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Binary black hole spacetimes with a helical
Killing vector

by

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# Binary black hole spacetimes with a helical Killing vector

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## Abstract

Binary black hole spacetimes with a helical Killing vector, which are discussed as an approximation for the early stage of a binary system, are studied in a projection formalism. In this setting the four dimensional Einstein equations are equivalent to a three dimensional gravitational theory with a $SL(2, \mathbb{C})/SO(1, 1)$ sigma model as the material source. The sigma model is determined by a complex Ernst equation. $2+1$ decompositions of the 3-metric are used to establish the field equations on the orbit space of the Killing vector. The two Killing horizons of spherical topology which characterize the black holes, the cylinder of light where the Killing vector changes from timelike to spacelike, and infinity are singular points of the equations. The horizon and the light cylinder are shown to be regular singularities, i.e. the metric functions can be expanded in a formal power series in the vicinity. The behavior of the metric at spatial infinity is studied in terms of formal series solutions to the linearized Einstein equations. It is shown that the spacetime is not asymptotically flat in the strong sense to have a smooth null infinity under the assumption that the metric tends asymptotically to the Minkowski metric. In this case the metric functions have an oscillatory behavior in the radial coordinate in a non-axisymmetric setting, the asymptotic multipoles are not defined. The asymptotic behavior of the Weyl tensor near infinity shows that there is no smooth null infinity.

## 1 Introduction

Binary black hole systems in the last stage before coalescence are the most promising sources of gravitational radiation to be detected with the first generation of gravitational wave detectors. From a theoretical point of view this is a difficult relativistic problem which can possibly only be solved numerically since there are no symmetries. The advantage of black hole systems is that no matter is involved and that only the vacuum equations have to be studied. It is generally expected that binary systems will have an

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early stage of quasi-circular motion for large separation of the binaries. Radiation damping will lead to almost circular orbits with the radius decreasing in time due to the emitted radiation. For smaller distances of the binaries this motion is expected to be followed by a rapid inspiral. The end result will be a single black hole which will settle to a stationary hole in a ring down phase which can be described by black hole perturbation theory. For a given binary system an important characteristic quantity is the innermost stable circular orbit (ISCO), the last almost circular orbit before the final inspiral. In general there is no unique definition of the ISCO, but it should be possible to define a characteristic scale where the quasi-stationary early phase of the binary system comes to an end.

In the case of the binary motion of two oppositely charged particles, Schönberg [1] and Schild [2] considered an approximation to the quasi-stationary phase of the system. The quasi-circular motion is approximated by a sequence of exactly circular orbits which are obtained by exactly compensating the outgoing radiation by ingoing radiation. The binding energy of the system as defined in [2] decreases with the distance of the charges up to some minimal value which can be taken as the definition of the ISCO: for smaller values of the distance, more and more incoming radiation is needed to stabilize the circular motion. The approximation thus predicts its own breakdown and allows for an unambiguous definition of the ISCO. The quasi-stationary approximation corresponds to a so-called helical Killing vector $\xi$ of the system which is in standard Minkowski coordinates given by $\xi = \partial_t + \Omega \partial_\phi$. The main features of such a vector can already be inferred from this case: the vector becomes null at the so-called light cylinder given by $\rho = 1/\Omega$ where the observer rotating with the angular velocity $\Omega$ rotates with the velocity of light. In the interior of the light cylinder, the Killing vector is timelike, in the exterior spacelike. In a general spacetime with a helical Killing vector, the light cylinder will be deformed, but will still have cylindrical topology.

Detweiler [3, 4, 5] suggested to use this concept to describe the quasi-circular regime of binary black holes which corresponds to studying spacetimes with a helical Killing vector. Since Einstein's theory is a nonlinear theory, the incoming radiation will lead to a spacetime which is not asymptotically flat in a strong sense (mass and angular momentum cannot be defined in the usual way asymptotically). It was shown in [6] that spacetimes with a helical Killing vector cannot have a smooth null infinity if there is no additional stationary Killing vector close to $\mathcal{I}$, see also [7]. Though the Arnowitt-Deser-Misner(ADM)-mass cannot be defined, Friedman et al. [8] could show that a thermodynamical treatment as in the single black hole case is possible and that there exists a first law. With the help of the first law, the ISCO can be defined for asymptotically flat spacetimes (which are possible for instance in the case of the first post-Newtonian order) as in the Maxwell case as the minimum of the binding energy which also marks the onset of dynamical instability. It is not clear how this result can be generalized to non-asymptotically flat spacetimes in full general relativity. An important conceptional advantage of spacetimes with a helical Killing vector is the fact shown in [8] that spatially compact Killing horizons are event horizons. This allows for a local characterization of the event horizons in these models. It is thus not necessary to use local concepts as an apparent horizon or the concepts

developed in [9].

