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für Mathematik
in den Naturwissenschaften
Leipzig

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(revised version: September 2006)

by

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Preprint no.: 61 2004

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Dedicated to Anatoly Fomenko on the occasion of his 60th birthday.

Abstract

In this note we prove that a positive multiple of each even-dimensional integral homology class of a compact symplectic manifold (M^{2n}, ω) can be represented as the difference of the fundamental classes of two symplectic submanifolds in (M^{2n}, ω). We also prove the realizability of some integral homology classes by symplectic submanifolds in (M^{2n}, ω).

1 Introduction.

In 1954 Rene Thom prove the following celebrated theorem which relates the topological structure with the differentiable structure on compact manifolds.

1.1. Theorem. [Thom1954, Theorem II.25]. *For each element $\alpha \in H_k(M^m, \mathbb{Z})$ of a compact differentiable manifold M^m there exists a positive number $N(k, m)$ such that the element $N(k, m) \cdot \alpha$ can be realized by a differentiable submanifold in M^m.*

Thom's theorem is optimal in the sense that we cannot replace $N(k, m) = 1$. Namely Thom showed that for each $k \geq 7$ there is a compact differentiable manifold M^m and an element $\alpha \in H_k(M^m, \mathbb{Z})$

such that α cannot be realized by the fundamental class of a submanifold in M^m [Thom1954].

As an immediate consequence of Thom's theorem 1.1 we get that the homology group $H_*(M^m, \mathbb{Q})$ is generated by the fundamental classes of differentiable submanifolds of M^m.

In this note we prove the following weak version of Thom's theorem for compact symplectic manifolds.

1.2. Theorem. *Suppose that (M^{2n}, ω) is a compact symplectic manifold. Then for each element $\alpha \in H_{2k}(M^{2n}, \mathbb{Z})$, $1 \le k \le n$, there exists a positive number $N(\alpha) \in \mathbb{N}^+$ such that $N(\alpha) \cdot \alpha = [S_1^{2k}] - [S_2^{2k}]$, where S_1^{2k} and S_2^{2k} are symplectic submanifolds in (M^{2n}, ω) .*

A particular case of Theorem 1.2 is

1.3. Theorem. [Donaldson1996] *Let (M^{2n}, ω) be a compact integral symplectic manifold, i.e. the cohomology class $[\omega]$ belongs to $H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})$. Then for each $1 \le k \le n$ there exists a number $N_1(k) \in \mathbb{N}^+$ such that $N \cdot PD([\omega^{n-k}])$ can be realized by a symplectic submanifold S_1^{2k} in (M^{2n}, ω), if $N \ge N_1(k)$.*

Actually in his paper [Donaldson1996] Donaldson has stated his result as in Theorem 1.3 only for $k = n - 1$. Kaoru Ono has noticed that using a simple argument we can easily get Theorem 1.3 for $1 \le k \le n - 2$ as a consequence of [Donaldson1996, Theorems 3, 5]. We shall represent Ono's argument in section 2. Theorem 1.3 also follows from a result of Auroux [Auroux2000] (see Proposition 2.4 below) which generalizes Donaldson's arguments.

We shall call a homology class $\alpha \in H_{2k}(M, \mathbb{Z})$ **symplectic**, if α is the fundamental class of a symplectic submanifold in M^{2n}. We shall call a homology class $\alpha \in H_{2k}(M, \mathbb{Z})$ **formal symplectic**, if the pairing $< \alpha, \omega^k >$ is positive.

1.4. Conjecture. *For each formal symplectic class $\alpha \in H_{2k}(M^{2n}, \mathbb{Z})$ there exists a positive number $N_1(\alpha) \in \mathbb{N}^+$ such that $N_1(\alpha) \cdot \alpha$ is symplectic.*

For $k = 1$ and $n \ge 3$ the conjecture 1.4 is true. More precisely we have

1.5. Theorem. *Let (M^{2n}, ω) be a compact symplectic manifold with $2n \ge 6$.*

a) If $\omega_{\pi_2(M^{2n})} \neq 0$, then there exists a symplectic sphere in (M^{2n}, ω).
b) The conjecture 1.4 is true for $\alpha \in H_2(M^{2n}, \mathbb{Z})$.

