (4.31) can be proved in the same manner as the formula (4.29) for smooth metrics. We shall follow the strategy of the original work [3] and the recent paper [11] devoted to determinants of Laplacians in exterior domains. [1]

As in [3] and [11] the proof consists of the establishing the following three statements.

1. Let $s > 0$. Denote by $R(s)$ the Neumann jump operator $R(s)f = \partial_\nu(V_s^- - V_s^+)$, where V_s^+ (respectively V_s^-) is the solution of the Dirichlet problem for the operator $\Delta^{\mathbf{g}^c} + s$ in the domain D (respectively $\Sigma = \mathcal{L} \setminus D$) with data f on the boundary. Then
$$\frac{d}{ds}\log\,\det\,(\Delta^{\mathbf{g}^c} + s) =$$
$$= \frac{d}{ds}\left\{\log\,\det\,(\Delta^{\mathbf{g}^c} + s|D) + \log\,\det\,(\Delta^{\mathbf{g}^c} + s|\Sigma) + \log\,\det R(s)\right\},$$
and, hence,
$$\log\,\det\,(\Delta^{\mathbf{g}^c} + s|D) + \log\,\det\,(\Delta^{\mathbf{g}^c} + s|\Sigma) + \log\,\det R(s) =$$
$$= \log\,\det\,(\Delta^{\mathbf{g}^c} + s) + C \tag{4.32}$$
with some constant C.

2. In equation (4.32) the constant C should be zero.

3. Taking the limit $s \to 0+$ in (4.32) with $C = 0$, one gets (4.30).

The proofs of statements 1 and 3 coincide almost verbatim with the proof of formula (3.1) in [11]) and the reasoning from section 4.9 in [3]. The effects connected with conical point appear only in the proof of statement 2.

To prove statement 2 we are to consider the asymptotic expansions of

$$\log\,\det\,(\Delta^{\mathbf{g}^c} + s|D) + \log\,\det\,(\Delta^{\mathbf{g}^c} + s|\Sigma) + \log\,\det R(s) \tag{4.33}$$

and

$$\log\,\det\,(\Delta^{\mathbf{g}^c} + s) \tag{4.34}$$

as $s \to \infty$, and show that the constant terms in these expansions are zero. To this end, as in the proof of proposition 3.3 from ([11]), we rewrite the zeta function for the operator $\Delta^{\mathbf{g}^c} + s$ as

$$\zeta_s(p) = \frac{1}{\Gamma(p)}\int_0^\infty t^p \mathrm{Tr}\, e^{-t\Delta^{\mathbf{g}^c}} e^{-ts}\frac{dt}{p}. \tag{4.35}$$

Due to theorem 2 from ([13]), there is the following asymptotic expansion for the heat kernel on a compact two dimensional manifold with conical point:

$$\mathrm{Tr}\, e^{-t\Delta^{\mathbf{g}^c}} = \sum_{j=-2}^{\infty} a_j t^{j/2} + \sum_{j=0}^{\infty} b_j t^{\frac{j}{2}} \log t \tag{4.36}$$

[1] We are grateful to Steve Zelditch for a very useful conversation and to Paul Loya and Andrew Hassell for sharing a valuable information on this subject.

as $t \to 0+$. Moreover, for our case (when the metric is flat) in this expansion there is no pure logarithmic term, i. e.

$$b_0 = 0.$$

(The proof of this fact can be found in [14].) To get the asymptotical expansion of (4.34) as $s \to \infty$ one needs only to substitute (4.36) into (4.35) and make the change of variable $\tau = ts$ in all the integrals. This results in the expansion

$$\zeta_s(p) \sim \sum_{j=-2}^{\infty} a_j \frac{\Gamma(p+\frac{j}{2})}{\Gamma(p)} s^{-p-j/2} + \sum_{j=1}^{\infty} b_j s^{-p-j/2} \frac{1}{\Gamma(p)} \int_0^{\infty} \tau^{p+j/2-1} e^{-\tau} \log \tau d\tau -$$

$$- \log s \sum_{j=1}^{\infty} b_j \frac{\Gamma(p+j/2)}{\Gamma(p)} s^{-p-j/2}.$$

Differentiating this expansion with respect to p and substituting $p = 0$, we get the expansion

$$\log\det(\Delta^{\mathbf{g}^c} + s) = \sum_j (p_j s^{-j/2} + q_j s^{-j/2} \log s + r_j s^{-j/2} \log^2 s)$$

with $p_0 = 0$. Due to ([3]), the analogous expansion with zero constant term (but without squares of logarithm) holds for the second and the third term in (4.33). The analysis of the asymptotical expansion of the first term in (4.33) essentially coincides with that of (4.34): the constant term in this expansion is also absent. $\square$

