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**On the structure of pseudo-Riemannian
symmetric spaces**

by

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Abstract

Following our approach to metric Lie algebras developed in a previous paper we propose a way of understanding pseudo-Riemannian symmetric spaces which are not semi-simple. We introduce cohomology sets (called quadratic cohomology) associated with orthogonal modules of Lie algebras with involution. Then we construct a functorial assignment which sends a pseudo-Riemannian symmetric space M to a triple consisting of

- (i) a Lie algebra with involution (of dimension much smaller than the dimension of the transvection group of M),
- (ii) a semi-simple orthogonal module of the Lie algebra with involution, and
- (iii) a quadratic cohomology class of this module.

That leads to a classification scheme of indecomposable non-simple pseudo-Riemannian symmetric spaces. In addition, we obtain a full classification of symmetric spaces of index 2 (thereby completing and correcting in part earlier classification results due to Cahen/Parker and Neukirchner).

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1 Introduction

In the present paper we will develop a classification scheme for pseudo-Riemannian symmetric spaces. A (pseudo-)Riemannian symmetric space is a connected (pseudo-)Riemannian manifold (M, g) satisfying the following symmetry condition. For each point $x \in M$ there is an involutive isometry θ_x of (M, g) such that x is an isolated fixed point of θ_x . Clearly, in this case the “symmetry” θ_x is uniquely defined. In a normal neighbourhood of x it coincides with the geodesic reflection at x . Riemannian symmetric spaces are well understood. They were classified by É. Cartan. Contrary to that the classification in the pseudo-Riemannian case is much more involved.

One can reformulate the classification problem for pseudo-Riemannian symmetric spaces in terms of purely algebraic objects. In order to do so we assign to each pseudo-Riemannian symmetric space M a distinguished subgroup G of its group of isometries, the transvection group of M (sometimes also called group of displacements) which still acts transitively on M . For a fixed base point $x_0 \in M$ the involution θ_{x_0} induces an involution θ on the Lie algebra \mathfrak{g} of G and the scalar product g_{x_0} on the tangent space $T_{x_0}M$ at x_0 induces a (non-degenerate) scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} , which is \mathfrak{g} - and θ -invariant. Moreover, the decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ of \mathfrak{g} into the ± 1 -eigenspaces of θ satisfies $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$ (see Section 2 for a more exact explanation of these facts). A triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ which consists of a Lie algebra \mathfrak{g} , a scalar product $\langle \cdot, \cdot \rangle$ and an involution θ satisfying these conditions will be called a symmetric triple. This gives us a one-to-one correspondence between simply-connected pseudo-Riemannian symmetric spaces and symmetric triples. Hence the classification of simply-connected pseudo-Riemannian symmetric spaces up to isometry is equivalent to the classification of symmetric triples up to isomorphism.

A classification of all semi-simple symmetric triples, i.e. of symmetric triples $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \theta)$ with semi-simple \mathfrak{g} , has been already obtained by M. Berger [B]. In particular, this includes all symmetric triples which correspond to non-flat irreducible pseudo-Riemannian symmetric spaces. However, contrary to the Riemannian case a pseudo-Riemannian symmetric space in general does not decompose into a product of irreducible symmetric spaces. Thus Berger’s result does not solve our classification problem. In fact, symmetric spaces with semi-simple transvection group constitute only a very small part of all pseudo-Riemannian symmetric spaces. The next step was done by M. Cahen and N. Wallach [CW] who classified all symmetric triples which correspond to Lorentzian symmetric spaces, i.e. to pseudo-Riemannian symmetric spaces with a metric of index 1. For the case of index 2 M. Cahen and M. Parker gave classification results in [CP 1] and [CP 2]. However the results in [CP 1] are not complete and also the revised version of these results by Th. Neukircher in [N] is not quite correct (see also the introduction to Section 7). For higher index the classification problem is unsolved. Note however, that there are recent classification results of pseudo-Riemannian symmetric spaces admitting very special additional geometric structures ((para-) hyper-Kähler [AC 1],[ABCV], (para-) quaternionic Kähler [AC 2]).

Looking at the results for index 1 and 2 it becomes rather obvious that one cannot expect to get a list of all symmetric triples for arbitrary index. Therefore the aim will be to find a reasonable description of the structure of symmetric triples which will

lead to a nice description of the moduli space of isomorphism classes of all symmetric triples. For obvious reasons one is mainly interested in indecomposable symmetric triples, i.e. in those which are not the non-trivial direct sum of symmetric triples. Moreover it suffices to consider only those symmetric triples $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ for which \mathfrak{g} does not contain semi-simple ideals. Indeed, if \mathfrak{g} has a semi-simple ideal, then it also has a θ -invariant semi-simple ideal \mathfrak{h} . The restriction of $\langle \cdot, \cdot \rangle$ to any semi-simple ideal is non-degenerate and therefore the symmetric triple decomposes into the direct sum $(\mathfrak{h}, \theta|_{\mathfrak{h}}, \langle \cdot, \cdot \rangle_{\mathfrak{h}}) \oplus (\mathfrak{h}^{\perp}, \theta|_{\mathfrak{h}^{\perp}}, \langle \cdot, \cdot \rangle_{\mathfrak{h}^{\perp}})$ of two symmetric triples the first one being in Berger's classification list mentioned above.

The structure theory for symmetric spaces which we will develop in this paper is completely parallel to our theory of metric Lie algebras presented in [KO 2]. Let us describe the main ideas and results in the present context of symmetric triples. For each symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ without semi-simple ideals there is a canonically defined θ -invariant isotropic ideal $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{g}$ such that $\mathfrak{i}(\mathfrak{g})^{\perp}/\mathfrak{i}(\mathfrak{g})$ is abelian (see Section 5, Formula (16)). Furthermore, for any θ -invariant isotropic ideal $\mathfrak{i} \subset \mathfrak{g}$ such that $\mathfrak{i}^{\perp}/\mathfrak{i}$ is abelian the following holds. Put $\mathfrak{l} = \mathfrak{g}/\mathfrak{i}^{\perp}$ and $\mathfrak{a} = \mathfrak{i}^{\perp}/\mathfrak{i}$. The involution θ induces involutions $\theta_{\mathfrak{l}}$ on \mathfrak{l} and $\theta_{\mathfrak{a}}$ on \mathfrak{a} . Then \mathfrak{a} inherits an inner product from \mathfrak{g} and an \mathfrak{l} -action being compatible with this inner product and the involutions $\theta_{\mathfrak{a}}$ and $\theta_{\mathfrak{l}}$, i.e., it inherits the structure of an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module. Moreover, $\mathfrak{i} \cong \mathfrak{l}^*$ as an $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module, and \mathfrak{g} can be represented as the result of two subsequent extensions of Lie algebras with involution

$$0 \rightarrow \mathfrak{a} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{l} \rightarrow 0, \quad 0 \rightarrow \mathfrak{l}^* \longrightarrow \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}} \rightarrow 0, \quad (1)$$

where $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{i}$. Vice versa, given a Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$, an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} and two extensions as in (1) which in addition satisfy certain natural compatibility conditions, the resulting Lie algebra \mathfrak{g} has a distinguished invariant inner product and an isometric involution. This construction will be formalised into the notion of a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} in Subsection 4.1. In particular, there is a natural equivalence relation on the set of quadratic extensions of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} . We will call such a quadratic extension of a Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ admissible if the resulting metric Lie algebra with involution $(\mathfrak{g}, \theta_{\mathfrak{l}}, \langle \cdot, \cdot \rangle)$ is a symmetric triple and the image of \mathfrak{l}^* in \mathfrak{g} coincides with the canonical ideal $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{g}$ (the crucial condition). Admissibility implies in particular that the $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} is semi-simple.

Summarising up to here we have established that any symmetric triple without semi-simple ideals has the structure of an admissible quadratic extension of a Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} in a canonical way. Geometrically, the ideal $\mathfrak{i}(\mathfrak{g})^{\perp} \subset \mathfrak{g}$ defines a foliation of the pseudo-Riemannian space M invariant under all transvections. Its leaves are coisotropic symmetric subspaces of M which are flat. This foliation is actually a fibration over an affine symmetric space N (without metric). The Lie algebra \mathfrak{l} is then a central extension of the Lie algebra of the transvection group of N . The theory discussed so far is essentially a formalisation of ideas of L. Berard-Bergery [BB 1],[BB 2] (unpublished).

This formalisation allows us to proceed further. First, equivalence classes of quadratic extensions of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal module \mathfrak{a} are conveniently described by a cer-

tain cohomology set $\mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})$ (which is the quotient of a real algebraic variety by an algebraic group action). We introduce these quadratic cohomology sets for orthogonal modules of Lie algebras with involution in Section 3. The relation to quadratic extensions is established in Section 4. In particular, Subsection 4.2 contains our standard construction which produces a metric Lie algebra with involution $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ out of a Lie algebra with involution (\mathfrak{l}, θ) , an orthogonal (\mathfrak{l}, θ) -module \mathfrak{a} , and a cocycle representing an element in $\mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})$.

Secondly, for semi-simple modules \mathfrak{a} , we describe the subset $\mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})_{\sharp} \subset \mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})$ corresponding to admissible quadratic extensions (Section 5). We call a Lie algebra with involution (\mathfrak{l}, θ) admissible if it possesses a semi-simple orthogonal module \mathfrak{a} such that $\mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})_{\sharp} \neq \emptyset$. As examples show this condition is rather strong. In particular, not every affine symmetric space can appear as the base N of the special fibration described above associated with some pseudo-Riemannian symmetric space M .

And thirdly, in Section 6 we establish that the correspondence

$$\{\text{symmetric triples } (\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)\} \implies \left\{ \text{quadruples } (\mathfrak{l}, \theta, \mathfrak{a}, \varphi \in \mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})_{\sharp}) \right\} ,$$

which sends each symmetric triple to the datum which defines the equivalence class of its associated canonical quadratic extension descends to a bijection of isomorphism classes. Here two quadruples $(\mathfrak{l}_i, \theta_i, \mathfrak{a}_i, \varphi_i)$, $i = 1, 2$, are called to be isomorphic if there is an isomorphism of triples (for the formal definition of this notion see the end of Section 3)

$$T : (\mathfrak{l}_1, \theta_1, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \theta_2, \mathfrak{a}_2) \text{ such that } T^* \varphi_2 = \varphi_1 .$$

Let us express this result in a slightly different way. We consider the automorphism group $G_{\mathfrak{l}, \theta, \mathfrak{a}}$ of a fixed triple $(\mathfrak{l}, \theta, \mathfrak{a})$. It is a subgroup of $\text{Aut}(\mathfrak{l}, \theta) \times O(\mathfrak{a})^{\theta}$. Then the moduli space of all symmetric triples without semi-simple ideals can be identified with the union of quotient spaces

$$\coprod_{(\mathfrak{l}, \theta, \mathfrak{a})} \mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})_{\sharp} / G_{\mathfrak{l}, \theta, \mathfrak{a}} , \quad (2)$$

where the union is taken over a set of representatives of isomorphism classes of triples $(\mathfrak{l}, \theta, \mathfrak{a})$ consisting of an admissible Lie algebra with involution (\mathfrak{l}, θ) and a semi-simple orthogonal (\mathfrak{l}, θ) -module \mathfrak{a} . As already mentioned, we are mainly interested in indecomposable symmetric triples. Definition 6.1 together with Proposition 6.2 provides a manageable description of the subset $\mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})_0 \subset \mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})_{\sharp}$ corresponding to indecomposable symmetric triples. Therefore the moduli space of non-semi-simple indecomposable symmetric triples is given by the following subspace of (2)

$$\coprod_{(\mathfrak{l}, \theta, \mathfrak{a})} \mathcal{H}_Q^2(\mathfrak{l}, \theta, \mathfrak{a})_0 / G_{\mathfrak{l}, \theta, \mathfrak{a}} . \quad (3)$$

This is the first main theorem of the present paper (see Theorem 6.1).

In order to approach a true classification one has to evaluate (3) further. By construction, the dimension of the symmetric space N corresponding to (\mathfrak{l}, θ) is bounded by the index of the symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$, i.e. the index of $\langle \cdot, \cdot \rangle|_{\mathfrak{g}_-}$. For this reason the

bijection (3) is extremely useful for a concrete classification of symmetric triples with small index, where all the ingredients of (3) can be explicitly computed. The situation concerning the classification results in index 2 which we discussed at the beginning of this introduction motivated us to redo this classification using (3). This is the content of Section 7. Theorem 7.1 which is the second main theorem of the present paper provides a full classification of indecomposable symmetric triples of index 2 which are not semi-simple. Moreover, we determine the surprisingly small subset of them which belongs to pseudo-Hermitian symmetric spaces (Corollary 7.1).

Note that the holonomy representation of a symmetric space with associated symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is given by the adjoint representation of \mathfrak{g}_+ on \mathfrak{g}_- . Therefore the structure results of the present paper imply structure results for (indecomposable, non-irreducible) holonomy representations of pseudo-Riemannian symmetric spaces. E.g., each such holonomy representation has a distinguished invariant isotropic subspace given by $i(\mathfrak{g}) \cap \mathfrak{g}_-$. As already observed by L. Berard-Bergery [BB 1] a similar canonical isotropic subspace can be assigned to any pseudo-orthogonal representation. It should be possible to exploit this fact in order to uncover new structural results for holonomy representations of non-symmetric pseudo-Riemannian manifolds as well.

2 Symmetric pairs and triples

In the following short survey of basic facts on symmetric spaces we will fix some notation used in this paper and clarify the relations between the defined objects. For a detailed introduction to symmetric spaces see e. g. [CP 2], [H], [KN].

Let (M, ∇) be a manifold with an affine connection. (M, ∇) is called an *affine symmetric space* if for each $x \in M$ there is an involutive affine transformation θ_x of (M, ∇) such that x is an isolated fixed point of θ_x .

For an affine symmetric space (M, ∇) the transvection group G defined by

$$G := \langle \theta_x \circ \theta_y \mid x, y \in M \rangle.$$

acts transitively on M . Let us fix a point $x_0 \in M$. The involutive affine transformation θ_{x_0} acts by conjugation on G . We will denote this conjugation by θ . It induces an involutive automorphism of the Lie algebra \mathfrak{g} of G , which we also denote by θ . We will call (\mathfrak{g}, θ) *symmetric pair associated with (M, ∇)* . Let \mathfrak{g}_+ and \mathfrak{g}_- be the eigenspaces of θ with eigenvalues 1 and -1 , respectively. Then we have

$$[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+, \quad [\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-. \quad (4)$$

Moreover, \mathfrak{g}_+ acts faithfully on \mathfrak{g}_- . Let $G_+ \subset G$ be the stabilizer of $x_0 \in M$ with respect to the action of G on M . Then $G_0^\theta \subset G_+ \subset G^\theta$ holds. In particular, \mathfrak{g}_+ is the Lie algebra of G_+ whereas \mathfrak{g}_- can be identified with the tangent space $T_{x_0}M$ of M in x_0 . By (4) the homogeneous space G/G_+ is reductive. In particular, the decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ defines a canonical invariant connection ∇^0 . There is an affine diffeomorphism from (M, ∇) to $(G/G_+, \nabla^0)$.

A *Lie algebra with involution* is a pair $(\mathfrak{l}, \theta_{\mathfrak{l}})$ consisting of a finite-dimensional Lie algebra \mathfrak{l} and an involutive automorphism $\theta_{\mathfrak{l}}$ of \mathfrak{l} .

Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a Lie algebra with involution and induced eigenspace decomposition $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_-$ such that

$$(S1) \quad [\mathfrak{l}_-, \mathfrak{l}_-] = \mathfrak{l}_+,$$

$$(S2) \quad \mathfrak{l}_+ \text{ acts faithfully on } \mathfrak{l}_-,$$

then there is a uniquely determined (up to isomorphism) simply connected affine symmetric space (M, ∇) , such that $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is the symmetric pair associated with (M, ∇) . Therefore we will call any Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ satisfying Conditions (S1) and (S2) a *symmetric pair*.

Now let (M, g) be a pseudo-Riemannian manifold and denote by ∇^{LC} the Levi-Civita connection on M . (M, g) is called a *pseudo-Riemannian symmetric space* if (M, ∇^{LC}) is an affine symmetric space. In this case the affine transformations θ_x are isometries and the transvection group G is a subgroup of the isometry group.

The metric g on M induces a \mathfrak{g}_+ -invariant (in general indefinite) scalar product $\langle \cdot, \cdot \rangle_-$ on \mathfrak{g}_- . We will call the triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$ *symmetric triple associated with (M, g)* . It has the following properties:

$$(S1) \quad [\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+,$$

$$(S2) \quad \mathfrak{g}_+ \text{ acts faithfully on } \mathfrak{g}_-,$$

$$(S3) \quad \langle \cdot, \cdot \rangle_- \text{ has an extension to a } \theta\text{-invariant and } \mathfrak{g}\text{-invariant (non-degenerate) scalar product } \langle \cdot, \cdot \rangle \text{ on } \mathfrak{g}.$$

Vice versa, let \mathfrak{g} be a finite-dimensional Lie algebra with involutive automorphism θ and let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be the decomposition into eigenspaces. Let $\langle \cdot, \cdot \rangle_-$ be a \mathfrak{g}_+ -invariant scalar product on \mathfrak{g}_- such that $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$ satisfies two of the three Conditions (S1), (S2) and (S3), then also the third condition is satisfied and there exists a uniquely defined simply connected symmetric space (M, g) such that $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$ is the symmetric triple associated with (M, g) . Therefore we will call every triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$ consisting of a finite-dimensional Lie algebra \mathfrak{g} , an involutive automorphism θ of \mathfrak{g} and an \mathfrak{g}_+ -invariant scalar product $\langle \cdot, \cdot \rangle_-$ on \mathfrak{g}_- which satisfies (two of) the Conditions (S1), (S2) and (S3) a *symmetric triple*. For a symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$ the extension $\langle \cdot, \cdot \rangle$ of $\langle \cdot, \cdot \rangle_-$ in Condition (S3) is uniquely determined. Therefore we will also call $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ a symmetric triple.

A *metric Lie algebra* $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ (or \mathfrak{g} in abbreviated notation) is a finite-dimensional Lie algebra \mathfrak{g} together with a (non-degenerate) invariant scalar product. A metric Lie algebra with involution is a triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ (also abbreviated by \mathfrak{g} or by (\mathfrak{g}, θ)), where $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a metric Lie algebra and θ is an involutive isometric automorphism. A metric Lie algebra with involution $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is a symmetric triple if and only if (S1) or (S2) is satisfied.