I am deeply thankful to Simon Donaldson and Jürgen Jost for their interests and supports. I am greatly indebted to Kaoru Ono for his quick help and criticism and I thank Dietmar Salamon for pointing out an error in the early version of this note. It is my pleasure to dedicate this note to my teacher Anatoly Fomenko, for he introduced me into algebraic topology and symplectic geometry. He gave me the book [Rass1958] from his library and urged me to study the classical works of the French topology school in the last century, and so he influenced me over this work directly.

2 Proof of Theorem 1.2.

Clearly Theorem 1.2 is a consequence of the following

2.1. Theorem. *Let (M^{2n}, ω) be a compact symplectic manifold and $\alpha \in H_{2k}(M^{2n}, \mathbb{Z})$. Then there exist an integral symplectic form $\bar{\omega}$ on M^{2n} and positive integral numbers $N_2(\alpha)$ and $N_3(\alpha)$ such that $[N_2(\alpha) \cdot \alpha + N_3(\alpha) PD[\bar{\omega}^{n-k}]]$ and $N_3(\alpha) \cdot PD[\bar{\omega}^{n-k}]$ can be realized by symplectic submanifolds, respectively S_i^{2k} of (M^{2n}, ω), $i = 1, 2$.*

In this section we shall give a proof of Theorem 2.1. First we prove Theorem 2.1 for compact integral symplectic manifolds (M^{2n}, ω) (and $\bar{\omega} = \omega$ for this case) with certain estimates for obtained symplectic submanifolds. Next for an arbitrary symplectic form ω we shall find a rational symplectic form $\bar{\omega}$ close enough to ω such that the obtained $\bar{\omega}$ -symplectic submanifolds are also ω-symplectic submanifolds.

Our needed estimates for symplectic submanfolds are expressed in terms of η-transversality and C^k-approximately (resp. C^k-asymptotically) homolomorphic maps. These notions have been first introduced by Donaldson for mappings from an almost complex manifold (M^{2n}, J) with a Riemannian metric g to $\mathbb{C}$ [Donaldson1996] and later extended by Auroux for mappings from (M^{2n}, J) with a Riemannian metric g to $\mathbb{C}^k$ (see e.g. [Auroux2000]).

2.2. Definition. [Donaldson1996], [Auroux2000]. Given a constant $\eta > 0$, we say that a section s of a vector bundle carrying a metric and a connection is **η-transverse to** 0, if at every point x

such that $|s(x)| \leq \eta$ the covariant derivative $\nabla s(x)$ is surjective and admits a right inverse of norm less than η^{-1}.

2.3. Definition. [Donaldson1996], [Auroux2000, Definition3.1]. Let (M, J) be an almost-complex manifold with a Riemannian metric and let s be a section of an almost complex vector bundle with metric and connection. Given two constants C and c we say that s is **C^2-approximately holomorphic with bounds** $\{(C, c)\}$, if it satisfies the following estimates for all $1 \leq r \leq k$

$$|s| + |\nabla s| + |\nabla^2 s| \leq C, \ |\bar{\partial} s| + |\nabla \bar{\partial} s| \leq C \cdot c^{-1/2},$$

where $\bar{\partial}$ is the $(0, 1)$-part of the connection.

Moreover, given constants $c_k \to \infty$, we say that a sequence $\{s_k\}$ of sections is **C^2-asymptotically holomorphic**, if there exists a fixed constant C such that each section c_k, $k >> 0$, is C^2-approximately holomorphic with bounds $\{(C, c_k)\}$.

Our proof of Theorem 2.1 is based on the following result by Auroux.