4.4.5 The relation between Laplacians in conical and Poincaré metrics

Here we establish the relation between variations of the Laplacians in Poincaré and conical metrics. This together with (4.5) will lead to a result which turns out to be equivalent to Theorem 11. For definiteness we consider here the case of Abelian differentials, using the remarks to comment on the changes which are needed in the case of quadratic differentials. Proposition 5 implies that

$$\partial_t \left(\log \frac{\det\Delta_{-1}}{\det\Im\mathbf{B}\mathrm{Vol}_{-1}(\mathcal{L})} - \log \frac{\det\Delta^{|w|^2}}{\det\Im\mathbf{B}\mathrm{Vol}(\mathcal{L}, |w|^2)} \right) \tag{4.37}$$

$$= \partial_t \left(\log \frac{\det\Delta_{-1}}{\det\Im\mathbf{B}\mathrm{Vol}_{-1}(\mathcal{L})} - - \log \frac{\det\Delta^{\mathbf{g}}}{\det\Im\mathbf{B}\mathrm{Vol}(\mathcal{L}, \mathbf{g})} \right),$$

where as usually ∂_t denotes the variation with respect to one of the coordinates on the space of Abelian differentials, $\mathbf{g}$ is the nonsingular metric defined in section 4.4.3.

We are going to apply Polyakov's formula (4.3) to the right-hand side of (4.37). For this purpose we introduce, as in section 4.4.1, the function $\phi^{ext}(z, \bar{z})$ of local parameter $z(P) = \int_{P_1}^{P} w$ outside the zeros of the differential w:

$$e^{\phi^{ext}(z,\bar{z})} |dz|^2 = \frac{|du|^2}{|\Im u|^2}, \tag{4.38}$$

where u is the Fuchsian coordinate on the universal covering $\mathbb{H}$ of $\mathcal{L}$. Define the functions $\phi^{int}(x_m)$ of the distinguished local parameter x_m near P_m by

$$e^{\phi^{int}(x_m,\bar{x}_m)} |dx_m|^2 = \frac{|du|^2}{|\Im u|^2}. \tag{4.39}$$

Let us choose in formula (4.3) $\mathbf{g}_1$ to be the Poincaré metric and $\mathbf{g}_0$ to be the metric $\mathbf{g}$. Inside the disk $D_m = \{|x_m| \leq \epsilon\}$ one has the relation

$$\frac{\rho_0}{\rho_1} = f \exp(\phi^{int}/2),$$

where the function f is defined before proposition 5. Outside the disks D_m one has the relation

$$\frac{\rho_0}{\rho_1} = \exp(\phi^{ext}/2).$$

Notice also that the Gaussian curvature K_0 vanishes outside the disks D_m. Moreover, the function $K_0\rho_0^{-2} = K_0 f^{-2}$ tends (in a weak sense) as $\epsilon \to 0$ to a linear combination of δ-functions supported at the zeros of w. In particular,

$$\lim_{\epsilon\to 0}\int_{D_m} \phi^{int} K_0 f^{-2}\widehat{dx_m} = -2\pi k_m \phi^{int}(x_m, \bar{x}_m)\Big|_{x_m=0}. \tag{4.40}$$

Let, as in Section 4.4.1 $\Lambda_\epsilon := \mathcal{L} \setminus \cup_{m=1}^M D_m$. Now the right hand side of (4.37) can be rewritten as

$$-\frac{1}{12\pi}\partial_t\left\{\int_{\Lambda_\epsilon} |\phi_z^{ext}|^2\widehat{dz} + \sum_{m=1}^M \int_{D_m} \left(4|\partial_{x_m}\log f|^2 + 2(\log f)_{x_m}\phi_{\bar{x}_m}^{int} + 2(\log f)_{\bar{x}_m}\phi_{x_m}^{int} + |\phi_{x_m}^{int}|^2\right)\widehat{dx_m} + \right.$$

$$\left. +2\sum_{m=1}^M \int_{D_m} K_0 f^{-2}\log f\widehat{dx_m} + \sum_{m=1}^M \int_{D_m} \phi^{int} K_0 f^{-2}\widehat{dx_m}\right\}.$$