Let (M, g) be a pseudo-Riemannian symmetric space and $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ the associated symmetric triple. If $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is the corresponding decomposition of \mathfrak{g} , then \mathfrak{g}_+ is the Lie algebra of the holonomy group of (M, g) and the adjoint representation of \mathfrak{g}_+ on \mathfrak{g}_- is the holonomy representation. According to de Rham's decomposition theorem a pseudo-Riemannian manifold is called indecomposable, if its holonomy representation has no proper non-trivial non-degenerate invariant subspace. Hence a simply-connected pseudo-Riemannian symmetric space is indecomposable if and only if the symmetric triple associated with this symmetric space is not the direct sum of two non-vanishing symmetric triples. We will call such symmetric triples *indecomposable*. In particular a simply-connected pseudo-Riemannian symmetric space is indecomposable if and only if it is not the product of two non-trivial pseudo-Riemannian symmetric spaces. However note that there are non simply-connected pseudo-Riemannian symmetric spaces which are not indecomposable in this sense, but which are not the product of two non-trivial pseudo-Riemannian symmetric spaces.

A pseudo-Riemannian symmetric space (M, g) as well as its associated symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ are called *semi-simple* if the Lie algebra \mathfrak{g} is semi-simple.

Finally let us introduce the notion of a *homomorphism (isomorphism)* for each of the above defined algebraic objects. An isomorphism of metric Lie algebras is a Lie algebra isomorphism which is also an isometry with respect to the given inner products. A homomorphism (isomorphism) $\phi : (\mathfrak{l}_1, \theta_{\mathfrak{l}_1}) \rightarrow (\mathfrak{l}_2, \theta_{\mathfrak{l}_2})$ of Lie algebras with involution is a Lie algebra homomorphism (isomorphism) satisfying $\phi \circ \theta_{\mathfrak{l}_1} = \theta_{\mathfrak{l}_2} \circ \phi$. An isomorphism of metric Lie algebras with involution is an isomorphism of the underlying metric Lie algebras which is also an isomorphism of Lie algebras with involution.

3 Quadratic cohomology of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules

Let us first recall the notion of quadratic cohomology introduced in [KO 2]. Let \mathfrak{l} be a finite-dimensional Lie algebra. An *orthogonal \mathfrak{l} -module* is a tuple $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ (also \mathfrak{a} or (ρ, \mathfrak{a}) in abbreviated notation) such that ρ is a representation of the Lie algebra \mathfrak{l} on the finite-dimensional real vector space \mathfrak{a} and $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ is a scalar product on \mathfrak{a} satisfying

$$\langle \rho(L)A_1, A_2 \rangle_{\mathfrak{a}} + \langle A_1, \rho(L)A_2 \rangle_{\mathfrak{a}} = 0$$

for all $L \in \mathfrak{l}$ and $A_1, A_2 \in \mathfrak{a}$.

For \mathfrak{l} and (any \mathfrak{l} -module) \mathfrak{a} we have the standard cochain complex $(C^*(\mathfrak{l}, \mathfrak{a}), d)$, where $C^p(\mathfrak{l}, \mathfrak{a}) = \text{Hom}(\wedge^p \mathfrak{l}, \mathfrak{a})$ and we have the corresponding cohomology groups $H^p(\mathfrak{l}, \mathfrak{a})$. Furthermore we have the standard cochain complex $(C^*(\mathfrak{l}), d)$, which arises from the one-dimensional trivial representation of \mathfrak{l} . Even if \mathfrak{a} is one-dimensional we will distinguish between $C^*(\mathfrak{l}, \mathfrak{a})$ and $C^*(\mathfrak{l})$.

We have a product

$$C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \longrightarrow C^{p+q}(\mathfrak{l})$$

defined by the composition

$$C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \xrightarrow{\wedge} C^{p+q}(\mathfrak{l}, \mathfrak{a} \otimes \mathfrak{a}) \xrightarrow{\langle \cdot, \cdot \rangle_{\mathfrak{a}}} C^p(\mathfrak{l}).$$

Let p be even. Then the group of quadratic $(p-1)$ -cochains is the group

$$\mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a}) = C^{p-1}(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-2}(\mathfrak{l})$$

with group operation defined by

$$(\tau_1, \sigma_1) * (\tau_2, \sigma_2) = (\tau_1 + \tau_2, \sigma_1 + \sigma_2 + \frac{1}{2}\langle \tau_1 \wedge \tau_2 \rangle).$$

We consider now the set

$$\mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a}) = \{(\alpha, \gamma) \in C^p(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-1}(\mathfrak{l}) \mid d\alpha = 0, d\gamma = \frac{1}{2}\langle \alpha \wedge \alpha \rangle\}$$

whose elements are called quadratic p -cocycles. Then the group $\mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})$ acts on $\mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})$ by

$$(\alpha, \gamma)(\tau, \sigma) = \left(\alpha + d\tau, \gamma + d\sigma + \langle (\alpha + \frac{1}{2}d\tau) \wedge \tau \rangle \right).$$

and we define the quadratic cohomology set $\mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a}) := \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a}) / \mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})$. As usual, we denote the equivalence class of $(\alpha, \gamma) \in \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})$ in $\mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a})$ by $[\alpha, \gamma]$.

Now we consider pairs $(\mathfrak{l}_i, \mathfrak{a}_i)$, $i = 1, 2$, where \mathfrak{l}_i are finite-dimensional Lie algebras and $\mathfrak{a}_i = (\rho_i, \mathfrak{a}_i)$ are orthogonal \mathfrak{l}_i -modules. We say that $(S, U) : (\mathfrak{l}_1, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \mathfrak{a}_2)$ is a *morphism of pairs* if $S : \mathfrak{l}_1 \rightarrow \mathfrak{l}_2$ is a Lie algebra homomorphism and $U : \mathfrak{a}_2 \rightarrow \mathfrak{a}_1$ is an isometric embedding such that

$$U \circ \rho_2(S(L)) = \rho_1(L) \circ U.$$

Note that U maps in the reverse direction.

Now let $T := (S, U) : (\mathfrak{l}_1, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \mathfrak{a}_2)$ be a morphism of pairs. For all $p \in \mathbb{N}_0$ we define maps

$$\begin{aligned} T^* : C^p(\mathfrak{l}_2) &\longrightarrow C^p(\mathfrak{l}_1), & T^*(\gamma) &:= S^* \gamma \\ T^* : C^p(\mathfrak{l}_2, \mathfrak{a}_2) &\longrightarrow C^p(\mathfrak{l}_1, \mathfrak{a}_1), & T^*(\alpha) &:= U \circ (S^* \alpha). \end{aligned}$$

Then T^* commutes with the differentials and

$$\langle T^* \alpha \wedge T^* \tau \rangle = T^* \langle \alpha \wedge \tau \rangle \tag{5}$$

holds for all $\alpha \in C^p(\mathfrak{l}, \mathfrak{a})$ and $\tau \in C^q(\mathfrak{l}, \mathfrak{a})$. In particular, $\mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})$ is invariant under $T^* \oplus T^*$. Moreover, $T^* \oplus T^*$ restricted to $\mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})$ is a group homomorphism, and

$$(T^* \oplus T^*)((\alpha, \gamma)(\tau, \sigma)) = (T^* \alpha, T^* \gamma)(T^* \tau, T^* \sigma) \tag{6}$$

holds for all $(\alpha, \gamma) \in \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})$ and $(\tau, \sigma) \in \mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})$. Hence,

$$T^* : \mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a}) \longrightarrow \mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a}), \quad T^*([\alpha, \gamma]) := [T^* \alpha, T^* \gamma]$$

is correctly defined.

Now let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a Lie algebra with involution.

Definition 3.1 *An orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module is a tuple $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}})$, where*

1. $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ is an orthogonal \mathfrak{l} -module for the Lie algebra \mathfrak{l} ,
2. $\theta_{\mathfrak{a}}$ is an involutive isometry of \mathfrak{a} compatible with ρ and $\theta_{\mathfrak{l}}$:

$$\theta_{\mathfrak{a}} \circ \rho(\theta_{\mathfrak{l}}(L)) = \rho(L) \circ \theta_{\mathfrak{a}} \quad (7)$$

for all $L \in \mathfrak{l}$.

Often we abbreviate the notation by \mathfrak{a} or (ρ, \mathfrak{a}) .

If \mathfrak{a} is an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module, then $(\theta_{\mathfrak{l}}, \theta_{\mathfrak{a}}) : (\mathfrak{l}, \mathfrak{a}) \rightarrow (\mathfrak{l}, \mathfrak{a})$ is a morphism of pairs. As explained above $(\theta_{\mathfrak{l}}, \theta_{\mathfrak{a}})$ defines involutions

$$\Theta := (\theta_{\mathfrak{l}}, \theta_{\mathfrak{a}})^* \quad (8)$$

on $C^*(\mathfrak{l}, \mathfrak{a})$, $C^*(\mathfrak{l})$. Let $C^p(\mathfrak{l}, \mathfrak{a}) = C^p(\mathfrak{l}, \mathfrak{a})_+ \oplus C^p(\mathfrak{l}, \mathfrak{a})_-$ and $C^p(\mathfrak{l}) = C^p(\mathfrak{l})_+ \oplus C^p(\mathfrak{l})_-$ be the eigenspace decomposition with respect to Θ . Obviously,

$$\mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})_+ := C^{p-1}(\mathfrak{l}, \mathfrak{a})_+ \oplus C^{2p-2}(\mathfrak{l})_+$$

is the space of fixed vectors of $\Theta \oplus \Theta : \mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a}) \rightarrow \mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})$. Moreover,

$$\mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})_+ := \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a}) \cap (C^p(\mathfrak{l}, \mathfrak{a})_+ \oplus C^{2p-1}(\mathfrak{l})_+)$$

is the set of fixed points of $\Theta \oplus \Theta : \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a}) \rightarrow \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})$. Using (6) we see that $\mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})_+$ acts on $\mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})_+$.

Definition 3.2 *Let p be even. Then the set*

$$\mathcal{H}_Q^p(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) := \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})_+ / \mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})_+$$

is called quadratic cohomology of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ with values in the orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} .

On the other hand, Θ acts on $\mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a})$. The next proposition compares the set of invariants of this action with $\mathcal{H}_Q^p(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$.

Proposition 3.1 *The map*

$$\begin{aligned} \mathcal{H}_Q^p(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) &\longrightarrow \mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a})^{\Theta} := \{ [\alpha, \gamma] \in \mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a}) \mid \Theta([\alpha, \gamma]) = [\alpha, \gamma] \} \\ [\alpha, \gamma] &\longmapsto [\alpha, \gamma] \end{aligned}$$

is a bijection.

Proof. Let pr_+ and pr_- denote the projections with respect to the decomposition $C^q(\mathfrak{l}, \mathfrak{a}) = C^q(\mathfrak{l}, \mathfrak{a})_+ \oplus C^q(\mathfrak{l}, \mathfrak{a})_-$. We abbreviate $\text{pr}_+\beta =: \beta_+$, $\text{pr}_-\beta =: \beta_-$ for $\beta \in C^q(\mathfrak{l}, \mathfrak{a})$. Then $d(\beta_+) = (d\beta)_+$ and we write only $d\beta_+$.

First we show that the map is injective. Suppose that $(\alpha_1, \gamma_1), (\alpha_2, \gamma_2) \in \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})_+$ and $[\alpha_1, \gamma_1] = [\alpha_2, \gamma_2] \in \mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a})$. Then we have $(\alpha_1, \gamma_1)(\tau, \sigma) = (\alpha_2, \gamma_2)$ for a suitable element $(\tau, \sigma) \in \mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})$. Applying pr_+ we obtain

$$\alpha_2 = \alpha_1 + d\tau_+ \quad (9)$$

$$\gamma_2 = \gamma_1 + d\sigma_+ + \langle (\alpha_1 + \frac{1}{2}d\tau)_+ \wedge \tau_+ \rangle + \langle (\alpha_1 + \frac{1}{2}d\tau)_- \wedge \tau_- \rangle. \quad (10)$$

Since $(\alpha_1, \gamma_1)(\tau, \sigma) = (\alpha_2, \gamma_2)$ implies $\alpha_1 + d\tau = \alpha_2$ we have $d\tau_+ = d\tau$ and $d\tau_- = 0$. Now (9) and (10) yield $(\alpha_1, \gamma_1)(\tau_+, \sigma_+) = (\alpha_2, \gamma_2)$. Since $(\tau_+, \sigma_+) \in \mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})_+$ we obtain $[\alpha_1, \gamma_1] = [\alpha_2, \gamma_2] \in \mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a})_+$.

Now we prove that the map is surjective. Suppose $[\Theta\alpha, \Theta\gamma] = [\alpha, \gamma] \in \mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a})$. Then there exists an element $(2\tau, 2\sigma) \in \mathcal{C}_Q^{p-1}(\mathfrak{l}, \mathfrak{a})$ such that

$$(\Theta\alpha, \Theta\gamma) = (\alpha, \gamma)(2\tau, 2\sigma). \quad (11)$$

Applying pr_+ and pr_- to the first component of (11) we obtain

$$d\tau_+ = 0, \quad d\tau_- = -\alpha_-.$$

Therefore, applying pr_- to the second component of (11) gives

$$-\gamma_- = \gamma_- + 2d\sigma_- + \langle (\alpha_- + d\tau_-) \wedge 2\tau_+ \rangle + \langle (\alpha_+ + d\tau_+) \wedge 2\tau_- \rangle = \gamma_- + 2d\sigma_- + \langle \alpha_+ \wedge 2\tau_- \rangle,$$

hence

$$\gamma_- = -d\sigma_- - \langle \alpha_+ \wedge \tau_- \rangle.$$

Consequently,

$$\begin{aligned} (\alpha, \gamma) &= (\alpha_+ + \alpha_-, \gamma_+ + \gamma_-) \\ &= (\alpha_+ - d\tau_-, \gamma_+ - d\sigma_- - \langle \alpha_+ \wedge \tau_- \rangle) \\ &= (\alpha_+ - d\tau_-, \gamma_+ - \frac{1}{2}d\tau_- \wedge \tau_- - d\sigma_- - \langle (\alpha_+ - \frac{1}{2}d\tau_-) \wedge \tau_- \rangle) \\ &= (\alpha_+, \gamma_+ - \frac{1}{2}d\tau_- \wedge \tau_-)(-\tau_-, -\sigma_-). \end{aligned}$$

We obtain $[\alpha, \gamma] = [\alpha_+, \gamma_+ - \frac{1}{2}d\tau_- \wedge \tau_-] \in \mathcal{H}_Q^p(\mathfrak{l}, \mathfrak{a})$. This proves the assertion since $(\alpha_+, \gamma_+ - \frac{1}{2}d\tau_- \wedge \tau_-) \in \mathcal{Z}_Q^p(\mathfrak{l}, \mathfrak{a})_+$. \square

Let $(\mathfrak{l}_i, \theta_i)$, $i = 1, 2$, be Lie algebras with involution and let \mathfrak{a}_i , $i = 1, 2$, be orthogonal $(\mathfrak{l}_i, \theta_i)$ -modules. We will say that $(S, U) : (\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ is a *morphism of triples* if

1. $(S, U) : (\mathfrak{l}_1, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \mathfrak{a}_2)$ is a morphism of pairs and
2. $S : \mathfrak{l}_1 \rightarrow \mathfrak{l}_2$ and $U : \mathfrak{a}_2 \rightarrow \mathfrak{a}_1$ are homomorphisms of Lie algebras with involution.

If in addition S and U are isomorphisms, then (S, U) is called *isomorphism* of triples. If $(S, U) : (\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ is a morphism of triples, then $(S, U)^* : C^p(\mathfrak{l}_2, \mathfrak{a}_2) \rightarrow C^p(\mathfrak{l}_1, \mathfrak{a}_1)$ induces a map

$$(S, U)^* : \mathcal{H}_Q^p(\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2) \longrightarrow \mathcal{H}_Q^p(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1).$$

4 Quadratic extensions

4.1 Definition

Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a Lie algebra with involution and let $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}})$ be an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module.

Definition 4.1 *A quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} is a tuple $(\mathfrak{g}, \theta, \mathfrak{i}, i, p)$, where*

1. (\mathfrak{g}, θ) is a metric Lie algebra with involution,
2. $\mathfrak{i} \subset \mathfrak{g}$ is a θ -invariant isotropic ideal of \mathfrak{g} and
3. i and p are homomorphisms of Lie algebras with involution constituting an exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{l} \longrightarrow 0,$$

such that

- (i) this exact sequence is consistent with the representation ρ of \mathfrak{l} on \mathfrak{a} and
- (ii) i is an isometry from \mathfrak{a} onto $\mathfrak{i}^{\perp}/\mathfrak{i}$,

Remark 4.1 1. If $(\mathfrak{g}, \theta, \mathfrak{i}, i, p)$ is a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} , then $(\mathfrak{g}, \mathfrak{i}, i, p)$ is a quadratic extension of the Lie algebra \mathfrak{l} by the orthogonal \mathfrak{l} -module \mathfrak{a} in the sense of [KO 2].

2. Let $(\mathfrak{g}, \theta, \mathfrak{i}, i, p)$ be a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} . Let $\tilde{p} : \mathfrak{g} \rightarrow \mathfrak{l}$ be the composition of the natural projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{i}$ with p . Now let $p^* := \mathfrak{l}^* \rightarrow \mathfrak{g}$ be the dual map of \tilde{p} , where we identify \mathfrak{g}^* with \mathfrak{g} using the non-degenerate inner product on \mathfrak{g} . This homomorphism is injective since \tilde{p} is surjective. Its image equals $(\ker \tilde{p})^{\perp} = \mathfrak{i}$. Hence p^* determines a second exact sequence of Lie algebras with involution

$$0 \longrightarrow \mathfrak{l}^* \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i} \longrightarrow 0,$$

where we consider \mathfrak{l}^* as abelian Lie algebra with involution $\theta_{\mathfrak{l}^*}$. In particular, (\mathfrak{g}, θ) can be considered as the result of two subsequent extensions of Lie algebras with involution which satisfy certain compatibility conditions: first we extend $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by $(\mathfrak{a}, \theta_{\mathfrak{a}})$ and then we extend the resulting Lie algebra by $(\mathfrak{l}^*, \theta_{\mathfrak{l}^*})$.

If (\mathfrak{g}, θ) is a metric Lie algebra with involution and if $\mathfrak{i} \in \mathfrak{g}$ is an isotropic θ -invariant ideal such that $\mathfrak{i}^{\perp}/\mathfrak{i}$ is abelian, then the sequence

$$0 \longrightarrow \mathfrak{i}^{\perp}/\mathfrak{i} \xrightarrow{i} \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{g}/\mathfrak{i}^{\perp} \longrightarrow 0 \tag{12}$$

defines a quadratic extension of $\mathfrak{g}/\mathfrak{i}^{\perp}$ (with the induced involution) by the orthogonal module $\mathfrak{i}^{\perp}/\mathfrak{i}$. We call (12) the canonical extension associated with $(\mathfrak{g}, \theta, \mathfrak{i})$.

