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by

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CONVEX INTEGRATION AND THE L^p THEORY OF ELLIPTIC EQUATIONS

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Abstract. We consider elliptic partial differential equations and provide a method constructing solutions with critical integrability properties. We illustrate the technique by studying isotropic equations and equations in non-divergence form in the plane.

1. introduction

In the theory of the elliptic partial differential equations with bounded measurable coefficients the solutions are basically assumed to have square summable derivatives; for equations of non-divergence form the assumptions concern the second derivatives. It is well known that there is a range of exponents beyond the $p = 2$ where the L^p-theory of derivatives is still valid. Recent developments in the theory of planar quasiconformal mappings, in particular the area distortion theorem obtained by the first author in [1] and the invertibility of Beltrami Operators proved in [4], have in two dimensions provided the precise range for these exponents, see [2], [20] and Theorem 1.1 below. For more information see also the monograph [3]. These ranges of exponents depend only on the ellipticity constants of the equation.

It is a natural question if restricting the range of the ellipticity matrix could yield higher integrability for the gradients of the solutions. We present in this article a general method for constructing examples which show that such improved regularity is not possible and that beyond the critical exponents the theory is not valid anymore.

Let us start by recalling the basic notations and the positive results.

Theorem 1.1. *Let $K \geqq 1$ and $\frac{2K}{K+1} \leqq q$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $\sigma(x) : \Omega \to \mathbb{R}^{2\times 2}_{sym}$ be a measurable function such that for almost every x,*

$$\frac{1}{K} I \leqq \sigma(x) \leqq K I \tag{1}$$

in the sense of quadratic forms.

Let $u \in W^{1,q}_{loc}(\Omega)$ be a weak solution of the equation

$$\operatorname{div} \sigma(x) \nabla u = 0 \tag{2}$$

in Ω. Then $u \in W^{1,p}_{loc}(\Omega)$ for all $p < \frac{2K}{K-1}$.

Here $\mathbb{R}^{2\times 2}_{\text{sym}}$ represents the space of symmetric matrices with real entries.

Theorem 1.1 was obtained by Leonetti and Nesi in [20] where they discovered that under the assumptions (1), (2) the solution $u = \Re f$, where f is a K-quasiregular mapping, with the same K. Since the work of the first author [1] says that a K quasiregular mapping in $W^{1,q}$ is immediately in $W^{1,p}$ whenever $\frac{2K}{K+1} < q < p < \frac{2K}{K-1}$, Theorem 1.1 follows. In addition, in the theorem we have included the end point $q = \frac{2K}{K+1}$. This case follows from the recent result of Petermichl and Volberg (see [4],[28],[10]).

The classical examples built on the radial stretching $u(x) = \Re(x|x|^{\frac{1}{K}-1})$ show that for general σ the range of exponents p, q can not be improved without extra assumptions. On the other hand, there was the hope that if the range of σ was restricted then the gradients would enjoy higher integrability. A basic example pointing towards this direction is the work of Piccinini and Spagnolo [29]. There it is shown that if $\sigma(x) = \rho(x)I$, where ρ is a real valued function with $1/K \le \rho(x) \le K$, then u has a better Hölder regularity than in the case of a general σ.

Our first theorems show, however, that for Sobolev regularity one can not improve any of the critical exponents $\frac{2K}{K+1}, \frac{2K}{K-1}$ even if the essential range of σ consists of only two matrices.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ and let $K > 1$. Then there exists a measurable function $\rho_1 : \Omega \to \{\frac{1}{K}, K\}$ such that the solution $u_1 \in W^{1,2}(\Omega)$ to the equation*

$$\begin{cases} \operatorname{div} \rho_1(x)\nabla u_1(x) = 0 & \text{in } \Omega \\ u_1(x) = x_1 & \text{on } \partial\Omega \end{cases} \tag{3}$$

satisfies for every $B(x,r) \subset \Omega$ the condition

$$\int_{B(x,r)} |\nabla u_1|^{\frac{2K}{K-1}} = \infty. \tag{4}$$

Theorem 1.3. *For every $\alpha \in (0,1)$ there exists a measurable function $\rho_2 : \Omega \to \{\frac{1}{K}, K\}$ and a function $u_2 \in C^\alpha(\overline{\Omega})$ such that $u_2(x) = x_1$ on $\partial\Omega$,*

$$\operatorname{div} \rho_2(x)\nabla u_2(x) = 0$$

in the sense of distributions and $u_2 \in W^{1,q}(\Omega)$ for all $q < \frac{2K}{K+1}$, but for every $B(x,r) \subset \Omega$

$$\int_{B(x,r)} |\nabla u_2|^{\frac{2K}{K+1}} = \infty. \tag{5}$$

As a particular consequence, Theorems 1.2 and Theorem 1.3 apply also to the quasiregular mappings since $u_1 = \Re f$ where f is quasiregular with

$$\partial_{\bar z} f = \pm k \overline{\partial_z f}, \qquad k = \frac{K-1}{K+1}. \tag{6}$$

The same ideas yield extremal solutions also to the Beltrami equation

$$\partial_{\bar z} f = \pm k \partial_z f, \tag{7}$$

for details see Remark 3.4.

Remark 1.1. Our methods do not imply that ρ_1 and ρ_2 in the above theorems could be equal. Surprisingly, in the analogous problem for the Beltrami equation (7) a simple argument shows that this choice can be made. To see this, let f_2 be the very weak solution of (7) constructed in Theorem 3.2, with dilatation $\mu : \Omega \to \{\pm k\}$ and $u_2 = \Re f_2$ satisfying (5). Let f_1 be the classical homeomorphic solution to the same equation, i.e. $f_1 \in W^{1,2}_{loc}$ and $\partial_{\bar z} f_1 = \mu \partial_z f_1$. Suppose that there exists a ball $B \subset \Omega$ where $\int_B |\nabla f_1|^{\frac{2K}{K-1}+\varepsilon} < \infty$ for some $\varepsilon > 0$. Since $\nabla f_2 \in L^q(B)$ for any $q < \frac{2K}{K+1}$, we deduce that $\nabla f_1 \in L^{p_0}(B)$ and $\nabla f_2 \in L^{q_0}(B)$ for some dual exponents p_0 and q_0. But then, for example by [19, Lemma 6.4], the composition $f = f_2 \circ f_1^{-1}$ is in $W^{1,1}(f_1 B)$ and f obeys the chain rule. Therefore $\partial_{\bar z} f = 0$ a.e. and by Weyl's lemma f is analytic. But then $f_2 = f \circ f_1$ is also quasiregular, which contradicts the fact that $f_2 \notin W^{1,2}(B)$.

Let us then consider linear elliptic equations in the non-divergence form. The following theorem follows from the recent work of Astala, Iwaniec and Martin in [2] where quasiconformal techniques are applied for developing the precise L^p theory for planar equations in non-divergence form.

Theorem 1.4. *Let $K \geq 1$, let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $A(x) : \Omega \to \mathbb{R}^{2\times 2}_{sym}$ be a measurable function such that for a.e. $x \in \Omega$,*

$$\frac{1}{\sqrt{K}} I \leqq A(x) \leqq \sqrt{K}\, I, \qquad \det A(x) \equiv 1. \tag{8}$$

Let $u \in W^{2,q}(\Omega)$, $q > \frac{2K}{K+1}$, be a solution of the equation

$$\operatorname{Tr}\big(A(x) D^2 u(x)\big) = 0 \tag{9}$$

where $D^2 u(x) = \big(\partial_{ij} u(x)\big)_{ij}$ is the Hessian matrix of u. Then $u \in W^{2,p}_{loc}(\Omega)$ for all $p < \frac{2K}{K-1}$.

The latter condition on $A(x)$ in (8) is a normalization which can always be made since A is bounded above and below and (9) is pointwise linear.

A key fact of the proof is that under the assumptions of the theorem, the complex gradient of u, $\partial_z u = (u_x, -u_y)$ is a K-quasiregular mapping. Therefore the ideas from [1],[4],[28] apply.

Concerning the sharpness of the theorem, an example due to Pucci, built on an appropriate radial function shows that there are no L^p estimates at the lower critical exponent $p = \frac{2K}{K+1}$. Nevertheless, examples built on radial functions do not seem to work with the upper critical exponent. We not only provide the required examples but again show that the range of $A(x)$ is as simple as one can ask for.

Theorem 1.5. *Let $K \geq 1$ and let $\Omega \subset \mathbb{R}^2$ be a bounded domain.*

There exists a measurable $A_3 : \Omega \to \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$ such that the solution $u_3 \in W^{2,2}(\Omega)$ to the equation

$$\begin{cases} \mathrm{Tr}(A_3(x)D^2u_3) = 0 & \text{in } \Omega \\ u_3(x) = x_1 & \text{on } \partial\Omega \end{cases} \tag{10}$$

satisfies for every $B(x,r) \subset \Omega$

$$\int_{B(x,r)} |D^2 u_3|^{\frac{2K}{K-1}} = \infty. \tag{11}$$

Theorem 1.6. . *For every $\alpha \in (0,1)$ there exists a measurable mapping $A_4 : \Omega \to \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$ and a function $u_4 \in W^{2,q}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for all $q < \frac{2K}{K+1}$, such that*

$$\mathrm{Tr}(A_4(x)D^2u_4) = 0 \tag{12}$$

in the sense of distributions but for every $B(x,r) \subset \Omega$

$$\int_{B(x,r)} |D^2 u_4|^{\frac{2K}{K+1}} = \infty. \tag{13}$$

Theorem 1.2 is a generalization of [11] which in turn has roots in [22] where Milton proposed as extremal configurations the so-called layered construction with infinitely many scales involved. In [11] the second author interpreted Milton's idea from a different point of view introducing the so-called staircase laminates and used Beltrami Operators to complete the technical details left open by Milton. Unfortunately, the method in [11] yield only a sequence of equations of the type (3), such that the corresponding solutions $\{u_j\}_{j=1}^{\infty}$ satisfy

$$\lim_{j\to\infty} \int_B |\nabla u_j|^{\frac{2K}{K-1}} = \infty.$$

Moreover, the method is not valid for showing that the lower critical exponent $\frac{2K}{K+1}$ is sharp.

In this work we replace the use of Beltrami Operators by versions of *convex integration*. It was discovered by V. Šverák and S. Müller that the method of convex integration, introduced by M. Gromov [14] for solving partial differential inclusions could be adapted to the Lipschitz setting. This method has then been successfully applied to produce minimizers to certain variational problems motivated by nonlinear elasticity where direct methods fail due to lack of lower semicontinuity ([24, 23]). The same authors also used convex integration in [25] to construct counterexamples to regularity for elliptic systems (see also [33]). The method has been intensively studied and adapted to produce weak solutions to various other problems [17, 32, 18, 35]. Another, similar method for solving partial differential inclusions, which followed more closely the classical Baire category approach to solving ordinary differential inclusions has been pursued by B. Dacorogna and P. Marcellini in [8] (see also [9, 36]).

Generally, convex integration is a method for solving differential inclusions of the type

$$\nabla f(x) \in E \tag{14}$$

where E is a given closed set of matrices. Roughly speaking the method consists of iteratively constructing layers within layers of oscillations, starting with an affine function whose gradient is in a suitably defined "hull" of E (see Section 2) and iteratively pushing the gradients towards E itself. During the iteration scheme it suffices to keep track of the gradient distribution of the approximating sequence, instead of controlling the pointwise behavior of the gradients.

Our plan of proof for Theorems 1.2, 1.3 and Theorem 1.5, 1.6 is as follows. We rewrite the equations as differential inclusions, in Lemmas 3.1 4.1, and then proceed with convex integration. The first step is to find a sequence of laminates (see Definition 2.1) with the required integrability properties. These will be called the staircase laminates, following [11, 12]. We remark that the construction of this type of laminates seems very flexible and adaptable to other situations. For example in [5] and [6] they have been used in relation with the problem of regularity of rank-one convex functions.

Once we find staircase laminates supported in the appropriate sets, we proceed in a different way for the lower and upper critical exponents. For the upper critical exponents we are dealing with honest quasiregular mappings. This allows us to fit the set of solutions of (3) into a natural metric space setting. Then we can adapt the elegant method of B. Kirchheim in [17] and use Baire category. This approach is based on the observation that points of continuity of the gradient are typically residual.

For the lower critical exponent we are not able to find a natural metric space setting. The reason is that the only norm which we are able to bound is the $W^{1,1}$ norm (see Remark 3.6). Instead we follow the lines of the original approach of Müller and Šverák [25] of successive modification on smaller and smaller sets via a sequence of in-approximations.

In our case the in-approximations take a particularly simple form, as the corresponding $p-$rank-one convex hull is the whole space, but we need to take care that the gradients diverge on sufficiently large sets in any ball of the domain. This approach also has the advantage that we get very precise information on the integrability of the gradient, namely that our solutions have gradient in the space $L^{\frac{2K}{K+1}}_{weak}$ (see Theorem 3.2).

2. Preliminaries

We start by introducing the following notation. For matrices $A \in \mathbb{R}^{2\times 2}$ write $A = (a_+, a_-)$ where $a_+, a_- \in \mathbb{C}$ denote the conformal coordinates. That is, if we identify the vector $v = (x, y) \in \mathbb{R}^2$ with the complex number $v = x + iy$, the relation

$$Av = a_+v + a_-\overline{v}$$

holds. For future reference we record that multiplication of matrices in conformal coordinates corresponds to

$$AB = (a_+b_+ + a_-\overline{b}_-,\ a_+b_- + a_-\overline{b}_+), \tag{15}$$

and that $\mathrm{Tr}A = 2\Re a_+$. Also

$$\begin{aligned} \det A &= |a_+|^2 - |a_-|^2, \\ |A|^2 &= 2|a_+|^2 + 2|a_-|^2, \\ \|A\| &= |a_+| + |a_-|, \end{aligned} \tag{16}$$

where $|A|$ and $\|A\|$ denote the Hilbert-Schmidt and the operator norm respectively.