2.4. Proposition. [Corollary 5.1, Auroux2000]. *Let (M^{2n}, ω) be a compact symplectic manifold endowed with an ω-tame almost-complex structure J, and let E_k be an asymptotically very ample sequence of locally splittable vector bundles over (M^{2n}, J). Then, for all large enough values of k there exist asymptotically holomorphic sections s_k of E_k which are uniformly transverse to 0 and whose zero sets are smooth symplectic submanifolds in M^{2n}.*

Recall that a sequence of complex vector bundles E_k with metrics and connections over a compact almost-Hermitian manifold (M, g, J) is **asymptotically very ample**, if there exist constants δ, $\{C_r\}_{,r\geq 0}$, and a sequence $c_k \to +\infty$ such that the curvature F_k of E_k satisfies the following properties
1) $< iF_k(v, Jv) \cdot u, u > \geq c_k g(v, v)|u|^2, \forall v \in TX, \forall u \in E_k$,
2) $\sup |F_k^{0,2}|_g \leq \delta_r c_k^{1/2}$,
3)$\sup |\nabla^r F_k|_g \leq C_r c_k$, $\forall r \geq 0$.

A sequence of asymptotically very ample complex vector bundles E_k with metrics $|.|_k$ and connections ∇_k is **locally splittable**, if given any point $x \in X$, there exists over a ball of of fixed g-radius around x a decomposition of E_k as a direct sum $L_{k,1} \oplus \cdots \oplus L_{k,m}$ of line bundles such that the following properties hold:
1) the $|.|_k$-determinant of a local frame consisting of unit length local

sections of $L_{k,1}, \cdots L_{k,m}$ is bounded from below by a fixed constant independently of x and k,
2)denoting by $\nabla_{k,i}$ the connection on $L_{k,i}$ obtained by projecting $\nabla_{k|L_{k,i}}$ to $L_{k,i}$ and by ∇'_k the direct sum of the $\nabla_{k,i}$, the 1-form $\alpha_k = \nabla_k - \nabla'_k$ (the non-diagonal part of ∇_k) satisifies the uniform bounds $|\nabla^r \alpha_k|_g = O(c_k^{r/2})$ for all $r \geq 0$ independently of x.

2.5. Remark-Example. [Auroux2000]. If E is a fixed complex vector bundle and L_k are asymptotically very ample line bundles, then the vector bundles $E \otimes L_k$ are locally splittable and asymptotically very ample, so are direct sums of vector bundles of this type.

Proposition 2.4 is a generalization of Theorem 5 in [Donaldson1996] which is a special case of Proposition 2.4 for $E^1 = L_\omega^{\otimes k}$. Here L_ω denotes the complex line bundle over (M^{2n}, ω) whose curvature is equal to ω.

Our proof of Theorem 2.1 also uses the following Proposition due to Thom [Thom1954]. Let G be a subgroup of $O(k)$ and M^n be a differentiable manifold. We say that a cohomology class $\alpha \in H_k(M^n, \mathbb{Z})$ is realizable w.r.t. G, if there is a G-vector bundle V^k of dimension k over M^n whose Euler class $e(V^k)$ is equal to α (see [Thom1954]).

2.6. Proposition. [Thom1954 Theorem II.25]. *For each cohomology class $z \in H^k(M^n, \mathbb{Z})$ there exists a number $N(k,n) \in \mathbb{N}^+$ such that the class $N(k,n) \cdot z$ is realizable w.r.t. $SO(k)$. If $k = 2l$, then there exists a number $N_1(k,n) \geq N(k,n)$ such that the class $N_1(k,n) \cdot z$ is realizable w.r.t. $U(l)$.*

Actually the proof of Thom in [Thom1954] brings more information on the vector bundle associated with $N(k,n) \cdot z$. We have namely the following

2.7. Proposition. *For each cohomology class $\alpha \in H^{2k}(M^n, \mathbb{Z})$ there exists a number $N_2(k,n) \in \mathbb{N}^+$ and a complex vector bundle E^k over M^n such that*

$$c_k(E^k) = N_2(k,n) \cdot \alpha,$$
$$c_i(E^k) = 0, \text{ for all } 1 \leq i \leq k-1.$$