Definition 4.2 Two quadratic extensions $(\mathfrak{g}_j, \theta_j, i_j, i_j, p_j)$, $j = 1, 2$, of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} are called equivalent if there exists an isomorphism of metric Lie algebras with involution $\Psi : (\mathfrak{g}_1, \theta_1) \rightarrow (\mathfrak{g}_2, \theta_2)$ which maps i_1 onto i_2 and satisfies

$$\overline{\Psi} \circ i_1 = i_2 \quad \text{and} \quad p_2 \circ \overline{\Psi} = p_1 ,$$

where $\overline{\Psi} : \mathfrak{g}_1/i_1 \rightarrow \mathfrak{g}_2/i_2$ is the induced map.

4.2 The standard model

Definition 4.3 Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a Lie algebra with involution and let $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}})$ be an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module. We consider the vector space

$$\mathfrak{d} := \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$$

and define an inner product $\langle \cdot, \cdot \rangle$ and an involutive endomorphism θ on \mathfrak{d} by

$$\begin{aligned} \langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2 \rangle &:= \langle A_1, A_2 \rangle_{\mathfrak{a}} + Z_1(L_2) + Z_2(L_1) \\ \theta(Z + A + L) &:= \theta_{\mathfrak{l}}^*(Z) + \theta_{\mathfrak{a}}(A) + \theta_{\mathfrak{l}}(L) \end{aligned}$$

for $Z, Z_1, Z_2 \in \mathfrak{l}^*$, $A, A_1, A_2 \in \mathfrak{a}$ and $L, L_1, L_2 \in \mathfrak{l}$. Now we choose $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})$ and $\gamma \in C^3(\mathfrak{l})$ and define an antisymmetric bilinear map $[\cdot, \cdot] : \mathfrak{d} \times \mathfrak{d} \rightarrow \mathfrak{d}$ by

$$\begin{aligned} [\mathfrak{l}^*, \mathfrak{l}^* \oplus \mathfrak{a}] &= 0 \\ [L_1, L_2] &= \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}} \\ [L, A] &= \rho(L)A - \langle A, \alpha(L, \cdot) \rangle \\ [L, Z] &= \text{ad}^*(L)(Z) \\ [A_1, A_2] &= \langle \rho(\cdot)A_1, A_2 \rangle \end{aligned}$$

for $Z \in \mathfrak{l}^*$, $A, A_1, A_2 \in \mathfrak{a}$ and $L, L_1, L_2 \in \mathfrak{l}$.

Remark 4.2 The triple $(\mathfrak{d}, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ coincides with the triple $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ defined in [KO 2].

Proposition 4.1 If $(\alpha, \gamma) \in \mathcal{Z}_{\mathbb{Q}}^2(\mathfrak{l}, \mathfrak{a})_+$, then $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) := (\mathfrak{d}, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \theta)$ is a metric Lie algebra with involution.

Proof. This is a direct calculation. See [KO 2], Section 3.2 for a proof that $(\alpha, \gamma) \in \mathcal{Z}_{\mathbb{Q}}^2(\mathfrak{l}, \mathfrak{a})$ implies that $(\mathfrak{d}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ is a metric Lie algebra. Obviously θ is an isometry. It remains to prove that θ is a Lie algebra homomorphism. Because of

$$\begin{aligned} \theta^{-1}[\theta(L_1), \theta(L_2)] &= \\ &= \theta_{\mathfrak{l}}^*(\gamma(\theta_{\mathfrak{l}}(L_1), \theta_{\mathfrak{l}}(L_2), \cdot)) + \theta_{\mathfrak{a}}(\alpha(\theta_{\mathfrak{l}}(L_1), \theta_{\mathfrak{l}}(L_2))) + \theta_{\mathfrak{l}}([\theta_{\mathfrak{l}}(L_1), \theta_{\mathfrak{l}}(L_2)]_{\mathfrak{l}}) \\ &= \gamma(\theta_{\mathfrak{l}}(L_1), \theta_{\mathfrak{l}}(L_2), \theta_{\mathfrak{l}}(\cdot)) + \theta_{\mathfrak{a}}(\alpha(\theta_{\mathfrak{l}}(L_1), \theta_{\mathfrak{l}}(L_2))) + [L_1, L_2]_{\mathfrak{l}} \\ &= (\Theta\gamma)(L_1, L_2, \cdot) + (\Theta\alpha)(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}} \\ &= \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}} = [L_1, L_2] \end{aligned}$$

we obtain $[\theta(L_1), \theta(L_2)] = \theta([L_1, L_2])$ for all $L_1, L_2 \in \mathfrak{l}$. The remaining identities $[\theta(L), \theta(A)] = \theta([L, A])$, $[\theta(L), \theta(Z)] = \theta([L, Z])$, and $[\theta(A_1), \theta(A_2)] = \theta([A_1, A_2])$ for $Z \in \mathfrak{l}^*$, $A, A_1, A_2 \in \mathfrak{a}$ and $L \in \mathfrak{l}$ can be proved in a similar way using the compatibility of $\theta_{\mathfrak{l}}, \theta_{\mathfrak{a}}$ and ρ . \square

We identify $\mathfrak{d}/\mathfrak{l}^*$ with $\mathfrak{a} \oplus \mathfrak{l}$ and denote by $i : \mathfrak{a} \rightarrow \mathfrak{a} \oplus \mathfrak{l}$ the injection and by $p : \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{l}$ the projection. Then we have

Corollary 4.1 *If $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$, then $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}), \theta, \mathfrak{l}^*, i, p)$ is a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by $(\mathfrak{a}, \theta_{\mathfrak{a}})$.*

We will denote the quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}), \theta, \mathfrak{l}^*, i, p)$ also by $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$.

Remark 4.3 Let $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_-$, and $\mathfrak{d} = \mathfrak{d}_+ \oplus \mathfrak{d}_-$ be the eigenspace decompositions with respect to the corresponding involutions. If the signature of the restriction of $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ to \mathfrak{a}_- equals (p, q) , then the signature of $\langle \cdot, \cdot \rangle$ restricted to \mathfrak{g}_- equals $(p + \dim \mathfrak{l}_-, q + \dim \mathfrak{l}_-)$. If $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is a symmetric triple, then this is also the signature of the metric on any pseudo-Riemannian symmetric space which is associated with this triple.

4.3 Classification by cohomology

Proposition 4.2 *For $(\alpha_1, \gamma_1), (\alpha_2, \gamma_2) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ the quadratic extensions $\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ and $\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} are equivalent if and only if $[\alpha_1, \gamma_1] = [\alpha_2, \gamma_2] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$.*

Proof. Assume first that $\Psi : \mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \rightarrow \mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is an equivalence. The following facts can be verified by direct calculations, see also [KO 2], Prop. 3.3 for a detailed proof. Since Ψ is an isometry and satisfies $\Psi(\mathfrak{l}^*) = \mathfrak{l}^*$, $\text{pr}_{\mathfrak{a}} \Psi|_{\mathfrak{a}} = \text{Id}$ and $\text{pr}_{\mathfrak{l}} \Psi|_{\mathfrak{l}} = \text{Id}$ it can be written as

$$\Psi = \begin{pmatrix} \text{Id} & -\tau^* & \bar{\sigma} - \frac{1}{2}\tau^*\tau \\ 0 & \text{Id} & \tau \\ 0 & 0 & \text{Id} \end{pmatrix} : \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \longrightarrow \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}, \quad (13)$$

where $\tau \in C^1(\mathfrak{l}, \mathfrak{a})$ and $\sigma(\cdot, \cdot) = \langle \bar{\sigma}(\cdot), \cdot \rangle \in C^2(\mathfrak{l})$. Moreover, since Ψ is a Lie algebra isomorphism the cochains $\tau \in C^1(\mathfrak{l}, \mathfrak{a})$ and $\sigma \in C^2(\mathfrak{l})$ satisfy $(\alpha_1, \gamma_1)(\tau, \sigma) = (\alpha_2, \gamma_2)$. Furthermore, τ and σ are Θ -invariant, since Ψ commutes with $\theta = \theta_{\mathfrak{l}^*} \oplus \theta_{\mathfrak{a}} \oplus \theta_{\mathfrak{l}}$. Hence $[\alpha_1, \gamma_1] = [\alpha_2, \gamma_2] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$. Conversely, if $[\alpha_1, \gamma_1] = [\alpha_2, \gamma_2] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$, then there exist cochains $\tau \in C^1(\mathfrak{l}, \mathfrak{a})_+$ and $\sigma \in C^2(\mathfrak{l})_+$ such that $(\alpha_1, \gamma_1)(\tau, \sigma) = (\alpha_2, \gamma_2)$ holds and we can define an equivalence $\Psi : \mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \rightarrow \mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ by (13). \square

Lemma 4.1 *Let $(\mathfrak{g}, \theta, i, p)$ be a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} and let $\tilde{p} : \mathfrak{g} \rightarrow \mathfrak{l}$ be the composition of the natural projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{i}$ with p . Then there exists an injective homomorphism of vector spaces $s : \mathfrak{l} \rightarrow \mathfrak{g}$ such that*

(i) $\tilde{p} \circ s = \text{Id}$,

(ii) $s \circ \theta_{\mathfrak{l}} = \theta \circ s$, and

(iii) $s(\mathfrak{l})$ is isotropic.

Proof. Let us consider the orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ into eigenspaces of θ . Since \mathfrak{i}^\perp is θ -invariant we have $\mathfrak{i}^\perp = \mathfrak{g}_+ \cap \mathfrak{i}^\perp \oplus \mathfrak{g}_- \cap \mathfrak{i}^\perp$. Moreover, $\mathfrak{i}_+^\perp := \mathfrak{g}_+ \cap \mathfrak{i}^\perp$ and $\mathfrak{i}_-^\perp := \mathfrak{g}_- \cap \mathfrak{i}^\perp$ are coisotropic subspaces of \mathfrak{g}_+ and \mathfrak{g}_- , respectively. Therefore we can choose isotropic vector space complements V_+ of \mathfrak{i}_+^\perp in \mathfrak{g}_+ and V_- of \mathfrak{i}_-^\perp in \mathfrak{g}_- . Then $V := V_+ \oplus V_-$ is a θ -invariant isotropic vector space complement of \mathfrak{i}^\perp in \mathfrak{g} . Hence, $s := (\tilde{p}|_V)^{-1} : \mathfrak{l} \rightarrow V \subset \mathfrak{g}$ satisfies Conditions (i), (ii), and (iii) of the lemma. \square

Proposition 4.3 *Let $(\mathfrak{g}, \theta, \mathfrak{i}, i, p)$ be a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by (ρ, \mathfrak{a}) and let $s : \mathfrak{l} \rightarrow \mathfrak{g}$ be as in Lemma 4.1. Let $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})$ and $\gamma \in C^3(\mathfrak{l})$ be defined by*

$$i(\alpha(L_1, L_2)) := [s(L_1), s(L_2)] - s([L_1, L_2]) + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i} \quad (14)$$

$$\gamma(L_1, L_2, L_3) := \langle [s(L_1), s(L_2)], s(L_3) \rangle. \quad (15)$$

Then $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ holds and the quadratic extension $(\mathfrak{g}, \theta, \mathfrak{i}, i, p)$ is equivalent to $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$.

Proof. By [KO 2], Proposition 3.4 we already know that $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ and that $(\mathfrak{g}, \mathfrak{i}, i, p)$ and the triple $\mathfrak{d} := (\mathfrak{d}, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ associated with (α, γ) by Definition 4.3 are equivalent as quadratic extensions of the Lie algebra \mathfrak{l} by the orthogonal \mathfrak{l} -module \mathfrak{a} (disregarding involutions). In fact we proved the following. If $p^* : \mathfrak{l}^* \rightarrow \mathfrak{i}$ is the isomorphism defined in Remark 4.1 and if we define the linear map $t : \mathfrak{a} \rightarrow (\mathfrak{i} \oplus s(\mathfrak{l}))^\perp \subset \mathfrak{g}$ by

$$i(A) = t(A) + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i},$$

then

$$\Psi = p^* + t + s : \mathfrak{d} = \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \longrightarrow \mathfrak{g},$$

is an equivalence of \mathfrak{d} and $(\mathfrak{g}, \mathfrak{i}, i, p)$.

It remains to show that $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})$ is Θ -invariant and that Ψ is compatible with the involutions $\theta_{\mathfrak{l}^*} \oplus \theta_{\mathfrak{a}} \oplus \theta_{\mathfrak{l}}$ on $\mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$ and θ on \mathfrak{g} . The first assertion follows easily from (14) and (15) using that i is a homomorphism of Lie algebras with involution and that s satisfies $s \circ \theta_{\mathfrak{l}} = \theta \circ s$. Let us now verify the second assertion. Since p is a homomorphism of Lie algebras with involution and s satisfies $s \circ \theta_{\mathfrak{l}} = \theta \circ s$ it remains to prove that $t \circ \theta_{\mathfrak{a}} = \theta \circ t$ holds. Recall that $t \circ \theta_{\mathfrak{a}}(A) \in (\mathfrak{i} \oplus s(\mathfrak{l}))^\perp$ for all $A \in \mathfrak{a}$. Since $\mathfrak{i} \oplus s(\mathfrak{l})$ is θ -invariant, also $\theta \circ t(A) \in (\mathfrak{i} \oplus s(\mathfrak{l}))^\perp$ for all $A \in \mathfrak{a}$. Hence it suffices to prove that $t \circ \theta_{\mathfrak{a}}(A) + \mathfrak{i} = \theta \circ t(A) + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i}$ for all $A \in \mathfrak{a}$. However, this is true by definition of t and the fact that i is a homomorphism of Lie algebras with involution. \square

The following fact follows from Propositions 4.2 and 4.3.

Corollary 4.2 *Let $(\mathfrak{g}, \theta, i, p)$ be a quadratic extension of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} and let $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})$ and $\gamma \in C^3(\mathfrak{l}, \mathfrak{a})$ be defined as in Proposition 4.3. Then the cohomology class $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ does not depend on the choice of s .*

Finally we obtain:

Theorem 4.1 *The equivalence classes of quadratic extensions of a Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} are in one-to-one correspondence with elements of $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$.*

5 Admissible extensions

In this section we will equip each symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ without semi-simple ideals with the structure of a quadratic extension in a canonical way. Therefore we are particularly interested in those (so-called admissible) quadratic extensions which come from this canonical procedure. The main result is Theorem 5.1 which describes the subset $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#} \subset \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ corresponding to admissible quadratic extensions of a Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by an orthogonal module \mathfrak{a} .

First we recall the construction of the canonical isotropic ideal of a metric Lie algebra \mathfrak{g} from [KO 2], Section 4. Let \mathfrak{g} be a finite-dimensional Lie algebra. Then there are chains of ideals

$$\begin{aligned} \{0\} &= S_0(\mathfrak{g}) \subset S_1(\mathfrak{g}) \subset S_2(\mathfrak{g}) \subset \dots \subset S_{l_+}(\mathfrak{g}) = \mathfrak{g} \\ \mathfrak{g} &= R_0(\mathfrak{g}) \supset R_1(\mathfrak{g}) \supset R_2(\mathfrak{g}) \supset \dots \supset R_{l_-}(\mathfrak{g}) = \{0\} \end{aligned}$$

which are defined inductively as follows: $S_k(\mathfrak{g})$ is the largest ideal of \mathfrak{g} containing $S_{k-1}(\mathfrak{g})$ such that the \mathfrak{g} -module $S_k(\mathfrak{g})/S_{k-1}(\mathfrak{g})$ is semi-simple. $R_k(\mathfrak{g})$ is the smallest ideal contained in $R_{k-1}(\mathfrak{g})$ such that the \mathfrak{g} -module $R_{k-1}(\mathfrak{g})/R_k(\mathfrak{g})$ is semi-simple. Then we set

$$i(\mathfrak{g}) := \sum_{k=1}^{l_- - 1} S_k(\mathfrak{g}) \cap R_k(\mathfrak{g}) . \quad (16)$$

If $i(\mathfrak{g})$ is a metric Lie algebra, then $i(\mathfrak{g})$ is isotropic and $i(\mathfrak{g})^\perp/i(\mathfrak{g})$ is abelian provided \mathfrak{g} does not contain non-trivial simple ideals ([KO 2], Lemma 4.2). By construction, $i(\mathfrak{g})$ is invariant under all automorphisms of \mathfrak{g} , in particular under all involutions.

Definition 5.1 *A quadratic extension $(\mathfrak{g}, \theta, i, p)$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by \mathfrak{a} is called balanced if $i = i(\mathfrak{g})$. It is called admissible if it is balanced and if $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is a symmetric triple.*

Proposition 5.1 *Any symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ without semi-simple ideals has the structure of a balanced, hence admissible, quadratic extension in a canonical way.*

Proof. Take the canonical extension associated with $(\mathfrak{g}, \theta, i(\mathfrak{g}))$ defined by (12). \square

Definition 5.2 Let $(\mathfrak{l}, \theta_\mathfrak{l})$ be a Lie algebra with involution such that

$$(T_1) \quad [\mathfrak{l}_-, \mathfrak{l}_-] = \mathfrak{l}_+ .$$

Let (ρ, \mathfrak{a}) be a semi-simple orthogonal $(\mathfrak{l}, \theta_\mathfrak{l})$ -module and let $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$. Then $\mathfrak{a} = \mathfrak{a}^\mathfrak{l} \oplus \rho(\mathfrak{l})\mathfrak{a}$, and we have a corresponding decomposition $\alpha = \alpha_0 + \alpha_1$. We consider the following conditions:

(A₀) Let $L_0 \in \mathfrak{z}(\mathfrak{l}) \cap \ker \rho$ be such that there exist elements $A_0 \in \mathfrak{a}$ and $Z_0 \in \mathfrak{l}^*$ satisfying for all $L \in \mathfrak{l}$

$$(i) \quad \alpha(L, L_0) = \rho(L)A_0,$$

$$(ii) \quad \gamma(L, L_0, \cdot) = -\langle A_0, \alpha(L, \cdot) \rangle_{\mathfrak{a}} + \langle Z_0, [L, \cdot]_{\mathfrak{l}} \rangle \text{ as an element of } \mathfrak{l}^*,$$

then $L_0 = 0$.

(B₀) The subspace $\alpha_0(\ker [\cdot, \cdot]_{\mathfrak{l}}) \subset \mathfrak{a}^\mathfrak{l}$ is non-degenerate.

$$(T_2) \quad \mathfrak{a}_+^\mathfrak{l} = \alpha_0(\ker [\cdot, \cdot]_{\mathfrak{l}_-}).$$

(A_k) ($k \geq 1$)

Let $\mathfrak{k} \subset S(\mathfrak{l}) \cap R_k(\mathfrak{l})$ be an \mathfrak{l} -ideal such that there exist elements $\Phi_1 \in \text{Hom}(\mathfrak{k}, \mathfrak{a})$ and $\Phi_2 \in \text{Hom}(\mathfrak{k}, R_k(\mathfrak{l})^*)$ satisfying for all $L \in \mathfrak{l}$ and $K \in \mathfrak{k}$

$$(i) \quad \alpha(L, K) = \rho(L)\Phi_1(K) - \Phi_1([L, K]_{\mathfrak{l}}),$$

$$(ii) \quad \gamma(L, K, \cdot) = -\langle \Phi_1(K), \alpha(L, \cdot) \rangle_{\mathfrak{a}} + \langle \Phi_2(K), [L, \cdot]_{\mathfrak{l}} \rangle + \langle \Phi_2([L, K]_{\mathfrak{l}}), \cdot \rangle \text{ as an element of } R_k(\mathfrak{l})^*,$$

then $\mathfrak{k} = 0$.