By $\mathcal{M}(\mathbb{R}^{m\times n})$ we denote the set of signed Radon measures on $\mathbb{R}^{m\times n}$ with finite mass. By the Riesz representation theorem, $\mathcal{M}(\mathbb{R}^{m\times n})$ can be identified with the dual of the space $C_0(\mathbb{R}^{m\times n})$ of continuous functions vanishing at infinity. Given $\nu \in M(\mathbb{R}^{m\times n})$ with finite first moment we use the notation

$$\overline{\nu} = \int_{\mathbb{R}^{m\times n}} \lambda d\nu(\lambda)$$

and call $\overline{\nu}$ the *barycenter* of ν

Now we turn to convex integration. The basic building block for solving partial differential inclusions is the following lemma. Here, and in the rest of the paper, we will say that a mapping $f : \Omega \to \mathbb{R}^2$, continuous up to the boundary, is *piecewise affine (or piecewise quadratic)* if there exists a decomposition of Ω into countable pairwise disjoint open subsets Ω_i with $|\partial\Omega_i| = 0$ and

$$\Big|\Omega \setminus \bigcup_i \Omega_i\Big| = 0,$$

such that f is affine (resp. quadratic) on each subset Ω_i.

Lemma 2.1. *Let $\alpha \in (0,1)$, $\varepsilon, \delta > 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain.*

(i) *Let $A, B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A-B) = 1$ and suppose $C = \lambda A + (1-\lambda)B$ for some $\lambda \in (0,1)$. There exists a piecewise affine Lipschitz mapping $f : \Omega \to \mathbb{R}^m$ such that*
  (a) *$f(x) = Cx$ if $x \in \partial\Omega$,*
  (b) *$[f - C]_{C^\alpha(\overline{\Omega})} < \varepsilon$,*
  (c) *$|\{x \in \Omega : |\nabla f(x) - A| < \delta\}| = \lambda|\Omega|$,*
  (d) *$|\{x \in \Omega : |\nabla f(x) - B| < \delta\}| = (1-\lambda)|\Omega|$.*

(ii) *If in addition to part (i) we assume that $A, B \in \mathbb{R}^{n\times n}_{sym}$, then the map f can be chosen so that $f = \nabla u$ for some piecewise quadratic $u \in W^{2,\infty}(\Omega)$.*

Proof. Part (i) of the lemma is standard in the literature (see [24]), but usually with C^0 instead of C^α approximation in (b). The estimate $[f - C]_{C^\alpha} < \varepsilon$ is obtained by rescaling the function. That is, fixing $r > 0$ we cover Ω by small copies of itself upto measure zero, so that

$$\Big|\Omega \setminus \bigcup_{i=1}^{\infty}(a_i + r_i\Omega)\Big| = 0$$

with $r_i < r$, and then place in each copy the rescaled function $f_{r_i,a_i}(x) = r_i f(r_i^{-1}(x - a_i)) + Ca_i$. The estimate follows immediately since the C^α norm decreases as $r \to 0$.

Part (ii) is an easy example of convex integration. Firstly, [17, Proposition 3.4] yields a piecewise quadratic scalar function u_1 such that

- $u_1(x) = \frac{1}{2}\langle Cx, x\rangle$ if $x \in \partial\Omega$,
- $\big|\{x \in \Omega : D^2u_1(x) = A\}\big| > (1-\varepsilon)\lambda|\Omega|$,
- $\big|\{x \in \Omega : D^2u_1(x) = B\}\big| > (1-\varepsilon)(1-\lambda)|\Omega|$,
- $\operatorname{dist}\big(D^2u_1(x), [A,B]\big) < \delta$ a.e. in Ω.

However, we need a function u such that $D^2u(x)$ belongs to a δ neighborhood of $\{A,B\}$ instead of a neighborhood of the whole segment $[A,B]$. To achieve this, we iterate the construction of u_1 to produce a sequence of piecewise quadratic functions $\{u_i\}_{i=1}^\infty$ with the following properties: let

$$U_i = \big\{x \in \Omega : \operatorname{dist}\big(D^2u_i, \{A,B\}\big) < \delta\big\}.$$

Then,

(1) $u_i(x) = \frac{1}{2}\langle Cx, x\rangle$ if $x \in \partial\Omega$,
(2) $\big|\{x \in \Omega : D^2u_i(x) = A\}\big| > (1-\varepsilon)\lambda|\Omega|$,
(3) $\big|\{x \in \Omega : D^2u_i(x) = B\}\big| > (1-\varepsilon)(1-\lambda)|\Omega|$,
(4) $u_j(x) = u_i(x)$ for $x \in U_i$ whenever $j \geqq i$,
(5) $|\Omega \setminus U_{i+1}| \le \frac{1}{4}|\Omega \setminus U_i|$ and $U_i \subset U_{i+1}$,
(6) $\operatorname{dist}\big(D^2u_i, [A,B]\big) < \delta$ a.e. in Ω.

It follows from (4) and (6) that the sequence $\{u_i\}$ converges strongly in $W^{2,\infty}$ to a function u such that $\operatorname{dist}(D^2u(x), \{A,B\}) < \delta$ almost everywhere in Ω.

The sequence $\{u_i\}$ is defined by induction. We have already defined u_1. Suppose that u_i is given. Then, since u_i is piecewise quadratic, $\Omega \setminus U_i$ has a decomposition

$$\Omega \setminus U_i = \bigcup_j \tilde{U}_j \cup N$$

where $\tilde{U}_j$ are open sets on which $D^2u_i = \tilde{C}$ for some $\tilde{C}$ with $\operatorname{dist}\big(\tilde{C}, [A,B]\big) < \delta$ and $|N| = 0$. Hence we can write

$$\tilde{C} = \tilde{\lambda}(A + \tilde{D}) + (1 - \tilde{\lambda})(B + \tilde{D}),$$

where $0 < \tilde{\lambda} < 1$, and $\tilde{D} = \tilde{\lambda}A + (1-\tilde{\lambda})B - \tilde{C}$ satisfies that $|\tilde{D}| \le \delta$. Therefore, if we set $\tilde{A} = A + \tilde{D}$ and $\tilde{B} = B + \tilde{D}$

$$\tilde{C} = \tilde{\lambda}\tilde{A} + (1 - \tilde{\lambda})\tilde{B}, \tag{17}$$

with $|\tilde{A} - A| < \delta$ and $|\tilde{B} - B| < \delta$ and

$$\operatorname{rank}(\tilde{A} - \tilde{B}) = \operatorname{rank}(A - B) = 1.$$

Thus, we can use again [17, Proposition 3.4] in each $\tilde{U}_j$ to produce $\tilde{u_j}$ such that

- $\tilde{u_j}(x) = \frac{1}{2}\langle \tilde{C}x, x\rangle$ if $x \in \partial \tilde{U}_j$
- $\big|\{x \in \tilde{U}_j : \operatorname{dist}(D^2\tilde{u_j}, \{A,B\}) < \delta\}\big| > \frac{3}{4}|\tilde{U}_j|$.

Then the function u_{i+1} obtained from u_i by replacing u_i by $\tilde{u_j}$ on $\tilde{U}_j$ (modulo an affine function) fulfills the properties (1)-(6).

It remains to show that we can get the correct volume proportions. To this end, for a given function u with $\operatorname{dist}(D^2u, \{A,B\}) < \delta$ let

$$\mu_u = \big|\{x \in \Omega : |D^2u(x) - A| < \delta\}\big|.$$

The construction outlined above shows that for every ϵ there exists a function u with

$$(1-\epsilon)\lambda < \mu_u < (1+\epsilon)\lambda.$$

Take $\epsilon = 1/2$. Then there is no loss of generality assuming the existence of a function $u_1$with

$$\frac{\lambda}{2} < \mu_{u_1} < \lambda. \tag{18}$$

Choose $\tilde{A} \in B(A,\delta)$ such that

$$C = (\lambda + \epsilon_1)\tilde{A} + (1 - \lambda - \epsilon_1)B,$$

for some small $0 < \epsilon_1 < \delta/2$. Now for arbitrary $\epsilon_2 > 0$ we repeat the above construction with $(\lambda + \epsilon_1)$ in the place of λ to obtain a function u_2 equal to $1/2\langle Cx, x\rangle$ on the boundary and such that dist $(D^2u_2(x), \{A, B\}) < \delta$ and

$$\mu_{u_2} > (\lambda + \epsilon_1)(1 - \epsilon_2).$$

Now chose $\epsilon_2 > 0$ so that

$$\mu_2 > \lambda. \tag{19}$$

From (18) and (19) we deduce that

$$t \stackrel{def}{=} \frac{\mu_{u_2} - \lambda}{\mu_{u_2} - \mu_{u_1}}$$

satisfies $t \in (0, 1)$, and $\lambda = t\mu_2 + (1 - t)\mu_1$. Finally we divide Ω in two regions Ω_1, Ω_2 with $|\Omega_1| = t|\Omega|$ and $|\Omega_2| = (1 - t)|\Omega|$, and place rescaled copies of u_1 and u_2 in Ω_1 and Ω_2 respectively. This defines our final mapping u. It follows that $\mu_u = t\mu_2 + (1 - t)\mu_1 = \lambda$ and this concludes the proof. □

The matrix A, B in the above lemma are said to be rank-one connected and the measure $\lambda\delta_A + (1-\lambda)\delta_B$ corresponding to A and B from Lemma 2.1 is called a laminate of first order. The construction in the lemma can be iterated by modifying the map f in the regions where is affine. For example, in the region where $\{|\nabla f(x) - B| \leq \delta\}$, f can be replaced it with a map(also given by the lemma) whose gradient oscillates, on a much smaller scale between neigborhoods of the two rank-one connected matrices C_1, C_2 such that

$$B = \lambda' C_1 + (1 - \lambda')C_2. \tag{20}$$

Notice that, as in (17), (20) implies that if $|\nabla f(x) - B| \leq \delta$, then $\nabla f(x) = \lambda'\tilde{C}_1 + (1 - \lambda)\tilde{C}_2$ where $\tilde{C}_1, \tilde{C}_2$ are rank-one connected and they lie in corresponding neigborhoods of C_1 and C_2. Thus, on each region where f is affine, we can apply Lemma 2.1 to obtain the new mapping. On the level of the gradient distribution this amounts to replacing δ_B by $\lambda'\delta_{C_1} + (1 - \lambda')\delta_{C_2}$. This type of iteration motivates the following definition ([7], [25],[26])

Definition 2.1. The family of *laminates of finite order* $\mathcal{L}$ is the smallest family of probability measures in $\mathcal{M}(\mathbb{R}^{m\times n})$ such that

(1) $\mathcal{L}$ contains all Dirac masses.
(2) Suppose $\sum_{i=1}^{N} \lambda_i\delta_{A_i} \in \mathcal{L}$ and $A_1 = \lambda B + (1-\lambda)C$ where $\lambda \in [0, 1]$ and rank$(B - C) = 1$. Then the probability measure

$$\sum_{i=2}^{N} \lambda_i\delta_{A_i} + \lambda_1(\lambda\delta_B + (1 - \lambda)\delta_C)$$

is also contained in $\mathcal{L}$.

Proposition 2.1. *Let* $\nu = \sum_{i=1}^{N} \alpha_i\delta_{A_i} \in \mathcal{L}(\mathbb{R}^{m\times n})$ *be a laminate of finite order with barycenter* $\overline{\nu} = A$. *Then, for every* $\alpha \in (0, 1)$, $0 < \delta < \min|A_i - A_j|$ *and every bounded open set* $\Omega \in \mathbb{R}^n$, *there exists a piecewise affine Lipschitz mapping* $f : \Omega \to \mathbb{R}^m$ *such that*

i) $f(x) = Ax$ *on* $\partial\Omega$,
ii) $[f - A]_{C^\alpha(\overline{\Omega})} < \delta$,
iii) $|\{x \in \Omega : |\nabla f(x) - A_i| < \delta\}| = \alpha_i |\Omega|$, *thus*
iv) $\operatorname{dist}(\nabla f(x), \operatorname{spt}\nu) < \delta$ *a.e. in* Ω.

Moreover, if $A_i \in \mathbb{R}^{n\times n}_{sym}$, *then the map* f *can be chosen so that* $f = \nabla u$ *for some* $u \in W^{2,\infty}(\Omega)$.

Proof. The proof is by induction using Lemma 2.1. The precise argument is given in [25, Lemma 3.2] with the C^0 norm instead of the C^α and matrices in $\mathbb{R}^{n\times n}$. The case of symmetric matrices is handled using part ii) of Lemma 2.1 instead of part i). $\square$

Finally, we recall the definition of certain semiconvex envelopes of sets of 2×2 matrices.

Definition 2.2. Let $E \subset \mathbb{R}^{2\times 2}$. The *polyconvex hull* of E is given by

$$E^{pc} = \Big\{ A \in \mathbb{R}^{2\times 2} : \text{there exists } \nu \in \mathcal{M}(\mathbb{R}^{2\times 2}) \text{ such that} \\ \overline{\nu} = A, \operatorname{spt}\nu \subset E \text{ and } \det A = \int_{\mathbb{R}^{2\times 2}} \det d\nu \Big\}$$

Similarly the *lamination hull* of E is given by

$$E^{lc} = \Big\{ A \in \mathbb{R}^{2\times 2} : \overline{\nu} = A \text{ for some laminate of finite order } \nu \in \mathcal{L}(\mathbb{R}^{2\times 2}) \Big\},$$

and in particular the *first lamination hull* is

$$E^{lc,1} = \Big\{ A \in \mathbb{R}^{2\times 2} : \overline{\nu} = A \text{ for some laminate of first order } \nu \Big\}.$$

3. Isotropic equations

We start by transforming the set of solutions to the isotropic equation into solutions to a certain differential inclusion.

