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by

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AN EVANS FUNCTION APPROACH TO SPECTRAL STABILITY OF SMALL-AMPLITUDE SHOCK PROFILES

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ABSTRACT. In recent work, the second author and various collaborators have shown using Evans function/refined semigroup techniques that, under very general circumstances, the problems of determining one- or multi-dimensional nonlinear stability of a smooth shock profile may be reduced to that of determining spectral stability of the corresponding linearized operator about the wave. It is expected that this condition should in general be analytically verifiable in the case of small amplitude profiles, but this has so far been shown only on a case-by-case basis using clever (and difficult to generalize) energy estimates. Here, we describe how the same set of Evans function tools that were used to accomplish the original reduction can be used to show also small-amplitude spectral stability by a direct and readily generalizable procedure. This approach both recovers the results obtained by energy methods, and yields new results not previously obtainable. In particular, we establish one-dimensional stability of small amplitude relaxation profiles, completing the Evans function program set out in Mascia and Zumbrun [MZ.1]. Multidimensional stability of small amplitude viscous profiles will be addressed in a companion paper [PZ], completing the program of Zumbrun [Z.3].

1. Introduction

In this paper, we study one-dimensional spectral stability in the small amplitude limit of smooth shock profiles, i.e., traveling wave solutions

$$u = \bar{u}(x - st), \qquad \lim_{z\to\pm\infty} \bar{u}(z) = u_\pm, \tag{1}$$

arising under various regularizations of a hyperbolic system of conservation laws

$$u_t + f(u)_x = 0 : \tag{2}$$

specifically, *viscous conservation laws* of form

$$u_t + f(u)_x = (B(u)u_x)_x, \tag{3}$$

where Re $\sigma(B) \geq 0$, and *relaxation systems* of form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} \tilde{f}(u,v) \\ \tilde{g}(u,v) \end{pmatrix}_x = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix},$$

where

$$\text{Re } \sigma\big(q_v(u, v^*(u))\big) < 0 \tag{4}$$

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along a smooth equilibrium manifold defined by

$$q(u, v^*(u)) \equiv 0, \tag{5}$$

and $f(u) = \tilde{f}(u, v^*(u))$. Here, $x, t \in \mathbb{R}^1$, $u, f, \tilde{f} \in \mathbb{R}^n$, $v, \tilde{g}, q \in \mathbb{R}^r$, and $B \in \mathbb{R}^{n\times n}$, where typically $n, r > 1$; we make the standard assumptions of strict hyperbolicity of Df, and genuine nonlinearity of the principal characteristic eigenvalue (the one associated with the approximate shock direction), along with further standard hypotheses on B and $\tilde{f}$, $\tilde{g}$, q, to be described later on: in particular, that systems (189) and (1) be dissipative in the sense of Kawashima and Zeng [Kaw, Ze]. These are generically satisfied for most models of physical interest; for discussions of applications, we refer the reader to the general surveys [Z.3, Z.6, MZ.1, N, Yo.4].

A recent development in the stability analysis of smooth shock profiles has been the successful adaptation of Evans function/dynamical system techniques to this nonstandard setting, with an associated explosion of new results; see, e.g., [GZ, H.1, H.2, H.3, ZH, BSZ, ZS, HZ, Z.2, Z.3, Z.4, HoZ.1, HoZ.2, OZ.1, OZ.2, MZ.1, MZ.2]. For the origins of these methods in the setting of reaction diffusion and singular perturbation problems, see [E.1, E.2, E.3, E.4, J, AGJ]; see also the important analyses of [Sat, K.1, K.2, JGK] in the scalar case. In particular, it has now been established under extremely general circumstances that both linearized and nonlinear stability are implied by spectral stability of the linearized operator about the profile, a fact that was left in doubt by earlier "direct" analyses [MN, Go.1, Go.2, KMN, SX.1, SX.2, S, FreL, L.1, L.2, L.3] carried out on a case-by-case basis. This reduces the question of stability to an ODE issue, and raises the hope that at least small amplitude stability can be treated in a uniform way across dimension and type of regularization.

On the other hand, spectral stability has up to now been verified only by energy estimates resembling those of the earliest direct analyses of [MN, Go.1, Go.2, KMN]; see [Z.1, HuZ, Liu, Hu, Z.4] for examples in various contexts. And, these have proven difficult to generalize: in particular, one-dimensional stability of general small amplitude relaxation profiles, and multi-dimensional stability of small amplitude viscous shock profiles remain outstanding open problems in the theory. In this, and a companion paper [PZ] treating the multidimensional case, we remove this troublesome gap in the theory, showing that the same Evans function tools that were used to effect the reduction to spectral stability may be used as the basis of a general procedure to verify spectral stability in the zero-shock-amplitude limit, applicable in particular in the two open cases mentioned just above. This both unifies and completes existing stability theory, at least as regards the small amplitude case; stability of large amplitude shock profiles remains as the outstanding open question in this area.

The basic idea behind our approach is a natural one suggested by Gardner and Jones [GJ.3] in the context of the one-dimensional strictly parabolic case. In the weighted energy method of Goodman [Go.1, Go.2], the approach on which most subsequent extensions have been based, the strict transversality of characteristic fields other than the principal one is used to "project out" behavior in transverse modes, reducing the problem to an approximate scalar conservation law in the principal characteristic field. Gardner and Jones pointed out that transverse modes correspond in the associated eigenvalue ODE roughly to "fast" and "super-slow" modes, which on a large portion of the relevant spectral domain may be projected out using dynamical system techniques to leave an approximately scalar "slow" manifold flow as in the argument of Goodman, and proposed this observation as the basis of a dynamical systems argument alternative to the one-dimensional stability argument of Goodman. However, Gardner and Jones did not examine the crucial small-frequency regime where super-slow and slow modes intermingle, and therefore eliminated only the possibility of eigenvalues with real part greater than a small, but fixed constant $\theta > 0$; by contrast, a standard Gärding-type energy estimate gives the much stronger result $\text{Re } \lambda \le C\epsilon^2$, $\text{Im } \lambda \le C\epsilon$, where $\epsilon := |u_+ - u_-|$ denotes shock amplitude. Thus, at a technical level, substantial issues still remain; indeed, virtually all of our effort will be directed to the understanding of the "mixing" regime complementary to that considered in [GJ.3].

In the remainder of the introduction, we give an overview of the analysis and main results, deferring technical aspects to the main body of the paper.

Spectral stability. Let us first make precise the notion of spectral stability that we consider here. Without loss of generality taking $s = 0$ in (1) (by shifting to an appropriate traveling frame), we have that $\bar{u}(x)$ is a stationary solution of the nonlinear evolution system

$$u_t = \mathcal{F}(u) \tag{6}$$

described by its associated regularized conservation law, of form (189) or (1). Linearizing this system about the stationary solution $\bar{u}$, we obtain a linearized evolution system

$$v_t = Lv, \tag{7}$$

where $L := d\mathcal{F}(\bar{u})$ is the formal derivative of operator $\mathcal{F}$ about $\bar{u}$.

Definition 1.1. Following [ZH, Z.3], we define strong spectral stability (condition (D1') of [Z.3]) as

$$\sigma_p(L) \subset \{\lambda : \text{Re } \lambda < 0\} \cup \{0\}. \tag{8}$$

where $\sigma_p(M)$ denotes point spectrum of a linear operator M: equivalently, there exist no L^2 solutions of the eigenvalue equation $(L - \lambda)w = 0$ for $\text{Re } \lambda \ge 0$ and $\lambda \ne 0$.

Remark 1.2. It has been shown in a variety of contexts, in particular, all contexts considered here, that strong spectral stability together with the "hyperbolic" conditions of inviscid stability of the corresponding ideal shock (u_-, u_+) of (2) (dynamical, or "outer" stability) and transversality of the traveling wave connection (structural, or "inner" stability) implies linearized and nonlinear stability; see [ZH, Z.2, Z.3, MZ.1, MZ.2]. In the small amplitude case, under reasonable assumptions, inviscid stability holds always; see [M.1, M.2, M.3] in the one-dimensional case, [Mé.1, Mé.2, Mé.3, Mé.4, Mé.5, FM, FZ, Z.5, Z.6] in the multidimensional case. Likewise, under reasonable assumptions, connections are always transverse; see discussion of existence theory given below. Thus, *strong spectral stability is sufficient to imply linearized and nonlinear stability.*

The canonical model. In the one-dimensional genuinely nonlinear context, it is well known that *Burgers equation*,

$$u_t + (u^2/2)_x = u_{xx}, \tag{9}$$

approximately describes small-amplitude viscous behavior in the principal characteristic mode. In particular, the family of exact solutions

$$\bar{u}^\epsilon(x) := -\epsilon \tanh(\epsilon x/2) \tag{10}$$

give an asymptotic description of the structure of weak viscous shock profiles in the principal direction, in the limit as amplitude ϵ goes to zero. Note that (9) is invariant under the parabolic scaling $x \to \epsilon x$, $t \to \epsilon^2 t$, $u \to u/\epsilon$, whence we may immediately conclude that any unstable spectrum of the linearized operator about the profile $\bar{u}^\epsilon$ must lie within a ball of radius $C\epsilon^2$. (By standard considerations, the unstable spectrum of a second order parabolic operator is at least bounded.)

Burgers profiles are strongly spectrally stable, as may be readily established by a number of different techniques: for example, by L^1 contraction, Sturm-Liouville considerations [Sat, He, Z.5], or direct energy estimates as in [MN, Go.1, Go.2] or Appendix A.6 in [Z.3]. We record this fact as:

Proposition 1.3. *Shock profiles* (10) *(of any amplitude ϵ, by scale-invariance) are strongly spectrally stable as solutions of* (9)*.*

The reduced profile equations. Existence/structure of small-amplitude profiles may be determined by center manifold reduction, as pointed out by Majda and Pego [MP] in their pioneering treatment of the general strictly parabolic case; extensions to general partially parabolic, or "real" viscosity, and relaxation systems have been carried out in [P], [YoZ, FZe, MZ.1], respectively. The result in all cases, is that the center manifold associated with the weak profile problem (appropriately chosen) is foliated by one-dimensional fibers, on which the reduced flow, in the genuinely nonlinear case, is approximately that of the profile equation for Burgers equation (9).

More precisely, the center manifold lies tangent to (i.e., within angle $\mathcal{O}(\epsilon)$ of) the principal eigendirection r_p of Df^1, and the reduced flow, parametrized by $\eta := l_p(u_-) \cdot (\bar{u} - (u_- + u_+)/2)$, l_p the corresponding left eigendirection, obeys the approximate Burgers profile equation

$$\beta \eta' = \frac{\Lambda}{2}(\eta^2 - \epsilon^2) + \mathcal{O}(|\eta, \epsilon|^3),$$

or, rescaling by

$$\eta \to \eta/\epsilon, \quad x \to \Lambda \epsilon x/\beta : \tag{11}$$

$$\eta' = \frac{1}{2}(\eta^2 - 1) + \epsilon O(\eta, \epsilon), \tag{12}$$

$O(\cdot, \cdot) \in C^1$, as compared to the exact profile equation

$$\bar{\eta}' = \frac{1}{2}(\bar{\eta}^2 - 1) \tag{13}$$

for (9)–(10) with $\epsilon = 1$; here and below, "$'$" denotes $\partial/\partial x$. The effective diffusion coefficient β is given by $\beta := l_p B r_p(u_-)$ in the viscous case, and $\beta := l_p B_* r_p(u_-)$ in the relaxation case, where

$$\begin{aligned} B_*(u) &:= -\tilde{f}_v q_v^{-1}(g_u - v_u^* f_u) \\ &= -\tilde{f}_v q_v^{-1}(\tilde{g}_u - \tilde{g}_v q_v^{-1} q_u + q_v^{-1} q_u (\tilde{f}_u - \tilde{f}_v q_v^{-1} q_u),) \end{aligned} \tag{14}$$

$g := \tilde{g}(u, v^*(u))$, denotes the effective "Chapman–Enskog" viscosity associated with the relaxation system, and $\Lambda := l_p^t D^2 f^1(r_p, r_p)(u_-) \neq 0$ denotes the coefficient of genuine nonlinearity for the hyperbolic system (2); for details, see [MP, Z.1, P, YoZ, FZe, MZ.1]. From the reduced, normal form (12) we obtain in standard fashion:

Proposition 1.4. *Under the rescaling* (11)*, there hold*

$$|\eta_\pm - \bar{\eta}_\pm| \leq C\epsilon \tag{15}$$

and

$$|(\eta - \eta_\pm) - (\bar\eta - \bar\eta_\pm)| \le C\epsilon e^{-\theta|x|} \tag{16}$$

for $x \lessgtr 0$ *for any fixed* $0 < \theta < 1$*, and some* $C = C(\theta) > 0$*, where* $\eta_\pm$ *and* $\bar\eta_\pm := \pm 1$ *denote the rest points of* (12) *and* (13)*, respectively.*

Proof. Assertion (15) follows by the Implicit Function Theorem from normal hyperbolicity of $\bar\eta_\pm$. Assertion (16) follows from standard stable/unstable manifolds estimates. □

Corollary 1.5. *Under the rescaling* (11)*, the principle eigenvalue* $a_p := a_p(\bar u)$ *converges to* $\bar\eta$ *at the same rates* (15)–(16) *as does* η.

Proof. This follows using tangency of the profile to direction r_p together with the fact [Sm] that $\nabla a_p \cdot r_p = \Lambda$. □

The reduced eigenvalue equations. Following [GJ.3], we carry out here a reduction of the generalized eigenvalue equations similar in spirit to that carried out by Majda and Pego [MP] and others for the profile equations. However, whereas those anlayses involved center manifold reduction of an autonomous nonlinear system, we shall work directly with the linear, nonautonomous system given by the eigenvalue equation $(L - \lambda)w = 0$ associated with (7), written as an appropriate first-order system

$$W' = \mathbb{A}(x, \lambda)W, \tag{17}$$

using the *tracking lemma* of [GZ, ZH] (specifically, a refined version introduced in [MZ.1]) to effect the reduction to a slow manifold; see Section 2. Here, (17) is dimension $2n$ in the strictly parabolic viscous case, $n + \mathrm{Rank}B$ in the general viscous case, and $n + r$ in the relaxation case; as above, "$'$" denotes $\partial/\partial x$. For the efficient coordinatization of the first-order system (17) in the relaxation and general viscous case (important for practical computation), we follow the scheme of [Z.3]; see sections 3–4 for details.

Of course the tracking lemma is itself an analytic formulation of classical invariant manifold techniques; see the original formulation in [GZ] in terms of projectivized flow/invariant sets of autonomous systems. The calculation of the reduced flow on the slow manifold can equally well be carried out by classical center manifold reduction; see the concurrent treatment of Plaza [Pl] in the one-dimensional case; see also the independent treatment of [FreS] (described further in the note following the introduction). Indeed, this may be preferable in the more complicated situations arising with various types of degeneracies in the underlying equations. However, there is some advantage in taking account of the linear nature of the underlying problem, in that the calculation of reduced equations simplifies considerably.

First reduction. A first application of the tracking lemma projects out "fast" transverse modes: $n - 1$ in the strictly parabolic viscous case, $\mathrm{Rank}B - 1$ in the general viscous case, and $r - 1$ in the relaxation case, in each case reducing to an $(n + 1)$-dimensional "slow manifold," of which $n - 1$ correspond to "super slow" transverse modes ρ_j, and the remaining two to "medium slow" Burgers modes which are roughly linearized versions of η, $\eta' =: z$ in the limiting equations above. For details, see the specific, equation-dependent calculations carried out in Sections 3 and 4 in these different contexts.

Rescaling by $x \to \Lambda\epsilon x/\beta$, $\lambda \to \beta\lambda/\Lambda^2\epsilon^2$, and (balancing by) $\eta' \to \beta\eta'/\Lambda\epsilon$, we obtain a normal form for the reduced flow that has bounded ($\mathcal{O}(1)$) coefficients for bounded λ and, modulo error terms $\Phi_j(x,\lambda)\tilde{W}$ ($\tilde{W} := (\rho,\eta,z)$), becomes a block triangular system in $\rho = (\rho_+,\rho_-)$ and (η,z), of form

$$\rho'_\pm = M^\epsilon_\pm(x,\lambda)\rho_\pm, \tag{18}$$

$$\begin{pmatrix}\eta\\ z\end{pmatrix}' = \bar{M}_0(x,\lambda)\begin{pmatrix}\eta\\ z\end{pmatrix} + N^\epsilon(x,\lambda)\rho, \tag{19}$$

where

$$|M_\pm| = \mathcal{O}(\epsilon|\lambda|), \qquad M_\pm \lessgtr \epsilon\theta(|\mathrm{Re}\ \lambda| + \epsilon|\mathrm{Im}\ \lambda|^2), \tag{20}$$

$\theta > 0$; $\bar{M}_0$ corresponds exactly to the coefficient

$$\bar{\mathbb{A}}_0 := \begin{pmatrix}0 & 1\\ \lambda & \bar{\eta}(x)\end{pmatrix} \tag{21}$$

of the eigenvalue equation for the standard Burgers equation (9)–(10), $\epsilon = 1$, written as a first-order system; and

$$N = \begin{pmatrix}0 & 0\\ n_- & n_+\end{pmatrix}$$

satisfies

$$|N| \le Ce^{-\theta|x|}. \tag{22}$$

More precisely, including error terms we obtain

$$\begin{pmatrix}\rho_-\\ \eta\\ z\\ \rho_+\end{pmatrix}' = \begin{pmatrix}M^\epsilon_{\rho_-} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ n^\epsilon_- & \lambda & \bar{\eta} & n^\epsilon_+\\ 0 & 0 & 0 & M^\epsilon_{\rho_+}\end{pmatrix}\begin{pmatrix}\rho_-\\ \eta\\ z\\ \rho_+\end{pmatrix} + \Phi^\epsilon\begin{pmatrix}\rho_-\\ \eta\\ z\\ \rho_+\end{pmatrix}, \tag{23}$$

where

$$\Phi_{(\rho_-,\rho_+),(\rho_-,\eta,z,\rho_+)\pm}(\pm\infty,\lambda) \equiv 0, \qquad \Phi_{(\eta,z),(\rho_-,\rho_+)\pm}(\pm\infty,\lambda) \equiv 0, \tag{24}$$

and

$$\Phi_{(\eta,z)(\eta,z)}(\pm\infty,\lambda) = \epsilon\mathcal{E}_\pm(\lambda,\epsilon), \tag{25}$$

where

$$|\Phi(x,\lambda) - \Phi(\pm\infty,\lambda)| \le C(1+|\lambda|)\epsilon e^{-\theta|x|} \quad \text{for } x \lessgtr 0, \tag{26}$$

and $\mathcal{E}_\pm(\cdot)$ is a smooth function satisfying the growth bounds

$$\mathcal{E}_\pm = \begin{pmatrix}\mathcal{O}(1+|\lambda|\epsilon) & \mathcal{O}(1+|\lambda|\epsilon)\\ \mathcal{O}(1+|\lambda|) & \mathcal{O}(1+|\lambda|)\end{pmatrix} \tag{27}$$

for $|\lambda| \le C^{-1}\epsilon^{-2}$, $C > 0$ sufficiently large. By standard considerations, the spectrum of L is in the unrescaled coordinates restricted to small frequencies $|\lambda| \le C^{-1}$, $C > 0$ arbitrary (see Proposition 2.9, below), hence this is the relevant frequency regime for stability. *This initial reduction* (besides the cited reduction to small frequencies in the unrescaled coordinates) *is the only part of the argument that is equation-dependent.* Once (18)–(27) have been verified, the rest of the argument proceeds from these facts alone, independent of the original context.