(B_k) ($k \geq 1$)

Let $\mathfrak{b}_k \subset \mathfrak{a}$ be the maximal \mathfrak{l} -submodule such that the system of equations

$$\langle \alpha(L, K), B \rangle_{\mathfrak{a}} = \langle \rho(L)\Phi(K) - \Phi([L, K]_{\mathfrak{l}}), B \rangle_{\mathfrak{a}} , \quad L \in \mathfrak{l}, K \in R_k(\mathfrak{l}), B \in \mathfrak{b}_k,$$

has a solution $\Phi \in \text{Hom}(R_k(\mathfrak{l}), \mathfrak{a})$. Then \mathfrak{b}_k is non-degenerate.

If the conditions (T₂), (A_k), (B_k), $0 \leq k \leq m$, where m is such that $R_{m+1}(\mathfrak{l}) = 0$, are satisfied, then the cohomology class $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})$ is called admissible. We denote the set of all admissible cohomology classes by $\mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_\#$. A Lie algebra with involution $(\mathfrak{l}, \theta_\mathfrak{l})$ is called admissible if it satisfies (T₁) and there is a semi-simple orthogonal $(\mathfrak{l}, \theta_\mathfrak{l})$ -module \mathfrak{a} such that $\mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_\# \neq \emptyset$.

Each of the above conditions depends only on the cohomology class $[\alpha, \gamma]$ and not on the particular cocycle (α, γ) representing it (compare the discussion in [KO 2], Section 4). Thus the notion of admissibility of a cohomology class is well-defined. In addition, we remark that the submodule \mathfrak{b}_k appearing in Condition (B_k) is automatically $\theta_\mathfrak{a}$ -invariant.

In the remainder of this section we want to prove the following theorem.

Theorem 5.1 *Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a Lie algebra with involution. Let (ρ, \mathfrak{a}) be an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module and let $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$. Then the quadratic extension $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is admissible if and only if*

- (i) $(\mathfrak{l}, \theta_{\mathfrak{l}})$ satisfies (T_1) ,
- (ii) the representation ρ is semi-simple, and
- (iii) $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\sharp}$.

In the following we abbreviate $\mathfrak{d} := \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$, $\mathfrak{d} = \mathfrak{d}_+ \oplus \mathfrak{d}_-$.

Lemma 5.1 *Let ρ be semi-simple. If $[\mathfrak{d}_-, \mathfrak{d}_-] = \mathfrak{d}_+$, then Conditions (T_1) and (T_2) are satisfied.*

Proof. The only non-trivial assertion to be proved is $\mathfrak{a}_+^{\mathfrak{l}} \subset \alpha_0(\ker[\cdot, \cdot]_{\mathfrak{l}_-})$. Suppose $A \in \mathfrak{a}_+^{\mathfrak{l}}$. By assumption there are elements $X_i^1 = Z_i^1 + A_i^1 + L_i^1$, $X_i^2 = Z_i^2 + A_i^2 + L_i^2$, $i = 1, \dots, k$, in \mathfrak{d}_- such that

$$A = \sum_{i=1}^k [X_i^1, X_i^2] = \sum_{i=1}^k [L_i^1, L_i^2]_{\mathfrak{l}} + \sum_{i=1}^k \left(\alpha(L_i^1, L_i^2) + \rho(L_i^1)(A_i^2) - \rho(L_i^2)(A_i^1) \right) + Z$$

with $Z \in \mathfrak{l}^*$. Obviously, $Z = 0$. Moreover, $\rho(L_i^1)(A_i^2) - \rho(L_i^2)(A_i^1) = 0$ since this term is in $\rho(\mathfrak{l})\mathfrak{a}$ and $\mathfrak{a} = \mathfrak{a}^{\mathfrak{l}} \oplus \rho(\mathfrak{l})\mathfrak{a}$. Hence, $A = \sum_{i=1}^k \alpha(L_i^1, L_i^2)$ and $\sum_{i=1}^k [L_i^1, L_i^2]_{\mathfrak{l}} = 0$. \square

Lemma 5.2 *Let ρ be semi-simple. If the Conditions (T_1) , (T_2) , (A_0) are satisfied, then the representation of \mathfrak{d}_+ on \mathfrak{d}_- is faithful.*

Proof. First we note that (T_1) implies

$$(\rho(\mathfrak{l})\mathfrak{a})_+ = \rho(\mathfrak{l}_-)\mathfrak{a}_-. \quad (17)$$

To verify this we have to prove that $L(A) \in \rho(\mathfrak{l}_-)\mathfrak{a}_-$ for all $L \in \mathfrak{l}_+$ and $A \in \mathfrak{a}_+$. Because of (T_1) it suffices to show that $[L_1, L_2](A) \in \rho(\mathfrak{l}_-)\mathfrak{a}_-$ for all $L_1, L_2 \in \mathfrak{l}_-$ and $A \in \mathfrak{a}_+$. But this is obvious since $[L_1, L_2](A) = L_1 L_2(A) - L_2 L_1(A)$ and $L_i(A) \in \mathfrak{a}_-$ for $i = 1, 2$.

Equation (17) implies

$$(\rho(\mathfrak{l})\mathfrak{a})_+ + \mathfrak{l}_+^* = [\mathfrak{l}_-, \mathfrak{a}_-] + \mathfrak{l}_+^* \subset [\mathfrak{d}_-, \mathfrak{d}_-] + \mathfrak{l}_+^*,$$

whereas (T_2) yields

$$\mathfrak{a}_+^{\mathfrak{l}} \subset [\mathfrak{l}_-, \mathfrak{l}_-] + \mathfrak{l}_+^* \subset [\mathfrak{d}_-, \mathfrak{d}_-] + \mathfrak{l}_+^*.$$

Consequently,

$$\mathfrak{a}_+ \subset [\mathfrak{d}_-, \mathfrak{d}_-] + \mathfrak{l}_+^*$$

and therefore

$$\mathfrak{d}_+ = [\mathfrak{l}_-, \mathfrak{l}_-] + \mathfrak{a}_+ + \mathfrak{l}_+^* = [\mathfrak{d}_-, \mathfrak{d}_-] + \mathfrak{l}_+^*. \quad (18)$$

Assume now, that $Z_+ \in \mathfrak{l}_+^*$, $A_+ \in \mathfrak{a}_+$ and $L_+ \in \mathfrak{l}_+$ are such that $[Z_+ + A_+ + L_+, \mathfrak{d}_-] = 0$. In particular this implies $[L_+, \mathfrak{l}_-]_{\mathfrak{l}} = 0$. Because of (T_1) this yields $L_+ \in \mathfrak{z}(\mathfrak{l})$. Using this and (18) we obtain

$$[Z_+ + A_+ + L_+, \mathfrak{d}_+] = [Z_+ + A_+ + L_+, \mathfrak{l}_+^*] = \text{ad}^*(L_+)(\mathfrak{l}_+^*) = 0,$$

hence $Z_+ + A_+ + L_+ \in \mathfrak{z}(\mathfrak{d})$. A straightforward computation shows that Condition (A_0) is equivalent to $\mathfrak{z}(\mathfrak{d}) \subset \mathfrak{l}^* \oplus \mathfrak{a}$. It follows that $L_+ = 0$. Hence we have $Z_+ + A_+ \in \mathfrak{z}(\mathfrak{d})$. In particular, $[Z_+ + A_+, \mathfrak{l}] = 0$ holds. This implies $A_+ \in \mathfrak{a}^{\mathfrak{l}}$ and

$$\langle A_+, \alpha(\cdot, \cdot) \rangle = \langle A_+, \alpha_0(\cdot, \cdot) \rangle = -Z_+([\cdot, \cdot]).$$

However the last equation yields $A_+ \perp \alpha_0(\ker[\cdot, \cdot]_{\mathfrak{l}_-}) = \mathfrak{a}_+^{\mathfrak{l}}$, hence $A_+ = 0$. We obtain $Z_+ \in \mathfrak{z}(\mathfrak{d})$ and, in particular, $[Z_+, \mathfrak{l}] = 0$, which yields $Z_+([\mathfrak{l}, \mathfrak{l}]_{\mathfrak{l}}) = 0$. This implies $Z_+(\mathfrak{l}_+) = 0$, thus $Z_+ = 0$. Consequently, the representation of \mathfrak{d}_+ on \mathfrak{d}_- is faithful. \square

Proof of Theorem 5.1. Theorem 4.1 in [KO 2] tells us that the quadratic extension $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is balanced if and only if (ρ, \mathfrak{a}) is semi-simple and the conditions (A_k) , (B_k) , $0 \leq k \leq m$, are satisfied. Recall from Section 2 that a metric Lie algebra with involution is a symmetric triple if and only if one of the Conditions (S1) and (S2) is satisfied. The theorem now follows from Lemma 5.1 and Lemma 5.2. \square

6 A classification scheme for indecomposable symmetric triples

By the results of the previous section the metric Lie algebras with involution $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ associated with semi-simple orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules \mathfrak{a} of an admissible Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ and $[\alpha, \gamma] \in \mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$ exhaust all isomorphism classes of symmetric triples without semi-simple ideals. It remains to decide which of these data lead to isomorphic symmetric triples. Since one is interested in the classification of *indecomposable* symmetric triples we also have to check indecomposability of $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ as a symmetric triple in terms of the defining data $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$, $[\alpha, \gamma] \in \mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$. A comprehensive discussion of the analogous questions for metric Lie algebras can be found in [KO 2], Section 5. This allows us to be rather brief here. We will conclude the section with a classification scheme for indecomposable symmetric triples.

Proposition 6.1 *Let $(\mathfrak{l}_i, \theta_{\mathfrak{l}_i})$, $i = 1, 2$, be admissible Lie algebras with involution. Let \mathfrak{a}_i , $i = 1, 2$, be orthogonal $(\mathfrak{l}_i, \theta_{\mathfrak{l}_i})$ -modules and suppose $(\alpha_i, \gamma_i) \in \mathcal{Z}_{\mathbb{Q}}^2(\mathfrak{l}_i, \mathfrak{a}_i)_+$ such that $[\alpha_i, \gamma_i] \in \mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}_i, \theta_{\mathfrak{l}_i}, \mathfrak{a}_i)_{\#}$.*

Then $\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1)$ and $\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ are isomorphic as symmetric triples, if and only if there is an isomorphism of triples $(S, U) : (\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ such that $(S, U)^[\alpha_2, \gamma_2] = [\alpha_1, \gamma_1] \in \mathcal{H}_{\mathbb{Q}}^2(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1)_{\#}$.*

Proof. If $(S, U) : (\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ is an isomorphism of triples, then the symmetric triple $\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ is isomorphic to $\mathfrak{d}_{U \circ (S^* \alpha_2), S^* \gamma_2}(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1)$. If in addition $(S, U)^*[\alpha_2, \gamma_2] = [\alpha_1, \gamma_1]$, then the latter quadratic extension is equivalent to

$\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1)$ by Proposition 4.2. Thus $\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1)$ and $\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ are isomorphic.

For the reverse direction we really need the admissibility assumption. Since both quadratic extensions are balanced, an isomorphism $\Psi : \mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow \mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ maps the canonical isotropic ideals into each other, hence is compatible with the filtrations given by $\mathfrak{l}_i^* \subset \mathfrak{l}_i^* \oplus \mathfrak{a}_i$, $i = 1, 2$. It therefore induces maps $S : \mathfrak{l}_1 \rightarrow \mathfrak{l}_2$, $U^{-1} : \mathfrak{a}_1 \rightarrow \mathfrak{a}_2$. Then (S, U) is a morphism of triples and the quadratic extensions $\mathfrak{d}_{U \circ (S^* \alpha_2), S^* \gamma_2}(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1)$ and $\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1)$ are equivalent. Now we apply Proposition 4.2.

For more details we refer to [KO 2], Section 5. \square

Definition 6.1 *A non-trivial decomposition of a triple $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ consists of two non-zero morphisms of triples*

$$(q_i, j_i) : (\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \longrightarrow (\mathfrak{l}_i, \theta_{\mathfrak{l}_i}, \mathfrak{a}_i), \quad i = 1, 2,$$

such that $(q_1, j_1) \oplus (q_2, j_2) : (\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \rightarrow (\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1) \oplus (\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ is an isomorphism.

A cohomology class $\varphi \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is called decomposable if it can be written as a sum

$$\varphi = (q_1, j_1)^* \varphi_1 + (q_2, j_2)^* \varphi_2$$

for a non-trivial decomposition (q_i, j_i) of $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ and $\varphi_i \in \mathcal{H}_Q^2(\mathfrak{l}_i, \theta_{\mathfrak{l}_i}, \mathfrak{a}_i)$, $i = 1, 2$. Here addition is induced by addition in the vector space $C^p(\mathfrak{l}, \mathfrak{a})_+ \oplus C^{2p-1}(\mathfrak{l})_+$.

A cohomology class which is not decomposable is called indecomposable. We denote the set of all indecomposable elements in $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$ by $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$.

Proposition 6.2 *Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be an admissible Lie algebra with involution. Let \mathfrak{a} be an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module and let $(\alpha, \gamma) \in \mathcal{Z}_Q^2(\mathfrak{l}, \mathfrak{a})_+$ be such that $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\#}$.*

Then $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is indecomposable if and only if $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$.

Proof. Assume that $[\alpha, \gamma] = (q_1, j_1)^*[\alpha_1, \gamma_1] + (q_2, j_2)^*[\alpha_2, \gamma_2]$ for some non-trivial decomposition of $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$. Note that

$$(q_1, j_1)^*[\alpha_1, \gamma_1] + (q_2, j_2)^*[\alpha_2, \gamma_2] = ((q_1, j_1) \oplus (q_2, j_2))^*[(\alpha_1, \alpha_2), (\gamma_1, \gamma_2)].$$

Arguing as in the first part of the proof of Proposition 6.1 we find that $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is isomorphic to the orthogonal direct sum $\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \theta_{\mathfrak{l}_1}, \mathfrak{a}_1) \oplus \mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \theta_{\mathfrak{l}_2}, \mathfrak{a}_2)$ and thus decomposable.

It is again only the reverse direction where we need the admissibility assumptions. Let $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ be decomposable. Thus it is isomorphic to an orthogonal direct sum $\mathfrak{d}_1 \oplus \mathfrak{d}_2$. Without loss of generality we may assume $\mathfrak{d}_i = \mathfrak{d}_{\alpha_i, \gamma_i}(\mathfrak{l}_i, \theta_{\mathfrak{l}_i}, \mathfrak{a}_i)$ for certain triples $(\mathfrak{l}_i, \theta_{\mathfrak{l}_i}, \mathfrak{a}_i)$ and $[\alpha_i, \gamma_i] \in \mathcal{H}_Q^2(\mathfrak{l}_i, \theta_{\mathfrak{l}_i}, \mathfrak{a}_i)_{\#}$. Then

$$[(\alpha_1, \alpha_2), (\gamma_1, \gamma_2)] \in \mathcal{H}_Q^2(\mathfrak{l}_1 \oplus \mathfrak{l}_2, \theta_{\mathfrak{l}_1} \oplus \theta_{\mathfrak{l}_2}, \mathfrak{a}_1 \oplus \mathfrak{a}_2)_{\#}.$$

Now Proposition 6.1 implies that there exists an isomorphism of triples

$$(S, U) : (\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \rightarrow (\mathfrak{l}_1 \oplus \mathfrak{l}_2, \theta_{\mathfrak{l}_1} \oplus \theta_{\mathfrak{l}_2}, \mathfrak{a}_1 \oplus \mathfrak{a}_2)$$

such that $(S, U)^*[(\alpha_1, \alpha_2), (\gamma_1, \gamma_2)] = [\alpha, \gamma]$. Setting $q_i := \text{pr}_{\mathfrak{a}_i} \circ S$, $j_i := U|_{\mathfrak{a}_i}$ we obtain the decomposition $[\alpha, \gamma] = (q_1, j_1)^*[\alpha_1, \gamma_1] + (q_2, j_2)^*[\alpha_2, \gamma_2]$. Thus $[\alpha, \gamma]$ is decomposable. \square

Let us fix an admissible Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ and a pseudo-Euclidean vector space $(\mathfrak{a}, \theta_{\mathfrak{a}})$ with involution. Conjugation by $\theta_{\mathfrak{a}}$ defines involutions on the Lie algebra $\mathfrak{so}(\mathfrak{a})$ and on the group $O(\mathfrak{a})$, also denoted by $\theta_{\mathfrak{a}}$. We set

$$\text{Aut}(\mathfrak{a}) := O(\mathfrak{a})^{\theta_{\mathfrak{a}}} = O(\mathfrak{a}_+) \times O(\mathfrak{a}_-).$$

We consider the set $\text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}$ of all orthogonal semi-simple representations of \mathfrak{l} on \mathfrak{a} which are compatible with the involutions $\theta_{\mathfrak{l}}$ and $\theta_{\mathfrak{a}}$. If $\rho \in \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}$ is fixed we denote the corresponding orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module by \mathfrak{a}_{ρ} . The group $G := \text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}) \times \text{Aut}(\mathfrak{a})$ acts from the right on $\text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}$ by

$$(S, U)^* \rho := \text{Ad}(U^{-1}) \circ S^* \rho, \quad S \in \text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}), \quad U \in \text{Aut}(\mathfrak{a}).$$

Then for any $\rho \in \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}$ an element $g = (S, U) \in G$ defines an isomorphism of triples $\bar{g} := (S, U^{-1}) : (\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{g^* \rho}) \rightarrow (\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\rho})$ and therefore induces a bijection

$$\bar{g}^* : \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\rho}) \rightarrow \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{g^* \rho}).$$

We obtain a right action of G on the disjoint union

$$\coprod_{\rho \in \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}} \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\rho}).$$

As in Definition 6.1 let $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\rho})_0 \subset \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a}_{\rho})$ be the subset of all admissible indecomposable elements (see Definition 5.2 for the admissibility conditions). Then the set

$$\coprod_{\rho \in \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}} \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\rho})_0$$

is G -invariant. Combining Proposition 5.1 and Theorem 5.1 with Propositions 6.1 and 6.2 we obtain

Theorem 6.1 *Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be an admissible Lie algebra with involution, and let $(\mathfrak{a}, \theta_{\mathfrak{a}})$ be a pseudo-Euclidean vector space with involution. We consider the class $\mathcal{A}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ of non-semisimple indecomposable symmetric triples $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ satisfying*

1. *The Lie algebras with involution $\mathfrak{g}/\mathfrak{j}(\mathfrak{g})$ and $(\mathfrak{l}, \theta_{\mathfrak{l}})$ are isomorphic.*
2. *$\mathfrak{j}(\mathfrak{g})/\mathfrak{i}(\mathfrak{g})$ is isomorphic to $(\mathfrak{a}, \theta_{\mathfrak{a}})$ as a pseudo-Euclidean vector space with involution.*

Then the set of isomorphism classes of $\mathcal{A}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ is in bijective correspondence with the orbit space of the action of $G = \text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}) \times \text{Aut}(\mathfrak{a})$ on

$$\coprod_{\rho \in \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}} \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\rho})_0 .$$

This orbit space can also be written as

$$\coprod_{[\rho] \in \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}/G} \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\rho})_0 / G_{\rho} ,$$

where $G_{\rho} = \{g \in G \mid g^* \rho = \rho\}$ is the automorphism group of the triple $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_{\rho})$.