Proof. [1] Let $K(\mathbb{Z}, 2k)$ be the Eilenberg-MacLane space and $K(\mathbb{Z}, 2k)^q$ be the q-dimensional skeleton of $K(\mathbb{Z}, 2k)$. We denote by f a map from

[1] for a new full proof of Proposition 2.7 see "Weak equivalence classes of complex vector bundles", arXiv:math.DG/0609074

M^n to $K(\mathbb{Z}, 2k)^{n+1}$ such that $f^*(\tau) = \alpha$, where $\tau \in H^{2k}(K(\mathbb{Z}, 2k), \mathbb{Z})$ is the fundamental class of $K(\mathbb{Z}, 2k)$. Let $G_{\mathbb{C}}(k)$ denote the classifying space of the group $U(k)$ and $c_k \in H^{2k}(G_{\mathbb{C}}(k), \mathbb{Z})$ be the top Chern class of the universal complex vector bundle V^k over $G_{\mathbb{C}}(k)$. In the proof of [Thom1954 Theorem II.25] Thom has showed that there is a map

$$g : K(\mathbb{Z}, 2k)^{n+1} \to Gr_{\mathbb{C}}(k)$$

such that $g^*(c_k(V^k)) = N_2(k, n) \cdot \tau$ for some positive number $N_2(k, n)$. Clearly the complex vector bundle $E^k = f^*(g^*(V^k))$ satisfies the condition

$$c_k(E^k) = N_2(k, n)\alpha.$$

Since the cohomology group $H^i(K(\mathbb{Z}, 2k), \mathbb{Q})$ is zero for all $i \leq 2k - 1$, (namely $H^*(K(\mathbb{Z}, 2k), \mathbb{Q}) = \mathbb{Q}[x], \dim x = 2k$, see e.g. [F-F1989 Chapter III, 25.2]), taking into account the equality

$$H^i(K(\mathbb{Z}, 2k)^{n+1}, \mathbb{Q}) = H^i(K(\mathbb{Z}, 2k), \mathbb{Q}),\ \forall i \leq n$$

it follows that

$$(c_i(f^*(V^k)) = 0 \text{ for all } 1 \leq i \leq k - 1.$$

Now it is easy to see that E^k satisfies the condition of Proposition 2.7. $\square$

Proof of Theorem 2.1. First we shall prove Theorem 2.1 for a compact integral symplectic manifold (M^{2n}, ω). Given an element $PD(\alpha) \in H^{2(n-k)}(M^{2n}, \mathbb{Z})$ we choose a complex vector bundle E^{n-k} and a number $N_2(n-k, 2n)$ satisfying the condition of Proposition 2.7. The following Lemma is an immediate consequence of Proposition 2.4 and Remark 2.5.

2.8. Lemma. *There exists a number $N_3(E^{n-k})$ such that for any $N \geq N_3(E^{n-k})$ there are asymptotically holomorphic sections of $E^{n-k} \otimes L^{\otimes N}$ which are uniformly transversal to* 0 *and whose zero set are smooth symplectic submanifolds in (M^{2n}, ω).*

Now we use the following well-known formula for the Chern classes of tensor products of complex vector bundles (see e.g. [Hartshorne1977, Appendix A.3]). Denote by $c_t(E^r)$ the Chern polynomial of a complex vector bundle E^r over X: $c_t(E^r) = c_0(E) + c_1(E)t + \cdots + c_r(E^r)t^r$. Using the Grothendieck splitting principle we write

$$c_t(E^r) = \Pi_{i=1}^r (1 + a_i t).$$

Let F^s be another complex vector bundle over X with

$$c_t(E^s) = \Pi_{j=1}^s(1 + b_j t).$$

Then we have

$$c_t(E^r \otimes F^s) = \Pi_{i,j}(1 + (a_i + b_j)t). \tag{2.9}$$

It follows from (2.9) that for the chosen E^{n-k} satisfying Proposition 2.7 the top Chern class

$$c_{n-k}(E^{n-k} \otimes L^{N_3}) = N_2(n-k,n) \cdot PD(\alpha) + N_3(\alpha)[\omega^{n-k}]. \tag{2.10}$$

Combining (2.10) with Lemma 2.8 we get Theorem 2.1 for compact integral symplectic manifold (M^{2n}, ω).