Remarks. The asymmetric bounds on $\mathcal{E}_\pm$ are a result of rescaling. The stronger bound in the upper righthand corner is actually important for our later analysis, specifically, in the large-frequency regime where we approximately undo the rescaling in the course of our argument; see the treatment of Regions II and III in Section 3.

Further reductions/normal forms. In the rescaled coordinates, there are three distinct frequency regimes: I. $|\lambda| \le C_1$; II. $C_1 \le |\lambda| \le C_2\epsilon^{-1}$; and III. $C_2\epsilon^{-1} \le |\lambda| \le C_2^{-1}\epsilon^{-2}$, $C_1 \gg C_2 > 0$ sufficiently large, corresponding in the original (unrescaled) coordinates to $|\lambda| \le C_1\epsilon^2$, $C_1\epsilon^2 \le |\lambda| \le C_2\epsilon$, and $C_2\epsilon \le |\lambda| \le C_2^{-1}$, respectively. (Recall, Proposition 2.9, the spectrum of L is restricted to small frequencies $|\lambda| \le C_2^{-1}$.) Each regime requires a slightly different treatment, and is associated with a different normal form.

Region III, the region considered by Gardner and Jones [GJ.3], we will denote as the "parabolic" regime, as this is the regime on which behavior is dominated by dissipativity of the underlying system (which yields "parabolic" behavior for small frequencies, as evidenced by the structure of the reduced system (18)–(19)), independent of any other structure of the traveling wave profile whatsoever. This regime may be treated by the standard "high-frequency" techniques of the usual Evans function theory (see [AGJ, GZ, ZH, MZ.1]), which are roughly equivalent to sectorial, or Gärding-type, energy estimates, to preclude altogether the possibility of eigenvalues.

Specifically, (η, z) may be resolved into decaying/growing modes $\eta_\pm$ such that there is a uniform spectral gap between decaying modes (ρ_-, η_-) and growing modes (ρ_+, η_+), to an order sufficient that a second application of the tracking lemma, to the once-reduced equations (18)–(19) yields a further reduction of the flow onto two transverse invariant manifolds tangent to the (ρ_-, η_-) and (ρ_+, η_+) directions. In particular, we find that the manifold of solutions decaying at $+\infty$ is transverse to that of solutions decaying at $-\infty$, yielding the result. In other words, the normal form for this regime is trivial, with growing and decaying modes decoupled.

Region II we denote as the "reduced parabolic" regime, since this is the regime for which similar tracking/energy estimates prohibit eigenvalues for the Burgers equation. Here also, we find that eigenvalues cannot exist, independent of the structure of the profile. However, the analysis is much more delicate; indeed, we regard this as the trickiest case in our argument. Here, we can separate off decay/growth modes $\eta_\pm$ (parabolic behavior of the Burgers part) both from each other and from $\rho_\pm$ modes, but $\rho_\pm$ are not sufficiently separated to apply the tracking lemma to these coordinates. In this case, therefore, our ultimate reduction is to the $(n-1)$-dimensional superslow equations (18), but with error terms now involving ρ alone and not (η, z). Note that the coefficients $M_\pm$ are bounded on regime II, so that this is indeed a valid normal form.

Region I we denote as the "gap regime," in reference to the fact that on this regime we rely solely on the *gap lemma* of [GZ, KS, ZH, Z.3] (specifically a refined version given in [MéZ]) in our analysis of the slow flow, rather than using the tracking lemma to carry out a further reduction as in Regions II and III. The "gap" in "gap lemma" refers in fact to absence of spectral gap, as is the case in this regime; the lemma asserts that *uniform exponential convergence* of coefficients as $x \to \pm\infty$ and *continuous extension* of the stable/unstable eigenspaces of the limiting coefficient matrices $\mathbb{A}_\pm(\lambda)$ at $x \to \pm\infty$ can substitute in this situation for uniform spectral gap, to yield estimates valid on $x \lessgtr 0$. In this regime, which is the crucial one determining behavior, the normal form is the entire $(n+1)$-dimensional system (18)–(19).

The gap lemma: convergence to limiting flows. We complete our reduction in each case by an application of the gap lemma (specifically, a refinement given in [MéZ]), showing that the flow of the normal forms we have obtained converges to that of the "limiting" normal forms obtained by dropping error terms $\Phi\tilde{W}$, etc.: more precisely, the flows associated with the stable manifold at $+\infty$ and the unstable manifold at $-\infty$, or in some cases their analytic continuation into the essential spectrum. The basic approach is to: (i) conjugate approximate and limiting equations on the half-lines $x \lessgtr 0$ to their asymptotic systems at $\pm\infty$ by a linear change of variables uniformly close to the identity (an application of the gap lemma, made possible by the uniform exponential decay estimate (24)), then (ii) check by direct linear algebraic computation that the stable and unstable subspaces of the respective asymptotic coefficient matrices are uniformly close. In the present applications, (ii) is straightforward, since the asymptotic matrices of approximate and limiting flows agree for the ρ equation, while for the (decoupled) (η, z) equation, the stable and unstable subspaces for the limiting equations are, under our strategically chosen rescaling, uniformly spectrally separated on the region I where it comes into play.

Convergence of the stable flow at $+\infty$ and unstable flow at $-\infty$ then implies uniform convergence as $\epsilon \to 0$ of suitably chosen *Evans functions* associated with the approximate and limiting normal forms. The Evans function is a Wronskian measuring solid angle between the stable and unstable subspaces of a given eigenvalue ODE written as a first-order system, constructed so as to be analytic in the frequency λ on a "region of consistent splitting" (defined (A1), Section 2). In each of the contexts considered here, the region of consistent splitting includes the entire set of frequencies $\{\mathrm{Re}\,\lambda \geq 0\} \setminus \{0\}$ of interest; see, e.g., [Z.3]. Moreover, both the Evans function and its component subspaces *extend continuously along rays* in frequency space, in both the one- and multidimensional case [ZS, Z.3] to the closure at the origin, making possible uniform estimates of transversality; indeed, in the one-dimensional case considered here, the Evans function may in fact be extended analytically through the origin, without restricting to rays [GZ]. (Note: in the multidimensional case, the Evans function may be holomorphically extended along rays [ZS, Z.3]; however, in certain "glancing" directions, there arise branch singularities at the origin.) On the region of consistent splitting, the zeroes of the Evans function correspond both in location and multiplicity with the eigenvalues of the associated linear operator; for details, see [GJ.1, GJ.2, ZH].

The multiplicity of the generalized eigenvalue at the origin $\lambda = 0$ is directly calculable and related to hyperbolic stability, as shown in the one- and multidimensional case in [GZ] and [ZS], respectively. In the present, small-amplitude case, it is readily seen to be the same as the multiplicity for the limiting normal form equations, namely one. In the one-dimensional, analytically extendable case, we may thus conclude immediately from uniform convergence to the limiting Evans function (by degree, i.e., winding number, considerations) that *strong spectral stability of the original system is equivalent to strong spectral stability for the limiting normal forms.* Alternatively, we may remove the zero of the Evans function at the origin by working instead with the "integrated" eigenvalue equations following [MN, Go.1] or with "flux variables" following [Go.2], to recover the same conclusion using only continuity at the origin of D.

In this paper, we follow the latter strategy exclusively. In the multidimensional analysis of [PZ], we use a modification of the same strategy, specifically, an interesting variation intermediate to flux and integrated variable methods, neither of which themselves directly generalize to multidimensions in a useful way, to achieve the same result; this "balanced flux form" is quite similar to that introduced in the treatment of relaxation systems in Section 4 of the present paper. The former strategy is perhaps viable as well, using computation of winding number on Riemann surfaces as suggested in [GZ]; however, this would involve extra bookkeeping (specifically, tracking of branch points in approximate vs. limiting systems) that we prefer to avoid.

Conclusions/summary of the main results. From this point, we may immediately deduce our main results. For, in each of the (nontrivial) limiting normal forms arising in regions I–II, the equations for $\rho_\pm$ completely decouple, and clearly support no "decaying" solutions other than the trivial, zero solution (in general, "decaying" is defined as lying in ρ_+ direction at $+\infty$ and ρ_- direction at $-\infty$; where the limiting equations satisfy consistent splitting, this is equivalent to actual decay at $\pm\infty$). Thus, we may conclude that $\rho \equiv 0$ for any generalized eigensolution, leaving a decoupled (η, z) equation in region I, *agreeing with the generalized eigenvalue equations for canonical model* (9), and the trivial flow in region II. Recall that the reduced flow was already trivial in region III. Likewise, $\nu \equiv 0$ in each of these regions, since fast growing and fast decaying modes completely decouple.

Noting that the canonical model, being among the class considered, is subject to the same reduction in regions II–III (hence cannot support eigenvalues there), we may conclude, finally, our main result: that *strong spectral stability is equivalent to strong spectral stability of the canonical model*, at least modulo the degenerate case that the canonical model yields nontrivial nonstable eigenvalues lying precisely on the imaginary axis; see Proposition 3.2 and Remark 3.3 below. As a corollary, recalling the result of Proposition 5.4, we obtain the stated results of strong spectral stability of small-amplitude profiles, across the general class of models considered in (189) and (1). More precisely, in this paper, we carry out the details of the one-dimensional case for general strictly parabolic viscosities, and for general relaxation systems. The case of general real viscosity may be carried out similarly as the relaxation case, following the dual framework set out in Appendix A.2 of [Z.3]; we omit these calculations, as satisfactory one-dimensional results already exist [HuZ].

Plan of the paper. In Section 2, we recall and slightly extend the basic Evans function tools we will need. In Section 3, we carry out the one-dimensional, strictly parabolic viscous case. For clarity, we first carry out in its entirety the simpler identity viscosity case, afterward indicating the necessary adjustments in the general case. In Section 4, we carry out the general relaxation case, for clarity first treating the 2×2 case by reduction to the scalar viscous case; we note that this gives an independent proof of stability in this case, different from that of Liu [L.2]. Finally, in Section 5, we briefly indicate the extension of our analysis to the multidimensional viscous case; details will be given in [PZ].

Note: Shortly before the completion of this analysis (precisely: after the completion of the one-dimensional viscous case and before the completion of the relaxation case), we have learned of closely related work of H. Freistühler and P. Szmolyan [FreS] in which they establish a reduction method similar (indeed, perhaps equivalent) to ours, but proceeding entirely from the point of view of geometric singular perturbation theory. In the cited work, the first in a planned series of three, they carry out a complete analysis of the one-dimensional identity viscosity case, announcing the intention to treat one-dimensional relaxation/real viscosity and multidimensional stability in papers two and three, respectively.

2. Evans function framework

We begin our analysis by recalling, and in some cases slightly extending, the Evans function tools we will need. For later reference, we carry out the analysis in generality sufficient for the multidimensional case as well.

2.1. **The gap lemma and convergence of approximate flows.** Consider a family of first-order systems

$$W' = \mathbb{A}^\epsilon(x, \tilde{\xi}, \lambda)W \tag{28}$$

indexed by parameter ϵ, where $(\tilde{\xi}, \lambda)$ vary within the set

$$\Omega := \{(\tilde{\xi}, \lambda) : \tilde{\xi} \in \mathbb{R}^{d-1}, \text{Re } \lambda \geq 0\} \tag{29}$$

of unstable–neutrally stable frequencies, ϵ varies within some given set $\mathcal{V}$, and x varies within $\mathbb{R}^1$. Equations (28) are to be thought of as generalized eigenvalue equations, with parameter λ representing a Laplace transform frequency in time, and $\tilde{\xi}$ representing a Fourier transform frequency in directions of spatial symmetry; in the present, one-dimensional case, $\tilde{\xi} \equiv 0$. In the applications of this paper and of [PZ], ϵ will be just the shock strength $|u_+ - u_-|$.

We make the following basic assumptions.

(A0) Coefficient $\mathbb{A}^\epsilon(\cdot, \tilde{\xi}, \lambda)$, considered as a function from $(\tilde{\xi}, \lambda, \epsilon)$ into $L^\infty(x)$ is analytic in $(\tilde{\xi}, \lambda)$ and C^k in ϵ for some $k \geq 0$ on $\Omega \times \mathcal{V}$. Moreover, $\mathbb{A}^\epsilon(\cdot, \tilde{\xi}, \lambda)$ approach exponentially to limits $\mathbb{A}_\pm$ as $x \to \pm\infty$, with uniform exponential decay estimates

$$|\mathbb{A}^\epsilon - \mathbb{A}^\epsilon_\pm| \leq C_1 e^{-|x|/C_2},\ C_j > 0, \quad \text{for } x \gtrless 0 \tag{30}$$

on compact subsets of $\Omega \times \mathcal{V}$.

(A1) On $(\Omega \backslash \{(0,0)\}) \times \mathcal{V}$, the limiting matrices $\mathbb{A}^\epsilon_+$ and $\mathbb{A}^\epsilon_-$ are both hyperbolic (have no center subspace), and the dimensions of their stable (resp. unstable) subspaces S^ϵ_+ and S^ϵ_- (resp. unstable subspaces U^ϵ_+ and U^ϵ_-) agree.

(A2) At the origin $(\tilde{\xi}, \lambda) = (0,0)$, subpaces $S^\epsilon_\pm$, $U^\epsilon_\pm$ have continuous (to some order, which may be even analytic, or holomorphic with branch point at the origin) limits along rays (i.e., for $(\tilde{\xi}, \lambda) = (r\tilde{\xi}_0, r\lambda_0)$ as $r \in \mathbb{R} \to 0^+$, $(\tilde{\xi}_0, \lambda_0) \in \Omega \setminus \{(0,0)\}$), which, moreover, are C^k in the parameter ϵ, where k is as in (A0) above.

These are satisfied in all the contexts considered in both this paper and [PZ]; see [Z.3]. Condition (A0) is induced, at least for ϵ bounded from zero, by the origins of (28) through the linearization about smooth shock profiles of (189) or (1). Uniform approach (30) as $\epsilon \to 0$ will be recovered through reduction/rescaling. Condition (A1) may be recognized as the standard hypothesis of "consistent splitting," as introduced in [AGJ], and follows by the assumption of dissipativity of systems (189) and (1). Condition (A2) is an extension suitable for multidimensions of the "geometric separation" hypothesis of [GZ] and is a consequence of the hyperbolic structure of (2). (Note: separation was not needed for the arguments of [GZ], and in fact does not hold in multidimensions; see discussion of Appendix A.4, [Z.3].)

The gap lemma. Under these circumstances, the "gap lemma" established in various degrees of generality in [GZ, KS, ZH, Z.3] asserts that behavior of the variable-coefficient equation (28) may be related to that of its constant-coefficient limiting systems on $x \gtrless 0$, while maintaining the assumed regularity in all parameters. This has recently been greatly improved in [MéZ], in the form of the following *conjugation lemma*, a version that is particularly convenient in applications.