Definition 3.1. For a set $\varDelta \subset \mathbb{C} \cup \{\infty\}$, let

$$E_\varDelta = \{A = (a_+, a_-) : a_- = \mu \overline{a}_+ \text{ for some } \mu \in \varDelta\}, \tag{21}$$

i.e. $E_\varDelta$ is the set of matrices with second complex dilatation in $\varDelta$. In particular E_0 denotes the set of conformal matrices, and E_∞ the set of anti-conformal matrices.

By (15) we get the right-conformal invariance of $E_\varDelta$, namely that

$$E_\varDelta = E_\varDelta X \quad \text{for all } X \in E_0 \setminus \{0\}. \tag{22}$$

Lemma 3.1. *Let* $K \geqq 1$ *and* $k = \frac{K-1}{K+1}$. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain. Then* $u : \Omega \to \mathbb{R}$ *is a weak solution to*

$$\operatorname{div} \rho(x) \nabla u(x) = 0$$

for some $\rho \in L^\infty(\Omega, \{K, \frac{1}{K}\})$ *if and only if* $u = f_1$ *for some* $f : \Omega \to \mathbb{R}^2$ *satisfying*

$$\nabla f \in E_{\{k,-k\}}. \tag{23}$$

Proof. It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$, so that $f_1 = \Re(f)$. Accordingly, let us write $f = u + iv$. Standard calculations show that

$$\begin{aligned} 2\overline{\partial_z f} &= \partial_x u + \partial_y v + i(\partial_y u - \partial_x v), \\ 2\partial_{\overline{z}} f &= \partial_x u - \partial_y v + i(\partial_y u + \partial_x v), \end{aligned}$$

hence the condition $\nabla f \in E_{\{k,-k\}}$, or $\partial_{\overline{z}} f = \mu \overline{\partial_z f}$, is equivalent to the system

$$\begin{aligned} (1-\mu)\partial_x u &= (1+\mu)\partial_y v, \\ (1-\mu)\partial_y u &= -(1+\mu)\partial_x v. \end{aligned}$$

In other words

$$\frac{1-\mu}{1+\mu}\nabla u = J\nabla v, \tag{24}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But in a simply connected 2-dimensional domain Ω a vector-field is divergence-free if and only if it has the form $J\nabla v$. Hence (24) is equivalent to

$$\operatorname{div} \frac{1-\mu}{1+\mu}\nabla u = 0.$$

□

3.1. **Upper critical exponent.** Our strategy is as follows. First, we define an appropriate closed and bounded subset $X \subset W^{1,2}$ which contains all strong solutions to (23). Since bounded subsets of $W^{1,2}$ are metrizable under the weak topology, we deduce that (X, w), where w denotes the weak $W^{1,2}$ topology is a metric space. Then we prove that functions in X are points of continuity of the map $\nabla : (X, w) \mapsto L^2$ only if they satisfy the inclusion (23). From this we deduce, using that ∇ is a Baire-1 mapping, that the set of solutions in X is residual. Finally we show that the set of functions in X for which (4) holds in any fixed ball $B(x,r)$ is also residual. For this we use the staircase laminate construction introduced in [11].

Definition 3.2. Let

$$\Delta_k = \left\{ re^{i\phi} : r < k,\, r^2\cos^2\phi > \frac{(1-r^2)(r^2-k^4)}{(1-k^2)^2} \right\},$$

shown in Figure 1, and let $\mathcal{U} = E_{\Delta_k}$ (c.f. (21)). Let X be the closure in the weak topology of $W^{1,2}$ of the set

$$X_0 = \left\{ f \in W^{1,\infty}(\overline{\Omega}, \mathbb{R}^2) : \begin{array}{l} \bullet\, f \text{ piecewise affine} \\ \bullet\, \nabla f(x) \in \mathcal{U} \text{ a.e.} \\ \bullet\, f(x) = x \text{ on } \partial\Omega \end{array} \right\}. \tag{25}$$

Note that Δ_k, and hence $\mathcal{U}$, is open, and that $E_{\{k,-k\}} \subset \overline{\mathcal{U}}$. In Lemma 3.3 we show that X is a bounded subset of $W^{1,2}$ and hence the weak topology is metrizable.

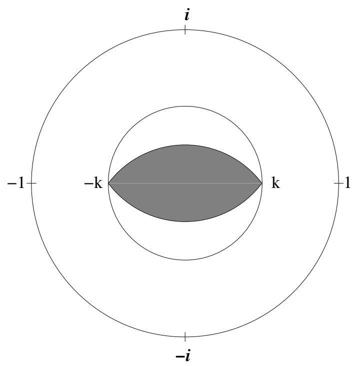

FIGURE 1. The set Δ_k in the complex plane.

The proofs here are very similar to the work of Kirchheim in [17]. The main difference is that we are dealing with unbounded sets, and so we need to make use of elliptic theory, in particular the relevant apriori estimates and higher integrability, in order to obtain a complete metric space X whose topology coincides with the weak $W^{1,2}$ topology.

Lemma 3.2. *Let $E_{\{k,-k\}}$ be as in Lemma 3.1 and let $E^{lc,1}_{\{k,-k\}}$ be the first lamination hull. Then*

$$E^{lc,1}_{\{k,-k\}} = E^{pc}_{\{k,-k\}} = E_{\overline{\Delta_k}}$$

with Δ_k given in Definition 3.2.

Proof. ,

First we will prove that $E^{pc}_{\{k,-k\}} = E^{lc,1}_{\{k,-k\}}$.

Recall that $X \in E^{pc}_{\{k,-k\}}$ if and only if $X = \overline{\nu}$ for some probability measure ν supported on $E_{\{k,-k\}}$ and satisfying

$$\det \overline{\nu} = \int \det \, d\nu$$

The crucial information to use is that $F \mapsto \det F$ is a convex function when restricted to $E_{\{k,-k\}}$. Let us write

$$\nu = \lambda \nu_k + (1 - \lambda)\nu_{-k}$$

where $\nu_{\pm k}$ are probability measures with $\operatorname{spt}\nu_k \subset E_k$ and $\operatorname{spt}\nu_{-k} \subset E_{-k}$ and barycenters Y and Z respectively. Then by Jensen's inequality we get

$$\det X \geq \lambda \det Y + (1-\lambda)\det Z.$$

Since $X = \lambda Y + (1-\lambda)Z$, expanding $\det X$ gives $\det(Y-Z) \leq 0$.

Now consider pairs tY and sZ with $t, s > 0$ such that for some $\lambda' \in (0,1)$ we have $\lambda'(tY) + (1-\lambda')(sZ) = X$. Linear independence gives

$$\lambda' = \frac{\lambda}{t} \text{ and } s = \frac{1-\lambda}{1-\lambda'},$$

so that such pairs are parametrized by $t \in (\lambda, \infty)$ with $s = s(t) \to \infty$ as $t \to \lambda$. Let $d(t) = \det(tY - s(t)Z)$. By the calculation above $d(1) \leq 0$ and since $\det Y, \det Z > 0$ we have $d(t) \to +\infty$ as $t \to \lambda$. Thus there exists $t_0 \in (\lambda, 1)$ such that $d(t_0) = 0$. But that implies that $X \in E^{lc,1}_{\{k,-k\}}$.

Next we obtain an explicit description of the lamination hull. A matrix $X = (x_+, x_-)$ is in the first lamination hull of $E_{\{k,-k\}}$ if and only if there exists $\lambda \in (0,1)$, and matrices $Y = (y_+, y_-) \in E_k$ and $Z = (z_+, z_-) \in E_{-k}$ such that

$$X = \lambda Y + (1-\lambda)Z \text{ and } |y_+ - z_+| = |y_- - z_-|$$

Substituting $y_- = k\overline{y}_+$, $z_- = -k\overline{z}_+$, $x_- = \mu\overline{x}_+$ and writing $t = (1-2\lambda)$ we see that $X \in E^{lc,1}_{\{k,-k\}}$ if and only if

$$|\mu + kt| = k|k + t\mu|$$

for some $t \in [-1,1]$.

Let $p(t) = |\mu+kt|^2 - k^2|k+t\mu|^2$. Then $p(t)$ is a quadratic polynomial in t, with leading term $k^2(1-|\mu|^2)t^2$.

Notice that

$$p(1) = (1-k^2)|\mu+k|^2 \geq 0, \;\; p(-1) = (1-k^2)|\mu-k|^2 \geq 0.$$

Therefore if p is concave it has no roots in $[-1,1]$. So we may assume $|\mu| < 1$ and then if p has a root in $[-1,1]$, the minimum of p also lies in $[-1,1]$. The discriminant of p is:

$$D = 4k^2 \left\{ (1-k^2)^2(\Re\mu)^2 - (1-|\mu|^2)(|\mu|^2-k^4) \right\}$$

and the minimum is at

$$t_0 = \frac{(1-k^2)\Re\mu}{k(1-|\mu|^2)}.$$

Suppose $D \geq 0$. Then if $|\mu| > k$,

$$t_0^2 = \frac{(1-k^2)^2(\Re\mu)^2}{k^2(1-|\mu|^2)^2} \geq \frac{|\mu|^2-k^4}{k^2(1-|\mu|^2)} > 1$$

whereas if $|\mu| \leq k$, then

$$|t_0| = \frac{(1-k^2)|\Re\mu|}{k(1-|\mu|^2)} \leq \frac{(1-k^2)}{(1-|\mu|^2)} \leq 1.$$

This proves that p has a root in $[-1,1]$ if and only if $\mu \in \overline{\Delta_k}$ and thus finishes the second part of the lemma.

□

Remark 3.1. The above lemma implies that the G-closure of $\{k,-k\}$ for the problem (6) is $\overline{\Delta}_k$. Similar computations can be found in [12] and [27].

3.1.1. *Points of continuity of* ∇.

Lemma 3.3. *The space (X,w) is metrizable, and for any $f \in X$ we have $\nabla f(x) \in \overline{\mathcal{U}}$ a.e. in Ω. Furthermore the set of continuity points of the map $\nabla : (X,w) \to L^2(\Omega, \mathbb{R}^{2\times 2})$ is residual in (X,w).*

Proof. The key information seems to be that we are dealing with elliptic equations, in particular with quasiregular mappings. To prove that (X,w) is metrizable we need to show that X_0 is bounded in $W^{1,2}$. There is no loss of generality in assuming that $\Omega \subset B(0,\frac{1}{2})$. Therefore for $f \in X_0$ the Lipschitz mapping $\tilde{f} = f\chi_\Omega + x\chi_{B(0,1)\setminus\Omega}$ is a well defined K-quasiregular mapping by the definition of $\mathcal{U}$. Thus, $\Re\tilde{f} = u$ is a solution to

$$\operatorname{div}\sigma(x)\nabla u(x) = 0 \qquad \text{in } B(0,1),$$

where σ is a matrix measurable function satisfying (1). Testing the equation with $v(x) = u(x) - x_1$ yields the estimate

$$\int_{B(0,1)} |\nabla u|^2 \leqq K^4\pi,$$

and using the same argument for $\Im(\tilde{f})$ we obtain that

$$\int_{B(0,1)} |\nabla\tilde{f}|^2 \leq C(K). \tag{26}$$

Finally, the Sobolev embedding theorem yields the required bound,

$$\|f\|_{W^{1,2}(\Omega)} \leq C(K,\Omega) \tag{27}$$

for f in X_0 and by the weak lower semicontinuity of the $W^{1,2}$ norm for f in X. This shows that (X,w) is metrizable with metric d. Let us remark that by the compactness of Sobolev embedding the weak topology in $W^{1,2}(\Omega)$ is equivalent to the the strong topology in $L^2(\Omega)$ and in fact, by the Hölder regularity of quasiregular mappings (see for example [19]) to the supremum norm topology. There is no problem in any of the statements with the boundary since $\Omega \subset B(0,\frac{1}{2})$.

To prove that $\nabla f(x) \in \overline{\mathcal{U}}$ a.e. for any $f \in X$ we use the weak continuity of $F \mapsto \det F$ in $W^{1,p}$ for $p > 2$. Note that if $f_n \rightharpoonup f$ in $W^{1,2}(\Omega)$, then also $f_n \rightharpoonup f$ in $W^{1,p}(\Omega)$ for some $p > 2$ by the higher integrability of quasiregular mappings. This means that $\nabla f(x) \in \mathcal{U}^{pc}$ for any $f \in X$, and $\mathcal{U}^{pc} = \overline{\mathcal{U}}$ by Lemma 3.2.

We now show that ∇ is a Baire-1 mapping. Let Δ_k be matrix of different quotients (Recall that we have extended functions in X by the identity outside Ω so that $\Delta_k f(x)$ is defined a.e. in Ω). For all Sobolev functions $f \in W^{1,2}$ we have that

$$\lim_{k\to\infty} \|\nabla f - \Delta_k(f)\|_{L^2(\Omega,\mathbb{R}^{2\times 2})} = 0.$$

On the other hand Δ_k is easily seen to be a continuous operator from $L^2(\Omega,\mathbb{R}^2) \to L^2(\Omega,\mathbb{R}^{2\times 2})$. By Sobolev embedding this implies that Δ_k is continuous as a map from $(X,w) \to L^2(\Omega,\mathbb{R}^{2\times 2})$. Therefore ∇ is a pointwise limit of continuous mappings, i.e a Baire-1 mapping. It is then a classical result that the points of continuity form a residual set in (X,w) (See [17, page 53]). □

Lemma 3.4. *The set of points of continuity in (X,d) of ∇ satisfy $\nabla f(x) \in E_{\{k,-k\}}$ almost everywhere.*

The proof is exactly as in [17] Proposition 3.17. Here we reproduce it for the reader's convenience. The main point is that if f is piecewise affine with an affine piece $A \in \overline{\mathcal{U}} \setminus E_{\{k,-k\}}$, then (since $\overline{\mathcal{U}} = E^{lc,1}_{\{k,-k\}}$) there exists a rank-one segment through A in $\overline{\mathcal{U}}$ of length proportional to the distance to $E_{\{k,-k\}}$. This permits us, with the help of Lemma 2.1, to produce a perturbation of f showing that it cannot be a point of continuity.