Proof of Theorem 2.1 for a general compact symplectic (M, ω). We need the following perturbation result.

2.9. Lemma. *Suppose that (M^{2n}, ω, J) is a compact symplectic manifold with a compatible almost complex structure J. Then for any given positive numbers $C > 0$, $\eta > 0$ there exist a positive number $\delta > 0$ and an integral symplectic form $\bar\omega$ together with its compatible almost complex structure $\bar J$ over M^{2n} such that the following statement holds. Let E^k be a complex vector bundle over $(M, \bar\omega, \bar J)$ with metric and connection and s be a section of $(M, g_{\bar J})$ to E^k which is η-transversal to 0 and which is C^2-approximately holomorphic with bounds (C, δ). Then the zero section $s^{-1}(0)$ is a symplectic submanifold w.r.t. both symplectic structures ω and $\bar\omega$.*

Proof. For any symplectic manifold (M, ω, J) with a compatible almost complex structure we denote by $G^s_{2k}(M, \omega)$ the Grassmanian of ω-symplectic $2k$-planes in T_*M and by $G^J_{2k}(M, J)$ the Grassmanian of J-invariant $2k$-planes in T_*M. Clearly $G^s_{2k}(M, \omega)$ is an open neighborhood of $G^J_{2k}(M, J)$. Hence, if M is compact and ω and J are given, there exists a positive number $\varepsilon > 0$ such that if $\rho^v(V^{2k}, G^J_{2k}(M, \omega)) < \varepsilon$, then V^{2k} is a symplectic plane. Here we identify V^{2k} with a point in $G_{2k}(M)$ with ρ^v being the induced metric on each fiber $G_{2k}(T_xM), x \in M$. Using this open property we can find a rational symplectic form ω_1 in a C^2-small neighborhood of a given symplectic form ω with a compatible almost complex structure J_1 which is C^2-close to J such that the following property holds. If V is a point in $G_{2k}(T_xM^{2n})$ and $\rho^v_1(V, G^{J_1}_{2k}(T_xM^{2n})) < (\varepsilon/2)$, then V is symplectic w.r.t. both the symplectic forms ω_1 and ω.

With this at hand it is easy to come to the conclusion of Lemma 2.9. Namely we take $\bar{\omega}$ to be a multiple of ω_1. Then the compatible almost complex structure J_1 is also compatible to $\bar{\omega}$. So we put $\bar{J} := J_1$. Next we note that the metrics g_1 and $\bar{g}$ associated respectively to (ω_1, J_1) and $(\bar{\omega}, \bar{J})$ induce the same fiber metric g_1^v on each $G_{2k}(T_xM)$, and any ω_1-symplectic plane (resp. J^1-invariant plane) is also $\bar{\omega}$-symplectic (resp. $\bar{J}$-invariant plane) and conversely.

Now given positive numbers η, C and ε we shall choose $\delta > 0$ so that from the η-transversality and C^2-approximate holomorphy of a section s from M^{2n} to a complex vector bundle E with bounds (C, δ) we shall get the estimate for tangential space of $s^{-1}(0)$:

$$\rho_1^v(T_x s^{-1}(0), G_{2n-2k}^{J_1}(T_x M^{2n})) < \frac{\varepsilon}{2}. \tag{2.10}$$