Lemma 2.1. [MéZ] *Under assumption (A0), there exists locally to any given base point* $(\tilde\xi, \lambda, \epsilon) \in \Omega \times \mathcal{V}$ *a pair of linear transformations* $P_+^\epsilon(x, \tilde\xi, \lambda)$ *and* $P_-^\epsilon(x, \tilde\xi, \lambda)$ *on* $x \geq 0$ *and* $x \leq 0$*, respectively, and possessing the same regularity as* $\mathbb{A}^\epsilon$ *in all arguments, such that: (i)* $|P_\pm^\epsilon|$ *and their inverses are uniformly bounded, with*

$$|P_\pm^\epsilon - I| \leq CC_1C_2 e^{-\theta|x|/C_2}$$

where $0 < \theta < 1$ *is an arbitrary fixed parameter, and* $C > 0$ *and the size of the neighborhood of definition depend only on* θ*, the modulus of the entries of* $\mathbb{A}^\epsilon$ *at* $(\tilde\xi, \lambda, \epsilon) \in \Omega \times \mathcal{V}$*, and the modulus of continuity of* $\mathbb{A}^\epsilon$ *on some neighborhood of* $(\tilde\xi, \lambda, \epsilon) \in \Omega \times \mathcal{V}$*. (ii) The change of coordinates* $W := P_\pm^\epsilon Z$ *reduces* (28) *on* $x \geq 0$ *and* $x \leq 0$*, respectively, to the asymptotic constant-coefficient equations*

$$Z' = \mathbb{A}_\pm^\epsilon(\tilde\xi, \lambda) Z. \tag{31}$$

Proof. As described in [MéZ, MZ.3], this is a straightforward corollary of the gap lemma as stated in [Z.3], applied to the "lifted" matrix-valued equations for the conjugating matrices $P_\pm^\epsilon$. □

Definition 2.2. Assuming (A0)–(A3), let $v_{1+}^\epsilon, \dots, v_{k+}^\epsilon$ and $v_{(k+1)-}^\epsilon, \dots, v_{N-}^\epsilon$ be bases for S_+^ϵ and U_-^ϵ, as defined in (A1)–(A2), chosen with the same regularity assumed on S_+^ϵ, U_-^ϵ (note: that such a choice is possible is a consequence of standard matrix perturbation theory [Kat]), and $P_\pm^\epsilon$ be transformations defined as above on some neighborhood of $(\tilde\xi, \lambda, \epsilon) \in \Omega \times \mathcal{V}$. Then, the *local Evans function* for (28) associated with these choices is defined as

$$D^\epsilon(\tilde\xi, \lambda) := \det \begin{pmatrix} P_+^\epsilon v_{1+}^\epsilon & \cdots & P_+^\epsilon v_{k+}^\epsilon & P_-^\epsilon v_{(k+1)-}^\epsilon & \cdots & P_-^\epsilon v_{N-}^\epsilon \end{pmatrix}_{|x=0}. \tag{32}$$

Remarks. 1. Defining $Z_{j\pm}^\epsilon$ as the solutions of limiting, constant-coefficient equations $Z' = \mathbb{A}_\pm^\epsilon Z$ with initial data $Z_{j\pm}^\epsilon(0) := v_{j\pm}^\epsilon$, and setting $W_{j\pm}^\epsilon := P_\pm^\epsilon Z_{j\pm}^\epsilon$, we have that $W_{1+}^\epsilon, \dots, W_{k+}^\epsilon$ and $W_{(k+1)-}^\epsilon, \dots, W_{N-}^\epsilon$ on the domain of consistent splitting are bases for the stable manifold at $+\infty$ and unstable manifold at $-\infty$ of the variable-coefficient equations (28), and on the boundary of the domain of consistent splitting are continuous extensions thereof. Observing that

$$D^\epsilon(\tilde\xi, \lambda) = \det \begin{pmatrix} W_{1+}^\epsilon & \cdots & W_{k+}^\epsilon & W_{(k+1)-}^\epsilon & \cdots & W_{N-}^\epsilon \end{pmatrix}_{|x=0}, \tag{33}$$

we recover the standard definition of the Evans function as given, e.g., in [AGJ, GZ, ZH, ZS]. Note that S_+^ϵ and U_-^ϵ are invariant subspaces of $\mathbb{A}_+^\epsilon$, $\mathbb{A}_-^\epsilon$, with the additional property that the only solutions of the limiting equations (31) that approach in angle to these subspaces as $x \to +\infty$ or $-\infty$, respectively, are solutions contained in these subspaces. (These properties hold by definition on the region of consistent splitting, and extend by continuity to points on the boundary.) Thus, $W_{j\pm}^\epsilon$ have an intrinsic description as bases for the unique manifolds of solutions of (28) approaching in angle as $x \to +\infty$ or $-\infty$, respectively, to S_+^ϵ and U_-^ϵ.

2. Evidently, the choice of local Evans function is highly nonunique; however, as indicated by Remark 1 just above, any two choices differ only by a nonvanishing factor with the same regularity assumed on $\mathbb{A}^\epsilon$, S_+^ϵ, and U_-^ϵ, i.e., the same regularity guaranteed for the Evans function itself. Though we shall not need it here, it can be shown that a globally analytic choice is possible in both the one- and multidimensional case; see [GZ, MZ.3] and [ZS, Z.3], respectively.

From (33) we immediately obtain the following result sufficient for our needs.

Proposition 2.3. *For $(\tilde{\xi}, \lambda)$ within the region of consistent splitting $\Omega \setminus \{(0,0)\}$, equation* (28) *admits a nontrivial solution $W \in L^2(x)$ if and only if $D^\epsilon(\tilde{\xi}, \lambda) = 0$.*

Proof. Evidently, the first K columns of the matrix on the righthand side of (32) are a basis for the stable manifold of (28) at $x \to +\infty$, while the final $N - K$ columns are a basis for the unstable manifold at $x \to -\infty$. Thus, its determinant vanishes if and only if these manifolds have nontrivial intersection, and the result follows. □

Remark. More detailed computation shows that the eigenvalues of L correspond with the zeroes of D not only in location but also in multiplicity; see [GJ.1, GJ.2]. See [ZH, MZ.1] for an appropriate extension to the boundary of the region of consistent splitting, i.e., to the case of eigenvalues embedded in essential spectrum.

Convergence of approximate flows. We now turn to the crucial question: Without loss of generality taking $\mathcal{V} = \{\epsilon \in \mathbb{R} : 0 < \epsilon < \epsilon_0\}$, and assuming that $\mathbb{A}^\epsilon$ has a limit $\mathbb{A}^0$ as $\epsilon \to 0^+$, under what circumstances and in what sense does the Evans function of (28) converge as $\epsilon \to 0^+$ to an Evans function of the limiting equations at $\epsilon = 0$? The following proposition gives a simple and sharp answer.

Proposition 2.4. *Suppose that (A0)–(A3) hold for $\mathcal{V} = \{\epsilon \in \mathbb{R} : 0 < \epsilon < \epsilon_0\}$ and, in an Ω-neighborhood of some $(\tilde{\xi}, \lambda)$:*

(i) As $\epsilon \to 0^+$, the asymptotic subspaces S_+^ϵ, U_-^ϵ converge uniformly in angle to some limits S_+^0, U_-^0, (not necessarily corresponding to stable, unstable subspaces of the limiting matrixes $\mathbb{A}_\pm^0$ defined below)[1] *with rate $\eta(\epsilon)$: equivalently, for $0 < \epsilon \le \epsilon_0$, their spanning bases satisfy*

$$|v_{j\pm}^\epsilon - v_{j\pm}^0| \le \eta(\epsilon). \tag{34}$$

(ii) The coefficient matrices $\mathbb{A}^\epsilon$ converge uniformly exponentially to limiting values $\mathbb{A}^0$, in the sense that, for $0 < \epsilon \le \epsilon_0$,

$$|(\mathbb{A}^\epsilon - \mathbb{A}_\pm^\epsilon) - (\mathbb{A}^0 - \mathbb{A}_\pm^0)| \le C_1 \eta(\epsilon) e^{-|x|/C_2}. \tag{35}$$

In particular, (A0) holds on $\Omega \times \bar{\mathcal{V}}$ with $k = 0$, so that the conjugating transformations $P_\pm^\epsilon$ guaranteed by Lemma 2.1 extend continuously to $\epsilon = 0$.

Then, for any specified P^0, there is at least one choice of P^ϵ, $\epsilon \in \bar{\mathcal{V}}$, satisfying the conclusions of Lemma 2.1, such that, on some (possibly smaller) Ω-neighborhood of $(\tilde{\xi}, \lambda)$, the local Evans functions D^ϵ defined as in Definition 2.2 (in the case of D^0, using the spanning bases $v_{j\pm}^0$ defined in (i) above) converge uniformly to D^0 as $\epsilon \to 0^+$, with rate

$$|D^\epsilon - D^0| \le CC_1C_2\eta(\epsilon); \tag{36}$$

more precisely, the columns of the defining matrix on the righthand side of (32) *converge uniformly, with the same rate $CC_1C_2\eta(\epsilon)$.*

Proof. Clearly, it is sufficient to prove the stronger assertion of convergence of individual columns, and for this we may restrict without loss of generality to the + columns, and the right half-line $x \ge 0$. Applying the conjugating transformation

[1] In some important applications, A^ϵ may lose hyperbolicity (consistent splitting) in the limit $\epsilon \to 0$, so that S_+^0, U_-^0 have not intrinsic meaning, but rather must be defined by continuous extension; see especially the treatment of Region I in Section 3, below.

$W \to (P^0_+)^{-1}W$ for the $\epsilon = 0$ equations, we may reduce to the case that $\mathbb{A}^0$ is constant, and $P^0_+ \equiv I$. In this case, (35) becomes just

$$|(\mathbb{A}^\epsilon - \mathbb{A}^\epsilon_\pm)| \le C_1\eta(\epsilon)e^{-|x|/C_2},$$

and we obtain directly from the conjugation lemma, Lemma 2.1, the estimate

$$|P^\epsilon_+ - P^0_+| = |P^\epsilon_+ - I| \le CC_1C_2\eta(\epsilon)e^{-\theta|x|/C_2},$$

and in particular

$$|P^\epsilon_+ - P^0_+|_{|x=0} \le CC_1C_2\eta(\epsilon). \tag{37}$$

The result now follows, from (32), (34), and (37). □

Remarks. 1. The same argument shows that the proposition is sharp, since easy examples show that the conjugation lemma itself is sharp. It is worth noting, however, that continuity of P^ϵ, and thus convergence of D^ϵ requires only continuity of $\mathbb{A}^\epsilon$, and not the stronger bound (35) needed to obtain rate (36).

2. In the case that (28) correspond to the eigenvalue equations of linearized operators about a family of traveling wave profiles, hypotheses (34)–(35) hold always for regular perturbations around $\epsilon = 0$ of the profile equations with hyperbolic rest points and λ in the region of consistent splitting, by standard ODE estimates. Thus, Proposition 2.4 includes the verification of spectral stability/continuity of the Evans function in this standard case. The main point here (and the reason for the somewhat convoluted statement of the theorem) is to allow the important situation that (A1)–(A2) fail at the limiting value $\epsilon = 0$. *Thus, we specifically exclude $\epsilon = 0$ from the domain $\mathcal{V}$.*

2.2. **The tracking lemma and reduction of the flow.** Next, consider the complementary situation of a family of equations of form (28) on an ϵ-neighborhood for which the coefficient $\mathbb{A}^\epsilon$ does not exhibit uniform exponential decay to its asymptotic limits, but instead is *slowly varying.* This occurs quite generally for rescaled eigenvalue equations arising in the study of the large-frequency regime; see, e.g., [GZ, ZH, MZ.1, Z.3]. In the present context, it arises for the *unrescaled equations* in the small-shock strength limit $\epsilon \to 0$.

In this situation, it frequently occurs that not only $\mathbb{A}^\epsilon$ but also certain of its invariant (group) eigenspaces are slowly varying with x, i.e., there exist matrices

$$L^\epsilon = \begin{pmatrix} L^\epsilon_1 \\ L^\epsilon_2 \end{pmatrix}(x), \quad R^\epsilon = \begin{pmatrix} R^\epsilon_1 & R^\epsilon_2 \end{pmatrix}(x) \tag{38}$$

for which $L^\epsilon R^\epsilon(x) \equiv I$ and $|LR'| = |L'R| \le C\delta(\epsilon)$, uniformly in ϵ, where $\delta(\epsilon)$ is small relative to

$$\mathbb{M}^\epsilon := L^\epsilon \mathbb{A}^\epsilon R^\epsilon(x) = \begin{pmatrix} M^\epsilon_1 & 0 \\ 0 & M^\epsilon_2 \end{pmatrix}(x) \tag{39}$$

and "$'$" as usual denotes $\partial/\partial x$. In this case, making the change of coordinates $W^\epsilon = R^\epsilon Z$, we may reduce (28) to the approximately block-diagonal equation

$$Z^{\epsilon\prime} = \mathbb{M}^\epsilon Z^\epsilon + \delta^\epsilon \Theta^\epsilon Z^\epsilon, \tag{40}$$

where $\mathbb{M}^\epsilon$ is as in (39), $\Theta^\epsilon(x)$ are uniformly bounded matrices, and $\delta^\epsilon(x) \le \delta(\epsilon)$ is a (relatively) small scalar. A sometimes crucial improvement may be obtained by arranging that error Θ vanish on diagonal blocks, i.e., that $L_j R_j' \equiv 0$; see [Go.1, Go.2, MZ.1]. Here, we shall make use of this observation only in the principal, 2×2 "Burgers" block.

Let us assume that such a procedure has been successfully carried out, and, moreover, that there exists an approximate *uniform spectral gap*, in the strong sense that

$$\min \sigma(\text{Re } M_1^\epsilon) - \max \sigma(\text{Re } M_2^\epsilon) \geq \eta(\epsilon) - \alpha^\epsilon(x) \text{ for all } x, \tag{41}$$

with $\eta(\epsilon) > 0$ and $\|\alpha^\epsilon\|_{L^1}$ uniformly bounded, where (here and elsewhere) Re $N := (1/2)(N + N^*)$ denotes the "real", or symmetric part of an operator N: more properly speaking, a gap in numerical range[2]. Then, there holds the following *reduction lemma*, a refinement established in [MZ.1] of the "tracking lemma" given in varying degrees of generality in [GZ, ZH, Z.3]. The new feature of the reduction lemma is that it asserts the existence of smooth invariant manifolds in the vicinity of the first and second block coordinates, whereas the tracking lemma(s) assert only the existence of forward and backward attracting cones in the same vicinity and do not give a reduction in the usual dynamical systems sense.

Proposition 2.5. [MZ.1] *Consider a system* (40) *under the gap assumption* (41)*, with Θ^ϵ uniformly bounded for ϵ sufficiently small. If, for $0 < \epsilon < \epsilon_0$, the ratio $\delta(\epsilon)/\eta(\epsilon)$ of formal error vs. gap is sufficiently small relative to the bounds on Θ (in particular if $\delta(\epsilon)/\eta(\epsilon) \to 0$ as $\epsilon \to 0^+$), then, for all $0 < \epsilon \leq \epsilon_0$, there exist (unique) linear transformations $\Phi_1^\epsilon(x, \tilde{\xi}, \lambda)$ and $\Phi_2^\epsilon(x, \tilde{\xi}, \lambda)$, possessing the same regularity with respect to the various parameters ϵ, x, $\tilde{\xi}$, λ as do coefficients $\mathbb{M}^\epsilon$ and $\delta(\epsilon)\Theta^\epsilon$, for which the graphs $\{(Z_1, \Phi_2^\epsilon(Z_1))\}$ and $\{(\Phi_1^\epsilon(Z_2), Z_2)\}$ are invariant under the flow of* (40)*, and satisfying*

$$|\Phi_1^\epsilon|, \, |\Phi_1^\epsilon| \leq C\delta(\epsilon)/\eta(\epsilon) \text{ for all } x.$$

Proof. This may be established by a contraction mapping argument, carried out for the projectivized "lifted" equations governing the flow of exterior forms as in Appendix A.3.2.2 (ii) of [MZ.1],[3] or (more directly) for lifted equations governing the flow of of conjugating matrices as in Appendix C of [MZ.3] (*Proof of Proposition 3.9*). □

From Proposition 2.5, we obtain reduced flows

$$Z_1^{\epsilon\prime} = M_1^\epsilon Z_1^\epsilon + \delta^\epsilon(\Theta_{11} + \Theta_{12}^\epsilon \Phi_2^\epsilon) Z_1^\epsilon = M_1^\epsilon Z_1^\epsilon + \mathcal{O}(\delta^\epsilon(x)) Z_1^\epsilon \tag{42}$$

and

$$Z_2^{\epsilon\prime} = M_2^\epsilon Z_2^\epsilon + \delta^\epsilon(\Theta_{22} + \Theta_{21}^\epsilon \Phi_1^\epsilon) Z_2^\epsilon = M_2^\epsilon Z_2^\epsilon + \mathcal{O}(\delta^\epsilon(x)) Z_2^\epsilon \tag{43}$$

on the two invariant manifolds described. Should we arrange that $\Theta_{11} \equiv 0$ or $\Theta_{22} \equiv 0$, then the error terms would become $\mathcal{O}(\delta^\epsilon(x)\delta(\epsilon)/\eta(\epsilon))Z_1^\epsilon$ or $\mathcal{O}(\delta^\epsilon(x)\delta(\epsilon)/\eta(\epsilon))Z_2^\epsilon$, respectively; that is, enforcing vanishing of Θ on a diagonal block reduces the error term by factor δ/η.

Remark 2.6. In the "standard" case that $\mathbb{M}_j^\epsilon$ are uniformly bounded from above and $\eta(\epsilon)$ uniformly bounded from below, with $(\mathbb{M}_j^\epsilon)' = \mathcal{O}(\delta)$, (41) may be arranged by a suitable coordinate transformation provided that there holds the weaker condition

$$\min \text{Re } \sigma(M_1) - \max \text{Re } \sigma(M_2) \geq \tilde{\eta} - \alpha^\epsilon(x) \text{ for all } x, \tag{44}$$

for some $\tilde{\eta} > \eta$; see, e.g., [ZH], proof of Proposition 7.3, or the proof of Proposition 2.9 below. However, in Proposition 2.5 *we specifically do not require uniform*

[2]Equation (41) corrects a misprint in (3.15) of [MZ.3].

[3]See, e.g., [Sat] for a corresponding argument in the case that M_1, M_2 are scalar; the lifting to exterior forms essentially reduces the problem to this case.

boundedness, either of $\mathbb{M}^\epsilon$ *from above or of* $\eta(\epsilon)$ *from below*, and in fact neither condition holds in general for our applications; see in particular the secondary reductions carried out in Section 3 on Regions II and III. Note further that *we neither assert nor require any limiting behavior as* $\epsilon \to 0^+$. Indeed, a typical application, as in Section 2.3 (specifically, the large-frequency analysis $|\lambda| \geq R$ in the proof of Proposition 2.9), is to eigenvalue equations that have been rescaled in the spatial variable, $x \to x/\nu(\epsilon)$, $\nu \to 0$ as $\epsilon \to 0^+$; in this case, the coefficients typically have no limit as $\epsilon \to 0^+$.

Remark 2.7. For the applications of this paper and of [PZ], as for those of [MZ.1, MZ.3], the gap condition need only be checked at $x = \pm\infty$, with variation in coefficients contributing an exponentially decaying, uniformly integrable error α^ϵ. That is, Proposition 2.5 through the inclusion of error term α^ϵ allows us to make (in this case crucial) use of the information that coefficients converge exponentially to constant states, even when that convergence, due to spatial rescaling, is no longer uniform.