Proof. Suppose for a contradiction that $\{x \in \Omega : \nabla f(x) \in \overline{\mathcal{U}} \setminus E_{\{k,-k\}}\}$ has positive measure, where $f \in X$ is a point of continuity. Then by Lusin's theorem there exists a compact $\Omega_0 \subset \Omega$ (with $|\Omega_0| = m > 0$) such that ∇f is continuous on Ω_0, and $\nabla f(\Omega_0) \cap E_{\{k,-k\}} = \emptyset$. Since $\nabla f(\Omega_0)$ is compact,

$$\varepsilon = \frac{1}{2}\mathrm{dist}\,(\nabla f(\Omega_0), E_{\{k,-k\}}) > 0. \tag{28}$$

Let

$$\mathcal{V} = N_\varepsilon\big(\nabla f(\Omega_0)\big) \cap \overline{\mathcal{U}}$$

be the $\varepsilon-$neighborhood of $\nabla f(\Omega_0)$ in $\overline{\mathcal{U}}$. Since $\overline{\mathcal{U}} = E^{lc,1}_{\{k,-k\}}$, to any $A \in \mathcal{V}$ there exists a rank-one segment connecting E_k to E_{-k} and containing A. Moreover, as $\mathrm{dist}\,(A, E_{\{k,-k\}}) \geqq \varepsilon$ by (28), there exists a rank-one matrix $C_A \in \mathbb{R}^{2\times 2}$ with $|C_A| \geqq \varepsilon$ such that

$$[A - C_A, A + C_A] \subset \overline{\mathcal{U}}. \tag{29}$$

Furthermore, since f is a point of continuity, there exists $\delta > 0$ such that

$$\|\nabla g - \nabla f\|_{L^2}^2 < \frac{1}{8}\varepsilon^2 m \text{ whenever } d(g,f) < \delta.$$

Take a sequence of piecewise affine functions $X_0 \ni f_n \to f$ in (X,d). Then in particular (since ∇ is continuous at f), $\nabla f_n \to \nabla f$ in measure, i.e. $|\{x : |\nabla f_n(x) - \nabla f(x)| > \varepsilon\}| \to 0$ as $n \to \infty$. Therefore there exists n and $\Omega_1 \subset \Omega_0$ with $|\Omega_1| > \frac{m}{2}$ so that

$$d(f_n, f) < \frac{\delta}{2}, \; \|\nabla f_n - \nabla f\|_{L^2}^2 < \frac{1}{8}\varepsilon^2 m$$

and for all $A \in \nabla f_n(\Omega_1)$ there exists a rank-one matrix C_A with $|C_A| > \varepsilon$ so that (29) holds. Since Ω is covered up to measure zero by open sets on which f_n is affine, there exists finite number of disjoint open sets $G_i \subset \Omega$ such that $f_n(x) = A_i x$ for $x \in G_i$,

$$[A_i - C_i, A_i + C_i] \subset \overline{\mathcal{U}}$$

for some rank-one matrix C_i with $|C_i| > \varepsilon$ and $|\bigcup G_i| > \frac{m}{2}$.

For each i Lemma 2.1 supplies a piecewise affine function ϕ_i such that $\phi_i \in W_0^{1,\infty}(G_i, \mathbb{R}^m)$,

$$\|\phi_i\|_\infty < \frac{\delta}{2} \text{ and } \int |\nabla \phi_i| > \varepsilon |G_i| \tag{30}$$

Let $g = f_n + \sum \chi_{G_i}\phi_i$. Then $g \in X$,

$$d(g,f) \le d(g, f_n) + d(f_n, f) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

but using Cauchy-Schwarz

$$\int_{\bigcup G_i} |\nabla g - \nabla f_n|^2 \ge \frac{1}{|\bigcup G_i|}\sum_i \Big(\int_{G_i} |\nabla \phi_i|\Big)^2 > \varepsilon^2 |\bigcup G_i|$$

and hence

$$\|\nabla g - \nabla f\|_{L^2}^2 \ge \frac{1}{2}\|\nabla g - \nabla f_n\|_{L^2}^2 - \|\nabla f_n - \nabla f\|_{L^2}^2 \ge \frac{1}{8}\varepsilon^2 m.$$

This gives the required contradiction. □

Corollary 3.1. *The set of mappings f in X such that $\nabla f(x) \in E_{\{k,-k\}}$ is residual.*

3.1.2. *Staircase laminates and integrability.*

Proposition 3.1 (The strong staircase)**.** *Every $A \in \mathcal{U}$ is the center of mass of a sequence of laminates of finite order $\nu_n \in \mathcal{L}$ supported in $\mathcal{U}$ with*

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2\times 2}} |\lambda|^{\frac{2K}{K-1}} d\nu_n = \infty \tag{31}$$

Proof. Let $A = (1, 0)$ in conformal coordinates. Then the claim was proved in [11], obtaining measures $\nu_n \in \mathcal{L}$ supported in $E_{\{k,-k\}} \cap \mathcal{D}$, where $\mathcal{D}$ denotes diagonal matrices. We shift the construction by considering the measures $\tilde{\nu_n}(\cdot) = \nu_n(\cdot + (1, 0))$. Recall that ν_n are supported on matrices of the form $\frac{j}{1+k}(\pm 1, k)$ and $(n, 0)$ (in conformal coordinates). Therefore $\operatorname{spt} \tilde{\nu}_n \subset \mathcal{U} \cap \mathcal{D}$. This proves the claim for $A = (2, 0)$.

Since the sets are invariant under precomposition with conformal matrices (see (22), the same is true whenever A is a conformal matrix. Finally we claim that for every $A \in \mathcal{U}$ there is a rank-one segment $[P, Q]$ contained in $\mathcal{U}$ with $Q \in E_0$ such that $A = \lambda P + (1 - \lambda)Q$ and $\lambda \in [0, 1)$. Indeed, writing $A = (a_+, a_-)$ in conformal coordinates, let $Q = (a_+ - a_-, 0) \in E_0$. Clearly $\operatorname{rank}(A-Q) \leqq 1$, since $A-Q = (a_-, a_-)$. Because $\mathcal{U}$ is lamination convex (see Lemma 3.2) and contains E_0, it also contains the whole segment $[A, Q]$. Furthermore, as $\mathcal{U}$ is open, the segment can be nontrivially extended to some $P \in \mathcal{U}$ so that $A \in [P, Q]$. Then the required laminates can be defined as

$$\nu_n = \lambda \delta_P + (1 - \lambda)\hat{\nu}_n,$$

where $\hat{\nu}_n$ are laminates with barycenter Q, constructed in the previous step. □

Proposition 3.2. *For any ball $B = B(x, r) \subset \Omega$ the set*

$$X_{B,M} = \{f \in X : \int_{B(x,r)} |\nabla f|^{\frac{2K}{K-1}} \leq M\}$$

is closed and has no interior in X.

Proof. By lower-semicontinuity of the norm $X_{B,M}$ is closed in the weak topology of $W^{1,2}(\Omega)$.

Suppose for a contradiction $X_{B,M}$ has nonempty interior for some B and M. Then there exists $f \in X_0 \cap X_{B,M}$ and $\varepsilon > 0$ such that

$$\int_B |\nabla g|^{\frac{2K}{K-1}} \leq M$$

whenever $g \in X$ with $d(g, f) < \varepsilon$.

Take any subdomain $\Omega_0 \subset B$ where f is affine, say $f(x) = Ax$ with $A \in \mathcal{U}$. By Proposition 3.1 there exists a laminate $\nu \in \mathcal{L}$ with barycenter $\overline{\nu} = A$ such that $\int_{\mathbb{R}^{2\times 2}} |\lambda|^{\frac{2K}{K-1}} d\nu > 2M$. Correspondingly, by Proposition 2.1 there exists a sequence $\phi_j \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ such that $\nabla\phi_j \rightharpoonup A$ in $W^{1,2}(\Omega_0)$, $\phi_j = A$ on $\partial\Omega_0$ and $\lim_{j\to\infty} \int_{\Omega_0} |\nabla\phi_j|^{\frac{2K}{K-1}} \geq 2M$. Then the sequence $f_j = f + \chi_{\Omega_0}(\phi_j - A)$ satisfies $f_j \in X \setminus X_{B,M}$ for sufficiently large j. On the other hand $d(f_j, f) \to 0$, and this is a contradiction. □

Corollary 3.2. *The set of points in X such that $\int_{B(x,r)} |\nabla f|^{\frac{2K}{K-1}} = \infty$ for all $B(x, r) \subset \Omega$ is residual in (X, d).*

Proof. This follows since

$$\{f \in X : \int_{B(x,r)} |\nabla f|^{\frac{2K}{K-1}} < \infty \text{ for some } B(x,r)\} = \bigcup_{M=1}^{\infty} \bigcup_{i=1}^{\infty} X_{B_i,M}$$

where B_i is an enumeration of balls in Ω with rational centers and radii. Therefore since each $X_{B_i,M}$ is of first category, the (countable) union is also of first category. □

Combining Corollaries 3.1 and 3.2 yields the following theorem:

Theorem 3.1. *Let $K > 1$ and $k = \frac{K-1}{K+1}$. For any bounded open set $\Omega \subset \mathbb{R}^2$ there exists a mapping $f \in W^{1,2}(\Omega; \mathbb{R}^2)$ with the following properties:*

(i) *$f(x) = x$ on $\partial\Omega$,*
(ii) *$\nabla f(x) \in E_{\{k,-k\}}$ a.e. in Ω,*
(iii) *For any ball $B \subset \Omega$ we have $\int_B |\nabla f(x)|^{\frac{2K}{K-1}} dx = \infty$.*

3.2. **Lower critical exponents.** In the following $J = (0, 1)$ in conformal coordinates, i.e.

$$J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}.$$

This subsection consists essentially of two parts. First we deal with the geometry of $E_{\{k,-k\}}$, in particular we show that any matrix A lies on a rank-one connection between $E_{\{k,-k\}}$ and E_∞ whose length is proportional to $|A|$. Then we define the staircase laminates in Lemma 3.7. The construction is much more explicit then the corresponding staircase for the upper critical exponents. The reason is that in contrast with the case of the upper exponent, there are no *apriori* bounds on the gradient, and so it becomes crucial to know precisely where the gradients of the approximating sequence lie.

In the second part we proceed with convex integration. The setting is quite general once the specific geometric properties have been established. In Proposition 3.3 we show the existence of piecewise affine maps f with the desired integrability property given by (39), which solve the inclusion up to a small L^∞ error. Moreover, the size of the error can be made to depend on $|\nabla f|$. Then in Theorem 3.2 we show that the L^∞ error can be successively removed. The main estimate is in (54), it guarantees that the limit mapping lies in the space $L^{q_K}_w$, where $q_K = \frac{2K}{K+1}$ is the critical exponent. Having made the size of the error depend on the size of the gradient enables us to show that (54) does not depend on the fact that $q_K < 2$. In fact, because $q_K < 2$, it would be sufficient for the error to be independent of $|\nabla f|$. This would allow for a substantial simplification of the proof of Proposition 3.3 and Lemma 3.7. However, we prefer this approach because it extends to higher dimensions.

We begin with two simple lemmas regarding the geometry of rank-one lines in $\mathbb{R}^{2\times 2}$:

Lemma 3.5. *Let* $A, B \in \mathbb{R}^{2\times 2}$ *with* $\det B \neq 0$ *such that* $\det(A-B) = 0$, *and let* $\sigma = \left|\frac{b_-}{b_+}\right|$. *Then*

$$|B| \leqq \sqrt{2}\left|\frac{1+\sigma}{1-\sigma}\right||A|,$$

where $\frac{1+\infty}{1-\infty} = -1$.

Proof. First we assume that B is not anti-conformal and write A and B in conformal coordinates:

$$A = (a_+, a_-), \quad B = (z, \mu z)$$

for some $\mu \in \mathbb{C}$ with $|\mu| = \sigma$. Since $\det(A - B) = 0$, we have

$$|z - a_+|^2 - |\mu z - a_-|^2 = 0,$$

which we can rewrite as

$$|z|^2 - 2\Re\Big(z\frac{\overline{a}_+ - \mu\overline{a}_-}{1-|\mu|^2}\Big) + \frac{|a_+|^2 - |a_-|^2}{1-|\mu|^2} = 0.$$

Completing the square yields

$$\left|z - \frac{a_+ - \overline{\mu}a_-}{1-|\mu|^2}\right| = \frac{|\mu a_+ - a_-|}{1-|\mu|^2}.$$

But then we estimate

$$\begin{aligned}|z| &\leqq \frac{1}{1-|\mu|^2}\big(|a_+ - \overline{\mu}a_-| + |\mu a_+ - a_-|\big)\\ &\leqq \frac{1}{1-|\mu|^2}\big((|a_+| + |a_-|)(1+|\mu|)\big) = \frac{|a_+| + |a_-|}{1-|\mu|}.\end{aligned}$$

Hence, using (16)

$$\begin{aligned}|B|^2 &= 2(1+\sigma^2)|z|^2\\ &\leqq 2\frac{1+\sigma^2}{(1-\sigma)^2}\big(|a_+| + |a_-|\big)^2\\ &\leqq 2\frac{(1+\sigma)^2}{(1-\sigma)^2}|A|^2.\end{aligned}$$

To treat the case of anti-conformal B we can repeat the above calculation with $B = (0, z)$. □

Lemma 3.6. *Let* $K > 1$ *and* $k = \frac{K-1}{K+1}$. *Every* $A \in \mathbb{R}^{2\times 2} \setminus \{0\}$ *lies on a rank-one segment connecting* E_∞ *and* E_k. *In particular there exist* $P \in E_\infty \setminus \{0\}$ *and* $Q \in E_k \setminus \{0\}$ *with* $\operatorname{rank}(P - Q) = 1$ *such that* $A \in [P, Q]$ *and*

$$\frac{1}{c_K}|A| < |P - Q| < c_K|A|, \tag{32}$$

where $c_K > 1$ depends only on K. The same holds if we replace E_k by E_{-k}.