This is possible since the Nijenhuis tensor which measures non-holomorphy of $s^{-1}(0)$ depends on the ratio of $\bar{\partial}s$ and ∂s up to the second order of s (see also [Auroux2000]). By our choice of $\bar{\omega}$ and $\bar{J}$ the equality (2.10) also implies that $s^{-1}(0)$ is both ω-symplectic and $\bar{\omega}$-symplectic. $\square$

Completion of the proof of Theorem 2.1. Now given a compact symplectic manifold (M, ω) with a compatible almost complex structure J and $\alpha \in H_2(M, \mathbb{Z})$ we shall choose $\bar{\omega}$ and $\bar{J}$ as in Lemma 9. Let F^{n-k} be a complex vector bundle satisfies the condition of Lemma 2.7 and $E_p = F^{n-k} \otimes L_{\bar{\omega}}^{\otimes p}$ be the sequence of asymptotically very ample complex vector bundles over $(M, \bar{\omega}, \bar{J})$. Here $L_{\bar{\omega}}$ is the associated complex line bundle over $(M, \bar{\omega}, g_{\bar{J}})$. Let s_p be sequence of asymptotically holomorphic sections of E_p from Proposition 2.4. By definition, for any given $\delta > 0$ there exists a number $K >> 0$ such that if $p \geq K$ then s_p is C^2-approximately holomorphic with bounds $\{(C, \delta), (C, \delta)\}$. It follows from Lemma 2.9 that the zero sections $s^{-1}(0)$ is also symplectic w.r.t. the symplectic form ω. $\square$

Another elementary proof of Theorem 1.3 for $1 \leq k \leq n-2$. [Ono2004]. Clearly Theorem 1.3 is a consequence of the following statement. There are positive integral numbers $n_1, \cdots, n_k$ and for each $i = \overline{1, k}$ a section s_i of the line bundle $L_\omega^{n_i}$ such that the section $\hat{s} := s_1 \oplus \cdots \oplus s_k$ intersects to the zero section of $\hat{L} := L_\omega^{N_1} \oplus \cdots \oplus L_\omega^{N_k}$ transversally and $\hat{s}^{-1}(0)$ is a symplectic submanifold in (M^{2n}, ω), if $N_i \geq n_i$ for $1 \leq i \leq k$. (Obviously this statement follows from Proposition 2.4 and Remark 2.5.)

Ono has noticed that Theorem 1.3 is also a consequence of Theorem 1 in [Donaldson1996]. Indeed, it suffices to prove the above statement inductively on k. For $k = 1$ the statement is exactly Theorem 1 in [Donaldson1996]. Assume that the above statement is valid for $k = K$. We shall prove its validity for $k = K+1$. We denote by S_K the common zero locus of sections $s_1, \cdots, s_K$ which is by the induction assumption a symplectic submanifold. We note that the restriction of the line bundle L_ω to S_K has the curvature $\omega_{|S_K}$ which is the symplectic form on S_K. According to Donaldson [Donaldson1996, Theorem 5 and Proposition 3] there exists a number N_0 such that if $n_{K+1} > N_0$ there is a section

$$\tilde{s}_{K+1} : S_K \to L_\omega^{n_{K+1}}$$

such that $\tilde{s}_{K+1}$ intersects with the zero section of $L_\omega^{n_{K+1}}$ transversally. Moreover the zero section $N := \tilde{s}_{K+1}^{-1}(0)$ is a symplectic submanifold in S_K.

Since the fiber $L_\omega^{n_{K+1}}$ is contractible, the section $\tilde{s}_{K+1}$ can be extended to a section $s_{K+1} : M^{2n} \to L_\omega^{n_{K+1}}$. The proof of the above statement is complete, if we can show that s_{K+1} can be chosen to be transversal to the zero section of a neighborhood $U_\varepsilon(S_K) \subset M^{2n}$. It is easy to see that the extension of the section from the submanifold to the ambient space is just done by pull back the section by the projection of the normal bundle of the symplectic submanifold. More explicit we choose $U_\varepsilon(S_K)$ to be a (geodesic) neighborhood of S_K in M^{2n}, such that there is a diffeomorphism f from $U_\varepsilon(S_K)$ to a neighborhood $V_\varepsilon(S_K)$ of the zero section of the normal bundle $V(S_K)$ of S_K in M. By using this diffeomorphism f we can work now on $V_\varepsilon(S_K)$. It is easy to see (see e.g. [Thom1954]) that

$$V(S_K) = (L_\omega^{n_1} \oplus \cdots \oplus L_\omega^{n_K})_{|S_K}. \tag{2.11}$$