Remark 2.8. As described in [MZ.1], provided that M_1 and M_2 are (uniformly) bounded and spectrally separated, one can in fact repeat the procedure used to construct (40) to obtain an approximate block-diagonalization to arbitrarily high order: i.e., an asymptotic expansion in powers of $\delta(\epsilon)$. In general, e.g., in the large-amplitude situation considered in [MZ.1], these higher order terms are not explicitly computable. However, we point out that, in the present small-amplitude situation, they may in principle be determined exactly to any desired order, using the error expansion for the traveling wave profile to approximate L'; indeed, this can be recognized as just a variation of the usual center manifold calculations used to approximate center manifold flow. In the calculations of this paper and of [PZ], we shall not need to perform any such higher-order corrections. However, this possibility may be useful in more degenerate situations: for example, when the principal characteristic eigenvalue associated with the shock is not genuinely nonlinear.

2.3. **Reduction to small frequency.** As a first application of Proposition 2.5, we obtain the following preliminary result analogous to that of [GJ.3], reducing the small-amplitude stability problem to the small frequency, or "diffusive," regime $|\lambda| \ll 1$.

Proposition 2.9. *In each of the one-dimensional contexts considered in this paper (viscous and relaxation systems, with hypotheses given below), the nonstable eigenvalues* $\mathrm{Re}\,\lambda \geq 0$ *of linearized operator* L *of* (7) *are restricted to* $|\lambda| \leq r(\epsilon)$, $r(\epsilon) \to 0$ *as* $\epsilon \to 0$, *where* $\epsilon := |u_+ - u_-|$ *denotes shock strength.*

Proof. Equivalently, we show, for arbitrary $r > 0$, that $|\lambda| \leq r$ for ϵ sufficiently small. For $R > |\lambda| > r$, R sufficiently large but fixed, and $\mathbb{A}^\epsilon$ as in (28) denoting the coefficient of the generalized eigenvalue problem (7) written as a first-order system, there is a uniform spectral gap between stable and unstable subpaces of $\mathbb{A}^\epsilon(x)$, independent of $0 < \epsilon \leq \epsilon_0$ and x; as discussed in [Z.3], this is a straightforward consequence of the assumed dissipativity of the systems (in the sense of Kawashima–Zeng [Kaw, Ze]). By standard matrix perturbation theory [Kat] plus boundedness (compactness) of $\mathbb{A}^\epsilon$, we may therefore deduce the existence of well-conditioned bases (53) for these subspaces varying smoothly with $\mathbb{A}^\epsilon$, and therefore satisfying

$$|LR'| = \mathcal{O}(|(\bar{u}^\epsilon)'|) = \mathcal{O}(\epsilon^2)$$

by Proposition 1.4. Moreover, by a proper choice of basis, we may arrange (41) with $\eta > 0$ uniformly bounded, independent of $0 < \epsilon \leq \epsilon_0$ and x; for details, see Appendix A4, [Z.3].

Applying Proposition 2.5, we obtain a pair of reduced systems (42) and (43) for which the coefficient matrices (with $\mathcal{O}(\epsilon^2)$ error term taken into account) are, respectively, uniformly positive and negative definite, and solutions therefore uniformly exponentially grow and decay, provided that ϵ is sufficiently small. It follows that the equations have no solutions bounded at plus and minus spatial infinity save for the trivial solution $W \equiv 0$.

For $|\lambda| \geq R$, $R > 0$ sufficiently large, on the other hand, the nonexistence of eigenvalues may be established independent of the size of ϵ by a rescaling followed by a similar reduction argument; see [ZH, MZ.1, MZ.2]. □

Remark 2.10. See Lemma 4.38, [Z.3] for a multidimensional version of Proposition 2.9, asserting that $|(\tilde{\xi}, \lambda)| \leq r(\epsilon)$, $r(\epsilon) \to 0$ as $\epsilon \to 0^+$, for nonstable eigenvalues $\mathrm{Re}\, \lambda \geq 0$ of the Fourier transformed linearized operator $L_{\tilde{\xi}}$, where $\tilde{\xi}$ denotes the Fourier transform frequency in directions parallel to the shock front, and λ as here denotes the Laplace transform frequency with respect to time (proof identical). The multidimensional case is discussed further in Section 5.

3. One-dimensional stability for parabolic systems

With these preparations, we are ready to carry out the stability analysis. We begin with a treatment of small-amplitude stationary profiles $\bar{u}^\epsilon(x)$ of one-dimensional strictly parabolic viscous conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x \tag{45}$$

in a neighborhood $\mathcal{U}$ of a particular state u_0. We make the assumptions:

(H0) $f, B \in C^2$ (regularity).

(H1) $\mathrm{Re}\, \sigma(B(\bar{u})) > 0$ (strict parabolicity).

(H2) $\sigma(Df)$ real, simple (strict hyperbolicity).

(H3) $\mathrm{Re}\, \sigma(-iDf\xi - B\xi^2) \leq -\theta|\xi|^2$, $\xi \in \mathbb{R}$, for some $\theta > 0$ (linearized stability of constant states).

(H4) $a_p = 0$ is a simple eigenvalue of $Df(u_0)$ with left and right eigenvectors l_p and r_p, and $l_p^t D^2 f(r_p, r_p) \neq 0$ (genuine nonlinearity of the principal characteristic field).

Here, $\epsilon > 0$ denotes shock strength $|u_+^\epsilon - u_-^\epsilon|$, and profiles $\bar{u}^\epsilon(\cdot)$ are assumed to converge as $\epsilon \to 0$ to u_0. Under these assumptions, the center-manifold argument of Majda and Pego [MP] verifies the assertion of Proposition 1.4, yielding convergence after rescaling of $\bar{u}^\epsilon$ to the standard Burgers profile (10). In the rest of this section, we shall establish our first main result:

Theorem 3.1. *Under assumptions (H0)–(H4), profiles $\bar{u}^\epsilon$ are strongly spectrally stable (and therefore linearly and nonlinearly orbitally stable* [ZH, Z.2]*) for ϵ sufficiently small.*

More precisely, we shall establish the following proposition, from which, together with Proposition 5.4, Theorem 3.1 follows as an immediate corollary.

Proposition 3.2. *Under assumptions (H0)–(H4), profiles $\bar{u}^\epsilon$ are strongly spectrally stable (and therefore linearly and nonlinearly orbitally stable* [ZH, Z.2]*) for ϵ sufficiently small, if the standard Burgers profile* (10) *is strongly spectrally stable for $\epsilon = 1$ and only if the Burgers profile is not strongly spectrally unstable for $\epsilon = 1$ in the sense that there exists an eigenvalue* $\text{Re}\,\lambda > 0$.

Remark 3.3. Of course, the "only if" part of Proposition 3.2 follows vacuously in the case at hand; the point is that we obtain this information by our method of proof, independent of any knowledge of behavior of the the limiting system. This distinction may be important in future applications to more general situations.

Remark 3.4. Theorem 3.1 recovers the spectral stability result obtained by energy methods in [Go.1, Go.2], and slightly extends it from the class of A, B such that $LBR > 0$ for some diagonalizing bases L, $R = L^{-1}$ of A to the class of all "stable" pairs A, B in the sense of Majda and Pego [MP]. (In the case that A, B are simultaneously symmetrizable, these two classes are equivalent; see [MP].) It is straightforward with some additional bookkeeping (see [Z.3, HuZ]) to extend our results to the case, to which the nonlinear analyses of [ZH, Z.2] also apply, that (H2) is relaxed to:

(H2') $\sigma(Df)$ real, semisimple.

3.1. **The case $B \equiv I$.** To illustrate the method, we first carry out the proof of Proposition 3.2 and Theorem 3.1 in the setting of identity viscosity $B \equiv I$, for which the associated linear algebra is particularly simple.

Linearizing about $\bar{u}^\epsilon$, we obtain the family of eigenvalue equations

$$\tilde{w}'' = (A^\epsilon \tilde{w})' + \lambda \tilde{w}, \tag{46}$$

where

$$A^\epsilon(x) := Df(\bar{u}^\epsilon(x)). \tag{47}$$

Following Goodman [Go.1], we consider instead the more favorable "integrated equations"

$$w'' = A^\epsilon w' + \lambda w \tag{48}$$

satisfied by the anti-derivative $w(x) := \int_{-\infty}^x \tilde{w}$ of $\tilde{w}$. For $\lambda \in \{\text{Re}\,\lambda \geq 0\} \setminus \{0\}$, it is easily seen that (46) possesses L^2 solutions if and only if does (48) [ZH]: indeed, the L^2 solutions of (46) consist exactly of the derivatives of those of (48). Unlike (46), which has a single zero eigenfunction $\tilde{w} = (\bar{u}^\epsilon)'$, (48) possesses no zero (L^2, i.e., decaying) eigenfunctions. For, the only candidate, $w = \bar{u}^\epsilon - u_-^\epsilon$, does not decay as $x \to +\infty$, but converges to $u_+^\epsilon - u_-^\epsilon$. Thus, for the integrated equations, we face the better-conditioned problem of showing that the limiting Evans function D^0 is simply bounded from below, rather than the problem for the unintegrated equations of showing that it vanishes to a certain specified multiplicity at the origin. For further comments on this point, see the the discussion of the introduction (just above *Conclusions*).

Remark 3.5. It is worth noting that equations (48) may be obtained by another route. As pointed out by Goodman [Go.2], the flux variable $w := \tilde{w}' - A\tilde{w}$ satisfies the same equation.

Writing (48) as the family of first-order systems

$$W' = \mathbb{A}^\epsilon(x,\lambda)W, \tag{49}$$

where

$$W := \begin{pmatrix} w \\ w' \end{pmatrix}, \quad \mathbb{A}^\epsilon := \begin{pmatrix} 0 & I \\ \lambda & A^\epsilon \end{pmatrix}, \tag{50}$$

we are ready to begin our analysis. By Proposition 2.9, it is sufficient to consider λ within an arbitrarily small ball around the origin. Accordingly, we hereafter restrict

$$|\lambda| \le 1/C_2, \tag{51}$$

with $C_2 > 0$ a large constant to be determined later.

Lemma 3.6. *The eigenvalues μ and associated left and right eigenvectors L, R of $\mathbb{A}^\epsilon(x)$ in (50) may be denoted as*

$$\mu_j^\pm(x) = \frac{a_j \pm \sqrt{a_j^2 + 4\lambda}}{2}, \tag{52}$$

$$L_j^\pm(x) = \Big(\lambda l_j/(\lambda + {\mu_j^\pm}^2), \mu_j^\pm l_j/(\lambda + {\mu_j^\pm}^2)\Big), \quad R_j^\pm(x) = \begin{pmatrix} r_j \\ \mu_j^\pm r_j \end{pmatrix}, \tag{53}$$

where $a_j(x)$ denote the eigenvalues of $A^\epsilon(x)$ and $l(x)$, $r(x)$ associated left and right eigenvectors, normalized so that $l_j r_k = \delta_k^j$ (here, we have suppressed the ϵ dependence for the sake of readability).

By (H2) and (H4), we may order the (real) eigenvalues of A^ϵ as

$$a_1, \dots, a_{p-1} \le -\theta < 0, \tag{54}$$

$$a_p \sim \epsilon\bar\eta(\Lambda\epsilon x/\beta) = \mathcal{O}(\epsilon), \tag{55}$$

and

$$0 < \theta \le a_{p+1}, \dots, a_n, \tag{56}$$

where Λ and β denote the genuine nonlinearity and effective diffusion coefficients described in the introduction, and $\bar\eta$ the standard Burgers profile (9). Like the principal eigenvalue a_p, the transverse eigenvalues a_j, $j \ne p$ vary within an $\mathcal{O}(\epsilon)$ neighborhood of the corresponding eigenvalues of $Df(u_0)$.

Evidently, for the indices $j \ne p$ such that a_j is bounded from zero, the eigenvalues $\mu_j^\pm$ are jointly analytic in a_j and thus A^ϵ and λ within neighborhoods of $Df(u_0)$, 0; a brief calculation shows that $L_j^\pm$ and $R_j^\pm$ are bounded, hence analytic in these parameters as well. The two-dimensional group eigenspace spanned by $L_p^\pm$ and $R_p^\pm$ likewise varies analytically *as subspaces*, as $\mathbb{A}^\epsilon$ and λ are varied; however, the individual eigenvectors that make it up do not (indeed, blow up for $a_p = 0$ and $\lambda \to 0$). We therefore do not attempt to resolve these subspaces, but just choose convenient analytically varying spanning bases

$$L_\eta := \begin{pmatrix} l_p & 0 \end{pmatrix}, L_z := \begin{pmatrix} 0 & l_p \end{pmatrix}, \tag{57}$$

and

$$R_\eta := \begin{pmatrix} r_p \\ 0 \end{pmatrix}, R_z := \begin{pmatrix} 0 \\ r_p \end{pmatrix}. \tag{58}$$

Summing up, we have:

Lemma 3.7. *In the transverse fields $j \neq p$, there hold expansions*

$$\begin{aligned} \mu_j^+ &= -\lambda/a_j + \lambda^2/a_j^3 + \dots \\ \mu_j^- &= a_j + \dots, \end{aligned} \tag{59}$$

for $j < p$ and

$$\begin{aligned} \mu_j^+ &= a_j + \dots, \\ \mu_j^- &= -\lambda/a_j + \lambda^2/a_j^3 + \dots \end{aligned} \tag{60}$$

for $j > p$, analytic in λ and smooth (indeed, analytic) in a_j, or, through a_j, in A^ϵ, on sufficiently small neighborhoods of $\lambda = 0$ and $a_j = a_j(u_0)$ or $A^\epsilon = Df(u_0)$. Likewise, the eigenvectors $L_j^\pm$, $R_j^\pm$, $j \neq p$ possess the same regularity, as do bases L_η, L_z and R_η, R_z (defined in (57)–(58)*) for their complementary two-dimensional invariant subspaces.*

Proof. Taylor expansion/direct calculation. □

Approximate block-diagonalization. Following the procedure outlined in Section 2.2, define now

$$Z = \begin{pmatrix} \nu_- \\ \rho_- \\ \eta \\ z \\ \rho_+ \\ \nu+ \end{pmatrix} := \mathbb{L}^\epsilon W, \tag{61}$$

$$\mathbb{L}^\epsilon := \begin{pmatrix} L_{\nu_-} \\ L_{\rho_-} \\ L_0 \\ L_{\rho_+} \\ L_{\nu_+} \end{pmatrix}, \quad \mathbb{R}^\epsilon := \Big(R_{\nu_-}, R_{\rho_-}, R_0, R_{\rho_+}, R_{\nu_+}\Big), \tag{62}$$

where

$$L_{\nu_-} := \begin{pmatrix} L_1^- \\ \vdots \\ L_{p-1}^- \end{pmatrix}, \quad R_{\nu_-} := \Big(R_1^-, \dots, R_{p-1}^-, \Big), \tag{63}$$

$$L_{\rho_-} := \begin{pmatrix} L_{p+1}^- \\ \vdots \\ L_n^- \end{pmatrix}, \quad R_{\rho_-} := \Big(R_{p+1}^-, \dots, R_n^-, \Big), \tag{64}$$

$$L_0 := \begin{pmatrix} L_\eta \\ L_z \end{pmatrix}, \quad R_0 := \Big(R_\eta, R_z, \Big) \tag{65}$$

$$L_{\rho_+} := \begin{pmatrix} L_1^+ \\ \vdots \\ L_{p-1}^+ \end{pmatrix}, \quad R_{\rho_+} := \Big(R_1^+, \dots, R_{p-1}^+, \Big), \tag{66}$$

and

$$L_{\nu_+} := \begin{pmatrix} L_{p+1}^+ \\ \vdots \\ L_n^+ \end{pmatrix}, \quad R_{\nu_+} := \Big(R_{p+1}^+, \dots, R_n^+\Big), \tag{67}$$

$\mathbb{L}^\epsilon \mathbb{R}^\epsilon \equiv I$, where all coefficients depend on x, ϵ through $A^\epsilon(\bar{u}^\epsilon(x))$.

This is just such an approximately block-diagonalizing transformation as described in Section 2.2: in particular, because $\mathbb{L}^\epsilon$, $\mathbb{R}^\epsilon$ depend smoothly on $\mathbb{A}^\epsilon$, we have that

$$\Theta^\epsilon := |\eta'|^{-1}\mathbb{L}^\epsilon(\mathbb{R}^\epsilon)' = \mathcal{O}(1), \tag{68}$$

where $\eta := l_p(u_-)\cdot(\bar u^\epsilon - (u_- + u_+)/2)$ as in the introduction. (Recall that $|\lambda|$ is uniformly bounded, (51).) Thus, in Z coordinates, (49) reduces to the form (40), with

$$\begin{aligned} &\delta^\epsilon(x) := |\eta'| = \mathcal{O}(\epsilon^2 e^{-\theta\epsilon|x|}), \quad \theta > 0,\\ &\delta(\epsilon) = \mathcal{O}(\epsilon^2), \end{aligned} \tag{69}$$

$$|\Theta^\epsilon| = \mathcal{O}(1), \tag{70}$$

and

$$\mathbb{M}^\epsilon := \operatorname{diag}(M_{\nu_-}, M_{\rho-}, M_0, M_{\rho_+}, M_{\nu_+}), \tag{71}$$

where, for Re $\lambda \geq 0$:

$$M_{\nu_-} := \operatorname{diag}(\mu_1^+, \dots, \mu_{p-1}^+) \leq -\eta < 0, \tag{72}$$

$$M_{\nu_+} := \operatorname{diag}(\mu_{p+1}^-, \dots, \mu_n^-) \geq \eta > 0, \tag{73}$$

$$M_{\rho-} := \operatorname{diag}(\mu_{p+1}^+, \dots, \mu_n^+) \leq -\eta(\text{Re }\lambda + |\text{Im }\lambda|^2) \leq 0, \tag{74}$$

$$M_{\rho+} := \operatorname{diag}(\mu_1^-, \dots, \mu_{p-1}^-) \geq \eta(\text{Re }\lambda + |\text{Im }\lambda|^2) \geq 0, \tag{75}$$

with also

$$|M_{\rho_\pm}| = \mathcal{O}(|\lambda|), \tag{76}$$

and

$$M_0 := \begin{pmatrix} 0 & 1 \\ \lambda & a_p \end{pmatrix}. \tag{77}$$

We make a final improvement, following [Go.1, Go.2, MZ.1], by making the rescaling $l_j(x) \to \alpha^{-1}(x) l_j(x)$, $r_j(x) \to r_j(x)\alpha(x)$, with α satisfying ODE

$$\alpha' = -l_j r_j' \alpha. \tag{78}$$

This achieves the normalization

$$l_j r_j' \equiv 0 \tag{79}$$

while preserving the previously stated properties (68)–(77). As a consequence, we obtain the key property that $L_0 R_0' \equiv 0$, i.e., error Θ^ϵ vanishes in the η, z block:

$$\Theta_{(\eta,z),(\eta,z)} \equiv 0. \tag{80}$$

We could carry out a similar normalization at the L_j, R_j level (with α_j now matrix-valued) to annihilate all diagonal blocks Θ_{jj}; see Lemma 4.9 of [MZ.1]. However, we do not require this for our argument. (In fact, we only require vanishing in the lower left hand corner $\Theta_{z,\eta} = 0$ of the η, z block, which is automatic; however, we wish to point out early on this more general procedure.)