7 Classification of symmetric triples of index 2

Our general classification principle yields more explicit results if one only considers symmetric triples of a (fixed) small index. Here the *index of a symmetric triple* $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$ is simply the index of $\langle \cdot, \cdot \rangle_-$ or, equivalently, the index of the metric of any pseudo-Riemannian symmetric space associated with $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$.

In the following section we will demonstrate this for symmetric triples of index 2. We will obtain a list of all isomorphism classes of indecomposable symmetric triples of index 2 or, equivalently, a list of all isometry classes of indecomposable simply-connected pseudo-Riemannian symmetric spaces of index 2. First classification results for this case were already obtained by M. Cahen and M. Parker in [CP 1] and [CP 2]. In [CP 1] symmetric spaces of index 2 with solvable transvection group were studied. Unfortunately, the classification of these spaces was not complete. In his diploma thesis Th. Neukirchner elaborated the claimed results, found the gaps and gave a revised classification of symmetric spaces of index 2 with solvable transvection group. Comparing the latter one with the classification which follows from our classification scheme we observed that also Neukirchner's classification is not quite correct (besides minor errors a series of spaces is missing and some of the normal forms contain too much parameters).

Now let $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$ be an indecomposable symmetric triple of index 2 which is not semi-simple. By Prop. 5.1 $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle_-)$ has the structure of a quadratic extension $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ in a canonical way, where $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is a suitable admissible Lie algebra with involution, \mathfrak{a} is a suitable orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module and $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$. Using Remark 4.3 we see that $\dim \mathfrak{l} \leq 2$. Since $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is admissible we have $\mathfrak{l} = [\mathfrak{l}_-, \mathfrak{l}_-] \oplus \mathfrak{l}_-$ and therefore $\dim \mathfrak{l} \leq 3$.

In particular it follows from dimensional reasons that either \mathfrak{l} is solvable or isomorphic to $\mathfrak{su}(2)$ or to $\mathfrak{sl}(2, \mathbb{R})$. If \mathfrak{l} is solvable, then also \mathfrak{g} is solvable and we are in the case discussed above. If \mathfrak{l} is isomorphic to $\mathfrak{su}(2)$ or to $\mathfrak{sl}(2, \mathbb{R})$, then \mathfrak{l} is the Levi factor of \mathfrak{g} . Symmetric triples with a Levi factor of this type were thoroughly studied in [CP 2]. In particular, Cahen and Parker obtained a classification of those triples whose Lie algebra structure satisfies in addition a certain minimality condition. In the special case of index 2 this minimality condition is always satisfied. For completeness we will reproduce also this classification of non-solvable indecomposable symmetric triples of index 2 using our classification scheme (taking only the rather elementary classification of indecomposable orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules from [CP 2]).

7.1 Semi-simple orthogonal representations of solvable Lie algebras with involution

Before we concentrate on symmetric triples of index 2 we will give the following description of $\text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}} / \text{Aut}(\mathfrak{a})$ for an arbitrary solvable Lie algebra with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ satisfying $[\mathfrak{l}_-, \mathfrak{l}_+] = \mathfrak{l}_+$ and the standard pseudo-Euclidean space $\mathfrak{a} = \mathbb{R}^{p+r, q+s}$ with involution $\theta_{\mathfrak{a}}$ given by $\mathfrak{a}_+ = \mathbb{R}^{p, q}$, $\mathfrak{a}_- = \mathbb{R}^{r, s}$. Here we have $\text{Aut}(\mathfrak{a}) = O(p, q) \times O(r, s)$. Since $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}] = R(\mathfrak{l})$ and $R(\mathfrak{l})$ is in the kernel of any semi-simple representation of \mathfrak{l} the map

$$\begin{aligned} \text{Hom}((\mathfrak{l}/\mathfrak{l}', \bar{\theta}_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}} &\longrightarrow \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}} \\ \bar{\rho} &\longmapsto \rho, \quad \rho(L) = \bar{\rho}(L + \mathfrak{l}) \end{aligned}$$

is a bijection. Here $\bar{\theta}_{\mathfrak{l}}$ is the involution induced by $\theta_{\mathfrak{l}}$ on the quotient. Since by assumption $\mathfrak{l}_+ \subset \mathfrak{l}'$ we see that $\bar{\theta}_{\mathfrak{l}} = -\text{Id}$. Hence for our purpose it is sufficient to determine $\text{Hom}((\mathfrak{l}, -\text{Id}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}} / \text{Aut}(\mathfrak{a})$ for any abelian Lie algebra \mathfrak{l} .

Let $\mathbb{R}^{p, q}$ be the standard pseudo-Euclidean space of dimension $n = p + q$. Then the standard basis e_1, \dots, e_n of $\mathbb{R}^{p, q}$ is orthogonal and satisfies $\langle e_k, e_k \rangle = -1$ for $k = 1, \dots, p$ and $\langle e_k, e_k \rangle = 1$ for $k = p + 1, \dots, n$.

We consider $\mathfrak{a} = \mathbb{R}^{p+r, q+s}$ with the involutive isometry $\theta_{\mathfrak{a}}$ defined above by $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, where $\mathfrak{a}_+ = \mathbb{R}^{p, q}$ and $\mathfrak{a}_- = \mathbb{R}^{r, s}$. We will use the notation $\mathfrak{a}_+^{p, q}$ for \mathfrak{a}_+ , $\mathfrak{a}_-^{r, s}$ for \mathfrak{a}_- and $\mathfrak{a}_+^{p, q} \oplus \mathfrak{a}_-^{r, s}$ for the pair $(\mathfrak{a}, \theta_{\mathfrak{a}})$. We will say that $A_1, \dots, A_{p+q+r+s}$ is a standard basis of $\mathfrak{a}_+^{p, q} \oplus \mathfrak{a}_-^{r, s}$ if A_1, \dots, A_{p+q} is the standard basis of $\mathfrak{a}_+^{p, q} = \mathbb{R}^{p, q}$ and $A_{p+q+1}, \dots, A_{p+q+r+s}$ is the standard basis of $\mathfrak{a}_-^{r, s} = \mathbb{R}^{r, s}$. given $n = p + q + r + s$, $n' = p' + q' + r' + s'$, is a standard basis of if A_1, \dots, A_n is the standard basis of $\mathfrak{a}_+^{p, q} \oplus \mathfrak{a}_-^{r, s}$ and $A_{n+1}, \dots, A_{n+n'}$ is the standard Often we identify $(\mathfrak{a} \oplus \mathfrak{a}', \theta_{\mathfrak{a}} \oplus \theta_{\mathfrak{a}'})$ with $\mathfrak{a}_+^{p+p', q+q'} \oplus \mathfrak{a}_-^{r+r', s+s'}$.

Now let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be an abelian Lie algebra with involution $\theta_{\mathfrak{l}} = -\text{Id}$. For $\lambda \in \mathfrak{l}^*$ we define orthogonal representations ρ_{λ}^+ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ on $\mathfrak{a}_+^{0,1} \oplus \mathfrak{a}_-^{0,1}$, ρ_{λ}^- of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ on $\mathfrak{a}_+^{1,0} \oplus \mathfrak{a}_-^{1,0}$, $\tilde{\rho}_{\lambda}^+$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ on $\mathfrak{a}_+^{1,0} \oplus \mathfrak{a}_-^{0,1}$, and $\tilde{\rho}_{\lambda}^-$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ on $\mathfrak{a}_+^{0,1} \oplus \mathfrak{a}_-^{1,0}$ by

$$\rho_{\lambda}^{\pm}(L) = \begin{pmatrix} 0 & -\lambda(L) \\ \lambda(L) & 0 \end{pmatrix}, \quad \tilde{\rho}_{\lambda}^{\pm}(L) = \begin{pmatrix} 0 & \lambda(L) \\ \lambda(L) & 0 \end{pmatrix}$$

w. r. t. the standard bases. Moreover, for $\mu, \nu \in \mathfrak{l}^*$ we define an orthogonal representation $\rho''_{\mu, \nu}$ of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ on $\mathfrak{a}_+^{1,1} \oplus \mathfrak{a}_-^{1,1}$ by

$$\rho''_{\mu, \nu}(L) = \begin{pmatrix} 0 & 0 & -\nu(L) & \mu(L) \\ 0 & 0 & \mu(L) & \nu(L) \\ \nu(L) & \mu(L) & 0 & 0 \\ \mu(L) & -\nu(L) & 0 & 0 \end{pmatrix}$$

w. r. t. the standard basis.

For $\lambda = (\lambda^1, \dots, \lambda^m)$, $\mu = (\mu^1, \dots, \mu^m)$, $\nu = (\nu^1, \dots, \nu^m) \in (\mathfrak{l}^*)^m$ we define semi-simple orthogonal representations

$$\rho_{\lambda}^+ \text{ of } (\mathfrak{l}, \theta_{\mathfrak{l}}) \text{ on } \mathfrak{a}_+^{0, m} \oplus \mathfrak{a}_-^{0, m} = \bigoplus_{i=1}^m \mathfrak{a}_+^{0, 1} \oplus \mathfrak{a}_-^{0, 1},$$

$$\begin{aligned}
\rho_{\lambda}^{-} \text{ of } (\mathfrak{l}, \theta_{\mathfrak{l}}) \text{ on } \mathfrak{a}_{+}^{m,0} \oplus \mathfrak{a}_{-}^{m,0} &= \bigoplus_{i=1}^m \mathfrak{a}_{+}^{1,0} \oplus \mathfrak{a}_{-}^{1,0}, \\
\tilde{\rho}_{\lambda}^{+} \text{ of } (\mathfrak{l}, \theta_{\mathfrak{l}}) \text{ on } \mathfrak{a}_{+}^{m,0} \oplus \mathfrak{a}_{-}^{0,m} &= \bigoplus_{i=1}^m \mathfrak{a}_{+}^{1,0} \oplus \mathfrak{a}_{-}^{0,1}, \\
\tilde{\rho}_{\lambda}^{-} \text{ of } (\mathfrak{l}, \theta_{\mathfrak{l}}) \text{ on } \mathfrak{a}_{+}^{0,m} \oplus \mathfrak{a}_{-}^{m,0} &= \bigoplus_{i=1}^m \mathfrak{a}_{+}^{0,1} \oplus \mathfrak{a}_{-}^{1,0}, \text{ and} \\
\rho_{\mu,\nu}'' \text{ of } (\mathfrak{l}, \theta_{\mathfrak{l}}) \text{ on } \mathfrak{a}_{+}^{m,m} \oplus \mathfrak{a}_{-}^{m,m} &= \bigoplus_{i=1}^m \mathfrak{a}_{+}^{1,1} \oplus \mathfrak{a}_{-}^{1,1}
\end{aligned}$$

by

$$\rho_{\lambda}^{\pm} = \bigoplus_{i=1}^m \rho_{\lambda^i}^{\pm}, \quad \tilde{\rho}_{\lambda}^{\pm} = \bigoplus_{i=1}^m \tilde{\rho}_{\lambda^i}^{\pm}, \quad \rho_{\mu,\nu}'' = \bigoplus_{i=1}^m \rho_{\mu^i, \nu^i}''.$$

Finally, let $(\rho_0)_{r,s}^{p,q}$ be the trivial representation of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ on $\mathfrak{a}_{+}^{p,q} \oplus \mathfrak{a}_{-}^{r,s}$.

The symmetric group \mathfrak{S}_m acts on $(\mathfrak{l}^*)^m$ by permuting coordinates and on $(\mathfrak{l}^*)^m \oplus (\mathfrak{l}^*)^m$ by permuting pairs of coordinates. The group $(\mathbb{Z}_2)^m$ acts on $(\mathfrak{l}^*)^m$ by changing the signs of the coordinates. We define the orbit spaces $\Lambda_m := (\mathfrak{l}^* \setminus 0)^m / \mathfrak{S}_m \times (\mathbb{Z}_2)^m$ and $\Lambda_m'' := \left(((\mathfrak{l}^* \setminus 0)^m / (\mathbb{Z}_2)^m) \oplus ((\mathfrak{l}^* \setminus 0)^m / (\mathbb{Z}_2)^m) \right) / \mathfrak{S}_m$. Finally we define an action of $\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}})$ on Λ_m and Λ_m'' by $S^*[\lambda] := [S^*\lambda]$ and $S^*[\mu, \nu] := [S^*\mu, S^*\nu]$.

Proposition 7.1 *Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be an abelian Lie algebra with involution $\theta_{\mathfrak{l}} = -\text{Id}$ and let $\mathfrak{a} = \mathfrak{a}_{+}^{p,q} \oplus \mathfrak{a}_{-}^{r,s}$ be as defined above. Then the map*

$$\begin{aligned}
\bigcup_{i \in I_{r,s}^{p,q}} \Lambda_{m_1} \times \dots \times \Lambda_{m_4} \times \Lambda_{m_5}'' &\longrightarrow \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}} / \text{Aut}(\mathfrak{a}) \\
([\lambda_1], \dots, [\lambda_4], [\mu, \nu]) &\longmapsto [\rho_{\lambda_1}^{+} \oplus \rho_{\lambda_2}^{-} \oplus \tilde{\rho}_{\lambda_3}^{+} \oplus \tilde{\rho}_{\lambda_4}^{-} \oplus \rho_{\mu,\nu}'' \oplus (\rho_0)_{r_0, s_0}^{p_0, q_0}],
\end{aligned}$$

where

$$I_{r,s}^{p,q} = \left\{ i = (m_1, \dots, m_5, p_0, q_0, r_0, s_0) \in \mathbb{Z}^9 \left| \begin{array}{l} m_2 + m_3 + m_5 + p_0 = p, \\ m_1 + m_4 + m_5 + q_0 = q, \\ m_2 + m_4 + m_5 + r_0 = r, \\ m_1 + m_3 + m_5 + s_0 = s \end{array} \right. \right\}$$

is a bijection. It is equivariant with respect to the action of $\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}})$.

7.2 Admissible 3-dimensional Lie algebras with involution

We know that any indecomposable, non-irreducible symmetric triple of index 2 has the structure of a quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ in a canonical way, where $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is an admissible Lie algebra with involution such that \mathfrak{l}_{-} is at most 2-dimensional, \mathfrak{a} is a suitable orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module and $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$. Therefore we will now determine all admissible Lie algebras with involution $(\mathfrak{l}, \theta_{\mathfrak{l}})$ such that \mathfrak{l}_{-} is at most 2-dimensional. Moreover we determine

$cH_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$ for an arbitrary semi-simple orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} .

Proposition 7.2 *If $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is an admissible Lie algebra with involution and if $\dim \mathfrak{l}_{-} \leq 2$, then $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is isomorphic to one of the following Lie algebras with involution (given by the induced decomposition into eigenspaces):*

1. $\mathfrak{l} = \mathbb{R}^k$ ($k \leq 2$), $\mathfrak{l}_+ = 0$, $\mathfrak{l}_- = \mathfrak{l}$,
2. $\mathfrak{l} = \mathfrak{n}(2) = \{[X, Y] = Z, [X, Z] = -Y\}$, $\mathfrak{l}_+ = \mathbb{R} \cdot Z$, $\mathfrak{l}_- = \text{span}\{X, Y\}$,
3. $\mathfrak{l} = \mathfrak{r}_{3,-1} = \{[X, Y] = Y, [X, Z] = -Z\}$, $\mathfrak{l}_+ = \mathbb{R} \cdot (Y + Z)$, $\mathfrak{l}_- = \text{span}\{X, Y - Z\}$,
4. $\mathfrak{l} = \mathfrak{h}(1) = \{[X, Y] = Z\}$, $\mathfrak{l}_+ = \mathbb{R} \cdot Z$, $\mathfrak{l}_- = \text{span}\{X, Y\}$
5. $\mathfrak{l} = \mathfrak{su}(2) = \{[H, X] = 2Y, [H, Y] = -2X, [X, Y] = 2H\}$
 $\mathfrak{l}_+ = \mathbb{R} \cdot H$, $\mathfrak{l}_- = \text{span}\{X, Y\}$,
6. $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R}) = \{[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H\}$
 $\mathfrak{l}_+ = \mathbb{R} \cdot H$, $\mathfrak{l}_- = \text{span}\{X, Y\}$,
7. $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{l}_+ = \mathbb{R} \cdot (X - Y)$, $\mathfrak{l}_- = \text{span}\{H, X + Y\}$.

Proof. Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a Lie algebra with involution satisfying (T_1) . Obviously, if $\dim \mathfrak{l}_- = 1$, then $\mathfrak{l} = \mathfrak{l}_- = \mathbb{R}$, and if $\dim \mathfrak{l}_- = 2$ and $[\mathfrak{l}_-, \mathfrak{l}_-] = 0$, then $\mathfrak{l} = \mathfrak{l}_- = \mathbb{R}^2$.