Proof. It suffices to prove the lemma for E_k. We can introduce coordinates in $\mathbb{R}^{2\times 2}$ related to E_∞ and E_k since they are linearly independent two-dimensional subspaces. Accordingly, to every matrix A there exist $z, w \in \mathbb{C}$ so that A lies in the plane $L_{z,w}$ spanned by the matrices $(0, z)$ and $(w, k\overline{w})$. Since $\det(0, z) < 0 < \det(w, k\overline{w})$ and the determinant is a quadratic form, there exists two linearly independent rank-one directions in $L_{z,w}$. It is then easy to see that for $A \in L_{z,w}$ there exist nonzero $P \in L_{z,w} \cap E_\infty$ and $Q \in L_{z,w} \cap E_k$ so that $A \in [P, Q]$. The upper estimate (32) follows from Lemma 3.5 and for the lower estimate observe that

$$\operatorname{dist}(A, E_\infty) + \operatorname{dist}(A, E_k) \leqq |A - P| + |A - Q| = |P - Q|,$$

and (32) follows by the linear independence of E_∞ and E_k. □

Lemma 3.7 (One weak step)**.** *Let $A \in B_r(nJ)$ for some $0 < r < 1/2$. There exists a laminate ν_A of third order with the following properties:*

- $\overline{\nu}_A = A,$
- $\operatorname{spt} \nu_A \subset E_{\{k,-k\}} \cup \big\{(n+1)J\big\},$
- $\operatorname{spt} \nu_A \subset \big\{\xi \in \mathbb{R}^{2\times 2} : c_K^{-1} n < |\xi| < c_K n\big\}$
- $(1 - c_K^{-1} \frac{r}{n})\beta_n < \nu_A\big(\{(n+1)J\}\big) < (1 + c_K \frac{r}{n})\beta_n,$

where

$$\beta_n = \frac{n}{n+1}\frac{1-k+2n}{1+k+2n} \tag{33}$$

and $c_K > 1$ is a constant only depending on K.

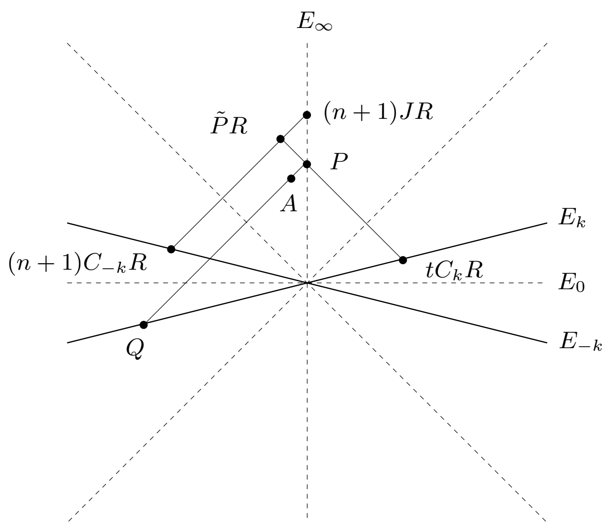

FIGURE 2. One weak step towards infinity.

Proof. Using Lemma 3.6 there exists $P \in E_\infty$ and $Q \in E_k$ with $\operatorname{rank}(P-Q) = 1$ such that $A = \lambda_1 P + (1-\lambda_1)Q$ for some $\lambda_1 \in [0,1]$. Hence $\lambda_1\delta_P + (1-\lambda_1)\delta_Q$ is a laminate. Since $|A - nJ| < r$, from Lemma 3.5 applied to $A - nJ$ and $P - nJ \in E_\infty$ we have $|P - nJ| < \sqrt{2}r$, hence

$$|P - A| < 3r \text{ and } |P - Q| > \frac{n}{c_K}.$$

But then for some $c_K > 1$

$$\lambda_1 > 1 - \frac{r}{c_K n}. \tag{34}$$

Now JP is conformal, so that $JP = tR$ for some $t > 0$ and $R \in SO(2)$. Moreover, $n|J| - \sqrt{2}r < |P| < n|J| + \sqrt{2}r$, and $P = tJR$, hence

$$|t - n| < r. \tag{35}$$

Define the matrices

$$C_k = \frac{1}{1+k}(1,k), \quad C_{-k} = \frac{1}{1+k}(-1,k) \tag{36}$$

in conformal coordinates. Note that $C_{\pm k} \in E_{\pm k}$ respectively and $\det(J - C_{\pm k}) = 0$. Moreover, let

$$\tilde{P} = \Big(-\frac{1-(t-n)}{2}, n + \frac{1+(t-n)}{2}\Big).$$

By direct calculation $P = \lambda_2 t C_k + (1-\lambda_2)\tilde{P}$ and $\tilde{P} = \lambda_3 (n+1) C_{-k} + (1-\lambda_3)(n+1)J$, where

$$\lambda_2 = \frac{1 + k - (t-n)(1+k)}{2n + 1 + k + (t-n)(1-k)} \tag{37}$$

$$\lambda_3 = \frac{(1 - t + n)(1+k)}{2(n+1)}. \tag{38}$$

In particular, by Definition 2.1

$$\begin{aligned}\nu_A =& \lambda_1\Big(\lambda_2\delta_{tC_k} + (1-\lambda_2)\big(\lambda_3\delta_{(n+1)C_{-k}} + (1-\lambda_3)\delta_{(n+1)J}\big)\Big)\\ &+ (1-\lambda_1)\delta_Q\end{aligned}$$

is a laminate with barycenter A, and

$$\nu_A\big(\{(n+1)J\}\big) = \lambda_1(1-\lambda_2)(1-\lambda_3).$$

Thus, by direct calculation using (34), (35), (37) and (38) we get

$$\big(1 - c_K^{-1}\frac{r}{n}\big)\beta_n < \nu_A\big(\{(n+1)J\}\big) < \big(1 + c_K\frac{r}{n}\big)\beta_n.$$

Finally, from Lemma 3.5 we get $c_K^{-1} n < |Q| < c_K n$, and this concludes the proof.

□

Proposition 3.3 (The weak staircase)**.** *Let $K > 1$ and $k = \frac{K-1}{K+1}$. Let $\alpha \in (0,1)$, $\delta > 0$ and $\tau : [0,\infty) \to (0,1/2]$ a continuous, non increasing function with $\tau(0) > 0$ and $\int_1^\infty \frac{\tau(t)}{t}dt < \infty$.*

For any bounded open set $\Omega \subset \mathbb{R}^2$ and any nonzero matrix F with $|F - J| < \tau(|F|)$ there exists a piecewise affine mapping $f \in W^{1,1}(\Omega;\mathbb{R}^2) \cap C^\alpha(\overline{\Omega};\mathbb{R}^2)$ with the following properties:

(i) $f(x) = Fx$ *on* $\partial\Omega$,
(ii) $[f - F]_{C^\alpha(\overline{\Omega})} < \delta$,
(iii) $\operatorname{dist}\big(\nabla f(x), E_{\{k,-k\}}\big) < \tau\big(|\nabla f(x)|\big)$ *a.e. in* Ω,

and there exists a constant $c_{K,\tau} > 0$ so that for all $t > 1$ we have

$$\frac{1}{c_{K,\tau}} t^{-\frac{2K}{K+1}} < \Big|\{x \in \Omega : \, |\nabla f(x)| > t\}\Big| < c_{K,\tau} t^{-\frac{2K}{K+1}}. \tag{39}$$

Proof. We define a sequence of piecewise affine mappings $\{f_n\}$ inductively using repeatedly Proposition 2.1. Let $f_1(x) = Fx$ in Ω. For the inductive step we assume the existence of a piecewise affine Lipschitz mapping $f_n : \Omega \to \mathbb{R}^2$ such that

(a) $f_n(x) = Fx$ on $\partial\Omega$,
(b) $[f_n - F]_{C^\alpha(\overline{\Omega})} < (1 - 2^{-n})\delta$,
(c) $\operatorname{dist}\big(\nabla f_n(x), E_{\{k,-k\}} \cup \{nJ\}\big) < \tau(|\nabla f_n(x)|)$ a.e. in Ω,

and with $\Omega_n = \{x \in \Omega : \, |\nabla f_n(x) - nJ| < \tau(n)\}$ we have

$$\prod_{j=1}^{n-1} \big(1 - c_K^{-1} \frac{\tau(j)}{j}\big)\beta_j < \frac{|\Omega_n|}{|\Omega|} < \prod_{j=1}^{n-1} \big(1 + c_K \frac{\tau(j)}{j}\big)\beta_j. \tag{40}$$

By assumption $|F - J| < \tau(|F|)$, so f_1 satisfies (c). Moreover, $|J| = \sqrt{2}$, so $\tau(n) \geqq \tau(|J|)$, hence $\Omega_1 = \Omega$. We deduce that f_1 satisfies the inductive hypothesis. To obtain f_{n+1} we modify f_n on the set Ω_n. Because f_n is piecewise affine, we have a decomposition into pairwise disjoint open subsets $\Omega_{n,i}$ such that

$$\Big|\Omega_n \setminus \bigcup_{i=1}^\infty \Omega_{n,i}\Big| = 0,$$

and $f_n(x) = A_i x + b_i$ in $\Omega_{n,i}$ for some $A_i \in B_{\tau(n)}(nJ)$ and $b_i \in \mathbb{R}^2$. For each i we use Proposition 2.1 with the laminate ν_{A_i} from Lemma 3.7 to obtain a piecewise affine Lipschitz mapping $g_i : \Omega_{n,i} \to \mathbb{R}^2$ with

(d) $g_i(x) = A_i x + b_i$ on $\partial\Omega_{n,i}$,
(e) $[g_i - f_n]_{C^\alpha(\Omega_{n,i})} < 2^{-(n+1+i)}\delta$,
(f) $c_K^{-1} n < |\nabla g_i(x)| < c_K n$ a.e. in $\Omega_{n,i}$,
(g) $\operatorname{dist}\big(\nabla g_i(x), E_{\{k,-k\}} \cup \{(n+1)J\}\big) < \tau(c_K n)$ a.e. in $\Omega_{n,i}$,

and

(41)

$$\big(1 - c_K^{-1}\frac{\tau(n)}{n}\big)\beta_n <$$

$$< \Big|\big\{x \in \Omega_{n,i} : |\nabla g_i(x) - (n+1)J| < \tau\big((n+1)\big)\big\}\Big| < \big(1 + c_K\frac{\tau(n)}{n}\big)\beta_n.$$

We then define

$$f_{n+1}(x) = \begin{cases} f_n(x) & \text{if } x \in \Omega \setminus \bigcup_{i=1}^{\infty} \Omega_{n,i} \\ g_i(x) & \text{if } x \in \Omega_{n,i} \end{cases}$$

It is clear that $f_{n+1}(x) = Fx$ on $\partial\Omega$, and from (e) we get $[f_{n+1} - f_n]_{C^\alpha(\overline{\Omega})} < 2^{-(n+1)}\delta$, hence (b) follows. Because τ is non increasing, (c) follows from (g). Finally (40) follows from (41).

It is clear that the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists pointwise almost everywhere. Observe that we obtain f_{n+1} from f_n by modifying on a nested sequence of open sets Ω_n whose measure

$$|\Omega_n| < |\Omega| \prod_{j=1}^{n-1}\big(1 + c_K\frac{\tau(j)}{j}\big)\beta_j < |\Omega|\frac{c_{K,\tau}}{n}\prod_{j=1}^{n}\frac{1-k+2j}{1+k+2j} < \frac{c_{K,\tau}}{n}|\Omega|$$

converges to zero, since by the condition on τ the product $\prod_j\big(1+c\frac{\tau(j)}{j}\big)$ converges. Thus the limit mapping f is piecewise affine. Moreover, it satisfies (i)-(iii) by our construction.