First reduction. Observing that coefficients M_{ν_-} and M_{ν_+} of the "fast transverse modes" ν_- and ν_+ are uniformly bounded and separated by a uniform spectral gap (in the standard sense (44)) both from each other, and from the "slow" modes $(\rho_-, \eta, z, \rho_+)^t$, so that (41) may be arranged by Remark 2.6 between each of these three groups and the others, we may apply Proposition 2.5 twice to reduce (40) to decoupled flows on three invariant manifolds, associated respectively with ν_-, ν_+, and $(\rho_-, \eta, z, \rho_+)^t$, appearing as

$$\nu_-' = M_{\nu_-}\nu_- + \mathcal{O}(\delta(\epsilon))\nu_-, \tag{81}$$

$$\nu_+' = M_{\nu_+}\nu_+ + \mathcal{O}(\delta(\epsilon))\nu_+, \tag{82}$$

and

$$\begin{pmatrix}\rho_-\\ \eta\\ z\\ \rho_+\end{pmatrix}' = \begin{pmatrix} M_{\rho_-} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & \lambda & a_p & 0\\ 0 & 0 & 0 & M_{\rho_+}\end{pmatrix}\begin{pmatrix}\rho_-\\ \eta\\ z\\ \rho_+\end{pmatrix} + \delta^\epsilon \tilde\Theta^\epsilon \begin{pmatrix}\rho_-\\ \eta\\ z\\ \rho_+\end{pmatrix}, \tag{83}$$

where, by (70),

$$\tilde\Theta^\epsilon := \Theta^\epsilon_{(\rho_-,\eta,z,\rho_+),(\rho_-,\eta,z,\rho_+)} + \mathcal{O}(\delta(\epsilon))\Theta^\epsilon_{(\rho_-,\eta,z,\rho_+),(\nu_-,\nu_+)} \tag{84}$$

is small on the principal block:

$$\tilde\Theta^\epsilon_{(\eta,z),(\eta,z)} = \mathcal{O}(\delta(\epsilon)) = \mathcal{O}(\epsilon^2). \tag{85}$$

Observing that the flow is uniformly exponentially decreasing for (81) and increasing for (82), we find that the only possible decaying solution are the trivial ones $\nu_\pm \equiv 0$, and so we may discard these "fast transverse" equations, leaving us with (83). Rescaling now by $x \to \Lambda\epsilon x$, $\lambda \to \lambda/\Lambda^2\epsilon^2$, $z \to z/\Lambda\epsilon$, we obtain the block-triangular system of equations (18)–(20) described in the introduction, modulo coefficient errors

$$\Phi := \bar\delta^\epsilon\bar\Theta^\epsilon + \epsilon\mathcal{E}^\epsilon, \tag{86}$$

where

$$\bar\delta^\epsilon(x) := \epsilon^{-1}\delta^\epsilon(x/\epsilon) = \mathcal{O}(\epsilon e^{-\theta|x|}), \quad \theta > 0, \tag{87}$$

with $\bar\Theta^\epsilon$ uniformly bounded and $\mathcal{O}(\epsilon)$ on the principal, (η, z) block), and $\mathcal{E}^\epsilon \equiv 0$ in the $\rho_\pm$ equations and in the $(\eta,z),\rho_\pm$ components, and in the $(\eta,z),(\eta,z)$ components is given by

$$\mathcal{E}^\epsilon_{(\eta,z),(\eta,z)} := \begin{pmatrix} 0 & 0\\ 0 & a_p - \bar\eta\end{pmatrix}, \tag{88}$$

giving

$$\mathcal{E}^\epsilon_{(\eta,z),(\eta,z)\pm} = \begin{pmatrix} 0 & 0\\ 0 & \mathcal{O}(1)\end{pmatrix}, \quad |\mathcal{E}^\epsilon_{(\eta,z),(\eta,z)} - \mathcal{E}^\epsilon_{(\eta,z),(\eta,z)\pm}| = \mathcal{O}(e^{-\theta|x|}), \tag{89}$$

by Corollary 1.5. Rearranging, we have the error bounds asserted in the introduction (indeed, slightly better). Note that the forcing term

$$\begin{aligned} N(x,\lambda) &= \begin{pmatrix} 0\\ n(x,\lambda)\end{pmatrix},\\ n &:= \epsilon^{-2}\delta^\epsilon\Theta_{z,\rho} = \mathcal{O}(e^{-\theta|x|}), \end{aligned} \tag{90}$$

arises in (19) as a result of the rescaling $z \to z/\epsilon$.

Further reductions/Normal forms. Loosely following the introduction, we now examine separately each of the three regimes (with respect to the rescaled variable λ) I. $|\lambda| \le C_1$, II. $C_1 \le |\lambda| \le C_2\epsilon^{-1}$: and III. $C_2\epsilon^{-1} \le |\lambda| \le C_2^{-1}\epsilon^{-2}$, where $C_1 >> C_2 > 0$ are sufficiently large constant to be determined later. (Recall that in Proposition 2.9 we have already disposed of the case $|\lambda| \ge C_2^{-1}\epsilon^{-2}$.)

Region I. On region I, the rescaled $\rho_\pm$, (η, z) equations evidently have bounded coefficients satisfying condition (ii) of Proposition 2.4, with $\eta(\epsilon) = \mathcal{O}(\epsilon)$. Moreover, the asymptotic coefficients at plus and minus spatial infinity are block-diagonal, with M_+ and M_- positive and negative definite, respectively, and $M_0 = \bar M_0 + \mathcal{O}(\epsilon)$, where

$$\bar M_0(\pm\infty) = \begin{pmatrix} 0 & 1\\ \lambda & \mp 1\end{pmatrix} \tag{91}$$

have uniform spectral gap between stable and unstable eigenvalues for $\mathrm{Re}\,\lambda \geq 0$, $\lambda \neq 0$. Evidently, the stable subspaces $S^\epsilon_\pm$ are the direct sum of the entire ρ_- subspace and the stable subspace of $M_0(\pm\infty)$, and the unstable subspaces $U^\epsilon_\pm$ are the direct sum of the ρ_+ subspace and the unstable subspace of $M_0(\pm\infty)$. Expansion $M_0 = \bar{M}_0 + \mathcal{O}(\epsilon)$ together with uniform spectral gap of (91) imply by standard matrix perturbation theory that the stable and unstable subspaces of $M_0(\pm\infty)$ lie within $\mathcal{O}(\epsilon)$ of those of $\bar{M}_0(\pm\infty)$. Thus, S^ϵ_+ and U^ϵ_- indeed approach limits S^0_+ and U^0_-, satisfying condition (i) of Proposition 2.4 with $\eta(\epsilon) = \mathcal{O}(\epsilon)$.

Applying Proposition 2.4, we may conclude, therefore, that the approximate, ($\epsilon \neq 0$) system has a local Evans function D^ϵ that is $\mathcal{O}(\epsilon)$ close to the local Evans function D^0 defined in the proposition for the limiting, ($\epsilon = 0$) system

$$\rho_\pm' \equiv 0, \qquad \begin{pmatrix}\eta\\ z\end{pmatrix}' = \bar{M}_0 \begin{pmatrix}\eta\\ z\end{pmatrix} + N\begin{pmatrix}\rho_-\\ \rho_+\end{pmatrix}. \tag{92}$$

Recall that we are free to choose any conjugating transformation P^0 satisfying the bounds of Lemma 2.1, and (by change of coordinates) limiting bases $v^0_{j\pm}$. Taking advantage of the decoupled, constant-coefficient nature of the $\rho_\pm$ equations, we may choose P^0 of form

$$P^0 = \begin{pmatrix} I & 0 & 0 & 0\\ * & * & * & *\\ * & * & * & *\\ 0 & 0 & 0 & I\end{pmatrix} \tag{93}$$

and take basis elements of form $(\rho_-, *, *, 0)$ and $(0, \eta, z, 0)$ for S^0_- and of form $(0, *, *, \rho_+)$ and $(0, \eta, z, 0)$ for U^0_-, to find after a brief computation that

$$D^0 = d^0, \tag{94}$$

where d^0 is an Evans function for the Burgers eigenvalue equation

$$\begin{pmatrix}\eta\\ z\end{pmatrix}' = \begin{pmatrix}0 & 1\\ \lambda & \bar{\eta}\end{pmatrix}\begin{pmatrix}\eta\\ z\end{pmatrix}, \tag{95}$$

i.e., the limiting (η, z) block with $\rho_\pm$ set to zero.

Since (95) is known to admit no decaying solutions for $\lambda = 0$, we thus have that the limiting Evans function D^0 has no zeroes on Region I if the Burgers equation is strongly stable (possesses no nonzero eigenvalue λ with nonnegative real part $\mathrm{Re}\,\lambda \geq 0$, Definition 5.1) on Region I, and has a zero of strictly positive real part if the Burgers equation is strongly unstable on Region I (possesses a positive real part eigenvalue). By uniform convergence/analyticity in λ, we may conclude the same regarding the zeroes of the approximate Evans function D^ϵ, and thus also the eigenvalues of the original operator about the wave.

Region II. On Region II, the eigenvalues

$$\bar{\mu}_p^\pm(x) = \frac{\bar{\eta} \pm \sqrt{\bar{\eta}^2 + 4\lambda}}{2}(x) \tag{96}$$

of the (η, z) ("Burgers") block $\bar{M}_0$ satisfy

$$\begin{aligned} \bar{\mu}_p^- &\leq -\eta|\lambda|^{1/2} \leq -C_1^{1/2}\eta < 0,\\ \bar{\mu}_p^+ &\geq +\eta|\lambda|^{1/2} \geq C_1^{1/2}\eta > 0, \end{aligned} \tag{97}$$

with also $|\bar{\mu}_p^\pm| = \mathcal{O}(|\lambda|^{1/2})$. Thus, "balancing" the reduced equations by the rescaling $z \to z/|\lambda|^{1/2}$, we obtain well-conditioned basis vectors for this block. (Note:

in the new coordinates, $|\lambda|^{-1/2}M_0 = \begin{pmatrix} 0 & 1 \\ 1 & \mp|\lambda|^{-1/2} \end{pmatrix}$ is a small perturbation of the diagonalizable matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.) At the same time, the forcing matrix N of (22) becomes now $\mathcal{O}(|\lambda|^{-1/2})$ (note: we have partially undone the original rescaling $z \to z/\epsilon$), and may be viewed as an error term of order $\mathcal{O}(C_1^{-1/2})$: uniformly small, but not vanishing as $\epsilon \to 0$. Indeed, checking the behavior under this rescaling of the various error terms in (24)–(27), we find that this is the dominant error term *within both of Regions II and III*, with error terms in the $\rho_\pm$ equations retaining their originally stated form.

Diagonalizing $\bar{M}_0$ by the above-mentioned well-conditioned change of basis, we thus obtain (in rescaled coordinates) an approximately block-diagonalized system

$$\begin{pmatrix} \rho_- \\ \eta_- \\ \eta_+ \\ \rho_+ \end{pmatrix} = \operatorname{diag}(M_-, \bar{\mu}_p^-, \bar{\mu}_p^+, M_+) \begin{pmatrix} \rho_- \\ \eta_- \\ \eta_+ \\ \rho_+ \end{pmatrix} + \mathcal{O}(|\lambda|^{-1/2}) \begin{pmatrix} \rho_- \\ \eta_- \\ \eta_+ \\ \rho_+ \end{pmatrix}, \tag{98}$$

where $\eta_\pm$ denote the now-separated Burgers modes (diagonalized versions of the rescaled (η, z) coordinates). Observing that the $\bar{\mu}_p^\pm$ are separated by uniform spectral gap $\eta|\lambda|^{1/2} \sim C_1^{1/2}$ (in either the weak sense of (44) or the strong sense of (41), since $\bar{\mu}_p^\pm$ are scalar) both from each other and from the $\mathcal{O}(\epsilon|\lambda|) = \mathcal{O}(C_2)$ blocks $M_\pm$ (recall, we chose $C_1 \gg C_2$) on all of Region II (indeed, they remain bounded from each other on Region III as well), we may therefore apply Lemma 2.5 a second time to further reduce to decoupled equations on three invariant manifolds, associated respectively with η_p^-, η_p^+, and (ρ_-, ρ_+), appearing as

$$(\eta_p^-)' = (\bar{\mu}_p^- + \mathcal{O}(C^{-1}))\eta_p^- \tag{99}$$

$$(\eta_p^+)' = (\bar{\mu}_p^+ + \mathcal{O}(C^{-1}))\eta_p^+. \tag{100}$$

and

$$\begin{pmatrix} \rho_- \\ \rho_+ \end{pmatrix}' = \begin{pmatrix} M_- & 0 \\ 0 & M_+ \end{pmatrix} \begin{pmatrix} \rho_- \\ \rho_+ \end{pmatrix} + \bar{\delta}^\epsilon \bar{\Theta}^\epsilon \begin{pmatrix} \rho_- \\ \rho_+ \end{pmatrix}, \tag{101}$$

$C > 0$ sufficiently large, where $\bar{\delta}^\epsilon$ *is still* $\mathcal{O}(\epsilon e^{-\theta|x|})$ *as in the original rescaling*: note that we are here using in a crucial way the fact that the error terms in $\rho_\pm$ coordinates alone are better than the $\mathcal{O}(|\lambda|^{-1/2}$ error terms in the $\eta_p^\pm$ blocks, which were the ones (recorded in the uniform description (98)) that led to the error/gap ratio $|\lambda|^{-1/2}/|\lambda|^{1/2} \sim |\lambda|^{-1}$ determining the tracking angle in Proposition 2.5.

Noting as usual that $\eta_p^\pm$ flows are uniformly exponentially growing/decaying, so support only trivial decaying solutions, we may discard the first two equations, leaving us with the (ρ_-, ρ_+) equation alone. Noting further that coefficients $M_\pm = \mathcal{O}(\epsilon\lambda)$ for these"superslow" modes are bounded on Region II, we may then rescale by $\lambda \to \lambda/\epsilon$ and apply Proposition 2.4 in the resulting new coordinates (for which coefficients now converge uniformly to limiting values as $\epsilon \to 0$) to obtain that suitably chosen local Evans functions for this superslow flow converge uniformly (as $\mathcal{O}(\epsilon)$) to that for the limiting equations. But, similarly as in the treatment of Region I, the limiting $\rho_\pm$ equations decouple into uniformly exponentially growing/decaying flows, which evidently support no nontrivial decaying solutions. Thus, we may conclude, finally, that there exist no decaying solutions of the reduced equations, and therefore no eigenvalues for the original equations, for λ within Region II, and ϵ sufficiently small.

Region III. The treatment of Region III is similar to but somewhat simpler than the treatment of Region II. For, as already pointed out, all steps up to the reduction to $\rho_\pm$ equations (101) remain valid on this region as well, with η_+ and η_- still maintaining a uniform spectral gap (in the sense of (41)) from one another, though no longer from $\rho_\pm$. Further, by (20), M_+ and M_- have uniform spectral gap $\bar\eta(\epsilon)$ (in the strong sense (41)) of order $\theta C_2^2 \epsilon$ on this region, $\theta > 0$ fixed, while $\bar\delta^\epsilon$ remains uniformly $\mathcal{O}(\epsilon)$. Choosing $C_2 > 0$ sufficiently large, therefore (recall, the treatments of regions I and II were valid for arbitrarily large $C_1 \gg C_2 > 0$), we may arrange that the ratio $\bar\delta^\epsilon/\bar\eta(\epsilon) = \mathcal{O}(C_2^{-2})$ of approximation error to spectral gap be (uniformly) arbitrarily small, and thus apply Lemma 2.5 to obtain a simpler reduction to decoupled equations on two invariant manifolds associated with modes (ρ_-, η_-) and $(\rho_+, \eta_+$, appearing as

$$\begin{pmatrix} \rho_- \\ \eta_- \end{pmatrix}' = \Big(\operatorname{diag}\{M_-, \mu_p^-\} + \mathcal{O}(C_2^{-2} \bar\delta^\epsilon \bar\Theta^\epsilon_{\rho_-, \eta_-}) \Big) \begin{pmatrix} \rho_- \\ \eta_- \end{pmatrix} \tag{102}$$

and

$$\begin{pmatrix} \rho_+ \\ \eta_+ \end{pmatrix}' = \Big(\operatorname{diag}\{M_+, \mu_p^+\} + \mathcal{O}(C_2^{-2} \bar\delta^\epsilon \bar\Theta^\epsilon_{\rho_+, \eta_+}) \Big) \begin{pmatrix} \rho_+ \\ \eta_+ \end{pmatrix}$$

But, these, having negative and positive definite coefficients

$$\begin{aligned} \operatorname{diag}\{M_\pm, \eta_\pm\} + \mathcal{O}(C_2^{-2} \bar\delta^\epsilon \bar\Theta^\epsilon_{\rho_\pm}, \eta_\pm) &\lesseqgtr \pm\theta C_2^2 \epsilon + \mathcal{O}(C_2^{-2}\epsilon) \\ &\lesseqgtr \pm\theta C_2^2 \epsilon/2, \end{aligned} \tag{103}$$

are uniformly exponentially decaying and growing, respectively, so support no nontrivial decaying solutions. We may thus conclude that there exist no eigenvalues in Region III, for ϵ sufficiently small.