If $\dim \mathfrak{l}_- = 2$ and $[\mathfrak{l}_-, \mathfrak{l}_-] \neq 0$, then $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is isomorphic to one of the Lie algebras with involution \mathfrak{l}_{φ} , $\varphi \in \mathfrak{sl}(2, \mathbb{R})$, which are defined as follows:

$$\begin{aligned} \mathfrak{l}_{\varphi} &= (\mathfrak{l}_{\varphi})_+ \oplus (\mathfrak{l}_{\varphi})_-, \quad (\mathfrak{l}_{\varphi})_- = \text{span}\{L_1, L_2\} = \mathbb{R}^2, \quad (\mathfrak{l}_{\varphi})_+ = \mathbb{R} \cdot L_3 = \mathbb{R} \\ &[L_1, L_2] = L_3, \quad [L_3, L_i] = \varphi(L_i) \text{ for all } L_i \in (\mathfrak{l}_{\varphi})_-. \end{aligned}$$

For $\varphi, \varphi' \in \mathfrak{sl}(2, \mathbb{R})$ the Lie algebras with involution \mathfrak{l}_{φ} and $\mathfrak{l}_{\varphi'}$ are isomorphic if and only if there is a map $g \in GL(2, \mathbb{R})$ such that $\varphi' = \det(g^{-1})g\varphi g^{-1}$. Consequently, $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is isomorphic to $\mathfrak{l}_i := \mathfrak{l}_{\varphi_i}$ for exactly one of the following $\varphi_i \in \mathfrak{sl}(2, \mathbb{R})$, $i = 1, \dots, 6$:

$$\varphi_1 = 0, \quad \varphi_{2,3} = \pm \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varphi_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varphi_{5,6} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is not hard to see that $\mathfrak{l}_1 \cong \mathfrak{h}(1)$, $\mathfrak{l}_2 \cong \mathfrak{r}_{3,-1}$, $\mathfrak{l}_3 \cong \mathfrak{n}(2)$, $\mathfrak{l}_5 \cong \mathfrak{su}(2)$, all with the involution given in the proposition, $\mathfrak{l}_4 \cong \mathfrak{sl}(2, \mathbb{R})$ with the involution given in 6. and $\mathfrak{l}_6 \cong \mathfrak{sl}(2, \mathbb{R})$ with the involution given in 7. We will see in this subsection that all these Lie algebras with involution are indeed admissible. \square

Next we will determine $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$ for all $(\mathfrak{l}, \theta_{\mathfrak{l}})$ listed in Prop. 7.2 and any semi-simple orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} . Let Θ be defined as in (8). By Prop. 3.1 we can identify $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ with $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^{\Theta}$. Therefore let us recall the following fact on quadratic cohomology sets of three-dimensional Lie algebras (see [KO 2], Lemma 7.2):

If \mathfrak{l} is a three-dimensional unimodular Lie algebra and \mathfrak{a} is an orthogonal \mathfrak{l} -module, then

$$\begin{aligned} \iota_Q : \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a}) &\longrightarrow (H^2(\mathfrak{l}, \mathfrak{a}) \setminus \{0\}) \cup C^3(\mathfrak{l}) \\ [\alpha, \gamma] &\longmapsto \begin{cases} [\alpha] \in H^2(\mathfrak{l}, \mathfrak{a}) & \text{if } [\alpha] \neq 0 \\ \gamma \in C^3(\mathfrak{l}) & \text{if } [\alpha] = 0 \end{cases} \end{aligned}$$

is a bijective map.

Now let $\theta_{\mathfrak{l}}$ be an involution on \mathfrak{l} and let \mathfrak{a} be an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module. Obviously, $\Theta \circ \iota_Q = \iota_Q \circ \Theta$. In particular, restricting ι_Q to $\mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^\Theta$ we obtain a bijection

$$\iota_Q : \mathcal{H}_Q^2(\mathfrak{l}, \mathfrak{a})^\Theta \longrightarrow (H^2(\mathfrak{l}, \mathfrak{a})^\Theta \setminus \{0\}) \cup C^3(\mathfrak{l})_+. \quad (19)$$

Here $H^2(\mathfrak{l}, \mathfrak{a})^\Theta$ denotes the space of invariants of the action of Θ on $H^2(\mathfrak{l}, \mathfrak{a})$.

7.2.1 The case $\mathfrak{l} = \mathfrak{n}(2)$ or $\mathfrak{l} = \mathfrak{r}_{3,-1}$

Now let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be a three-dimensional admissible Lie algebra with involution such that $\dim \mathfrak{l}' = 2$, i.e. $\mathfrak{l} = \mathfrak{n}(2)$ or $\mathfrak{l} = \mathfrak{r}_{3,-1}$ with $\theta_{\mathfrak{l}}$ as given in Proposition 7.2. The adjoint representation of \mathfrak{l} induces a semi-simple representation ad_0 of $\mathfrak{l}_0 := \mathfrak{l}/\mathfrak{l}'$ on \mathfrak{l}' . Let $\lambda^i \in (\mathfrak{l}_{\mathbb{C}}/\mathfrak{l}'_{\mathbb{C}})^*$, $i = 1, 2$, be the weights of the complexification of ad_0 , and let

$$V_{\lambda^i} = \{U \in \mathfrak{l}'_{\mathbb{C}} \mid \text{ad}_0(L)(U) = \lambda^i(L) \cdot U \text{ for all } L \in \mathfrak{l}\}$$

be the corresponding weight spaces. Let (ρ, \mathfrak{a}) be a semi-simple orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module. Then $\mathfrak{l}' \subset \ker \rho$ since $\mathfrak{l}' = R(\mathfrak{l})$. Hence, ρ induces a representation of $\mathfrak{l}/\mathfrak{l}'$ on \mathfrak{a} . In particular, the complexified module $(\rho, \mathfrak{a}_{\mathbb{C}})$ decomposes into weight spaces

$$E_{\lambda} = \{A \in \mathfrak{a}_{\mathbb{C}} \mid \rho(L)(A) = \lambda(L) \cdot A \text{ for all } L \in \mathfrak{l}\},$$

where $\lambda \in (\mathfrak{l}_{\mathbb{C}}/\mathfrak{l}'_{\mathbb{C}})^*$. We will identify λ with $\lambda(X) \in \mathbb{C}$. For $\mathfrak{l} = \mathfrak{n}(2)$ we also define $E := E_i \oplus E_{-i}$.

Proposition 7.3 *Suppose $\mathfrak{l} \in \{\mathfrak{r}_{3,-1}, \mathfrak{n}(2)\}$. Let $Z_{\mathfrak{l}} \subset C^2(\mathfrak{l}, \mathfrak{a})_+$ be defined by*

$$Z_{\mathfrak{l}} = \{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(Y, Z) \in \mathfrak{a}_+^{\mathfrak{l}}, \alpha(X, Y) \in E_1, \theta_{\mathfrak{a}}(\alpha(X, Y)) = -\alpha(X, Z)\}$$

if $\mathfrak{l} = \mathfrak{r}_{3,-1}$ and by

$$Z_{\mathfrak{l}} = \{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(Y, Z) \in \mathfrak{a}_+^{\mathfrak{l}}, \alpha(X, Y) \in E \cap \mathfrak{a}_+, \alpha(X, Z) = \rho(X)\alpha(X, Y)\}$$

if $\mathfrak{l} = \mathfrak{n}(2)$. Then

$$\begin{aligned} (Z_{\mathfrak{l}} \setminus \{0\}) \cup C^3(\mathfrak{l}) &\longrightarrow \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}) \\ (Z_{\mathfrak{l}} \setminus \{0\}) \ni \alpha &\longmapsto [\alpha, 0] \\ C^3(\mathfrak{l}) \ni \gamma &\longmapsto [0, \gamma] \end{aligned}$$

is a bijection and

$$[\alpha, 0] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0 \Leftrightarrow \mathfrak{a}^{\mathfrak{l}} = \mathbb{R} \cdot \alpha(Y, Z) \text{ and} \\ \text{span}\{\alpha(X, Y), \alpha(X, Z)\} \text{ is non-degenerate}$$

$$[0, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0 \Leftrightarrow \mathfrak{a}^{\mathfrak{l}} = 0 \text{ and } \gamma \neq 0$$

for all $\alpha \in Z_{\mathfrak{l}} \setminus \{0\}$ and $\gamma \in C^3(\mathfrak{l})$.

Proof. In [KO 2], Lemma 7.1 we proved that

$$\begin{aligned} \{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(\mathfrak{l}', \mathfrak{l}') \subset \mathfrak{a}^{\mathfrak{l}}, \alpha(X, V_{\lambda^i}) \subset E_{\lambda^i}, i = 1, 2\} &\longrightarrow H^2(\mathfrak{l}, \mathfrak{a}) \\ \alpha &\longmapsto [\alpha] \end{aligned} \quad (20)$$

is correctly defined and an isomorphism.

By Equation (7) we have $\rho(-X)(\theta_{\mathfrak{a}}(A)) = \theta_{\mathfrak{a}}(\rho(X)(A))$ for all $A \in \mathfrak{a}$, which implies

$$\theta_{\mathfrak{a}}(E_{\lambda}) = E_{-\lambda}.$$

Now let $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})$ be such that $\alpha(X, V_{\lambda^i}) \subset E_{\lambda^i}$, $i = 1, 2$. Then

$$(\Theta\alpha)(X, V_{\lambda^i}) = \theta_{\mathfrak{a}}(\alpha(X, \theta_{\mathfrak{l}}(V_{\lambda^i}))) = \theta_{\mathfrak{a}}(\alpha(X, V_{-\lambda^i})) \subset \theta_{\mathfrak{a}}(E_{-\lambda^i}) = E_{\lambda^i}.$$

Moreover, if $\alpha(\mathfrak{l}', \mathfrak{l}') \subset \mathfrak{a}^{\mathfrak{l}}$, then

$$(\Theta\alpha)(\mathfrak{l}', \mathfrak{l}') = \theta_{\mathfrak{a}}(\alpha(\theta_{\mathfrak{l}}(\mathfrak{l}'), \theta_{\mathfrak{l}}(\mathfrak{l}'))) = \theta_{\mathfrak{a}}(\alpha(\mathfrak{l}', \mathfrak{l}')) \subset \mathfrak{a}^{\mathfrak{l}}.$$

It follows that $\{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(\mathfrak{l}', \mathfrak{l}') \subset \mathfrak{a}^{\mathfrak{l}}, \alpha(X, V_{\lambda^i}) \subset E_{\lambda^i}, i = 1, 2\}$ is Θ -invariant. Hence the restriction of the map defined in (20) to

$$\{\alpha \in C^2(\mathfrak{l}, \mathfrak{a})_+ \mid \alpha(\mathfrak{l}', \mathfrak{l}') \subset \mathfrak{a}^{\mathfrak{l}}, \alpha(X, V_{\lambda^i}) \subset E_{\lambda^i}, i = 1, 2\} = Z_{\mathfrak{l}}$$

is an isomorphism onto $H^2(\mathfrak{l}, \mathfrak{a})^{\Theta}$. The first assertion of the proposition now follows from (19) and $C^3(\mathfrak{l})_+ = C^3(\mathfrak{l})$. It remains to prove the statement on admissibility.

Assume first that $[\alpha, 0] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_0$ for $\alpha \in Z_{\mathfrak{l}} \setminus \{0\}$. Since $[\alpha, 0]$ is indecomposable we have $\mathfrak{a}^{\mathfrak{l}} = \mathbb{R} \cdot \alpha(Y, Z)$. In particular, $\mathbb{R} \cdot \alpha(Y, Z) \subset \mathfrak{a}$ is a non-degenerate subspace. Assume that $\text{span}\{\alpha(X, Y), \alpha(X, Z)\} \neq 0$ and that $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ restricted to this subspace degenerates. Then $\text{span}\{\alpha(X, Y), \alpha(X, Z)\} \neq 0$ is totally isotropic. Let \mathfrak{b}_1 be as in Condition (B_1) of Definition 5.2. Since obviously the submodule $\alpha(Y, Z)^{\perp} \cap \{\alpha(X, Y), \alpha(X, Z)\}^{\perp}$ is contained in \mathfrak{b}_1 and \mathfrak{b}_1 is non-degenerate we obtain $\alpha(Y, Z)^{\perp} \subset \mathfrak{b}_1$. This yields $\alpha(X, Y) = \alpha(X, Z) = 0$, a contradiction.

Now assume that $\alpha \in Z_{\mathfrak{l}} \setminus \{0\}$ satisfies $\mathfrak{a}^{\mathfrak{l}} = \mathbb{R} \cdot \alpha(Y, Z)$ and that the subspace $\text{span}\{\alpha(X, Y), \alpha(X, Z)\} \subset \mathfrak{a}$ is non-degenerate. Since $\alpha_0(\ker[\cdot, \cdot]_{\mathfrak{l}}) = \mathbb{R} \cdot \alpha(Y, Z)$ Condition (B_0) is satisfied. We prove that (B_1) is also satisfied. We use that the submodule $\mathfrak{m} := \alpha(Y, Z)^{\perp} \cap \{\alpha(X, Y), \alpha(X, Z)\}^{\perp} = (\mathfrak{a}^{\mathfrak{l}})^{\perp} \cap \{\alpha(X, Y), \alpha(X, Z)\}^{\perp}$ is contained in \mathfrak{b}_1 and that $\mathfrak{b}_1 \subset (\mathfrak{a}^{\mathfrak{l}})^{\perp}$ itself is a submodule. In case $\mathfrak{l} = \mathfrak{n}(2)$ this implies that \mathfrak{b}_1 is equal to $(\mathfrak{a}^{\mathfrak{l}})^{\perp}$ or to \mathfrak{m} . Since both submodules are non-degenerate (B_1) holds. Now we consider the case $\mathfrak{l} = \mathfrak{r}_{3,-1}$. Assume that \mathfrak{b}_1 contains the submodule $\mathbb{R} \cdot \alpha(X, Y)$. Then

$$\langle \alpha(X, Z), \alpha(X, Y) \rangle = \langle (\rho(X) + \text{Id})(\Phi(Z)), \alpha(X, Y) \rangle = 0$$

since $\alpha(X, Y) \in E_1$. Since on the other hand $\alpha(X, Z) \in E_{-1}$ and by assumption $\text{span}\{\alpha(X, Y), \alpha(X, Z)\} \subset \mathfrak{a}$ is non-degenerate we obtain $\alpha(X, Y) = \alpha(X, Z) = 0$. Similarly $\mathbb{R} \cdot \alpha(X, Z) \subset \mathfrak{b}_1$ implies $\alpha(X, Y) = \alpha(X, Z) = 0$. Again we obtain that \mathfrak{b}_1 is equal to $(\mathfrak{a}^{\mathfrak{l}})^{\perp}$ or to \mathfrak{m} and (B_1) holds. Now let again \mathfrak{l} be $\mathfrak{r}_{3,-1}$ or $\mathfrak{n}(2)$. Since $R_2(\mathfrak{l}) = 0$ Conditions (A_k) and (B_k) hold for $k \geq 2$. Moreover, (A_0) holds since $\mathfrak{z}(\mathfrak{l}) = 0$ and (A_1) holds since no $\alpha \in Z_{\mathfrak{l}} \setminus \{0\}$ satisfies Assumption (i) of (A_1) . Since also Conditions (T_1)

and (T_2) are satisfied $[\alpha, 0]$ is admissible by Theorem 5.1. Since \mathfrak{l} does not decompose into the direct sum of two non-vanishing Lie algebras and $\mathfrak{a}^\mathfrak{l} = \mathbb{R} \cdot \alpha(Y, Z)$ Proposition 6.2 yields $[\alpha, 0] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_0$.

For $[0, \gamma]$ with $\gamma \in C^3(\mathfrak{l})$ Assumption (ii) of (A_1) is satisfied if and only if $\gamma = 0$. Hence (A_1) holds if and only if $\gamma \neq 0$. Since all other Conditions (A_k) and (B_k) are trivially satisfied $[0, \gamma]$ is admissible if and only if $\gamma \neq 0$ and $\mathfrak{a}_+^\mathfrak{l} = 0$ (Condition (T_2)). Applying Proposition 6.2 we obtain $[0, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_0$ if and only if $\mathfrak{a}^\mathfrak{l} = 0$ and $\gamma \neq 0$. \square

Proposition 7.4 *If $\mathfrak{l} = \mathfrak{n}(2)$, then*

$$\text{Aut}(\mathfrak{l}, \theta_\mathfrak{l}) = \left\{ \begin{pmatrix} u & 0 & 0 \\ v & a & 0 \\ 0 & 0 & ua \end{pmatrix} \mid u = \pm 1, a, v \in \mathbb{R}, a \neq 0 \right\}$$

For $\mathfrak{l} = \mathfrak{r}_{3,-1}$ we have

$$\text{Aut}(\mathfrak{l}, \theta_\mathfrak{l}) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ v & a & 0 \\ -v & 0 & a \end{pmatrix} \mid a, v \in \mathbb{R}, a \neq 0 \right\} \cup \left\{ \begin{pmatrix} -1 & 0 & 0 \\ v & 0 & b \\ -v & b & 0 \end{pmatrix} \mid b, v \in \mathbb{R}, b \neq 0 \right\}.$$

Here all automorphisms are written with respect to the basis X, Y, Z of \mathfrak{l} .

7.2.2 The case $\mathfrak{l} = \mathfrak{h}(1)$

Now we suppose that $\mathfrak{l} = \mathfrak{h}(1)$ and that $\theta_\mathfrak{l}$ is given as in Proposition 7.2, 4.

Proposition 7.5 *For $\mathfrak{l} = \mathfrak{h}(1)$ and*

$$Z_\mathfrak{l} := \{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(X, Y) = 0, \alpha(Z, \mathfrak{l}) \subset \mathfrak{a}_-^\mathfrak{l}\}$$

the map

$$\begin{aligned} (Z_\mathfrak{l} \setminus \{0\}) \cup C^3(\mathfrak{l}) &\longrightarrow \mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a}) \\ Z_\mathfrak{l} \setminus \{0\} \ni \alpha &\longmapsto [\alpha, 0] \\ C^3(\mathfrak{l}) \ni \gamma &\longmapsto [0, \gamma] \end{aligned}$$

is a bijection and $\mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_0$ is the image of

$$\{\alpha \in Z_\mathfrak{l} \setminus \{0\} \mid \alpha(Z, \mathfrak{l}) = \mathfrak{a}^\mathfrak{l}\}$$

under this bijection. In particular, $\mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_0 = \emptyset$ if $\mathfrak{a}_+^\mathfrak{l} \neq 0$.

Proof. In [KO 2], Lemma 7.5 we proved that

$$\begin{aligned} \{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(X, Y) = 0, \alpha(Z, \mathfrak{l}) \subset \mathfrak{a}^\mathfrak{l}\} &\longrightarrow H^2(\mathfrak{l}, \mathfrak{a}) \\ \alpha &\longmapsto [\alpha] \end{aligned}$$

is an isomorphism. The domain of this isomorphism is Θ -invariant. Hence it restricts to an isomorphism from $\{\alpha \in C_+^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(X, Y) = 0, \alpha(Z, \mathfrak{l}) \subset \mathfrak{a}^\perp\} = Z_\mathfrak{l}$ to $H^2(\mathfrak{l}, \mathfrak{a})^\Theta$.

The cohomology classes $[0, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})$ are not admissible, neither (A_0) nor (A_1) is satisfied (see also [KO 2], Lemma 7.6). If $\alpha \in Z_\mathfrak{l} \setminus \{0\}$, then neither assumption (i) of (A_0) nor assumption (i) of (A_1) is satisfied. Hence (A_0) and (A_1) hold. Furthermore, both (B_0) and (B_1) are equivalent to the condition that $\alpha(Z, \mathfrak{l})$ is non-degenerate. Condition (T_2) is equivalent to $\mathfrak{a}_+^\perp = 0$. Hence, $[\alpha, 0]$ is admissible if and only if $\mathfrak{a}_+^\perp = 0$ and $\alpha(Z, \mathfrak{l})$ is non-degenerate. Proposition 6.2 now yields $[\alpha, 0] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_0$ if and only if $\alpha(Z, \mathfrak{l}) = \mathfrak{a}^\perp$. \square

Proposition 7.6 *For $\mathfrak{l} = \mathfrak{h}(1)$ we have*

$$\text{Aut}(\mathfrak{l}, \theta_\mathfrak{l}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & u \end{pmatrix} \mid A \in GL(2, \mathbb{R}), \det A = u \right\},$$

where the automorphisms are written with respect to the basis X, Y, Z of \mathfrak{l} .