Finally, the distribution function of ∇f can be estimated as follows. Using (g), for $n \in \mathbb{N}$ we have

$$|\nabla f(x)| > \frac{n}{c_K} \text{ in } \Omega_n, \text{ and } |\nabla f(x)| < c_K n \text{ in } \Omega \setminus \Omega_n.$$

Hence, for given $t > c_K$ let n_1 be the integer part of $c_K t$ and n_2 the integer part of $c_K^{-1}t$. Then

$$\Omega_{n_1+1} \subset \{x \in \Omega : |\nabla f(x)| > t\} \subset \Omega_{n_2},$$

so it remains to estimate $|\Omega_n|$ using (40). As observed already, the product $\prod_j\big(1 + c\frac{\tau(j)}{j}\big)$ converges for any c to a finite positive number. Moreover, taking logarithms we see that for some constant $c > 0$ independent of n

$$\Big|\log\big(\prod_{j=1}^{n}\beta_j\big) + \log n + \sum_{j=1}^{n}\frac{2k}{1+k+2j}\Big| < c,$$

hence for a possibly different constant $c > 0$

$$\Big|\log\big(\prod_{j=1}^{n}\beta_j\big) + \frac{2K}{K+1}\log n\Big| < c, \tag{42}$$

since $k+1 = \frac{2K}{K+1}$. But this implies the estimate (39). Finally (39) implies that ∇f_n is uniformly bounded in L^1, hence $f \in W^{1,1}$ by dominated convergence. □

Theorem 3.2. *Let $K > 1$, $k = \frac{K-1}{K+1}$ and let $F \in \mathbb{R}^{2\times 2} \setminus \{0\}$. For any $\alpha \in (0,1)$, $\delta > 0$ and for any bounded open set $\Omega \subset \mathbb{R}^2$ there exists a mapping $f \in W^{1,1}(\Omega;\mathbb{R}^2) \cap C^\alpha(\overline{\Omega};\mathbb{R}^2)$ with the following properties:*

(i) $f(x) = Fx$ *on* $\partial\Omega$,
(ii) $[f - F]_{C^\alpha(\overline{\Omega})} < \delta$,
(iii) $\nabla f(x) \in E_{\{k,-k\}}$ *a.e. in* Ω,
(iv) *For any ball $B \subset \Omega$ there exists a constant $c_B > 1$ such that*

$$\frac{1}{c_B} t^{-\frac{2K}{K+1}} < \left|\{x \in B : |\nabla f(x)| > t\}\right| < c_B t^{-\frac{2K}{K+1}}$$

for all $t \geqq 1$.

In particular $f \in W^{1,q}(\Omega)$ for all $q < \frac{2K}{K+1}$ and for any ball $B \subset \Omega$ we have $\int_B |\nabla f(x)|^{\frac{2K}{K+1}} dx = \infty$.

Proof. We construct a sequence of piecewise affine mappings f_n inductively, similarly to the proof of Proposition 3.3, but with one important difference: this time we will obtain f_{n+1} from f_n by modifying the mapping almost everywhere, and so in order to guarantee the pointwise convergence of the gradients, in addition we need to control the L^1-norm of the difference $\nabla f_{n+1} - \nabla f_n$.

The basic construction in each step is as follows: Let A be a nonzero matrix and let $\varepsilon = \mathrm{dist}\,(A, E_k)$. Furthermore, let $\omega \subset \Omega$ be any open subset. Using Lemma 3.6 there exist matrices P, Q_0 with $\mathrm{rank}(P - Q_0) = 1$, $P \in E_\infty$, $Q_0 \in E_k$ with $|P - Q_0| > c_K^{-1}|A|$ such that $A \in [P, Q_0]$. Moreover

$$\mathrm{dist}\,(A, E_k) = \varepsilon \leqq |A - Q_0| < c_K \varepsilon \tag{43}$$

using Lemma 3.5. Let $Q \in [A, Q_0]$ with

$$|A - Q| = (c_K - \frac{1}{4})\varepsilon. \tag{44}$$

Then $|Q - Q_0| \leq \frac{1}{4}\varepsilon$. Moreover

$$|P - Q| \geqq |P - Q_0| - \frac{\varepsilon}{4} \geqq \frac{3}{4}|P - Q_0| > \frac{3}{c_K 4|A|} \tag{45}$$

since $|P - Q_0| \geqq |A - Q_0| \geqq \varepsilon$.

For any $\eta > 0$ Lemma 2.1 yields a piecewise affine Lipschitz mapping $g : \omega \to \mathbb{R}^2$ with $g(x) = Ax$ if $x \in \partial\omega$,

$$[g - A]_{C^\alpha(\overline{\omega})} < \eta/2, \tag{46}$$

and for some $\tilde\varepsilon \in (0, \varepsilon/4)$ to be chosen later

$$\big|\{x \in \omega :\, |\nabla g(x) - P| < \tilde\varepsilon\}\big| = \frac{|A-Q|}{|P-Q|}|\omega|$$
$$\big|\{x \in \omega :\, |\nabla g(x) - Q| < \tilde\varepsilon\}\big| = \frac{|A-P|}{|P-Q|}|\omega|.$$

Observe that (44) and (45) imply that for some $c_K > 1$

$$\frac{1}{c_K}\frac{\varepsilon}{|A|}\frac{|A-Q|}{|P-Q|} < c_K\frac{\varepsilon}{|A|}. \tag{47}$$

Let $\tilde\omega = \{x \in \omega :\, |\nabla g(x) - P| < \tilde\varepsilon\}$. We modify g in $\tilde\omega$ using Proposition 3.3, to yield a piecewise affine mapping $h : \omega \to \mathbb{R}^2$ with the following properties:

(a) $h(x) = Ax$ on $\partial\omega$,
(b) $[h - A]_{C^\alpha(\overline{\omega})} < \eta$,
(c) $\operatorname{dist}\big(\nabla h(x), E_{\{k,-k\}}\big) < \varepsilon/2 \min\big(|\nabla h(x)|^{-k}, 1\big)$ a.e. in ω,
(d) $\int_\omega |\nabla h(x) - A|dx < C\varepsilon|\omega|$

and for all $t > |A|$

$$\frac{1}{c_K}|A|^{\frac{2K}{K+1}}t^{-\frac{2K}{K+1}} < \big|\{x \in \Omega :\, |\nabla h(x)| > t\}\big| < c_K|A|^{\frac{2K}{K+1}}t^{-\frac{2K}{K+1}}. \tag{48}$$

More precisely, on each subset of $\tilde\omega$ where g is affine, that is $Dg = \tilde A$ a.e in $\tilde\omega$ for a matrix $\tilde A$ s we replace it with a map of the form

$$\tilde h(x) = \tilde g(Rx).$$

Here, $R = J^{-1}P$ (so that R is conformal) and $\tilde g : R(\tilde\omega) \to \mathbb{R}^2$ is given by Proposition 3.3 with $\tilde g(y) = \tilde A R^{-1}y$ on the boundary of $R(\tilde\omega)$ and

$$\tau_{\tilde A}(t) = \frac{\varepsilon}{2|R|}\min\Big(\big(t|R|\big)^{-k}, 1\Big).$$

At this point we choose $\tilde\varepsilon$ so that $\tilde\varepsilon|R^{-1}| < \tau_{\tilde A}(|\tilde A R^{-1}|)$, because then

$$|\tilde A R^{-1} - J| = |\tilde A R^{-1} - PR^{-1}| \leqq \tilde\varepsilon|R^{-1}| < \tau_{\tilde A}(|\tilde A R^{-1}|),$$

so the conditions of the proposition are fulfilled. Recall that $E_{\{k,-k\}}$ is invariant under pre-multiplication with conformal matrices (see (22)), hence

$$\begin{aligned}\operatorname{dist}\big(\nabla\tilde h, E_{\{k,-k\}}\big) &\leqq \operatorname{dist}\big(\nabla\tilde g R, E_{\{k,-k\}}R\big)\\ &< |R|\frac{\varepsilon}{2|R|}\min\Big(\big(|R||\nabla\tilde g|\big)^{-k}, 1\Big)\\ &\leqq \frac{\varepsilon}{2}\min\Big(|\nabla\tilde h|^{-k}, 1\Big).\end{aligned}$$

The properties (a) and (48) also follow directly from our construction and Proposition 3.3, and (b) follows from (46) and the fact that in getting h from g, we apply the proposition at most countably many times (in each subdomain of $\tilde{\omega}$ where g is affine). Moreover,

$$\begin{aligned}\int_\omega |\nabla h(x) - A|dx &\leqq \int_{\tilde{\omega}} |\nabla h(x)| + |A|dx + \int_{\omega\setminus\tilde{\omega}} |\nabla g(x) - A|dx \\ &\leqq 2|A||\tilde{\omega}| + \int_{|A|}^\infty \big|\{x \in \tilde{\omega} : |\nabla h(x)| > t\}\big|dt + \frac{\varepsilon}{2}|\omega| \\ &\leqq (2 + c_K)|A|\frac{|A-Q|}{|P-Q|}|\omega| + c_K\varepsilon|\omega| \\ &\leqq c_K\varepsilon|\omega|,\end{aligned}$$

for some $c_K > 1$ from (47) and (48).

Now we proceed with defining the sequence $\{f_n\}$ such that

$$\operatorname{dist}\big(\nabla f_n(x), E_{\{k,-k\}}\big) \leqq |F|^{1+k}2^{-n}\min\big(|\nabla f_n(x)|^{-k}, 1\big). \tag{49}$$

Let $f_0(x) \equiv Fx$. To obtain f_{n+1} from f_n, decompose Ω into a union of pairwise disjoint open sets of diameter no more than $\frac{1}{n}$ with

$$\Big|\Omega \setminus \bigcup_i \Omega_i^n\Big| = 0,$$

so that f_n is affine in each Ω_i^n, with $\nabla f_n = A_i^n$. In each Ω_i^n we apply the above construction with $\eta = 2^{-(n+i+1)}\delta$ and $\epsilon_n = |F|^{1+k}2^{-n}$ in order to obtain f_{n+1} with

$$[f_{n+1} - f_n]_{C^\alpha(\overline{\Omega})} < 2^{-(n+1)}\delta \tag{50}$$

$$\int_\Omega |\nabla f_{n+1} - \nabla f_n|dx < c_K 2^{-n}. \tag{51}$$

Observe that (49) follows automatically from property (c). Thus the sequence converges to some limit f in $W^{1,1}(\Omega)$ and $C^\alpha(\overline{\Omega})$. Clearly $f(x) = Jx$ on $\partial\Omega$, $[f - Fx]_{C^\alpha} < \delta$, and from (3) it follows that $\nabla f(x) \in E_{\{k,-k\}}$ almost everywhere in Ω. To conclude with the proof of the theorem, we provide estimates from above and below for the distribution function of the gradient ∇f.

To get an estimate from above, recall that f_{n+1} is obtained from f_n using the construction outlined in (43)-(48) above. In particular, let $\tilde{\Omega}_i^n \subset \Omega_i^n$ denote the open subset corresponding to $\tilde{\omega}$ and assume $\nabla f_n(x) = A_i^n$ on Ω_i^n with $|A_i^n| > 1$. From (47) and (49) we deduce

$$\big|\tilde{\Omega}_i^n\big| < c_K \frac{\operatorname{dist}\big(A_i^n, E_{\{k,-k\}}\big)}{\big|A_i^n\big|}\big|\Omega_i^n\big| < c_K 2^{-n}\big|A_i^n\big|^{-\frac{2K}{K+1}}|F|^{1+k}, \tag{52}$$

and

$$\big|\tilde{\Omega}_i^n\big| > \frac{1}{c_K}\frac{\operatorname{dist}\big(A_i^n, E_{\{k,-k\}}\big)}{\big|A_i^n\big|}\big|\Omega_i^n\big| > \frac{1}{c_K}2^{-n}\big|A_i^n\big|^{-\frac{2K}{K+1}}|F|^{1+k} \tag{53}$$

Recall that $k+1 = \frac{2K}{K+1}$. Then for $t > 1$ we have

$$\big|\{x \in \Omega :\, |\nabla f_{n+1}(x)| > t\}\big| = \sum_{i=1}^{\infty} \big|\{x \in \Omega_i^n :\, |\nabla f_{n+1}(x)| > t\}\big|$$

$$= \sum_{i=1}^{\infty} \big|\{x \in \tilde{\Omega}_i^n : |\nabla f_{n+1}(x)| > t\}\big| + \big|\{x \in \Omega_i^n \setminus \tilde{\Omega}_i^n : |\nabla f_{n+1}(x)| > t\}\big|,$$

which by (48) and (52) is controlled by,

$$\begin{aligned} &\leqq c_K\Big(\sum_{i=1}^{\infty} |\tilde{\Omega}_i^n| |A_i^n|^{\frac{2K}{K+1}} t^{-\frac{2K}{K+1}} + \big|\{x \in \Omega :\, |\nabla f_n(x)| > t\}\big|\Big) \\ &< c_K\Big(|F|^{\frac{2K}{K+1}} 2^{-n} \Big(\sum_{i=1}^{\infty} |\Omega_i^n|\Big) t^{-\frac{2K}{K+1}} + \big|\{x \in \Omega :\, |\nabla f_n(x)| > t\}\big|\Big) \\ &\leqq c_K\Big(|F|^{\frac{2K}{K+1}} 2^{-n} |\Omega| t^{-\frac{2K}{K+1}} + \big|\{x \in \Omega :\, |\nabla f_n(x)| > t\}\big|,\Big) \end{aligned}$$

where the sums are over all i such that $|A_i^n| > 1$. But then, passing to the limit, we see that

$$\big|\{x \in \Omega :\, |\nabla f(x)| > t\}\big| < 2|F|^{\frac{2K}{K+1}} c_K t^{-\frac{2K}{K+1}}. \tag{54}$$

For the estimate from below, we argue in a similar way. Let $B \subset \Omega$ be an open ball. For large enough $n_0 \in \mathbb{N}$ there exists i such that $\Omega_i^{n_0} \subset B$ and $\nabla f_{n_0} = A_i^{n_0}$ with $|A_i^{n_0}| \geq |F|$. But then by the construction there exists a constant c_K such that for $t > |A_i^{n_0}|$

$$\begin{aligned} \big|\{x \in B :\, |\nabla f_{n_0+1}(x)| > t\}\big| &\geqq c_K \big|\{x \in \tilde{\Omega}_i^{n_0} :\, |\nabla f_{n_0+1}(x)| > t\}\big| \\ &> c_K |F|^{\frac{2K}{K+1}} t^{-\frac{2K}{K+1}} |\Omega_i^{n_0}|. \end{aligned}$$

Next, arguing exactly as we did for bounding the distribution function from above, but with (53) instead of (52), we obtain that

$$\big|\{x \in \tilde{\Omega}_i^{n_0} : |\nabla f(x)| > t\}\big| \geqq c_K 2^{-N_0} |F|^{\frac{2K}{K+1}} t^{-\frac{2K}{K+1}} |\Omega_i^{n_0}|. \tag{55}$$

This concludes the proof of (iv), and the proof of the theorem. □

Remark 3.2. In Theorem 3.2 one could also take $F = 0$. Indeed, let $\tilde{F}$ be a nonzero rank-one matrix. By Lemma 2.1 there exists a piecewise affine Lipschitz mapping $f_0 : \Omega \to \mathbb{R}^2$ such that $f_0(x) = 0$ on $\partial\Omega$ and

$$\nabla f_0(x) \in B_{1/2|\tilde{F}|}(\tilde{F}) \cup B_{1/2|\tilde{F}|}(-\tilde{F}),$$

in other words there exist pairwise disjoint open subsets $\Omega_i \subset \Omega$ such that $\nabla f_0 \equiv F_i$ in Ω_i for some $F_i \in \mathbb{R}^{2\times 2} \setminus \{0\}$ and $|\Omega \setminus \cup_i \Omega_i| = 0$. Let

$$f(x) \stackrel{\text{def}}{=} f_i(x) \text{ in } \Omega_i,$$

where f_i is the mapping corresponding to Ω_i and boundary value F_i provided by Theorem 3.2. It follows that $f \in W^{1,1} \cap C^\alpha$ and satisfies the conditions (i)-(iv) of the theorem, with $F = 0$.