Notice that

$$\partial_t\left\{4\sum_{m=1}^M \int_{D_m} |\partial_{x_m}\log f|^2\widehat{dx_m} + 2\sum_{m=1}^M \int_{D_m} K_0 f^{-2}\log f\widehat{dx_m}\right\} = 0,$$

since the expression in the braces is independent of moduli. We have also the asymptotics

$$\sum_{m=1}^M \int_{D_m} \left(2(\log f)_{x_m}\phi_{\bar{x}_m}^{int} + 2(\log f)_{\bar{x}_m}\phi_{x_m}^{int} + |\phi_{x_m}^{int}|^2\right)\widehat{dx_m} = o(1)$$

and

$$\sum_{m=1}^M \int_{D_m} \phi^{int} K_0 f^{-2}\widehat{dx_m} = -2\pi\sum_{m=1}^M k_m\phi^{int}(x_m, \bar{x}_m)\Big|_{x_m=0} + o(1)$$

as $\epsilon \to 0$. As in section 4.4.1, let the regularized Dirichlet integral $\mathbb{D}_F^{reg}$ be defined as (4.17).

Since the left hand side of (4.37) is independent of ϵ we may take the limit $\epsilon \to 0$ in (4.37) and get the following lemma.

Lemma 6 *The following variational formula holds*

$$\partial_t\left(\log\frac{\det\Delta^{\mathbf{g}^P}}{\det\Im\mathbf{B}\mathrm{Vol}(\mathcal{L}, \mathbf{g}^P)} - \log\frac{\det\Delta^{|w|^2}}{\det\Im\mathbf{B}\mathrm{Vol}(\mathcal{L}, |w|^2)}\right) = -\frac{1}{12\pi}\partial_t\left\{\mathbb{D}_F^{reg} - 2\pi\sum_{m=1}^M k_m\phi^{int}(x_m, \bar{x}_m)\Big|_{x_m=0}\right\}, \tag{4.41}$$

where $\mathrm{reg}\,\mathbb{D}$ *is defined by (4.17).*

Remark 4 In the case of metric $|W|$ the analog of (4.41) will contain expression (4.22) in the braces at the r. h. s.

4.4.6 Reduction of Theorem 11 to variational formula for Dirichlet integral

Substituting in (4.5) instead of μ the basic Beltrami differentials, we derive from Lemma 6 the following proposition.

Proposition 6 *As in section 4.4.1, introduce the quantity $\mathcal{D}$ by*

$$\mathcal{D} = \left\{ \operatorname{reg}\mathbb{D} - 2\pi \sum_{m=1}^{M} k_m \phi^{int}(x_m, \bar{x}_m)\Big|_{x_m=0} \right\}.$$

Then the variational formulas hold true

$$\frac{\partial T(\mathcal{L}, |w|^2)}{\partial A_\alpha} = \frac{1}{12\pi i} \oint_{b_\alpha} \frac{S_B - S_F}{w} + \frac{1}{12\pi} \frac{\partial \mathcal{D}}{\partial A_\alpha}, \quad \alpha = 1, \dots, g; \tag{4.42}$$

$$\frac{\partial T(\mathcal{L}, |w|^2)}{\partial B_\alpha} = -\frac{1}{12\pi i} \oint_{a_\alpha} \frac{S_B - S_F}{w} + \frac{1}{12\pi} \frac{\partial \mathcal{D}}{\partial B_\alpha}, \quad \alpha = 1, \dots, g; \tag{4.43}$$

$$\frac{\partial T(\mathcal{L}, |w|^2)}{\partial z_m} = -\frac{1}{6} \operatorname{res}\Big|_{P_m} \frac{S_B - S_F}{w} + \frac{1}{12\pi} \frac{\partial \mathcal{D}}{\partial z_m}, \quad m = 2, \dots M\,. \tag{4.44}$$

Theorem 11 in its part concerning Abelian differentials follows from this proposition and Lemma 4.

Remark 5 To formulate the analog of proposition 6 for the case of quadratic differentials one has to change $\mathcal{D}$ for $\mathcal{D}_Q$ from (4.22) and add formulas for the derivatives with respect to coordinates A_m and B_m. Theorem 11 in its part concerning quadratic differentials then follows from Lemma 5.