7.2.3 The case $\mathfrak{l} = \mathbb{R}^k$, $k = 1, 2$

Proposition 7.7 *If $\mathfrak{l} = \mathbb{R}^k$, $k = 1, 2$, then we can identify*

$$\mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a}) = H^2(\mathfrak{l}, \mathfrak{a})^\Theta = C_+^2(\mathfrak{l}, \mathfrak{a}^\perp) = C^2(\mathfrak{l}, \mathfrak{a}_+^\perp).$$

and we have

$$\mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_0 = \begin{cases} C^2(\mathfrak{l}, \mathfrak{a}_+^\perp) \setminus \{0\} & \text{if } \dim \mathfrak{a}_+^\perp = 1, \dim \mathfrak{a}^\perp = 0 \\ \{0\} & \text{if } \dim \mathfrak{a}^\perp = 0, (\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a}) \text{ indecomposable} \\ \emptyset & \text{otherwise .} \end{cases}$$

Here the triple $(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})$ is indecomposable, if it has not any non-trivial decomposition in the sense of Definition 6.1. The proof of this proposition is easy, so we will omit it.

7.2.4 The case $\mathfrak{l} = \mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{R})$

Lemma 7.1 *Let $\mathfrak{l} \in \{\mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{R})\}$. Let $\theta_\mathfrak{l}$ be an involution of \mathfrak{l} . Then we have for all semi-simple orthogonal $(\mathfrak{l}, \theta_\mathfrak{l})$ -modules \mathfrak{a}*

$$\mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a}) = C^3(\mathfrak{l}) .$$

Moreover,

$$\mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a})_0 = \begin{cases} C^3(\mathfrak{l}) & \text{if } \mathfrak{a}^\perp = 0 \\ \emptyset & \text{if } \mathfrak{a}^\perp \neq 0 \end{cases} .$$

Proof. Since \mathfrak{l} is semi-simple we have $H^2(\mathfrak{l}, \mathfrak{a}) = 0$. In order to obtain the first assertion we now combine Proposition 3.1 with (19). Observe that $C^3(\mathfrak{l})_+ = C^3(\mathfrak{l})$. The second assertion is then easy to check. \square

Next we introduce certain orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules for $\mathfrak{l} \in \{\mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{R})\}$. Since we are interested in admissible quadratic extensions of $(\mathfrak{l}, \theta_{\mathfrak{l}})$ by such orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules \mathfrak{a} which yield symmetric triples of index 2 and since here $\dim \mathfrak{l}_- = 2$ we restrict ourselves to those \mathfrak{a} for which $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ restricted to \mathfrak{a}_- is positive definite (see Remark 4.3).

Let $\mathfrak{l} = \mathfrak{su}(2)$. For $k \in \mathbb{N}$ let ρ_k^{\pm} be the irreducible \mathfrak{l} -representation on a real vector space \mathfrak{a} of dimension $2k + 1$ which is equipped with a positive definite \mathfrak{l} -invariant scalar product and an involution $\theta_{\mathfrak{a}}$ uniquely characterized by (7) and $\theta_{\mathfrak{a}}|_{\mathfrak{a}^{\pm}} = \pm(-1)^k \text{Id}_{\mathfrak{a}^{\pm}}$. Then ρ_k^+ acts on $\mathfrak{a}_+^{0, k+1} \oplus \mathfrak{a}_-^{0, k}$, and ρ_k^- acts on $\mathfrak{a}_+^{0, k} \oplus \mathfrak{a}_-^{0, k+1}$. By ρ'_k we denote the irreducible orthogonal representation of \mathfrak{l} on a real vector space \mathfrak{a} of dimension $4k$. Then \mathfrak{a} carries an isometric involution $\theta_{\mathfrak{a}}$ satisfying (7) which is uniquely determined up to a sign. A different choice of the sign would produce an equivalent orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module. Thus there is no need to fix it here. We have $\mathfrak{a} = \mathfrak{a}_+^{0, 2k} \oplus \mathfrak{a}_-^{0, 2k}$.

Let $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$, and let $\theta_{\mathfrak{l}}$ as in Proposition 7.2,7. For $k \in \mathbb{N}$ let ρ_k^{\pm} be the irreducible \mathfrak{l} -representation on a real vector space \mathfrak{a} of dimension $2k + 1$ which is equipped with an involution $\theta_{\mathfrak{a}}$ uniquely characterized by (7) and $\theta_{\mathfrak{a}}|_{\mathfrak{a}^{\pm}} = \pm(-1)^k \text{Id}_{\mathfrak{a}^{\pm}}$. Then there is an \mathfrak{l} -invariant scalar product on \mathfrak{a} which is positive definite on \mathfrak{a}_- . Then ρ_k^+ acts on $\mathfrak{a}_+^{k+1, 0} \oplus \mathfrak{a}_-^{0, k}$, and ρ_k^- acts on $\mathfrak{a}_+^{k, 0} \oplus \mathfrak{a}_-^{0, k+1}$. We also consider the real irreducible \mathfrak{l} -representation acting on a $2k$ -dimensional real vector space V_k . The natural \mathfrak{l} -representation ρ'_k on $\mathfrak{a} := V_k \oplus V_k^*$ carries an invariant scalar product of signature $(2k, 2k)$ induced by the dual pairing. Moreover, there is exactly one involution $\theta_{\mathfrak{a}}$ which satisfies (7) and has a positive definite (-1) -eigenspace. Note that $\theta_{\mathfrak{a}}$ switches the two summands V_k and V_k^* . We have $\mathfrak{a} = \mathfrak{a}_+^{2k, 0} \oplus \mathfrak{a}_-^{0, 2k}$.

For $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$ and $\theta_{\mathfrak{l}}$ as in Proposition 7.2,6. we only consider the orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module given by $(\rho_{\mathfrak{l}}^+, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, \theta_{\mathfrak{a}}) := (\text{ad}, \mathfrak{l}, B, -\theta_{\mathfrak{l}})$, where B denotes the Killing form on \mathfrak{l} . We have $\mathfrak{a} = \mathfrak{a}_+^{1, 1} \oplus \mathfrak{a}_-^{0, 1}$.

We call an orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module \mathfrak{a} indecomposable if it has no proper non-trivial non-degenerate $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -invariant submodule. A semi-simple indecomposable module is either irreducible or the direct sum of two irreducible totally isotropic $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules in duality. The latter case does not occur if the restriction of the scalar product to \mathfrak{a}_+ or \mathfrak{a}_- is definite. All the $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules defined above are irreducible.

Lemma 7.2 *Let $(\mathfrak{l}, \theta_{\mathfrak{l}})$ be as in cases 5.-7. of Proposition 7.2. Then the orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules just defined exhaust the equivalence classes of those indecomposable orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules satisfying $\mathfrak{a}^{\mathfrak{l}} = \{0\}$ and \mathfrak{a}_- is positive definite. They are pairwise inequivalent.*

Proof. See [CP 2], Ch.V, §3. □

If $k = (k_1, \dots, k_p) \in \mathbb{N}^p$ for some $p \geq 0$, then we denote by ρ_k^+ the direct sum module $\rho_{k_1}^+ \oplus \dots \oplus \rho_{k_p}^+$. By convention $\mathbb{N}^0 = \emptyset$, and the corresponding direct sum is the zero module. In the same way we define ρ_k^- and ρ'_k , $k \in \mathbb{N}^p$. Moreover we set $|k| := k_1 + \dots + k_p$.

7.3 The classification result

Now we can formulate our classification of symmetric triples of index 2. As above we will use the notation \mathfrak{l}_0 for $\mathfrak{l}/R(\mathfrak{l})$ for a given solvable Lie algebra \mathfrak{l} . Furthermore let $\tilde{\mathfrak{S}}_{p,q}$ be the group $(\mathfrak{S}_p \times (\mathbb{Z}_2)^p) \times (\mathfrak{S}_q \times (\mathbb{Z}_2)^q)$ which acts on $(\mathfrak{l}_0^* \setminus 0)^p \times (\mathfrak{l}_0^* \setminus 0)^q$.

Theorem 7.1 *If $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \theta)$ is a symmetric triple associated with an indecomposable non-semi-simple symmetric space of index 2, then it is isomorphic to $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta, \mathfrak{a})$ for exactly one of the data in the following list (which contains only data giving rise to such triples):*

$$1. \mathfrak{l} = \mathbb{R}^1 = \mathbb{R} \cdot X, \mathfrak{l}_+ = 0, \mathfrak{l}_- = \mathfrak{l},$$

$$(a) \mathfrak{a} = \mathfrak{a}_+^{0,1} \oplus \mathfrak{a}_-^{1,0} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}, \quad p, q \geq 0,$$

$$\rho = \tilde{\rho}_1^- \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+,$$

$$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}^*)^p = \mathbb{R}^p, \quad \mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}^*)^q = \mathbb{R}^q,$$

$$0 < \lambda^1 \leq \dots \leq \lambda^p, \quad 0 < \mu^1 \leq \dots \leq \mu^q,$$

$$\alpha = 0, \quad \gamma = 0;$$

$$(b) \mathfrak{a} = \mathfrak{a}_+^{1,0} \oplus \mathfrak{a}_-^{1,0} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}, \quad p, q \geq 0,$$

$$\rho = \rho_1^- \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$$

$$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}^*)^p = \mathbb{R}^p, \quad \mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}^*)^q = \mathbb{R}^q,$$

$$0 < \lambda^1 \leq \dots \leq \lambda^p, \quad 0 < \mu^1 \leq \dots \leq \mu^q,$$

$$\alpha = 0, \quad \gamma = 0;$$

$$(c) \mathfrak{a} = \mathfrak{a}_+^{1,1} \oplus \mathfrak{a}_-^{1,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}, \quad p, q \geq 0,$$

$$\rho = \rho_{1,\nu}'' \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$$

$$\nu \in \mathfrak{l}^* \setminus 0 = \mathbb{R}^1 \setminus 0, \quad \lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}^*)^p = \mathbb{R}^p, \quad \mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}^*)^q = \mathbb{R}^q,$$

$$0 < \lambda^1 \leq \dots \leq \lambda^p, \quad 0 < \mu^1 \leq \dots \leq \mu^q,$$

$$\alpha = 0, \quad \gamma = 0;$$

$$2. \mathfrak{l} = \mathbb{R}^2 = \text{span}\{Y, Z\}, \mathfrak{l}_+ = 0, \mathfrak{l}_- = \mathfrak{l},$$

$$(a) \mathfrak{a} = \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}, \quad p, q \geq 0, \quad p+q \geq 3,$$

$$\rho = \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+,$$

$$[\lambda, \mu] \in ((\mathfrak{l}^* \setminus 0)^p \times (\mathfrak{l}^* \setminus 0)^q) / \sim, \text{ such that}$$

$$\{(\lambda^i(Y), \lambda^i(Z)) \mid i = 1, \dots, p\} \cup \{(\mu^j(Y), \mu^j(Z)) \mid j = 1, \dots, q\}$$

is not contained in the union of two one-dimensional subspaces of \mathbb{R}^2 ,
and

$$(\lambda_1, \mu_1) \sim (\lambda_2, \mu_2) \Leftrightarrow \text{span}\{y_1, z_1\} = \text{span}\{y_2, z_2\} \pmod{\tilde{\mathfrak{S}}_{p,q}},$$

where $y_i := (\lambda_i^1(Y), \dots, \lambda_i^p(Y), \mu_i^1(Y), \dots, \mu_i^q(Y))$ and
 $z_i := (\lambda_i^1(Z), \dots, \lambda_i^p(Z), \mu_i^1(Z), \dots, \mu_i^q(Z))$ for $i = 1, 2$,

$$\alpha = 0, \quad \gamma = 0;$$

$$(b) \mathfrak{a} = \mathfrak{a}_+^{0,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}, \quad p, q \geq 0,$$

$$\rho = (\rho_0)_{0,0}^{0,1} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+,$$

$[\lambda, \mu] \in ((\mathfrak{l}^* \setminus 0)^p \times (\mathfrak{l}^* \setminus 0)^q) / \sim$, with
 $(\lambda_1, \mu_1) \sim (\lambda_2, \mu_2) \Leftrightarrow$

$$(\text{span}\{y_1, z_1\}, y_1 \wedge z_1) = (\text{span}\{y_2, z_2\}, \pm y_2 \wedge z_2) \bmod \bar{\mathfrak{S}}_{p,q},$$

where $y_i, z_i, i = 1, 2$, are as in 2. (a),

$\alpha(Y, Z) = A_0$, where A_0 is the standard basis of $\mathfrak{a}_+^{0,1}$,

$\gamma = 0$;

(c) $\mathfrak{a} = \mathfrak{a}_+^{1,0} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = (\rho_0)_{0,0}^{1,0} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$[\lambda, \mu] \in ((\mathfrak{l}^* \setminus 0)^p \times (\mathfrak{l}^* \setminus 0)^q) / \sim$, with

$(\lambda_1, \mu_1) \sim (\lambda_2, \mu_2) \Leftrightarrow$

$$(\text{span}\{y_1, z_1\}, y_1 \wedge z_1) = (\text{span}\{y_2, z_2\}, \pm y_2 \wedge z_2) \bmod \bar{\mathfrak{S}}_{p,q},$$

where $y_i, z_i, i = 1, 2$, are as in 2. (a),

$\alpha(Y, Z) = A_0$, where A_0 is the standard basis of $\mathfrak{a}_+^{1,0}$,

$\gamma = 0$;

3. $\mathfrak{l} = \mathfrak{n}(2)$, $\mathfrak{l}_+ = \mathbb{R} \cdot Z$, $\mathfrak{l}_- = \text{span}\{X, Y\}$,

(a) $\mathfrak{a} = \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_0^*)^p = \mathbb{R}^p$, $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_0^*)^q = \mathbb{R}^q$,

$0 < \lambda^1 \leq \dots \leq \lambda^p$, $0 < \mu^1 \leq \dots \leq \mu^q$,

$\alpha = 0$, $\gamma(X, Y, Z) = \kappa$, $\kappa \in \{1, -1\}$;

(b) $\mathfrak{a} = \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = (\rho_0)_{0,1}^{0,0} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_0^*)^p = \mathbb{R}^p$, $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_0^*)^q = \mathbb{R}^q$,

$0 < \lambda^1 \leq \dots \leq \lambda^p$, $0 < \mu^1 \leq \dots \leq \mu^q$,

$\alpha(Y, Z) = A_0$, where A_0 is the standard basis of $\mathfrak{a}_-^{0,1}$,

$\gamma = 0$;

(c) $\mathfrak{a} = \mathfrak{a}_+^{0,1} \oplus \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = \rho_1^+ \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_0^*)^p = \mathbb{R}^p$, $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_0^*)^q = \mathbb{R}^q$,

$0 < \lambda^1 \leq \dots \leq \lambda^p$, $0 < \mu^1 \leq \dots \leq \mu^q$,

$\alpha(X, Y) = A_1$, $\alpha(X, Z) = A_2$, $\alpha(Y, Z) = 0$,

where A_1, A_2 is the standard basis of $\mathfrak{a}_+^{0,1} \oplus \mathfrak{a}_-^{0,1}$,

$\gamma = 0$;

(d) $\mathfrak{a} = \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{0,1} \oplus \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = (\rho_0)_{0,1}^{0,0} \oplus \rho_1^+ \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_0^*)^p = \mathbb{R}^p$, $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_0^*)^q = \mathbb{R}^q$,

$0 < \lambda^1 \leq \dots \leq \lambda^p$, $0 < \mu^1 \leq \dots \leq \mu^q$,

$\alpha(X, Y) = rA_1$, $\alpha(X, Z) = rA_2$, $\alpha(Y, Z) = A_0$, $r \in \mathbb{R}$, $r > 0$,

where A_0, A_1, A_2 is the standard basis of $\mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{0,1} \oplus \mathfrak{a}_-^{0,1}$,

$\gamma = 0$;

4. $\mathfrak{l} = \mathfrak{r}_{3,-1}$, $\mathfrak{l}_+ = \mathbb{R} \cdot (Y + Z)$, $\mathfrak{l}_- = \text{span}\{X, Y - Z\}$,

(a) $\mathfrak{a} = \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_0^*)^p = \mathbb{R}^p$, $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_0^*)^q = \mathbb{R}^q$,

$0 < \lambda^1 \leq \dots \leq \lambda^p$, $0 < \mu^1 \leq \dots \leq \mu^q$,

$\alpha = 0$, $\gamma(X, Y, Z) = \kappa$, $\kappa \in \{1, -1\}$;

(b) $\mathfrak{a} = \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = (\rho_0)_{0,1}^{0,0} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_0^*)^p = \mathbb{R}^p$, $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_0^*)^q = \mathbb{R}^q$,

$0 < \lambda^1 \leq \dots \leq \lambda^p$, $0 < \mu^1 \leq \dots \leq \mu^q$,

$\alpha(Y, Z) = A_0$, where A_0 is the standard basis of $\mathfrak{a}_-^{0,1}$,

$\gamma = 0$;

(c) $\mathfrak{a} = \mathfrak{a}_+^{1,0} \oplus \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = \tilde{\rho}_1^+ \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_0^*)^p = \mathbb{R}^p$, $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_0^*)^q = \mathbb{R}^q$,

$0 < \lambda^1 \leq \dots \leq \lambda^p$, $0 < \mu^1 \leq \dots \leq \mu^q$,

$\alpha(X, Y - Z) = A_1$, $\alpha(X, Y + Z) = A_2$, $\alpha(Y, Z) = 0$,

where A_1, A_2 is the standard basis of $\mathfrak{a}_+^{1,0} \oplus \mathfrak{a}_-^{0,1}$,

$\gamma = 0$;

(d) $\mathfrak{a} = \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{1,0} \oplus \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = (\rho_0)_{0,1}^{0,0} \oplus \tilde{\rho}_1^+ \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$\lambda = (\lambda^1, \dots, \lambda^p) \in (\mathfrak{l}_0^*)^p = \mathbb{R}^p$, $\mu = (\mu^1, \dots, \mu^q) \in (\mathfrak{l}_0^*)^q = \mathbb{R}^q$,

$0 < \lambda^1 \leq \dots \leq \lambda^p$, $0 < \mu^1 \leq \dots \leq \mu^q$,

$\alpha(X, Y - Z) = rA_1$, $\alpha(X, Y + Z) = rA_2$, $\alpha(Y, Z) = A_0$, $r \in \mathbb{R}$, $r > 0$,

where A_0, A_1, A_2 is the standard basis of $\mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{1,0} \oplus \mathfrak{a}_-^{0,1}$,

$\gamma = 0$;

5. $\mathfrak{l} = \mathfrak{h}(1)$, $\mathfrak{l}_+ = \mathbb{R} \cdot Z$, $\mathfrak{l}_- = \text{span}\{X, Y\}$