Conclusions. Collecting the results for each regime, we obtain the result of Proposition 3.2, and therefore of Theorem 3.1. This completes the analysis of the case $B \equiv I$. □

3.2. **The general strictly parabolic case.** The general case follows quite similarly as in the case $B \equiv I$, in terms of overall argument structure. However, there is an important new technical aspect introduced here, in the way that we carry out the linear algebra underlying the initial reduction to "slow" coordinates $(\rho_\pm, \eta, z)$.

For general B, the linearized eigenvalue equations (46) become

$$(B^\epsilon \tilde w')' = (A^\epsilon \tilde w)' + \lambda \tilde w, \tag{104}$$

where

$$B^\epsilon(x) := B(\bar u^\epsilon(x)), \tag{105}$$

and

$$A^\epsilon(x) v := Df(\bar u^\epsilon(x)) v - \Big(DB(\bar u^\epsilon(x)) v \Big) (\bar u^\epsilon)'(x), \tag{106}$$

and the associated integrated equations become

$$w'' = (B^\epsilon)^{-1} A^\epsilon w' + \lambda (B^\epsilon)^{-1} w. \tag{107}$$

We express these as a family of first-order systems

$$W' = \mathbb{A}^\epsilon(x, \lambda) W, \tag{108}$$

where

$$\mathbb{A}^\epsilon := \begin{pmatrix} 0 & I \\ \lambda (B^\epsilon)^{-1} & (B^\epsilon)^{-1} A^\epsilon \end{pmatrix}. \tag{109}$$

Note that A^ϵ lies within $\mathcal{O}(|(\bar u^\epsilon)'|)$ of $dF(\bar u^\epsilon)$, hence is likewise strictly hyperbolic, with eigenvalues a_j and eigenvectors l_j and r_j lying within $\mathcal{O}(|(\bar u^\epsilon)'|)$ of those of $dF(\bar u^\epsilon)$. For purposes of our argument, therefore, A^ϵ and $Df(\bar u^\epsilon)$ are indistinguishable; in particular, $a_j \le -\eta < 0$ for $j < p$, $a_j \ge \eta > 0$ for $j > p$, and the principle eigenvalue a_p after rescaling approaches to the Burgers profile $\bar\eta$ with the same rate predicted in Proposition 1.5. (This last is the main point, and holds also in the more general setting alluded to in Remark 3.4, with (H2) replaced by (H2').)

For simplicity of exposition, we make the provisional hypothesis:

$$\sigma(B^{-1}Df(u_0))\,\text{distinct.} \tag{110}$$

This concerns only the transverse modes $j \ne p$, since the argument of [MP] shows that zero is a simple eigenvalue under assumptions (H1), (H3), and is easily removed at the expense of further bookkeeping. The next lemma furnishes the linear algebra necessary to carry out the reduction to slow coordinates in the general case; for similar arguments, see [ZH, Z.3, MZ.1].

Lemma 3.8. *Associated with each of the transverse fields $j \ne p$ of A^ϵ, there exist eigenvalues $\mu_j^\pm$ and associated left and right eigenvectors $L_j^\pm$ and $R_j^\pm$ of $\mathbb{A}^\epsilon$, analytic in λ and (A^ϵ, B^ϵ) on sufficiently small neighborhoods of $\lambda = 0$ and $(Df(u_0), B(u_0))$, with expansions*

$$\mu_j^+ = -\lambda/a_j + \lambda^2\beta_j^2/a_j^3 + \dots, \tag{111}$$

$$L_j^+(x) = \Big(l_j, -l_jB/a_j\Big) + \dots, \quad R_j^\pm(x) = \begin{pmatrix} r_j \\ 0 \end{pmatrix} + \dots, \tag{112}$$

and

$$\mu_j^- = \gamma_j + \dots, \tag{113}$$

$$L_j^-(x) = \Big(0, \tilde s_j/\gamma_j\Big) + \dots, \quad R_j^-(x) = \begin{pmatrix} s_j \\ \gamma_j s_j \end{pmatrix} + \dots, \tag{114}$$

for $j < p$, and expansions

$$\mu_j^+ = \gamma_j + \dots, \tag{115}$$

$$L_j^+(x) = \Big(0, \tilde s_j/\gamma_j\Big) + \dots, \quad R_j^\pm(x) = \begin{pmatrix} s_j \\ \gamma_j s_j \end{pmatrix} + \dots, \tag{116}$$

and

$$\mu_j^- = -\lambda/a_j + \lambda^2\beta_j^2/a_j^3 + \dots \tag{117}$$

$$L_j^-(x) = \Big(l_j, -l_jB/a_j\Big) + \dots, \quad R_j^\pm(x) = \begin{pmatrix} r_j \\ 0 \end{pmatrix} + \dots, \tag{118}$$

for $j > p$. Here,

$$\gamma_1, \dots, \gamma_{p-1} \le -\eta < 0 < \eta \le \gamma_{p+1}, \dots, \gamma_n \tag{119}$$

denote eigenvalues of $(B^\epsilon)^{-1}A^\epsilon$, with associated left and right eigenvectors $\tilde s_j$, s_j, and $\beta_j := l_jB^\epsilon r_j$.

Dual to the right- and left-invariant subspaces $\text{Span}_{\,j\ne p}\{R_j\}$ *and* $\text{Span}_{\,j\ne p}\{L_j\}$ *are two-dimensional left- and right-invariant subspaces spanned by vectors L_η, L_z and R_η, R_z that are analytic in λ and (A^ϵ, B^ϵ) on the same neighborhoods of $\lambda = 0$ and $(Df(u_0), B(u_0))$, with expansions*

$$L_\eta = \Big(l_p, l_*\Big) + \dots, \quad L_z = \Big(0, \tilde s_p\Big) + \dots, \tag{120}$$

and

$$R_\eta = \begin{pmatrix} r_p \\ 0 \end{pmatrix} + \dots, \quad R_z = \begin{pmatrix} r_* \\ s_p \end{pmatrix} + \dots, \tag{121}$$

where

$$l_j r_* = \gamma_j l_j s_p, \quad l_* s_j = -l_p s_j/\gamma_j, \qquad \text{for } j \neq p \tag{122}$$

and

$$l_p r_* + l_* s_p = 0. \tag{123}$$

Proof. By standard matrix perturbation theory, we have analytic dependence in (A^ϵ, B^ϵ) and ξ of the eigenvalues λ_j and eigenvectors $\tilde{l}_j$, $\tilde{r}_j$ of the symbol

$$P^\epsilon(i\xi) := -i\xi A^\epsilon - \xi^2 B^\epsilon, \tag{124}$$

with expansions

$$\begin{aligned} \lambda_j &= -i\xi a_j - \xi^2 \beta_j + \dots, \\ \tilde{l}_j &= l_j + \dots, \\ \tilde{r}_j &= r_j + \dots, \end{aligned} \tag{125}$$

about $(Df(u_0), B(u_0), 0$, where $\beta_j := l_j B^\epsilon r_j$. For each $j \neq p$, a_j is bounded from zero, and so (176) may be inverted, by the Analytic Implicit Function Theorem, to yield analytic series for $\mu := i\xi$ and therefore $\tilde{l}_j$, $\tilde{r}_j$ in terms of λ, (A^ϵ, B^ϵ). This verifies expansions (111), (117) for the "slow" transverse modes; the corresponding eigenvector expansions (112), (118) then follow by the relations

$$\begin{aligned} L_j &= \Big(\lambda \tilde{l}_j/(\lambda + \mu_j^2 \beta_j)\Big), \mu_j \tilde{l}_j B^\epsilon/(\lambda + \mu_j^2 \beta_j), \\ R_j &= (\tilde{r}_j, \mu_j \tilde{r}_j), \end{aligned} \tag{126}$$

between $\tilde{l}_j$, $\tilde{r}_j$ and L_j, R_j; for similar arguments.

Expansions (113), (114) and (115), (116) for the "fast" transverse modes follow in more straightforward fashion by (110) and the analytic dependence of isolated eigenvalues/eigenvectors of $\mathbb{A}^\epsilon$ at $\lambda = 0$. Finally, analytic dependence of the complementary two-dimensional invariant subspaces follows by duality, whence (see [Kat]) we obtain the existence of analytic bases (L_η, L_z), (R_η, R_z). By inspection, the choices described are valid representatives at $\lambda = 0$. □

Now, similarly as in the case $B \equiv I$, we make the definitions (61)–(67), to obtain an approximately block-diagonal system satisfying (68)–(76), but with (77) now replaced by

$$\begin{aligned} M_0 &:= L_0 \mathbb{A}^\epsilon R_0 \\ &= L_{0|\lambda=0} \mathbb{A}^\epsilon R_{0|\lambda=0} + (L_0 - L_{0|\lambda=0}) \mathbb{A}^\epsilon_{|\lambda=0} R_{0|\lambda=0} \\ &\quad + L_{0|\lambda=0} \mathbb{A}^\epsilon_{|\lambda=0} (R_0 - R_{0|\lambda=0}) + \mathcal{O}(|\lambda|^2) \\ &= \begin{pmatrix} \lambda l_* B^{-1} r_p & l_p s_p + \lambda l_* B^{-1} r_* + \gamma_p l_* s_p) \\ \lambda \tilde{s}_p B^{-1} r_p & \gamma_p + \lambda \tilde{s}_p B^{-1} r_* \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathcal{O}(|\lambda|) & \mathcal{O}(|\lambda|) \\ \mathcal{O}(|\lambda||\gamma_p| + |\lambda|^2) & \mathcal{O}(|\lambda|) \end{pmatrix}, \end{aligned} \tag{127}$$

where, in the crucial lower left hand corner of the final (error) matrix, we have used the special properties

$$\begin{aligned} \mathbb{A}^\epsilon_{|\lambda=0} R_{\eta\,|\lambda=0} &= 0, \\ L_{z\,|\lambda=0} \mathbb{A}^\epsilon_{|\lambda=0} &= \gamma_p L_{z\,|\lambda=0} \end{aligned} \tag{128}$$

together with the usual

$$|R_0 - R_{0|\lambda=0}|, \, |L_0 - L_{0|\lambda=0}| = \mathcal{O}(|\lambda|) \tag{129}$$

to obtain the stated bound. The $\mathcal{O}(\cdot)$ terms in (127) may be expressed more precisely as

$$\begin{pmatrix} O_1\lambda & O_2\lambda \\ O_3\lambda^2 + O_4\gamma_p\lambda & O_5\lambda \end{pmatrix}, \tag{130}$$

with coefficients O_j analytic in all parameters.

Finally, we observe as in [MZ.1] that we may achieve $L_0R_0' \equiv 0$ by a renormalization $L_0 \to \alpha^{-1}L_0$, $R_0 \to R_0\alpha$, where α is a 2×2 matrix defined by ODE

$$\alpha' = -L_0R_0'\alpha, \quad \alpha|_{x=0} = I. \tag{131}$$

At $\lambda = 0$, L_0R_0' is initially upper-triangular, hence α is as well. In other words, the form of L_0, R_0 are preserved, and (131) amounts to the pair of vector normalizations

$$l_pr_p' \equiv \tilde{s}_ps_p' \equiv 0 \tag{132}$$

together with the condition

$$l_pr_*' + l_*s_p' \equiv 0 \tag{133}$$

restricting choices of l_*, r_*. This does not affect any of the previously stated equations or bounds, but only accomplishes again that error Θ vanish on the η, z block:

$$\Theta_{(\eta,z),(\eta,z)} \equiv 0. \tag{134}$$

Lemma 3.9. *Matrix M_0 defined in* (127) *satisfies*

$$M_0 - \tilde{M}_0 = \begin{pmatrix} \mathcal{O}(|\lambda|) & \mathcal{O}(|\lambda| + \epsilon) \\ \mathcal{O}(|\lambda|\epsilon) & \mathcal{O}(|\lambda| + \epsilon^2) \end{pmatrix} \tag{135}$$

and

$$(M_0 - M_0(\pm\infty)) - (\tilde{M}_0 - \tilde{M}_0(\pm\infty)) = \begin{pmatrix} \mathcal{O}(|\lambda|\epsilon e^{-\theta\epsilon|x|}) & \mathcal{O}((|\lambda|+1)\epsilon e^{-\theta\epsilon|x|}) \\ \mathcal{O}(|\lambda|\epsilon e^{-\theta\epsilon|x|}) & \mathcal{O}((|\lambda|\epsilon + \epsilon^2)e^{-\theta\epsilon|x|}) \end{pmatrix} \tag{136}$$

for $x \lessgtr 0$, where

$$\tilde{M}_0 := \begin{pmatrix} 0 & 1 \\ \lambda/\beta & a_p/\beta \end{pmatrix}, \tag{137}$$

β as in the introduction denoting $\beta_p(u_0) = \beta_p + \mathcal{O}(\epsilon)$.

Proof. Taylor expanding about $a_p = 0$, we have

$$s_p = r_p + O_1(a_p)a_p, \tag{138}$$

$$\tilde{s}_p = l_pB/\beta_p + O_2(a_p)a_p, \tag{139}$$

(up to scalar multipliers) and

$$\gamma_p = a_p/\beta_p + O_3(a_p)a_p^2, \tag{140}$$

with O_j analytic. From these expansions, together with the fact that A, B, l_p, l_*, r_p, r_*, $\tilde{s}_p$, s_p, and a_p, as well as coefficients O_j in (130), all approach their asymptotic values as $x \to \pm\infty$ at rate proportional to $|\bar{u}^\epsilon - u_\pm| \sim \epsilon e^{-\theta\epsilon|x|}$, the result readily follows. □

¿From this point on, the argument goes almost exactly as in the case $B \equiv I$. Carrying out the first reduction eliminating fast transverse modes, and rescaling by $x \to \Lambda\epsilon x/\beta$, $\lambda \to \beta\lambda/\Lambda^2\epsilon^2$, $z \to \beta z/\Lambda\epsilon$, we obtain again the block-triangular system (18)–(27) described in the introduction, with the errors there described. From there on, the arguments agree, word for word. (Recall, we obtained slightly smaller errors in the case $B \equiv I$; however, we only remarked this fact, and did not make use of it in the analysis). This completes the proof of one-dimensional spectral stability in the general strictly parabolic case. □

Remark 3.10. The crucial estimate of the lower left hand entry in the final (error) matrix of (127) is a generalization of the standard observation, in the case of a one-parameter bifurcation in λ from an isolated *semisimple block* $L(0)$, $R(0)$ of matrix $\mathbb{A}(\lambda)$, $L(0)$ and $R(0$ consisting of genuine eigenvectors with a single common eigenvalue $\mu(0)$, that

$$L\mathbb{A}R = L(0)\mathbb{A}R(0) + \mathcal{O}(\lambda^2), \tag{141}$$

which follows from the observation that

$$(d/d\lambda)L\mathbb{A}(0)R(0) + L(0)\mathbb{A}(0)(d/d\lambda)R = \mu(0)(d/d\lambda)(LR) \equiv 0. \tag{142}$$

In the case at hand, which is essentially a two-parameter bifurcation in λ and a_p from a Jordan block at $\lambda = 0$, $a_p = 0$, the computation (142) of course breaks down in general, but *does remain valid* for the single, lower left hand entry of the block that corresponds to the pairing of genuine left and right eigenvectors L_z and R_η, respectively, to give the stated result of validity to order $\mathcal{O}(|\lambda|^2 + |\lambda||a_p|)$. This explains the success of the calculation. Indeed, the entire calculation could have been done by expansion about $\lambda = 0$, $a_p = 0$, giving slightly degraded error terms $\mathcal{O}(|\lambda| + |a_p|)$ in other entries, since these would in fact have been sufficient for the analysis; however, we prefer to carry a_p within the computation, since it is no more difficult and gives sharper estimates.

4. One-dimensional stability for relaxation systems

Next, we treat small-amplitude stationary profiles of one-dimensional relaxation systems

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} \tilde f(u,v) \\ \tilde g(u,v) \end{pmatrix}_x = \begin{pmatrix} 0 \\ q(u,v) \end{pmatrix}, \tag{143}$$

with $q(u, v^*(u)) \equiv 0$, Re $\sigma\big(q_v(u, v^*(u))\big) < 0$ as in (5)–(4), and $f(u) = \tilde f(u, v^*(u))$. We make the assumptions:

(H0) $\tilde f$, $\tilde g$, $q \in C^2$ (regularity).

(H1) $\sigma\left((D\tilde f, D\tilde g)^t(u,v)\right)$ real, semi-simple with constant multiplicity, and uniformly bounded away from 0 (nonstrict hyperbolicity of relaxation system, plus weak subcharacteristic condition).

(H2) $\sigma\left(Df(u_\pm)\right)$ real, simple (strict hyperbolicity of equilibrium system).

(H3) Re $\sigma\left(i\xi\begin{pmatrix} D\tilde f \\ D\tilde g\end{pmatrix}(u_\pm, v_\pm) + \begin{pmatrix} 0 \\ dq \end{pmatrix}(u_\pm, v_\pm)\right) \le \dfrac{-\theta|\xi|^2}{1+|\xi|^2}$, $\theta > 0$, for all $\xi \in \mathbb{R}$ (linearized stability of constant states).

(H4) $a_p = 0$ is a simple eigenvalue of $Df(u_0)$ with left and right eigenvectors l_p and r_p, and $l_p^t D^2 f(r_p, r_p) \neq 0$ (genuine nonlinearity for the equilibrium system of the principal characteristic field).