5 Explicit formulas for $\det \Delta$ in Strebel metrics of finite volume

5.1 Determinant of $\Delta^{|w|^2}$ in terms of tau-function

Theorem 13 *Let a pair $(\mathcal{L}, w)$ be a point of the space $\mathcal{H}(k_1, \dots, k_M)$. Then the determinant of the Laplacian $\Delta^{|w|^2}$ acting in the trivial line bundle over the Riemann surface $\mathcal{L}$ admits the following explicit expression*

$$\det \Delta^{|w|^2} = C \operatorname{Vol}(\mathcal{L}, |w|^2) \det \Im \mathbf{B}\, |\tau(\mathcal{L}, w)|^2, \tag{5.1}$$

where $\operatorname{Vol}(\mathcal{L}) := \int_{\mathcal{L}} |w|^2$ is the area of $\mathcal{L}$; $\mathbf{B}$ is the matrix of b-periods; constant C is independent of a point of connected component of $\mathcal{H}(k_1, \dots, k_M)$.

Proof. The proof immediately follows from the definition of the Bergman τ-function and Theorem 11. □

5.1.1 Genus 1: Kronecker's formula

When the genus of $\mathcal{L}$ equals 1, holomorphic differentials have no zeroes and, due to the rescaling formula for analytic torsion ([9], formula (2.39)), it is sufficient to find $\det\Delta^{|dz|^2}$, where dz is the normalized holomorphic differential with periods 1 and σ. We have (see [9], p. 20)

$$S_B(\zeta) - S_{dz}(\zeta) = S_B(\zeta) - \{z, \zeta\} = -24\pi i \frac{d \log \eta(\sigma)}{d\sigma} \left(\frac{dz}{d\zeta}\right)^2,$$

where $\eta(\sigma)$ is the Dedekind eta-function,

$$\eta(\sigma) = \exp\left(\frac{\pi i \sigma}{12}\right) \prod_{n \in \mathbf{N}} \Big(1 - \exp(2\pi i n \sigma)\Big).$$

Obviously, $\mathrm{Vol}(\mathcal{L}, |dz|^2) = \Im\sigma$ and Rauch's formula (see [9], p.57) together with Theorem 11 imply that for any Beltrami differential μ

$$\delta_\mu \log \frac{\det \Delta^{|dz|^2}}{(\Im\sigma)^2} = 2\frac{d \log \eta(\sigma)}{d\sigma} \int_{\mathcal{L}} \mu (dz)^2 = \delta_\mu \log \eta^2(\sigma) = \delta_\mu \log |\eta(\sigma)|^4.$$

This gives the well-known equality

$$\det \Delta^{|dz|^2} = C |\Im\sigma|^2 |\eta(\sigma)|^4 \tag{5.2}$$

with some constant C independent of σ; we recall that expression (5.2) was found in [27], later it was discovered that an equivalent statement was known already to Kronecker.

The formula (5.1) is a natural generalization of (5.2) to higher genus.

5.2 Explicit formulas for $\det\Delta^{|W|}$

The following theorem follows from the definition of the Bergman τ-function on the space $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}[-1]^L)$ and Theorem 11.

Theorem 14 *Let a pair $(\mathcal{L}, W)$ belong to the space $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}[-1]^L)$. Then the determinant of the Laplacian $\Delta^{|W|}$ acting in the trivial line bundle over the Riemann surface $\mathcal{L}$ admits the following explicit expression*

$$\det \Delta^{|W|} = C\, \mathrm{Vol}(\mathcal{L}, |W|) \det \Im \mathbf{B}\, |\tau(\mathcal{L}, W)|^2, \tag{5.3}$$

where $\mathrm{Vol}(\mathcal{L}) := \int_{\mathcal{L}} |W|$ is the area of $\mathcal{L}$; constant C which is independent of a point of a connected component of $\mathcal{Q}_g(k_1, \dots, k_{M_1}, l_1, \dots, l_{M_2}, [-1]^L)$. Here the Bergman tau-function $\tau(\mathcal{L}, W)$ is given by (3.61).

5.2.1 Sphere with four conical singularities

Here we illustrate the general framework in the case of the space $\mathcal{Q}_0([-1]^4)$.