Remark 3.3. Theorem 3.2 answers a question of B.Yan on the existence of very weak quasiregular mappings with arbitrary affine boundary values. Namely, in the papers [35, 36] Yan constructed very weak quasiregular mappings $f:\Omega\to\mathbb{R}^n$ for $n\geqq 3$ such that $\nabla f(x)$ satisfies that $\|\nabla f\|^n=\rho\det\nabla f$ where $\rho(x)\in\{1,K\}$ and $f-Ax\in W_0^{1,p}(\Omega)$ for $p<\frac{nK}{K+1}$ where A is any matrix in 2×2. A question raised in [35] is whether such mappings exist fulfilling the more demanding condition

$$\|\nabla f\|^n = K\det\nabla f \ \text{ a.e.} \tag{56}$$

For $n=2$ Theorem 3.2 answers this in positive and in fact the control on the range of the gradient is substancially more precise than (56). It is an interesting question what happens in higher dimensions (see [13]).

Remark 3.4. It can be easily seen that by minor modifications Theorems 3.1 and 3.2 yield very weak solutions with the same properties to the classical Beltrami equation. We just need to replace the definition of E_Δ with

$$\{A=(a_+,a_-):\, a_-=\mu a_+ \text{ for some } \mu\in\mathit{\Delta}\}$$

and observe that the geometric properties necessary for the proof, Lemmas 3.5, 3.6 and 3.7 still hold.

Remark 3.5. Very weak solutions which fail to be solutions are really false solutions in the sense that they do not enjoy any of the special properties of honest weak solutions, like openess and discreteness, maximum principles and so forth. The investigation of this type of pathological solutions to elliptic equations started with the classical example by Serrin, [30], see also [16] for the concept of weak minimizer. Other types of very weak quasiregular mappings can be found in [15, Theorem 6.5.1,Theorem 11.6.1]. It is interesting however that our mappings are Hölder continuous for any exponent $0<\alpha<1$. A different type of Hölder continuous very weak quasiregular mappings haven been constructed by Jan Maly [21] using radial functions.

Remark 3.6. We conclude the section by discussing why we were not able to use the Baire Category argument. Theorem 3.2 implies that for any $F\in\mathbb{R}^{2\times 2}$ we can find a sequence f_j such that $Df_j(x)\in E_{k,-k}$ a.e in Ω and Df_j converge to F weakly in $W^{1,p}$ for every $p<\frac{2K}{K-1}$. Therefore the natural set $\mathcal{U}$ as in definition 3.2 is the entire $\mathbb{R}^{2\times 2}$, which yield no bound in the corresponding X. A way to go around this is to restrict the attention to a subset of E, which still supports the appropriate laminate. The difficulty is that then the set $\mathcal{U}$ will contain rank-one lines, which prevent us to have a bound in any $W^{1,p}$ for $p>1$.

4. Equations in non-divergence form

We follow the same lines as in the case of isotropic equations, since from the point of view of differential inclusions the structure of both problems is very similar. The entire construction lies in the set of 2 by 2 symmetric matrices $\mathbb{R}^{2\times 2}_{\text{sym}}$. Therefore during the whole section we use the topology of this 3 dimensional vector space. For a set $\Delta \in \mathbb{C}\cup\{\infty\}$ we use the notation

$$\mathbb{E}_\Delta = \{A = (a_+, a_-) : 2a_+ = \mu a_- + \overline{\mu a_-} \text{ for some } \mu \in \Delta\}. \tag{57}$$

Notice that $\mathbb{E}_\Delta \subset \mathbb{R}^{2\times 2}_{\text{sym}}$ by definition. Then $\mathbb{E}_0$ are anticonformal matrices and $\mathbb{E}_\infty$ are symmetric conformal matrices. That is $\mathbb{E}_\infty$ is the real one dimensional subspace spanned by the identity.

We start by reinterpreting the equation as a differential inclusion. Recall that for symmetric matrices, (15) implies that

$$\text{Tr}(AB) = 2\Re(a_+b_+ + a_-\overline{b}_-) = 2a_+b_+ + (a_-\overline{b}_- + \overline{a}_-b_-). \tag{58}$$

Lemma 4.1. *Let $K \geqq 1$ and $k = \frac{K-1}{K+1}$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then $u : \Omega \to \mathbb{R}$ is a solution to*

$$Tr\big(A(x)D^2u(x)\big) = 0 \text{ in } \Omega \tag{59}$$

with some measurable $A(x) : \Omega \to \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$
if and only if for almost every $x \in \Omega$

$$D^2u(x) \in \mathbb{E}_{\{k,-k\}}.$$

Proof. Let $A = (a_+, a_-), B = (b_+, b_-) \in \mathbb{R}^{2\times 2}_{\text{sym}}$ with $A > 0$ and put $\mu_A = \frac{a_-}{a_+}$. The lemma follows from writing

$$\text{Tr}(AB) = 0 \tag{60}$$

in conformal coordinates. In fact, by (58), we obtain that (60) is equivalent to

$$B \in \mathbb{E}_{-\overline{\mu}_A}. \tag{61}$$

Hence, u solves (59) for a general $A(x)$ if and only if $D^2u(x) \in \mathbb{E}_{-\overline{\mu}_{A(x)}}$ almost everywhere. Putting $A(x) \in \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$ finishes the proof. □

4.1. **Upper exponents.** As in the isotropic case, the first step is to define the appropriate complete metric space.

Definition 4.1. Let

$$\mathcal{U} = \{A \in \mathbb{R}^{2\times 2}_{\text{sym}} : a_+ < k|\Re a_-|\}$$

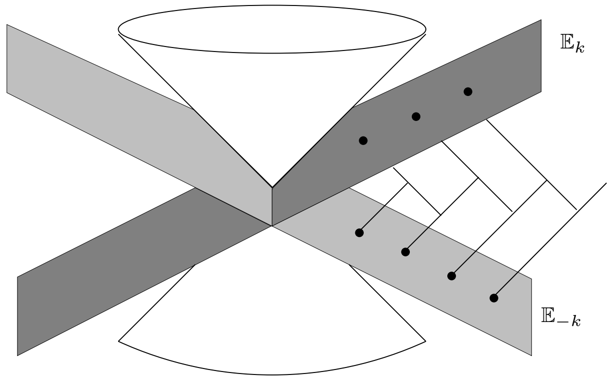

FIGURE 3. The set $\mathbb{E}_{\{k,-k\}}$ and the rank-one cone in $\mathbb{R}^{2\times 2}_{sym}$.

Let X be the closure in the weak topology of $W^{2,2}$ of the set

$$X_0 = \left\{ u \in W^{2,\infty}(\overline{\Omega}, \mathbb{R}^2) : \begin{array}{l} \bullet\, u \text{ piecewise affine} \\ \bullet\, D^2u(x) \in \mathcal{U} \text{ a.e.} \\ \bullet\, u(x) = \frac{|x|^2}{2} \text{ on } \partial\Omega \\ \bullet\, \nabla u(x) = x \text{ on } \partial\Omega \end{array} \right\}. \tag{62}$$

Lemma 4.2. *With the above definitions,*

$$\mathbb{E}^{lc,1}_{\{k.-k\}} = \mathbb{E}^{pc}_{\{k,-k\}} = \overline{\mathcal{U}}$$

Proof. Let $f : \mathbb{R}^{2\times 2}_{\text{sym}} \to \mathbb{R}$ be $f(A) = \Im(a_-)^2 + (1-k^2)\Re(a_+)^2 + \det(A)$. Then it is easy to see that if $A \in \mathbb{E}, f(A) = 0$. Therefore by definition 2.2 if $A \in \mathbb{E}^{pc}$ $f(A) \le 0$. But this implies that $A \in \overline{\mathcal{U}}$. On the other hand any $A \in \overline{\mathcal{U}}$ lies in a rank-one segment $[B, C]$ with $b_+ = k\Re(b_-), c_+ = -k\Re(c_-)$ and $B - C = t(1,1)$ □

Remark 4.1. The above lemma is equivalent to proving that for equations of the type

$$2\partial_z f = \mu \partial_{\bar z} f + \overline{\mu \partial_{\bar z} f} \tag{63}$$

the corresponding G-closure is given by $G(k,-k) = [-k, k]$.

Now we can repeat the arguments in Section 3 word by word. The only difference is that we need to use the part *ii*) of Lemma 2.1 to stay in symmetric matrices.

Lemma 4.3. *The space (X, w) is metrizable, with metric d, and for any $f \in X$ we have $D^2u(x) \in \overline{\mathcal{U}}$ a.e. in Ω. Furthermore the set of continuity points of the map $D^2 : (X, w) \to L^2(\Omega, \mathbb{R}^{2\times 2}_{sym})$ is second category in (X, w).*

Proof. We can use elliptic regularity here as well to obtain that X is metrizable. Indeed by [2, Theorem 3.6] there exists an uniform constant $c = c(K, \Omega)$ such that

$$\int_\Omega |D^2u|^2 \le c$$

for every $u \in X$. Since $f_u = (u_x, -u_y)$ is K-quasiregular and affine in the boundary of Ω we obtain that $Df_u \in W^{1,p}(\Omega)$ for $p > 2$. Thus, we obtain continuity of the determinant respect to the weak topology in $W^{2,2}$, which implies as in Lemma 3.3 that $D^2u(x) \in \mathcal{U}^{pc}$. The rest of the proof is exactly the same as in Lemma 4.3. □

Lemma 4.4. *The set of points of continuity in (X, d) of D^2 satisfy that $D^2u(x) \in \mathbb{E}_{\{k,-k\}}$ almost everywhere.*

Proof. We can repeat line by line the proof of Lemma 3.4. The only difference is at (30) when we use part *ii*) of Lemma 2.1 instead of part *i*). □

Corollary 4.1. *The set of mappings in X such that $D^2u(x) \in \mathbb{E}_{\{k,-k\}}$ is of second category*

4.1.1. *Laminates and Integrability.*

Proposition 4.1 (The symmetric strong staircase)**.** *Let $A \in \mathcal{U}$. Then there exists a sequence $\{\nu_n\}_{n=1}^{\infty} \in \mathcal{L}(\mathcal{U})$ such that*

$$\int_{\mathbb{R}^{2\times 2}_{sym}} \lambda d\nu_n = A,$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^{2\times 2}_{sym}} |\lambda|^{\frac{2K}{K-1}} d\nu_n = \infty. \tag{64}$$

Proof. We give the building blocks of the appropriate staircase laminate. Let $e_k = \frac{1}{1+k}(k, 1)$ and $e_{-k} = \frac{1}{1+k}(-k, 1)$ in conformal coordinates. We define

$$\nu^n = \lambda_n^1 \delta_{ne_k} + \lambda_n^2 \delta_{(n+1)e_{-k}} + \lambda_n^3 \delta_{(n+1)J}, \tag{65}$$

where $\lambda_n^1 = \frac{1+k}{2nk-1-k}$, $\lambda_n^2 = (1 - \frac{1+k}{2nk-1-k})\frac{1+k}{2nk}$ and $\lambda_n^3 = (1 - \frac{1+k}{2nk-1-k})(1 - \frac{1+k}{2nk})$. Then ν^n is a second order laminate $\int \lambda d\nu^n = (0, n)$. Moreover,

$$\Pi_{i=1}^n \lambda_i^3 \approx n^{-\frac{2K}{K-1}}. \tag{66}$$

Then the measures ν_n are obtained from ν^n exactly as in [11]. We briefly recall the process. To start with we put $\nu_1 \stackrel{def}{=} \nu^1$. The first step is to construct a new measure $\nu_2 \in \mathcal{L}(\mathbb{E})$ by replacing δ_{2J} in the definition of ν_1 by ν^2. Then ν_2 has an atom δ_{3J}. This is further replaced by ν^3 yielding a new measure ν_3. The process is continued till infinity replacing always δ_{nJ} by the laminate ν^n obtaining a measure ν_n. The center of mass ν_n will be always J and (64) follows from (66). To achieve that the support of ν_n is contained in $\mathcal{U}$ we shift the construction considering the measures $\tilde{\nu_n}(\cdot) = \nu_n(\cdot + J)$.