As in the previous section, $\epsilon > 0$ denotes shock strength $|u_+^\epsilon - u_-^\epsilon|$, and profiles $(\bar u^\epsilon, \bar v^\epsilon)(\cdot)$ are assumed to converge as $\epsilon \to 0$ to (u_0, v_0) (where necessarily $v_0 = v_*(u_0)$), under which assumptions the center-manifold argument of [MZ.1] verifies the assertion of Proposition 1.4, yielding convergence after rescaling of $\bar u^\epsilon$ to the standard Burgers profile (10); see also related analyses of [YoZ, FZe]. In the rest of this section we establish the following analogs of Theorem 3.1 and Proposition 3.2.

Theorem 4.1. *Under assumptions (H0)–(H4), profiles $\bar{u}^\epsilon$ are strongly spectrally stable (and therefore linearly and nonlinearly orbitally stable* [MZ.1]*) for ϵ sufficiently small.*

Proposition 4.2. *Under assumptions (H0)–(H4), profiles $\bar{u}^\epsilon$ are strongly spectrally stable (and therefore linearly and nonlinearly orbitally stable* [MZ.1]*) for ϵ sufficiently small, if the standard Burgers profile* (10) *is strongly spectrally stable for $\epsilon = 1$ and only if the Burgers profile is not strongly spectrally unstable for $\epsilon = 1$ in the sense that there exists an eigenvalue* $\mathrm{Re}\,\lambda > 0$.

Theorem 4.1 completes the program of Mascia and Zumbrun [MZ.1], resolving the open question of stability of general small-amplitude relaxation profiles.

Remark 4.3. The "real," or partially parabolic case is closely related to the relaxation case, and may be treated by quite similar analysis, following the framework set up in Appendix A2 of [Z.3]; see [Z.6]. In the real viscous case, satisfactory small-amplitude stability results have already been obtained by energy methods [HuZ].

4.1. **The 2×2 case.** For clarity, we first carry out the 2×2 case $\tilde{f}$, $\tilde{g}$, q, $f \in \mathbb{R}^1$, which was treated by energy methods in [L.3]. The linearized eigenvalue equations about profile $(\bar{u}^\epsilon, \bar{v}^\epsilon)$ are

$$\left(\mathcal{A}^\epsilon \begin{pmatrix} u \\ v \end{pmatrix}\right)' = (\mathcal{Q}^\epsilon - \lambda I) \begin{pmatrix} u \\ v \end{pmatrix}, \tag{144}$$

where

$$\mathcal{A}^\epsilon(x) := \begin{pmatrix} \tilde{f}_u & \tilde{f}_v \\ \tilde{g}_u & \tilde{g}_v \end{pmatrix} (\bar{u}^\epsilon, \bar{v}^\epsilon)(x), \quad \mathcal{Q}^\epsilon(x) := \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix} (\bar{u}^\epsilon, \bar{v}^\epsilon)(x). \tag{145}$$

Following [Z.3], we make the (nonsingular, by (H1)) change of variables

$$\begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix} := \mathcal{A}^\epsilon \begin{pmatrix} u \\ v \end{pmatrix} \tag{146}$$

to obtain a first-order system

$$\begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix}' = (\mathcal{Q}^\epsilon - \lambda I)(\mathcal{A}^\epsilon)^{-1} \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix} \tag{147}$$

in nondivergence form.

So far, this is analogous to the "flux transform" of [Go.2]. If we were now to convert to a single, second-order equation for $\tilde{f}$, then the analogy would be exact, since the equations would then clearly not support any L^2 (decaying) solutions at $\lambda = 0$. Instead, we take a different approach, keeping the convenient variables $\tilde{f}, \tilde{g}$, and rescaling as

$$\tilde{f} \to \tilde{f}/\lambda, \quad \tilde{g} \to \tilde{g}, \tag{148}$$

to obtain

$$\begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix}' = \operatorname{diag}(\lambda^{-1}, 1)(\mathcal{Q}^\epsilon - \lambda I)(\mathcal{A}^\epsilon)^{-1} \operatorname{diag}(\lambda, 1) \begin{pmatrix} \tilde{f} \\ \tilde{g} \end{pmatrix} \tag{149}$$

in place of (147). Since $\tilde{f}' = \lambda u$ in the original equations (144), we see that the rescaled $\tilde{f}$ variable is, in the case of decaying solutions, given by $\tilde{f}(x) = \int_{-\infty}^x u(z)dz$, i.e., is exactly the *integrated variable* of standard energy analyses [L.3, Liu, HuZ]; in particular, the rescaled equations (149) have the favorable property that they do not support L^2 (decaying) solutions at $\lambda = 0$.

Remark 4.4. Equations (149) are neither pure flux nor pure integrated form, but a new form intermediate to the two, which we might be called "balanced flux form." This form will prove useful again in the multidimensional case treated in [PZ]. (All three forms (flux, integrated, and balanced flux) coincide in the one-dimensional viscous case, so this distinction is not apparent, or necessary, here.)

By direct calculation, we have

$$\mathcal{A}^{-1} = (\det \mathcal{A})^{-1} \begin{pmatrix} \tilde g_v & -\tilde f_v \\ -\tilde g_u & \tilde f_u \end{pmatrix}, \tag{150}$$

$$\begin{aligned} \mathcal{Q}\mathcal{A}^{-1} &= (\det \mathcal{A})^{-1} \begin{pmatrix} 0 & 0 \\ q_u \tilde g_v - q_v \tilde g_u & -q_u \tilde f_v + q_v \tilde f_u \end{pmatrix} \\ &= \frac{q_v}{\det \mathcal{A}} \begin{pmatrix} 0 & 0 \\ -g_u & a_1 \end{pmatrix}, \end{aligned} \tag{151}$$

where $g(u) := g(u, v_*(u))$. Combining, we find that

$$\begin{aligned} (\mathcal{Q}-\lambda)\mathcal{A}^{-1} &= (\det \mathcal{A})^{-1} \Big(q_v \begin{pmatrix} 0 & 0 \\ -g_u & a_1 \end{pmatrix} - \lambda \begin{pmatrix} \tilde g_v & -\tilde f_v \\ -\tilde g_u & \tilde f_u \end{pmatrix} \Big) \\ &= \frac{q_v}{\det \mathcal{A}} \begin{pmatrix} 0 & \lambda \tilde f_v / q_v \\ -g_u & a_1 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(|\lambda|) & 0 \\ \mathcal{O}(|\lambda|) & \mathcal{O}(|\lambda|) \end{pmatrix}, \end{aligned} \tag{152}$$

and therefore

$$\begin{aligned} &\operatorname{diag}(\lambda^{-1}, 1)(\mathcal{Q}-\lambda)\mathcal{A}^{-1} \operatorname{diag}(\lambda, 1) = \\ &\qquad \frac{q_v}{\det \mathcal{A}} \begin{pmatrix} 0 & \tilde f_v / q_v \\ -g_u \lambda & a_1 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(|\lambda|) & 0 \\ \mathcal{O}(|\lambda|^2) & \mathcal{O}(|\lambda|) \end{pmatrix} \end{aligned} \tag{153}$$

in (149).

Let

$$\beta_* := -\tilde f_v q_v^{-1} (g_u - q_v^{-1} q_u f_u) = -\tilde f_v q_v^{-1} (g_u - q_v^{-1} q_u a_1) \tag{154}$$

denote the effective, "Chapman-Enskog viscosity" defined in (14). A consequence of (H3) is that $\beta_* > 0$; see, e.g. [Ze], or Appendix A.2 of [MZ.1].

Lemma 4.5. *There hold* $\det \mathcal{A} = -\tilde f_v gu + \tilde g_v a_1 = -\tilde f_v g_u + \mathcal{O}(\epsilon)$ *and*

$$\begin{aligned} \beta_* = -q_v^{-1} \tilde f_v gu + \tilde f_v q_v^{-2} q_u a_1 &= -q_v^{-1} \det \mathcal{A} + (q_v^{-1} \tilde g_v + \tilde f_v q_v^{-2} q_u) a_1 \\ &= -q_v^{-1} \det \mathcal{A} + \mathcal{O}(\epsilon). \end{aligned} \tag{155}$$

In particular, $\det \mathcal{A} < 0$ *and* $\operatorname{sgn} \tilde f_v = \operatorname{sgn} g_u \neq 0$*, for* ϵ *sufficiently small.*

Proof. A column operation reduces $\det \mathcal{A}$ to $\det \begin{pmatrix} f_u & \tilde f_v \\ g_u & \tilde g_v \end{pmatrix}$, from which the first assertion follows. Combining with (154), we obtain the second assertion. □

Remark 4.6. Condition $\det \mathcal{A} < 0$ for a_1 near zero may be recognized as the subcharacteristic condition, which in the 2×2 case is equivalent to (H3). Condition $\tilde f_v \neq 0$ is the "genuine coupling" assumption of Liu [L.3]; as we have just shown, it is in fact a consequence of (H3), and not a separate assumption.

Now, setting

$$\beta := \beta_* \det \mathcal{A} / q_v(0), \tag{156}$$

rescaling $x \to \Lambda\epsilon x/\beta$, $\lambda \to \beta\lambda/\Lambda^2\epsilon^2$, and balancing $\tilde f \to -g_u\beta\tilde f/\Lambda\epsilon$ (Λ the constant of genuine nonlinearity defined below (14)), we obtain the normal form

$$\begin{pmatrix}\tilde f\\ \tilde g\end{pmatrix}' = \begin{pmatrix}0 & 1\\ \lambda & \bar a_1\end{pmatrix}\begin{pmatrix}\tilde f\\ \tilde g\end{pmatrix} + \Phi\begin{pmatrix}\tilde f\\ \tilde g\end{pmatrix}, \tag{157}$$

where, by Lemma 4.5, error term Φ satisfies bounds (26)–(27) of the introduction: that is, just an approximate Burgers eigenvalue equation. From this form we may immediately conclude the stability results, by a simplified (because transverse, ρ terms are absent) version of the arguments of the previous section. This completes the analysis of the 2×2 case. □

4.2. **The general case.** Now, consider the case of general n, $r \geq 1$, starting as in the previous subsection with the linearized eigenvalue equation written in the "balanced flux variables" introduced in the previous subsection:

$$W' = \mathbb{A}^\epsilon W, \tag{158}$$

$$\begin{aligned}\mathbb{A}^\epsilon :=&\ \mathrm{diag}\,(\lambda^{-1}I_n, I_r)(\mathcal{Q}^\epsilon - \lambda I)(\mathcal{A}^\epsilon)^{-1}\,\mathrm{diag}\,(\lambda I_n, I_r)\\ =&\begin{pmatrix}I_n & 0\\ q_u & q_v\end{pmatrix}(\mathcal{A}^\epsilon)^{-1}\begin{pmatrix}\lambda I_n & 0\\ 0 & I_r\end{pmatrix}\end{aligned} \tag{159}$$

$\mathcal{A}^\epsilon$, $\mathcal{Q}^\epsilon$ as defined in (145).

Define

$$(\tilde H, H) := (q_u, q_v)\mathcal{A}^{-1}, \tag{160}$$

$\tilde H \in \mathbb{R}^{r\times n}$, $H \in \mathbb{R}^{r\times r}$. As discussed in the treatment of existence given in Appendix A.1 of [MZ.1], the $r \times r$ matrix H has a special role as the coefficient matrix of the linearized traveling wave ODE expressed in flux coordinates. For simplicity of exposition, similarly as in the viscous case, we make the provisional hypothesis:

(A1) $\sigma H(u_0)$ distinct; again, this is easily removed at the expense of further

bookkeeping. A consequence of dissipativity, (H3), is that on a sufficiently small neighborhood of the base point u_0, $H(u)$ has a single eigenvalue

$$\gamma_k = \mathcal{O}(\epsilon), \tag{161}$$

with the $(r-1)$ remaining eigenvalues uniformly bounded from zero as $\epsilon \to 0$:

$$\gamma_1, \dots, \gamma_{k-1} \leq -\eta < 0 < \eta \leq \gamma_{k+1}, \dots, \gamma_r; \tag{162}$$

we denote the associated left and right eigenvectors by $\tilde s_j$ and s_j, respectively.

At $\lambda = 0$, $\mathbb{A}^\epsilon$ reduces to

$$\begin{pmatrix}0 & E\\ 0 & H\end{pmatrix}, \tag{163}$$

where

$$(\tilde E, E) := \Big(-I_n, 0\Big)(\mathcal{A}^\epsilon)^{-1}. \tag{164}$$

More generally, we have

$$\mathbb{A}^\epsilon(x,\lambda) = \begin{pmatrix}\lambda\tilde E & E\\ \lambda\tilde H + \lambda^2\tilde F & H + \lambda F\end{pmatrix}, \tag{165}$$

where $\tilde E$, E, $\tilde H$, H are as defined in (164) and (160), and

$$(\tilde F, F) := \Big(0, -I_r\Big)(\mathcal{A}^\epsilon)^{-1}. \tag{166}$$

Lemma 4.7. *Associated with each eigenvalue γ_j, $j \neq k$ of H, there exist "fast" eigenvalues μ_j^f and associated left and right eigenvectors L_j^f and R_j^f of $\mathbb{A}^\epsilon$, analytic in λ and $(\mathcal{A}^\epsilon, \mathcal{Q}^\epsilon)$ on sufficiently small neighborhoods of $\lambda = 0$ and $(\mathcal{A}^\epsilon, \mathcal{Q}^\epsilon)(u_0, v_0)$, with expansions*

$$\mu_j^f = \gamma_j + \dots, \tag{167}$$

and

$$L_j^f = \begin{pmatrix} 0, \tilde{s}_j \end{pmatrix} + \dots, \quad R_j^f = \begin{pmatrix} \gamma_j^{-1} E s_j \\ s_j \end{pmatrix} + \dots, \tag{168}$$

where γ_j, $\tilde{s}_j$, s_j $\tilde{s}_j$ are eigenvalues and associated left and right eigenvectors of H, and E is as defined in (164)*.*

Likewise, associated with each of the transverse fields $j \neq p$ of A^ϵ, there exist "slow" eigenvalues μ_j^s and associated left and right eigenvectors L_j^s and R_j^s of $\mathbb{A}^\epsilon$, analytic in λ and $(\mathcal{A}^\epsilon, \mathcal{Q}^\epsilon)$ on sufficiently small neighborhoods of $\lambda = 0$ and $(\mathcal{A}^\epsilon, \mathcal{Q}^\epsilon)(u_0, v_0)$, of $\lambda = 0$ and $(\mathcal{A}, \mathcal{Q})(u_0, v_0))$, with expansions

$$\mu_j^s = -\lambda/a_j + \lambda^2 \beta_j^2 / a_j^3 + \dots, \tag{169}$$

$$L_j^s(x) = \begin{pmatrix} l_j, \tilde{t}_j \end{pmatrix} + \dots, \quad R_j^s(x) = \begin{pmatrix} r_j \\ 0 \end{pmatrix} + \dots, \tag{170}$$

where $\beta_j := l_j B_* r_j$, with B_* the effective, "Chapman-Enskog" viscosity defined in (14) *of the introduction, and $\tilde{t}_j$ satisfies*

$$l_j E + \tilde{t}_j H = 0. \tag{171}$$

Dual to the right- and left-invariant subspaces $\text{Span}_{\,j\neq k}\{R_j^f\} \oplus \text{Span}_{\,j\neq p}\{R_j^s\}$ *and* $\text{Span}_{\,j\neq k}\{L_j^f\} \oplus \text{Span}_{\,j\neq p}\{L_j^s\}$ *are two-dimensional left- and right-invariant subspaces spanned by vectors L_η, L_z and R_η, R_z analytic in λ and $(\mathcal{A}^\epsilon, \mathcal{Q}^\epsilon)$ on the same neighborhoods of $\lambda = 0$ and $(\mathcal{A}^\epsilon, \mathcal{Q}^\epsilon)(u_0, v_0)$, with expansions*

$$L_\eta = \begin{pmatrix} l_p, l_* \end{pmatrix} + \dots, \quad L_z = \begin{pmatrix} 0, \tilde{s}_p \end{pmatrix} + \dots, \tag{172}$$

and

$$R_\eta = \begin{pmatrix} r_p \\ 0 \end{pmatrix} + \dots, \quad R_z = \begin{pmatrix} r_* \\ s_p \end{pmatrix} + \dots, \tag{173}$$

where

$$l_j r_* = -\tilde{t}_j s_p, \quad l_* s_j = -\gamma_j^{-1} l_p E s_j, \qquad \text{for } j \neq p \tag{174}$$

and

$$l_p r_* + l_* s_p = 0. \tag{175}$$

Proof. This follows similarly as in the proof of Lemma 3.8, namely, analytic variation of the fast transverse modes follows by their spectral separation, while analytic variation of slow transverse modes follows by inversion (and appropriate conjugation) of the associated dispersion relations

$$\begin{aligned} \lambda_j &= -i\xi a_j - \xi^2 \beta_j + \dots, \\ \tilde{\mathcal{L}}_j &= \mathcal{L}_j + \dots, \\ \tilde{\mathcal{R}}_j &= \mathcal{R}_j + \dots, \end{aligned} \tag{176}$$

of the symbol $P^\epsilon(i\xi) := \mathcal{Q}^\epsilon - i\xi\mathcal{A}^\epsilon$. The calculation of expansions (176) may be found, e.g., in Appendix A.2 of [Z.3] or Appendix A.2 of [MZ.1]. Analytic variation of the two-dimensional complementary subspaces again follows by duality. □