This space can be considered as a space of equivalence classes of quadratic differentials on the Riemann sphere with four simple poles. Two such differentials W_1 and W_2 are called equivalent if there exists a Möbius transformation Θ such that $\Theta^* W_1 = W_2$. Take the quadratic differential

$$W = \frac{b\,(dz)^2}{(z - a_1)(z - a_2)(z - a_3)(z - a_4)}$$

with $b, a_1, \dots, a_4 \in \mathbb{C}$ as a representative of a class $[W]$. Then there is the following expression for the tau-function on $\mathcal{Q}_0([-1]^4)$:

$$\tau(W) = b^{-1/2} \prod_{i<j} (a_i - a_j)^{1/6}. \tag{5.4}$$

The determinant of the Laplacian on the Riemann sphere corresponding to the metric $|W|$ is given by the expression

$$\det\Delta^{|W|} = C\text{Vol}(\mathbb{C}P^1, |W|)|\tau|^2$$

or, equivalently,

$$\det\Delta^{|W|} = C\prod_{i<j}|a_i - a_j|^{1/3}\int\int_{\mathbb{C}}\frac{|dz|^2}{|z-a_1||z-a_2||z-a_3||z-a_4|}\,, \tag{5.5}$$

which coincides with expression previously obtained in ([1]). Thus, formula (5.3) can be considered as a generalization of (5.5) to higher genus.

5.3 Formulas of Polyakov type for Strebel metrics of finite volume

5.3.1 Dependence of $\tau(\mathcal{L}, w)$ and $\tau(\mathcal{L}, W)$ on the choice of differential: the case of simple poles

The following theorem shows how the Bergman tau-functions computed at the points $(\mathcal{L}, w)$ and $(\mathcal{L}, \tilde{w})$ are related to each other on the stratum of highest dimension, when both differentials w and $\tilde{w}$ have only simple zeros. We denote these zeros by P_m and $\tilde{P}_m$, respectively.

Theorem 15 *Let w and $\tilde{w}$ be two holomorphic 1-forms with simple poles on the same Riemann surface $\mathcal{L}$. Intruduce their divisors $(w) := \sum_{m=1}^{2g-2} P_m$ and $(\tilde{w}) := \sum_{m=1}^{2g-2} \tilde{P}_m$. Then*

$$\frac{\tau(\mathcal{L}, w)}{\tau(\mathcal{L}, \tilde{w})} = \prod_{m=1}^{2g-2}\left\{\frac{\text{res}|_{\tilde{P}_m}\{w^2/\tilde{w}\}}{\text{res}|_{P_m}\{\tilde{w}^2/w\}}\right\}^{1/24}\,. \tag{5.6}$$

Proof. The local parameter in a neighbourhood of P_m we choose to be

$$x_m(P) := \left[\int_{P_m}^P w\right]^{1/2}\;;$$

in a neighbourhood of $\tilde{P}_m$ we shall use the local parameter

$$\tilde{x}_m(P) := \left[\int_{\tilde{P}_m}^P w\right]^{1/2}\,.$$

Then the formula (5.6) can be alternatively rewritten as follows:

$$\tau(\mathcal{L}, w)\prod_{m=1}^{2g-2}\tilde{w}^{1/12}(P_m) = \tau(\mathcal{L}, \tilde{w})\prod_{m=1}^{2g-2}w^{1/12}(\tilde{P}_m)$$

where we use the following standard convention for "evaluation" of the differentials w and $\tilde{w}$:

$$\tilde{w}(P_m) := \frac{\tilde{w}(P)}{dx_m(P)}\Big|_{P=P_m}\,,\qquad w(\tilde{P}_m) := \frac{w(P)}{d\tilde{x}_m(P)}\Big|_{P=\tilde{P}_m}\,. \tag{5.7}$$