(a) $\mathfrak{a} = \mathfrak{a}_-^{0,1} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,

$\rho = (\rho_0)_{0,1}^{0,0} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,

$[\lambda, \mu] \in ((\mathfrak{l}_0^* \setminus 0)^p \times (\mathfrak{l}_0^* \setminus 0)^q) / \sim$, with

$(\lambda_1, \mu_1) \sim (\lambda_2, \mu_2) \Leftrightarrow (\lambda_1, \mu_1), (\lambda_2, \mu_2)$ satisfy (i) or (ii):

(i) $\dim \text{span}\{x_1, y_1\} = \dim \text{span}\{x_2, y_2\} = 1$ and

$$(\text{span}\{x_1, y_1\}, \mathbb{R} \cdot y_1) = (\text{span}\{x_2, y_2\}, \mathbb{R} \cdot y_2) \text{ mod } \bar{\mathfrak{E}}_{p,q},$$

(ii) $\dim \text{span}\{x_1, y_1\} = \dim \text{span}\{x_2, y_2\} = 2$ and

$$\exists r \in \mathbb{R}, r \neq 0 : (x_2 \wedge y_2, y_2) = (rx_1 \wedge y_1, r^2 y_1) \text{ mod } \bar{\mathfrak{E}}_{p,q},$$

where $x_i := (\lambda_i^1(X), \dots, \lambda_i^p(X), \mu_i^1(X), \dots, \mu_i^q(X))$ and

$y_i := (\lambda_i^1(Y), \dots, \lambda_i^p(Y), \mu_i^1(Y), \dots, \mu_i^q(Y))$ for $i = 1, 2$,

$\alpha(X, Z) = A_0$, $\alpha(X, Y) = \alpha(Y, Z) = 0$,

where A_0 is the standard basis of $\mathfrak{a}_-^{0,1}$,

$\gamma = 0$;

- (b) $\mathfrak{a} = \mathfrak{a}_-^{0,2} \oplus \mathfrak{a}_+^{p,q} \oplus \mathfrak{a}_-^{0,p+q}$, $p, q \geq 0$,
 $\rho = (\rho_0)_{0,2}^{0,0} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$,
 $[\lambda, \mu] \in ((\mathfrak{t}_0^* \setminus 0)^p \times (\mathfrak{t}_0^* \setminus 0)^q) / \sim$, with
 $(\lambda_1, \mu_1) \sim (\lambda_2, \mu_2) \Leftrightarrow M_1 M_1^\top = M_2 M_2^\top \pmod{\bar{\mathfrak{S}}_{p,q}}$,
for the $((p+q) \times 2)$ -matrices $M_i := (x_i^\top, y_i^\top)$, $i = 1, 2$, where x_i, y_i are defined
as in 5. (a),
 $\alpha(X, Z) = A_1$, $\alpha(Y, Z) = A_2$, $\alpha(X, Y) = 0$,
where A_1, A_2 is the standard basis of $\mathfrak{a}_-^{0,2}$,
 $\gamma = 0$;

6. $\mathfrak{l} = \mathfrak{su}(2)$, $\mathfrak{l}_+ = \mathbb{R} \cdot H$, $\mathfrak{l}_- = \text{span}\{X, Y\}$,

$$\begin{aligned} \mathfrak{a} &= \mathfrak{a}_+^{0,|k|+|l|+2|m|+p} \oplus \mathfrak{a}_-^{0,|k|+|l|+2|m|+q}, \quad p, q, r \geq 0, \\ k &\in \mathbb{N}^p, \quad l \in \mathbb{N}^q, \quad m \in \mathbb{N}^r, \\ k_1 &\leq k_2 \leq \dots \leq k_p, \quad l_1 \leq \dots \leq l_q, \quad m_1 \leq \dots \leq m_r, \\ \rho &= \rho_k^+ \oplus \rho_l^- \oplus \rho_m', \\ \alpha &= 0, \\ \gamma(H, X, Y) &= c, \quad c \in \mathbb{R}. \end{aligned}$$

7. $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{l}_+ = \mathbb{R} \cdot H$, $\mathfrak{l}_- = \text{span}\{X, Y\}$,

$$\begin{aligned} \mathfrak{a} &= \mathfrak{a}_+^{p,p} \oplus \mathfrak{a}_-^{0,p}, \quad p \geq 0, \quad \rho = \bigoplus_{i=1}^p \rho_1^+, \\ \alpha &= 0, \quad \gamma(H, X, Y) = c, \quad c \in \mathbb{R}. \end{aligned}$$

8. $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{l}_+ = \mathbb{R} \cdot (X - Y)$, $\mathfrak{l}_- = \text{span}\{H, X + Y\}$.

$$\begin{aligned} \mathfrak{a} &= \mathfrak{a}_+^{|k|+|l|+2|m|+p,0} \oplus \mathfrak{a}_-^{0,|k|+|l|+2|m|+q}, \quad p, q, r \geq 0, \\ k &\in \mathbb{N}^p, \quad l \in \mathbb{N}^q, \quad m \in \mathbb{N}^r, \\ k_1 &\leq k_2 \leq \dots \leq k_p, \quad l_1 \leq \dots \leq l_q, \quad m_1 \leq \dots \leq m_r, \\ \rho &= \rho_k^+ \oplus \rho_l^- \oplus \rho_m', \\ \alpha &= 0, \\ \gamma(H, X, Y) &= c, \quad c \in \mathbb{R}. \end{aligned}$$

Proof. We already know that for a given symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ the Lie algebra $\mathfrak{g}/i(\mathfrak{g})^\perp$ with the involution induced by θ is isomorphic to one of the Lie algebras with involution in Prop. 7.2. Moreover, if we consider $\mathfrak{a} := i(\mathfrak{g})^\perp/i(\mathfrak{g})$ with the induced involution and the induced scalar product, then \mathfrak{a}_- is positive definite if $\dim \mathfrak{l}_- = 2$ and the induced scalar product on \mathfrak{a}_- has index 1 if $\dim \mathfrak{l}_- = 1$.

According to Theorem 6.1 we have to determine the orbit space of the action of $G = \text{Aut}(\mathfrak{l}, \theta_\mathfrak{l}) \times \text{Aut}(\mathfrak{a})$ on

$$H := \coprod_{\rho \in \text{Hom}((\mathfrak{l}, \theta_\mathfrak{l}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}} \mathcal{H}_Q^2(\mathfrak{l}, \theta_\mathfrak{l}, \mathfrak{a}_\rho)_0$$

for these combinations of $(\mathfrak{l}, \theta_\mathfrak{l})$ and \mathfrak{a} .

We begin with $\mathfrak{l} \in \{\mathfrak{n}(2), \mathfrak{r}_{3,-1}\}$. We fix an arbitrary pseudo-Euclidean space \mathfrak{a} with involution $\theta_{\mathfrak{a}}$ such that \mathfrak{a}_- is positive definite, i.e. $\mathfrak{a} = \mathfrak{a}_+^{\bar{p},\bar{q}} \oplus \mathfrak{a}_-^{0,\bar{s}}$. Here we use the description of the orbit space as

$$\coprod_{[\rho] \in \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}/G} \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_\rho)_0/G_\rho.$$

From Propositions 7.4 and 7.6 we know that

$$\begin{aligned} \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}/G &= \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}/(\text{Aut}(\mathfrak{a}) \times \mathbb{Z}_2) \\ &= \text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}/\text{Aut}(\mathfrak{a}) \end{aligned}$$

and by Prop. 7.1 we can identify $\text{Hom}((\mathfrak{l}, \theta_{\mathfrak{l}}), \mathfrak{so}(\mathfrak{a}))_{\text{ss}}/\text{Aut}(\mathfrak{a})$ with

$$\{ (\rho_0)_{0,s_0}^{p_0,q_0} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+ \mid \lambda \in \Lambda_p, \mu \in \Lambda_q, p_0 + p = \bar{p}, q_0 + q = \bar{q}, p + q + s_0 = \bar{s} \}.$$

Furthermore, we identify $\lambda \in (\mathfrak{l}_0^*)^p$ with $\lambda(X) \in \mathbb{R}^p$ and therefore Λ_p with

$$\{ \lambda = (\lambda^1, \dots, \lambda^p) \in \mathbb{R}^p \mid 0 < \lambda^1 \leq \dots \leq \lambda^p \}.$$

Now let $\rho = (\rho_0)_{0,s_0}^{p_0,q_0} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+$ be fixed. Recall that we know $\mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_\rho)_0$ (Prop. 7.3) and $\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}})$ (Prop. 7.4). Consider $\alpha_1, \alpha_2 \in Z_{\mathfrak{l}}$. Then $[\alpha_1, 0]$ and $[\alpha_2, 0]$ are on the same G_ρ -orbit if and only if there exists a real number $a > 0$ such that the conditions

$$\alpha_1(Y, Z) = \pm a \alpha_2(Y, Z)$$

and

$$\begin{aligned} \langle \alpha_1(X, Y), \alpha_1(X, Y) \rangle &= a \langle \alpha_2(X, Y), \alpha_2(X, Y) \rangle \quad \text{if } \mathfrak{l} = \mathfrak{n}(2) \\ \langle \alpha_1(X, Y), \alpha_1(X, Z) \rangle &= a \langle \alpha_2(X, Y), \alpha_2(X, Z) \rangle \quad \text{if } \mathfrak{l} = \mathfrak{r}_{3,-1} \end{aligned}$$

are satisfied. Furthermore, $(C^3(\mathfrak{l}) \setminus 0)/G_\rho = (C^3(\mathfrak{l}) \setminus 0)/\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}) = \mathbb{Z}_2$. This yields the assertion for $\mathfrak{l} \in \{\mathfrak{n}(2), \mathfrak{r}_{3,-1}\}$.

Now we consider $\mathfrak{l} = \mathfrak{h}(1)$. We define subsets $H_i^{p,q} \subset H$, $i = 1, 2$, $p, q \geq 0$ by

$$H_i^{p,q} := \{ [\alpha_i, 0] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a}_\rho)_0 \mid \rho = (\rho_0)_{0,i}^{0,0} \oplus \tilde{\rho}_\lambda^+ \oplus \rho_\mu^+, \lambda \in (\mathfrak{l}_0^* \setminus 0)^p, \mu \in (\mathfrak{l}_0^* \setminus 0)^q \},$$

where α_1 is given by $\alpha_1(X, Z) = A_0$, $\alpha_1(X, Y) = \alpha_1(Y, Z) = 0$ and α_2 is given by $\alpha_2(X, Z) = A_1$, $\alpha_2(Y, Z) = A_2$, $\alpha_2(X, Y) = 0$ for standard bases A_0 and A_1, A_2 of $\mathfrak{a}_\rho^{\mathfrak{l}}$, respectively. Note that $\alpha_1, \alpha_2 \in Z_{\mathfrak{l}}$ for all such ρ . By our description of $\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}})$ (Prop. 7.6) we know that for each orbit \mathcal{O} of the G -action on H the intersection $\mathcal{O} \cap H_i^{p,q}$ is non-empty for exactly one triple (i, p, q) . Moreover, $\mathcal{O} \cap H_i^{p,q}$ is a $G_i \times \tilde{\mathfrak{S}}_{p,q}$ -orbit in $H_i^{p,q}$, where $G_i = (\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}}) \times \text{Aut}(\mathfrak{a}^{\mathfrak{l}}))_{\alpha_i}$ is the stabilizer of $\alpha_i \in C^2(\mathfrak{l}, \mathfrak{a}^{\mathfrak{l}})$. Using again our description of $\text{Aut}(\mathfrak{l}, \theta_{\mathfrak{l}})$ from Prop. 7.6 it is not hard to compute G_i and then $H_i^{p,q}/G_i$.

In the cases $\mathfrak{l} = \mathbb{R}^1$ and $\mathfrak{l} = \mathbb{R}^2$ we proceed as in the case of $\mathfrak{l} = \mathfrak{h}(1)$.

The result for $\mathfrak{l} \in \{\mathfrak{su}(2), \mathfrak{sl}(2, \mathbb{R})\}$ is a straightforward consequence of Lemma 7.2 and Lemma 7.1. Note that in this case any automorphism of \mathfrak{l} acts trivially on $C^3(\mathfrak{l})$. \square

Recall that a *pseudo-Hermitian symmetric space* is a pseudo-Riemannian symmetric space (M, g) equipped with an almost complex structure J which is compatible with g and the involutions θ_x , $x \in M$. Then J is automatically integrable and (M, g, J) is a Kähler manifold (see [CP 2], Ch.I,§6).

Corollary 7.1 *If $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is a symmetric triple associated with a simply connected indecomposable pseudo-Hermitian symmetric space of complex signature $(1, q)$, $q \geq 0$, which is neither semi-simple nor flat, then $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ is isomorphic to $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ for exactly one of the data in the following list (which contains only data giving rise to such triples):*

1. $q = 1 : \mathfrak{l} = \mathfrak{l}_- = \mathbb{R}^2 = \text{span}\{Y, Z\}$,

- (a) $\mathfrak{a} = \mathfrak{a}_+^{0,1}$, $\rho = (\rho_0)_{0,0}^{0,1}$, $\alpha(Y, Z) = A_0$, where A_0 is the standard basis of $\mathfrak{a}_+^{0,1}$, $\gamma = 0$;

- (b) $\mathfrak{a} = \mathfrak{a}_+^{1,0}$, $\rho = (\rho_0)_{0,0}^{1,0}$, $\alpha(Y, Z) = A_0$, where A_0 is the standard basis of $\mathfrak{a}_+^{1,0}$, $\gamma = 0$;

2. $q = 2 : \mathfrak{l} = \mathfrak{h}(1)$, $\mathfrak{l}_+ = \mathbb{R} \cdot Z$, $\mathfrak{l}_- = \text{span}\{X, Y\}$

$$\mathfrak{a} = \mathfrak{a}_-^{0,2}, \rho = (\rho_0)_{0,2}^{0,0}, \alpha(X, Z) = A_1, \alpha(Y, Z) = A_2, \alpha(X, Y) = 0,$$

where A_1, A_2 is the standard basis of $\mathfrak{a}_-^{0,2}$,
 $\gamma = 0$;

3. $q = 1 + p$, $p \geq 0 : \mathfrak{l} = \mathfrak{su}(2)$, $\mathfrak{l}_+ = \mathbb{R} \cdot H$, $\mathfrak{l}_- = \text{span}\{X, Y\}$,

$$\mathfrak{a} = \mathfrak{a}_+^{0,2p-r} \oplus \mathfrak{a}_-^{0,2p}, \quad 0 \leq r \leq p, \quad \rho = \bigoplus_{i=1}^r \rho_1^- \oplus \bigoplus_{i=1}^{p-r} \rho_1',$$

$$\alpha = 0, \quad \gamma(H, X, Y) = c, \quad c \in \mathbb{R}.$$

4. $q = 1 + p$, $p \geq 0 : \mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{l}_+ = \mathbb{R} \cdot (X - Y)$, $\mathfrak{l}_- = \text{span}\{H, X + Y\}$.

$$\mathfrak{a} = \mathfrak{a}_+^{2p-r,0} \oplus \mathfrak{a}_-^{0,2p}, \quad 0 \leq r \leq p, \quad \rho = \bigoplus_{i=1}^r \rho_1^- \oplus \bigoplus_{i=1}^{p-r} \rho_1',$$

$$\alpha = 0, \quad \gamma(H, X, Y) = c, \quad c \in \mathbb{R}.$$

Proof. A symmetric triple $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle)$ associated with a simply connected non-flat indecomposable pseudo-Hermitian symmetric space M is indecomposable ([CP 2], Ch.I, Prop. 6.6). Moreover, it carries an automorphism F satisfying $F^2 = \theta$, $F|_{\mathfrak{g}_+} = \text{Id}$, and any such automorphism induces a pseudo-Hermitian structure J on M ([CP 2], Ch.I, Prop. 6.4). If $(\mathfrak{g}, \theta, \langle \cdot, \cdot \rangle) = \mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ for $[\alpha, \gamma] \in \mathcal{H}_Q^2(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})_{\sharp}$, then the existence of such an automorphism F is equivalent to the existence of an automorphism $\bar{F} = (F_{\mathfrak{l}}, F_{\mathfrak{a}}^{-1})$ of the triple $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ satisfying

$$F_{\mathfrak{l}}^2 = \theta_{\mathfrak{l}}, \quad F_{\mathfrak{l}}|_{\mathfrak{l}_+} = \text{Id}, \quad F_{\mathfrak{a}}^2 = \theta_{\mathfrak{a}}, \quad F|_{\mathfrak{a}_+} = \text{Id} \quad (21)$$

such that $(\bar{F}^* \alpha, \bar{F}^* \gamma) = (\alpha, \gamma)$. The corollary now follows by inspection of the classification list in Theorem 7.1 provided one takes the following fact into account:

Let (ρ, \mathfrak{a}) be a semi-simple orthogonal $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -module such that the triple $(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ admits an automorphism \bar{F} satisfying (21).

- (i) ([CP 2], Ch.V, Prop.3.3) If $(\mathfrak{l}, \theta_{\mathfrak{l}})$ is as in cases 6. or 8. of Theorem 7.1 and \mathfrak{a}_+ is positive definite, then ρ is the direct sum of irreducible $(\mathfrak{l}, \theta_{\mathfrak{l}})$ -modules equivalent to $\rho_{\mathfrak{l}^-}$ or $\rho'_{\mathfrak{l}^-}$. All these modules admit an automorphism $F_{\mathfrak{a}}$ with the required properties.
- (ii) If \mathfrak{l} is solvable and satisfies (T_1) , then ρ is the trivial representation on \mathfrak{a} .

It remains to prove (ii). Let $\lambda \in \mathfrak{l}_0^* \setminus 0$ and let $E_{\lambda} \subset \mathfrak{a}_{\mathbb{C}}$ be the corresponding weight space. We have to show that $E_{\lambda} = \{0\}$. The automorphism $F_{\mathfrak{l}}$ induces a complex structure j on the real vector space \mathfrak{l}_0^* . The elements $j^k(\lambda)$, $k = 0, 1, 2, 3$, are pairwise different. Therefore the sum of the weight spaces $E_{j^k(\lambda)}$, $k = 0, 1, 2, 3$, is direct. Take $v \in E_{\lambda}$. Then $F_{\mathfrak{a}}^k(v) \in E_{j^k(\lambda)}$, and $v^- := v - F_{\mathfrak{a}}(v) + F_{\mathfrak{a}}^2(v) - F_{\mathfrak{a}}^3(v)$ satisfies $F_{\mathfrak{a}}(v^-) = -v^-$. However, the only eigenvalues of $F_{\mathfrak{a}}$ on $\mathfrak{a}_{\mathbb{C}}$ are 1, i and $-i$. We conclude that $v^- = 0$, hence $v = 0$. This finishes the proof of (ii). \square

Note that Assertion (ii) of the above proof has strong implications for the structure of arbitrary pseudo-Hermitian symmetric spaces with solvable transvection group.

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