The set $\mathbb{E}_{\{k,-k\}}$ it is invariant under multiplication for scalars and addition of matrices with conformal coordinates $(0, ti)$. Therefore we can do the previous constructions with any matrix in the plane $\mathbb{E}_0$ as center of mass. Finally any $A \in \mathcal{U}$ is rank-one connected to $\mathbb{E}_0$ along the rank one line $(1,1)$. Thus, we can argue as in the Proposition 3.1 to have the laminates with center of mass A. □

Since the proofs are again exactly the same we just quote the final outcome.

Proposition 4.2. *The set of points in X such that $\int_{B(x,r)} |D^2u|^{\frac{2K}{K-1}} = \infty$ for all $B(x,r) \subset \Omega$ is second category in (X,d)*

Theorem 4.1. *Let $K > 1$ and $k = \frac{K-1}{K+1}$. For any bounded open set $\Omega \subset \mathbb{R}^2$ there exists a function $u \in W^{2,2}(\Omega;\mathbb{R}^2)$ with the following properties:*

(i) *$u(x) = x$ on $\partial\Omega$,*
(ii) *$D^2u(x) \in \mathbb{E}_{\{k,-k\}}$ a.e. in Ω,*
(iii) *For any ball $B \subset \Omega$ we have $\int_B |D^2(x)|^{\frac{2K}{K-1}} dx = \infty$.*

4.2. **Lower critical exponent.** As in the case of the upper exponent the proofs here are very similar. We start by a geometric lemma describing the rank-one connections of any matrix to $\mathbb{E}_{\{k,-k\}}$

Lemma 4.5. *For every $A \in \mathbb{R}^{2\times 2}_{sym}$ there exists $P \in E_\infty$ and and $Q \in \mathbb{E}_{\{k,-k\}}$ with rank $(P-Q) = 1$ such that $A \in [P,Q]$ and*

$$\frac{1}{c_K}|A| \le |P-Q| \le c_K|A| \tag{67}$$

Proof. Let $A = (a_+, a_-) \in \mathbb{R}^{2\times 2}_{\text{sym}}$. Then for $P = (a_+ \pm |a_-|, 0) \in E_\infty, \det(P-A) = 0$. The line $P + t(\pm|a_-|, a_-)$ is rank one and eventually it will hit the planes $\mathbb{E}_k$ and $\mathbb{E}_{-k}$. The estimates (67) follow from Lemma 3.5 □

Next, we find that laminates with the required integrability exists also in this setting.

Lemma 4.6 (One symmetric weak step)**.** *Let $A \in B_r(nI)$ for some $0 < r < 1/2$. There exists a laminate ν_A of third order with the following properties:*

- *$\overline{\nu}_A = A$,*
- *$\operatorname{spt} \nu_A \subset \mathbb{E}_{\{k,-k\}} \cup \big\{(n+1)I\big\}$,*
- *$\operatorname{spt} \nu_A \subset \big\{\xi \in \mathbb{R}^{2\times 2} : c_K^{-1} n < |\xi| < c_K n\big\}$,*
- *$(1 - c_K^{-1}\frac{r}{n})\beta_n < \nu_A\big(\{(n+1)I\}\big) < (1 + c_K\frac{r}{n})\beta_n$,*

where

$$\beta_n = \frac{n}{n+1}\frac{1-k+2n}{1+k+2n}, \tag{68}$$

and $c_K > 1$ is a constant only depending on K.

Proof. Let

$$C_k = \frac{1}{1+k}(k,1), \quad C_{-k} = \frac{1}{1+k}(k,-1) \tag{69}$$

in conformal coordinates. If $A = nI$ the claim follows by considering the laminate

$$\nu_A = \Big(\lambda_1\delta_{nC_k} + (1-\lambda_1)\big(\lambda_2\delta_{(n+1)C_{-k}} + (1-\lambda_2)\delta_{(n+1)I}\big)\Big),$$

with

$$\lambda_1 = \frac{1+k}{2n+1+k}, \tag{70}$$

$$\lambda_2 = \frac{(1+k)}{2(n+1)}. \tag{71}$$

This is by definition a laminate and $(1-\lambda_2)(1-\lambda_3) = \beta_n$. The argument for $A \in B(nI, r)$ different for nI combines this observation with Lemma 4.5 just as in the case of isotropic equations, Lemma 3.7. □

Proposition 4.3 (The symmetric weak staircase). *Let $K > 1$ and $k = \frac{K-1}{K+1}$. Let $\alpha \in (0,1)$, $\delta > 0$ and $\tau : [0,\infty) \to (0,1]$ a continuous, non increasing function with $\tau(0) > 0$ and $\int_1^\infty \frac{\tau(t)}{t} dt < \infty$.*

For any bounded open set $\Omega \subset \mathbb{R}^2$ there exists a piecewise affine function $u \in W^{2,1}(\Omega;\mathbb{R}) \cap C^{1,\alpha}(\overline{\Omega};\mathbb{R})$ with the following properties:

(i) *$u(x) = x$ on $\partial\Omega$,*
(ii) *$[u - \frac{1}{2}\langle Cx, x\rangle]_{C^{1,\alpha}(\overline{\Omega})} < \delta$,*
(iii) *$\operatorname{dist}\big(D^2u(x), \mathbb{E}_{\{k,-k\}}\big) < \tau\big(|D^2u(x)|\big)$ a.e. in Ω,*

and there exists a constant $c_{K,\tau} > 0$ so that for all $t > 1$ we have

$$\frac{1}{c_{K,\tau}} t^{-\frac{2K}{K+1}} < \big|\{x \in \Omega : |D^2u(x)| > t\}\big| < c_{K,\tau} t^{-\frac{2K}{K+1}}. \tag{72}$$

Proof. The proof mimics once more the corresponding situation for isotropic equations. The difference is that the step laminates are those from Lemma 4.6 and that since the laminate is supported in symmetrics we can approximate the laminate by the distributions of Hessians. □

Theorem 4.2. *Let $K > 1$, $k = \frac{K-1}{K+1}$ and let $F \in \mathbb{R}^{2\times 2} \setminus \{0\}$. For any $\alpha \in (0,1)$, $\delta > 0$ and for any bounded open set $\Omega \subset \mathbb{R}^2$ there exists a function $u \in W^{2,1}(\Omega;\mathbb{R}) \cap C^{\alpha}(\overline{\Omega};\mathbb{R})$ with the following properties:*

(i) *$u(x) = \frac{1}{2}\langle Fx, x\rangle$ on $\partial\Omega$,*
(ii) *$[u - \frac{1}{2}\langle Cx, x\rangle]_{C^{1,\alpha}(\overline{\Omega})} < \delta$,*
(iii) *$D^2u(x) \in E_{\{k,-k\}}$ a.e. in Ω,*
(iv) *For any ball $B \subset \Omega$ there exists a constant $c_B > 1$ such that*

$$\frac{1}{c_B} t^{-\frac{2K}{K+1}} < \big|\{x \in B : |D^2u(x)| > t\}\big| < c_B t^{-\frac{2K}{K+1}}$$

for all $t \geqq 1$.

In particular $u \in W^{2,q}(\Omega)$ for every $q < \frac{2K}{K+1}$ but for any ball $B \subset \Omega$ we have $\int_B |D^2u(x)|^{\frac{2K}{K+1}} dx = \infty$.

Proof. The scheme of the proof for the corresponding theorem for isotropic equations, Theorem 3.2 can be followed line by line, replacing always Lemma 3.6 by Lemma 4.5 and part (i) of Lemma 2.1 by part (ii). □

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References

[1] Astala, K. Area distortion of quasiconformal mappings. *Acta Math. 173*, 1 (1994), 37–60.

[2] Astala, K., Iwaniec, T., and Martin, G. Pucci's conjecture and the alexandrov inequality for elliptic pdes in the plane. *Preprint* (2004).

[3] Astala, K., Iwaniec, T., and Martin, G. *Quasiconformal mappings and PDE in the plane*. In preparation.

[4] Astala, K., Iwaniec, T., and Saksman, E. Beltrami operators in the plane. *Duke Math. J. 107*, 1 (2001), 27–56.

[5] Conti, S., Faraco, D., and Maggi, F. A new approach to counterexamples to l^1 estimates: Korn's inequality, geometric rigidity, and regularity for gradients of separately convex functions. Preprint, MPI-MIS, 93/2003.

[6] Conti, S., Faraco, D., Maggi, F., and Müller, S. Rank-one convex functions on 2×2 symmetric matrices and laminates on rank-three lines. Preprint, MPI-MIS, 50/2004.

[7] Dacorogna, B. *Direct methods in the calculus of variations*, vol. 78 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 1989.

[8] Dacorogna, B., and Marcellini, P. General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases. *Acta Math. 178* (1997), 1–37.

[9] Dacorogna, B., and Marcellini, P. *Implicit partial differential equations*, vol. 37 of *Progress in Nonlinear Differential Equations and their Applications.* Birkhäuser Boston Inc., Boston, MA, 1999.

[10] Dragičević, O., and Volberg, A. Sharp estimate of the Ahlfors-Beurling operator via averaging martingale transforms. *Michigan Math. J. 51*, 2 (2003), 415–435.

[11] Faraco, D. Milton's conjecture on the regularity of solutions to isotropic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire 20*, 5 (2003), 889–909.

[12] Faraco, D. Tartar conjecture and Beltrami operators. *Michigan Math. J. 52*, 1 (2004), 83–104.

[13] Faraco, D., and Székelyhidi, Jr., L. *in preparation.*

[14] Gromov, M. *Partial differential relations*, vol. 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1986.

[15] Iwaniec, T., and Martin, G. *Geometric function theory and non-linear analysis*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2001.

[16] Iwaniec, T., and Sbordone, C. Weak minima of variational integrals. *J. Reine Angew. Math. 454* (1994), 143–161.

[17] Kirchheim, B. Rigidity and Geometry of microstructures. Habilitation thesis, University of Leipzig, 2003.

[18] Kirchheim, B., Müller, S., and Šverák, V. Studying nonlinear PDE by geometry in matrix space. In *Geometric analysis and Nonlinear partial differential equations*, S. Hildebrandt and H. Karcher, Eds. Springer-Verlag, 2003, pp. 347–395.

[19] Lehto, O., and Virtanen, K. I. *Quasiconformal mappings in the plane*, second ed. Springer-Verlag, New York, 1973. Translated from the German by K. W. Lucas, Die Grundlehren der mathematischen Wissenschaften, Band 126.

[20] Leonetti, F., and Nesi, V. Quasiconformal solutions to certain first order systems and the proof of a conjecture of G. W. Milton. *J. Math. Pures Appl. (9) 76*, 2 (1997), 109–124.

[21] Malý, J. Examples of weak minimizers with continuous singularities. *Exposition. Math. 13*, 5 (1995), 446–454.

[22] Milton, G. W. Modelling the properties of composites by laminates. In *Homogenization and effective moduli of materials and media (Minneapolis, Minn., 1984/1985)*, vol. 1 of *IMA Vol. Math. Appl.* Springer, New York, 1986, pp. 150–174.

[23] Müller, S. Variational models for microstructure and phase transitions. In *Calculus of variations and geometric evolution problems (Cetraro, 1996)*, vol. 1713 of *Lecture Notes in Math.* Springer, Berlin, 1999, pp. 85–210.

[24] Müller, S., and Šverák, V. Attainment results for the two-well problem by convex integration. In *Geometric analysis and the calculus of variations*. Internat. Press, Cambridge, MA, 1996, pp. 239–251.

[25] Müller, S., and Šverák, V. Convex integration for Lipschitz mappings and counterexamples to regularity. *Ann. of Math. (2) 157*, 3 (2003), 715–742.

[26] Pedregal, P. Laminates and microstructure. *European J. Appl. Math. 4* (1993), 121–149.

[27] Pedregal, P. Fully explicit quasiconvexification of the mean-square deviation of the gradient of the state in optimal design. *Electron. Res. Announc. Amer. Math. Soc. 7* (2001), 72–78 (electronic).

[28] Petermichl, S., and Volberg, A. Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular. *Duke Math. J. 112*, 2 (2002), 281–305.

[29] Piccinini, L. C., and Spagnolo, S. Una valutazione della regolarità delle soluzioni di sistemi ellittici variazionali in due variabili. *Ann. Scuola Norm. Sup. Pisa (3) 27* (1973), 417–429 (1974).

[30] Serrin, J. Pathological solutions of elliptic differential equations. *Ann. Scuola Norm. Sup. Pisa (3) 18* (1964), 385–387.

[31] Sychev, M. A. Comparing two methods of resolving homogeneous differential inclusions. *Calc. Var. Partial Differential Equations 13*, 2 (2001), 213–229.

[32] Sychev, M. A. Few remarks on differential inclusions. Preprint, MPI-MIS, 2001.

[33] Székelyhidi, Jr., L. The regularity of critical points of polyconvex functionals. *Arch. Ration. Mech. Anal. 172*, 1 (2004), 133–152.

[34] Tartar, L. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, vol. 39 of *Res. Notes in Math.* Pitman, Boston, Mass., 1979, pp. 136–212.

[35] Yan, B. A linear boundary value problem for weakly quasiregular mappings in space. *Calc. Var. Partial Differential Equations 13*, 3 (2001), 295–310.

[36] Yan, B. A Baire's category method for the Dirichlet problem of quasiregular mappings. *Trans. Amer. Math. Soc. 355*, 12 (2003), 4755–4765 (electronic).

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