Let us assume that the fundamental cell $\hat{\mathcal{L}}$ is chosen in such a way that the Abel maps of divisors (w) and $(\tilde{w})$ equal $2K^P$; this choice is always possible (see Lemma 3) in our present case, when all points of these divisors have multiplicity 1. Then vectors $\mathbf{r}$ and $\tilde{\mathbf{r}}$ (2.69), corresponding to tau-functions $\tau(\mathcal{L}, w)$ and $\tau(\mathcal{L}, \tilde{w})$, vanish, and we get, according to the formulas (2.71), (2.74) (all products below are taken from 1 to $2g-2$):

$$\frac{\tau^{12}(\mathcal{L},w)\prod_m \tilde{w}(P_m)}{\tau^{12}(\mathcal{L},\tilde{w})\prod_m w(\tilde{P}_m)} = \prod_m \frac{\tilde{w}(P_m)}{w(\tilde{P}_m)} \prod_{m<n} \frac{E^2(P_m,P_n)}{E^2(\tilde{P}_m,\tilde{P}_n)} \left\{ \frac{w(P)\prod_m E(P,\tilde{P}_m)}{\tilde{w}(P)\prod_m E(P,P_m)} \right\}^{4g-4} \tag{5.8}$$

Since this expression is independent of P, we can split the power $4g-4$ of the expression in the braces into product over arbitrary $4g-4$ points, in particular, into product over $P_1, \dots, P_{2g-2}$ and $\tilde{P}_1, \dots, \tilde{P}_{2g-2}$. Then most of the terms in (5.8) cancel each other. The only terms left are due to the fact that the prime-forms vanish at coinciding arguments; this compensates vanishing of w and $\tilde{w}$ at their zeros. As a result we can rewrite (5.8) as follows:

$$\prod_m \left\{ \lim_{P\to P_m} \frac{w(P)}{E(P,P_m)(dx_m(P)^{3/2}} \lim_{P\to\tilde{P}_m} \frac{E(P,\tilde{P}_m)(d\tilde{x}_m(P))^{3/2}}{\tilde{w}(P)} \right\} \tag{5.9}$$

which equals 1, since, say, in a neighbourhood of P_m we have $w(P) = 2x_m(P)dx_m(P)$ and $E(P,P_m) = x_m(P)/\sqrt{dx_m(P)}$.

Remark 6 If differentials w and $\tilde{w}$ have zeros of arbitrary multiplicities, i.e. $(w) = \sum_{m=1}^M k_m P_m$ and $(\tilde{w}) = \sum_{m=1}^M \tilde{k}_m \tilde{P}_m$, the formula (5.6) turns into

$$\frac{\tau(\mathcal{L},w)}{\tau(\mathcal{L},\tilde{w})} = C \left\{ \frac{\prod_{m=1}^M [\mathrm{res}|_{\tilde{P}_m}\{w^2/\tilde{w}\}]^{\tilde{k}_m}}{\prod_{m=1}^{\tilde{M}} [\mathrm{res}|_{P_m}\{\tilde{w}^2/w\}]^{k_m}} \right\}^{1/24} , \tag{5.10}$$

where C is a constant depending on $\{k_m, \tilde{k}_m\}$, and, possibly, on the choice of connected components in the strata $\mathcal{H}_g(k_1,\dots,k_M)$ and $\mathcal{H}_g(\tilde{k}_1,\dots,\tilde{k}_M)$. The proof of (5.10) is parallel to the proof of (5.6).

5.3.2 Analog of Polyakov formula for metrics $|w|^2$ and $|W|$

The natural version of Polyakov formula (1.16) for the metrics $|w|^2$ should say how the determinant $\Delta^{|w|^2}$ depends on the choice of holomorphic differential w on a given Riemann surface $\mathcal{L}$. From the formula (5.3) for $\Delta^{|w|^2}$ and the formula (5.6) which relates tau-functions corresponding to two holomorphic differentials with simple zeros, we get the following "singular" version of Polyakov formula:

$$\frac{\det\Delta^{|w|^2}}{\det\Delta^{|\tilde{w}|^2}} = C\frac{\int_{\mathcal{L}}|w|^2}{\int_{\mathcal{L}}|\tilde{w}|^2} \prod_{k=1}^{2g-2} \left| \frac{\mathrm{res}_{\tilde{P}_k}\{w^2/\tilde{w}\}}{\mathrm{res}_{P_k}\{\tilde{w}^2/w\}} \right|^{1/12} \tag{5.11}$$

where $\{P_k\}$ are zeros of w; $\tilde{P}_k$ are zeros of $\tilde{w}$. If differentials w and $\tilde{w}$ have arbitrary multiplicities of their zeros, the corresponding version of Polyakov formula follows from the link (5.10) between Bergman tau-functions on two arbitrary strata of the space $\mathcal{H}_g$.

The version of Polyakov formula for the spaces of quadratic differentials which is stated in introduction can be proved similarly.

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