Then we let the extension of $\tilde{s}_{K+1} : S_K \to L^{n_{K+1}}$ to a section $\bar{s}_{K+1} : V_\varepsilon(S_K) \to (f^{-1})^* L^{n_{K+1}}$ be defined by

(2.12). $$\bar{s}_{K+1}(x, l_1, \cdots, l_K) = (x, l_1, \ldots, l_K, \tilde{s}_{K+1}(x))$$

Finally we extend $\bar{s}_{K+1}$ to a section s_{K+1} over M^{2n}. This section s_{K+1} satisfies the condition of the statement at the beginning of the (elementary) proof of Theorem 1.3. □

3 Proof of Theorem 1.5.

To prove the part (a) of Theorem 1.5 we need the following

3.1. Lemma. *Suppose that the condition a of Theorem 1.5 is satisfied. Then there is an embedding $f : S^2 \to M$ such that $f^*([\omega]) = k[\omega_0]$ for some $k > 0$. Here we denote by ω_0 the standard symplectic form on S^2.*

Proof. Let f be a smooth map such that $< [\omega], [f(S^2)] >= k > 0$. Since the space of immersions from S^2 to $M^{2n}, n \geq 3$, is dense in the space of smooth maps [Hirsch1959] we can assume that f is an immersion. We can also assume that the intersection point is generic. Using an isotopy and the dimension condition we move the second intersection point from the first one. □

Proof of Thereom 1.5.a. First we find a map covering $F : T_*S^2 \to T_*M^{2m}$ which is fiber-wise symplectic. The existence of such a map is equivalent to the existence of a section of the bundle $Iso_{sym}(T_*S^2, T_*M^{2m})$ over S^2. The fiber of this bundle is $Symp(2m)/Symp(2m-2)$ which is $2m-1$ connected [Gromov1986]. So there is no obstruction for a section. Now Theorem 1.5.a follows immediately from the h-principle of Gromov and the observation, that we can make a C^1-perturbation of a symplectic immersion of S^2 to get a symplectic embedding, since the dimension of M^{2n} is at least 6.

3.2. H-principle for symplectic immersions. [Gromov1986, 3.4.2.A] *Let g be an arbitrary (possibly singular) closed smooth 2-form on a smooth manifold V and let $F_0 : (T_*V, g) \to (T_*M, \omega)$ be a fiber-wise injective isometric homomorphism for which $(\pi F)^*[\omega] = [g]$. Let us denote by i the embedding $V \to T^*V$ as the zero section, and by π the projection $T^*M \to M$. If $\dim V < \dim M$, then the map $\pi \circ F \circ i : V \to M^{2n}$ admits a fine C^0- approximation by isometric smooth immersions $f : (V, g) \to (M^{2n}, \omega)$ whose differentials $Df : T_*V \to T_*M$ are homotopic to F_0 in the space of fiberwise injective isometric homomorphisms.*

To prove (b) we replace Lemma 3.1 by the Thom existence theorem [Thom1954, Theorem II.25]. The rest of our argument repeats the proof of (a). □

We close this note with the following questions. It is easy to see that any symplectic curve Σ in a symplectic manifold (M^{2n}, ω) is a

J-holomorphic curve for some compatible almost complex structure J on (M^{2n}, ω). When can we ensure that the holomorphic curves in Theorem 3 are generic in Gromov's sense? If it is the case, we can obtain the existence of true holomorphic curves representing different homology classes in certain algebraic projective manifolds.

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