As in the treatment of the viscous case, we make the definitions (61)–(67), to obtain an approximately block-diagonal system satisfying (68)–(76), with (77) replaced by the following analog of (127):

$$\begin{aligned} M_0 &:= L_0\mathbb{A}^\epsilon R_0 \\ &= L_{0|\lambda=0}\mathbb{A}^\epsilon R_{0|\lambda=0} + (L_0 - L_{0|\lambda=0})\mathbb{A}^\epsilon_{|\lambda=0} R_{0|\lambda=0} \\ &\quad + L_{0|\lambda=0}\mathbb{A}^\epsilon_{|\lambda=0}(R_0 - R_{0|\lambda=0}) + \mathcal{O}(|\lambda|^2) \\ &= \begin{pmatrix} \mathcal{O}(\lambda) & l_pEs_p + \gamma_p l_* s_p + \mathcal{O}(\lambda) \\ \lambda \tilde{s}_p \tilde{H} r_p & \gamma_p + \mathcal{O}(\lambda) \end{pmatrix}, \\ &\quad + \begin{pmatrix} \mathcal{O}(|\lambda|) & \mathcal{O}(|\lambda|) \\ \mathcal{O}(|\lambda||\gamma_p| + |\lambda|^2) & \mathcal{O}(|\lambda|) \end{pmatrix}, \end{aligned} \tag{177}$$

where all $\mathcal{O}(\cdot)$ terms are analytic in λ, $\mathcal{A}$, $\mathcal{Q}$. Here, as in the viscous case, we have used (128) to improve the crucial bound in the lower left hand corner of the final, error matrix; more precisely, we have description (130) for this error bound, with coefficients O_j analytic in all parameters. As in the viscous case, we may achieve $L_0R_0' \equiv 0$ by a renormalization leaving the form of the equations otherwise unchanged; thus, error Θ vanishes on the η, z block:

$$\Theta_{(\eta,z),(\eta,z)} \equiv 0. \tag{178}$$

Lemma 4.8. *Matrix M_0 defined in* (177) *satisfies*

$$M_0 - \tilde{M}_0 = \begin{pmatrix} \mathcal{O}(|\lambda|) & \mathcal{O}(|\lambda| + \epsilon) \\ \mathcal{O}(|\lambda|\epsilon) & \mathcal{O}(|\lambda| + \epsilon^2) \end{pmatrix} \tag{179}$$

and

$$(M_0 - M_0(\pm\infty)) - (\tilde{M}_0 - \tilde{M}_0(\pm\infty)) = \begin{pmatrix} \mathcal{O}(|\lambda|\epsilon e^{-\theta\epsilon|x|}) & \mathcal{O}((|\lambda|+1)\epsilon e^{-\theta\epsilon|x|}) \\ \mathcal{O}(|\lambda|\epsilon e^{-\theta\epsilon|x|}) & \mathcal{O}((|\lambda|\epsilon + \epsilon^2)e^{-\theta\epsilon|x|}) \end{pmatrix} \tag{180}$$

for $x \lesseqgtr 0$, where

$$\tilde{M}_0 := \begin{pmatrix} 0 & 1 \\ \lambda/\beta & a_p/\beta \end{pmatrix}, \tag{181}$$

$\beta := \beta_p(0)$, and as usual $\beta_j := l_jB_*r_j$ with B_* as defined in (14).

Proof. Taylor expanding about $a_p = 0$, we have up to scalar multipliers (see Appendix A.1, [MZ.1], Claim just below equation (8.7)):

$$s_p = g_u r_p + O_1(|a_p|)a_p, \tag{182}$$

$$\tilde{s}_p = -l_p\tilde{f}_v q_v^{-1}/\beta_p + O_2(|a_p|)a_p, \tag{183}$$

and (by a standard matrix perturbation calculation for a simple eigenvector: see (8.14)–(8.15) of Appendix A.1, [MZ.1]).

$$\gamma_p = a_p/\beta_p + O_3(|a_p|)a_p^2, \tag{184}$$

where O_j are analytic. Furthermore, at $a_p = 0$, there hold the relations:

$$\left(\tilde{E}, E\right)\begin{pmatrix} 0 \\ s_p \end{pmatrix} = r_p, \tag{185}$$

$$\left(\tilde{F}, F\right)\begin{pmatrix} 0 \\ s_p \end{pmatrix} = -q_v^{-1}q_u r_p \tag{186}$$

(see (8.8), Appendix A.1, [MZ.1]), and

$$\tilde{s}_p\left(\tilde{H}, H\right) = \left(-l_p/\beta, 0\right) \tag{187}$$

(see (8.10), Appendix A.1, [MZ.1]), from which we obtain the key facts:

$$Es_p = r_p, \quad Fs_p = -q_v^{-1} q_u r_p, \quad \tilde{s}_p \tilde{H} = -l_p/\beta. \tag{188}$$

From expansions (182)–(184), the relations (188), and the fact that A, β_p, l_p, l_*, r_p, r_*, $\tilde{s}_p$, s_p, and a_p, as well as coefficients O_j in (130), all approach their asymptotic values as $x \to \pm\infty$ at rate proportional to $|\bar{u}^\epsilon - u_\pm| \sim \epsilon e^{-\theta\epsilon|x|}$), the result readily follows. □

Remark. Note that the natural choice of basis vectors L_z and R_z given in Lemma 4.7 automatically accomplishes the desired rescaling to the viscous case, so that we need not do this "by hand" as in the treatment of the 2×2 case above.

From this point, the argument goes through exactly as in the general viscous case. Carrying out the first reduction eliminating the $r-1$ fast transverse modes, and rescaling by $x \to \Lambda\epsilon x/\beta$, $\lambda \to \beta\lambda/\Lambda^2\epsilon^2$, $z \to \beta z/\Lambda\epsilon$, we obtain as in the general viscous case the block-triangular system (18)–(22) described in the introduction, with errors (23)–(27), and so the remaining arguments go through as before. This completes the proof of one-dimensional spectral stability in the general relaxation case. □

5. Multidimensional stability for parabolic systems

Finally, we briefly discuss the extension of the above analysis to the multidimensional viscous case; details will be given in [PZ]. Consider a sequence of planar stationary profiles $\bar{u}^\epsilon(x_1)$ of a strictly parabolic viscous conservation law of form

$$u_t + \sum_j f^j(u)_{x_j} = \sum_{jk} (B^{jk}(u) u_{x_k})_{x_j}, \tag{189}$$

lying in a neighborhood $\mathcal{U}$ of a particular state u_0, where $x \in \mathbb{R}^d$, u, $f^j \in \mathbb{R}^n$, and $B^{jk} \in \mathbb{R}^{n\times n}$. We make the assumptions:

(H0) $f^j, B^{jk} \in C^2$ (regularity).

(H1) Re $\sigma(\sum B^{jk}(\bar{u})\xi_j\xi_k) > 0$ for $\xi \in \mathbb{R}^d$ (strict parabolicity).

(H2) There exists $A^0(\cdot)$, symmetric positive definite and smoothly depending on u, such that $A^0 Df^j(u)$ is symmetric for all $1 \le j \le d$. (simultaneous symmetrizability, $\Rightarrow$ nonstrict hyperbolicity).

(H3) Re $\sigma(-i\sum Df^j(u_\pm)\xi_j - \sum B^{jk}(u_\pm)\xi_j\xi_k) \le -\theta|\xi|^2$, $\xi \in \mathbb{R}^d$, for some $\theta > 0$ (linearized stability of constant states).

(H4) (i) $a_p = 0$ is a simple eigenvalue of $Df^1(u_0)$ with left and right eigenvectors l_p and r_p, and $l_p^t D^2 f^1(r_p, r_p) \neq 0$ (genuine nonlinearity of the principal characteristic field in the normal spatial direction x_1).

(ii) r_p is not an eigenvector of any $Df^{\tilde{\xi}} := \sum_{j\neq 1} \xi_j Df^j$ for $\tilde{\xi} \in \mathbb{R}^{d-1} \neq 0$, and, in the intermediate case $1 < p < n$:

$$\langle r_p, A^0 A^{\tilde{\xi}} \tilde{\Pi} (A^1)^{-1} \tilde{\Pi} A^{\tilde{\xi}} r_p \rangle \neq 0 \tag{190}$$

for all $\tilde{\xi} \in \mathbb{R}^{d-1} \neq 0$, where $A^{\tilde{\xi}} := \sum_{j\neq p} Df^j \xi_j$, $A^1 := Df^1$, and $\tilde{\Pi}$ denotes the projection complementary to the eigenprojection of A^1 onto r_p ("nonresonance," or genuine coupling in the sense of [Mé.1, Mé.2, Mé.3, Mé.4, FZ, Z.5, Z.6]).

Here, $\epsilon > 0$ as usual denotes shock strength $|u_+^\epsilon - u_-^\epsilon|$, and profiles $\bar{u}^\epsilon(\cdot)$ are assumed to converge as $\epsilon \to 0$ to u_0. Since the profile problem is restricted to the normal direction, the one-dimensional analysis of Majda&Pego [MP] again yields

Proposition 1.4, with convergence after rescaling of $\bar u^\epsilon$ to the standard Burgers profile (10). Linearizing about $\bar u^\epsilon$, we obtain a linearized evolution system (7) with coefficients depending only on x_1; taking the Fourier transform in directions $\tilde x := (x_2, \dots, x_d)$, we then obtain for each fixed ϵ a *family* of one-dimensional equations

$$\hat v_t = L_{\tilde\xi} \hat v, \tag{191}$$

indexed by $\tilde\xi \in \mathbb{R}^{d-1}$.

Definition 5.1. Following [Z.3], we define strong spectral stability (condition (D1') of the reference) in the multidimensional case as

$$\text{Re}\ \sigma_p(L_\xi) < 0 \text{ for } (\tilde\xi, \lambda) \neq (0,0): \tag{192}$$

equivalently, the generalized eigenvalue equation $(L_{\tilde\xi} - \lambda)w = 0$ admits no L^2 solutions for $\text{Re}\ \lambda \geq 0$ and $(\tilde\xi, \lambda) \neq (0,0)$.

Under hypotheses (H0)–(H4), the results of [Z.3] show that strong spectral spectral stability implies linearized and nonlinear stability with sharp rates of decay (as discussed in [PZ], our conditions (H0)–(H4) imply conditions (H0)–(H7) assumed in the reference). In [PZ] we establish:

Theorem 5.2. *Under assumptions (H0)–(H4), profiles $\bar u^\epsilon$ are strongly spectrally stable (and therefore linearly and nonlinearly orbitally stable* [ZH, Z.2]*) for ϵ sufficiently small.*

Remark 5.3. As in the inviscid small amplitude case [Mé.1, Mé.2, Mé.3, Mé.4, Mé.5, FM, Z.5, Z.6, FZ], the condition of simultaneous symmetrizability (H2) may be relaxed under appropriate alternative structural conditions; however, so far there has been identified no such conditions of simple and general application. Likewise, the nonresonance condition (H4)(ii) might possibly be relaxed; however, the degenerate case that this condition fails would appear to require a considerably more detailed analysis than the one carried out here. In any case, these conditions are satisfied for most systems arising in physical applications; see [Mé.5, Z.3, Z.5, Z.6]. In the extreme shock case, (H4)(ii) is equivalent to a simpler and easily verified condition identified by Métivier [Mé.1, Mé.2, Mé.3, Mé.4, Mé.5]: that the principal eigenvalue a_p of the full symbol $\sum_{j=1}^d \xi_j Df^j$ be a strictly convex, resp. concave, function of $\xi = (\xi_1, \dots, \xi_d)$, according as $p = 1$ or n; see [FZ] or (implicitly) [Mé.1]. Note that (190) is automatically satisfied in the extreme shock case. The case of "real," or partially parabolic, viscosity, or relaxation may be treated similarly, following the model of the present, one-dimensional analysis.

Similarly as in the one-dimensional case, Theorem 5.2 is established by showing that stability is equivalent, modulo pure imaginary eigenvalues as in Theorems 3.2 and 4.2, to stability of an appropriate canonical system. However, this system is not a 2×2 system as in the scalar multidimensional case, but a 3×3 system corresponding to the full slow flow of an appropriate 2×2 viscous conservation law, namely, the *coupled Burgers-linear degenerate equations*,

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} u^2/2 \\ av \end{pmatrix}_{x_1} + \begin{pmatrix} v \\ u \end{pmatrix}_{x_2} = \sum B^{jk} \begin{pmatrix} u \\ v \end{pmatrix}_{x_j x_k}, \tag{193}$$

considered with the family of profiles

$$\begin{pmatrix} \bar u^\epsilon \\ \bar v^\epsilon \end{pmatrix}(x) = \begin{pmatrix} -\epsilon \tanh(\epsilon x/2) \\ 0 \end{pmatrix}, \tag{194}$$

where $a > 0$ and B^{jk} are constant, diagonal, and satisfy the uniform parabolicity condition

$$\sum_{jk} \xi_j \xi_k B^{jk} \geq \theta |\xi|^2, \quad \theta > 0, \tag{195}$$

for all $\xi := (\xi_1, \xi_2) \in \mathbb{R}^2$, with $B^{11}_{11} = 1$.

That is, *in multiple dimensions, small amplitude behavior is not captured by any scalar model, but generically requires a* 2×2 *model for its expression.* This fundamental observation was made in the inviscid setting by Métivier [Mé.1, Mé.2, Mé.3, Mé.4, Mé.5, FM] in his study of the weak shock limit, and is due to the new phenomenon of "glancing modes" arising in the multi-dimensional case; see also [Z.5, Z.6, FZ] for related discussions in the inviscid case. Accordingly, the analysis of the reduced equations on the "slow–superslow" manifold becomes considerably more complicated in multiple dimensions; in particular, we can no longer rely on simple Sturm–Liouville, or maximum principles to conclude stability of the reduced system. The following proposition, though it does not (because the equations (193) are not scale-invariant) directly imply stability of the reduced system, nonetheless indicates at an intuitive level why the analysis can still be carried out, to yield a full stability result and not only a reduction. Note that the reduction to constant coefficients a, A^2, B^{jk} (achieved as in the one-dimensional analysis by an application of Proposition 2.4) is crucial in the proof.

Proposition 5.4. *Shock profiles* (194) *of any amplitude* ϵ *are strongly spectrally stable as solutions of* (193)*, assuming only the partial parabolicity condition*

$$\sum_{jk} \xi_j \xi_k B^{jk} \geq \theta |\xi|^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \theta > 0. \tag{196}$$

Proof. We use a straightforward energy estimate modifying that used in [MN, Go.1] to treat the one-dimensional genuinely nonlinear scalar case. Linearizing (193) about a stationary profile (194), we obtain the generalized eigenvalue equation

$$(\lambda + i\xi_2 A^2 + \xi_2^2 B^{22})w + (Aw)' + i\xi_2(B^{12} + B^{21})w' = B^{11}w'', \tag{197}$$

where

$$A^1 := \begin{pmatrix} \bar{u}(x_1) & 0 \\ 0 & a \end{pmatrix} \qquad A^2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \tag{198}$$

here, as usual, "$'$" denotes $\partial/\partial x_1$.

Examining the symbols

$$P_\pm(i\xi) := i\xi_1 A^1_\pm + i\xi_2 A^2 + \xi_1^2 B^{11} + i\xi_2(B^{12} + B^{21}) + \xi_2^2 B^{22} \tag{199}$$

of the constant coefficient limiting equations of (197) as $x_1 \to \pm\infty$, we find from (196) that

$$\text{Re } P_\pm \geq \theta |\xi|^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{200}$$

whence, for $\xi \neq 0$, $P_\pm$ has no pure imaginary eigenvalues λ unless associated with an eigenvector of form $(0, 1)^t$, in which case we find by direct calculation that $\xi_2 = 0$ and $\lambda = -a\xi_1$ and (197) decouples into the one-dimensional Burgers eigenvalue equation and a trivial, constant–coefficient equation. In either case, we find from standard considerations (see, e.g., [He]) that w and all derivatives decay exponentially at spatial infinity for $w \in L^2$ and $(\xi, \lambda) \neq (0, 0)$. Similarly, from

$$\text{Re } (\lambda + i\xi_2 A^2 + \xi_2^2 B^{22}) = \text{Re } (\lambda) I + \xi_2^2 \text{Re } (B^{22}) \geq \theta_2 \xi_2^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{201}$$

for $\theta_2 > 0$, by (195), we easily find that $(\lambda + i\xi_2 A^2 + \xi_2^2 B^{22})$ is invertible for all real ξ_2 and Re $\lambda \geq 0$ such that $(\xi_2, \lambda) \neq (0,0)$, whence $\int_{-\infty}^{+\infty} w(z)dz = 0$; for, this implies that the kernel, if it exists, is spanned by $(0,1)^t$), yielding $\xi_2 = \lambda = 0$ by direct calculation.

Combining these observations, and integrating (197) from $-\infty$ to $+\infty$, we obtain

$$(\lambda + i\xi_2 A^2 + \xi_2^2 B^{22}) \int_{-\infty}^{+\infty} w(z)dz = 0 \tag{202}$$

and therefore $\int_{-\infty}^{+\infty} w(z)dz = 0$.

Thus, we may introduce the integrated variable $W(x_1) := \int_{-\infty}^{x_1} w(z)dz$, with $W \in L^2$ satisfying the integrated eigenvalue equation

$$(\lambda + i\xi_2 A^2 + \xi_2^2 B^{22})W + AW' + i\xi_2(B^{12} + B^{21})W' = B^{11}W'', \tag{203}$$

with W and derivatives again decaying exponentially at spatial infinity. Taking the real part of the complex inner product of W against (203), integrating by parts, and using (195), we obtain the estimate

$$\begin{aligned} \text{Re}\ \lambda|W|^2 - \int \bar{u}'|W1|^2 dx &= -\langle W', B^{11}W'\rangle + \langle i\xi_2 W, B^{12}W'\rangle \\ &\quad + \langle W', B^{12} i\xi_2 W\rangle + \langle i\xi_2 W, B^{22} i\xi_2 W\rangle \\ &\leq -\theta(|W_1'|^2 + \xi_2^2|W_1|^2), \end{aligned} \tag{204}$$

$\theta > 0$, a contradiction of the fact that $\bar{u}' < 0$, unless $|W_1'| \equiv \xi_2|W_1| \equiv 0$, hence $|W_1| \equiv 0$, and Re $\lambda = 0$. But, this case may be disposed of by observing that (197) then reduces to a single equation in W_2, which is constant-coefficient and therefore supports no L^2 solutions. □

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