In the study of binary black hole system, mainly approximative and numerical methods have been used so far. Post-Newtonian calculations have been carried out up to the third post-Newtonian order including resummation techniques (see [10, 11, 12, 13, 14]). Within this approximation the determination of the ISCO appears to be self-consistent since corrections to its value due to higher order terms can be shown to be negligible. The post-Newtonian metric, however, cannot be used close to the horizons since black holes are the strongest relativistic objects known. Numerical calculations so far have been mainly performed within the Isenberg-Wilson-Matthews (IWM) theory [15, 16], an alternative theory of gravitation without radiation. It follows from the Einstein equations in a standard $3 + 1$-splitting for a conformally flat spatial metric on the $t = const$ hypersurfaces. Only the trace of the 6 time evolution equations is considered. By construction the theory coincides with the Einstein theory for spacetimes with conformally flat spatial slices and thus reproduces exactly the first post-Newtonian approximation. It has to be noted that the Kerr solution does not allow conformally flat spatial hypersurfaces, see [17]. In [18, 19, 20] initial values were constructed from the constraint equations via so called Bowen-York initial data by solving the Lichnerowicz equation numerically. This resulted in a significant discrepancy with post-Newtonian results for the ISCO. In [21, 22], complete IWM binary black hole spacetimes with a helical Killing vector were constructed numerically. The results were in good agreement with post-Newtonian results, but suffered from an inconsistency in the model in the form of non-regular horizons. This non-regularity appears to be unavoidable for IWM spacetimes. To answer the question whether the Einstein equations allow for binary black hole spacetimes with smooth disconnected horizons in the presence of a helical Killing vector, it therefore appears necessary to study the fully relativistic situation. In [23] the constraint equations are solved in the presence of an approximate helical Killing vector on a background which is the superposition of two Kerr-Schild metrics.

In general relativity, a helical Killing vector will lead to Einstein equations which are of a mixed type, i.e. elliptic in the interior of the light cylinder and hyperbolic in the exterior. Such equations are studied in aerodynamics [24] in the case of transonic flows. For a review on mixed problems arising in gravitation, see [25]. It was shown by Torre [26] that the resulting equations can be written in the form of a first order differential system which belongs to the symmetric positive systems of Friedrichs [27] and Lax and Phillips [28]. Classes of boundary conditions compatible with these equations are given in [26]. Numerical studies of two-dimensional equations of this type were performed in [29, 30]. Three-dimensional toy models for the helically reduced Einstein equations were considered numerically in [31].

The purpose of this paper is to study the Einstein equations in the presence of a helical Killing vector for a vacuum spacetime with two disconnected Killing horizons of spherical topology. The idea is to derive a set of equations which is well suited for a numerical treatment by taking full advantage of the Killing vector. In the vicinity of the critical points of the equations as the black hole horizons, the metric functions will be given in

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terms of formal power series which should be useful for the numerical implementation. The approach is similar as in the study of cosmological singularities, see e.g. [32] and references therein. These methods are purely local and do not imply an existence proof for a spacetime with a helical Killing vector and two regular horizons. The use of so-called Fuchsian methods (see [33]) in the context of cosmological singularities would not change the situation since even an existence proof for one regular horizon would not imply the existence of the second regular horizon one is interested in here. Therefore we will not discuss the question of existence and convergence radii of the formal series solutions in this paper, to show global existence different methods will have to be applied. Function counting, i.e. the identification of free functions in the formal series solutions indicates, however, that spacetimes with a helical Killing vector and two regular horizons could exist: the series in the vicinity of the horizon contains two free functions which could be fixed in a way to allow for two regular horizons. Such an approach is suitable for a numerical treatment which could give – if successful – a strong indication of the existence for corresponding solutions to the Einstein equations. Though formal expansions do not provide existence proofs for solutions, they proved to be very useful and reliable to establish the behavior of cosmological solutions near singularities and served as a guide to prove existence and uniqueness, see [32, 33].

To establish the Einstein equations in the presence of a helical Killing vector we use a projection formalism [34, 35] which leads to equations of 3-dimensional gravity with a sigma model as material source. The sigma model is determined by a generalized Ernst equation. The equations are discussed on the space of orbits of the Killing vector in a $2 + 1$-decomposition of the 3-space. The fixed points of the Killing vector, the horizons and the light cylinder where the Killing vector changes from being timelike to spacelike are singular points of the equations. They will be studied in terms of formal expansions of the metric functions in a chosen gauge. In this approach the horizons are regular horizons as in the case of the Kerr metric, but the series solutions contain two free functions which cannot be fixed locally. At the light cylinder the metric shows a similar behavior as at an ergosphere. If the constraints in the $2 + 1$-approach are satisfied by the regularity conditions, the only equations to be solved are the two Ernst equations and the three 'evolution' (with respect to the radial coordinate) equations.

The behavior at infinity for spacetimes with a helical Killing vector is not yet well understood. There are indications from spherically symmetric models with equal amounts of ingoing and outgoing null dust [36], which can be seen as a spherically symmetric analog to a helically symmetric spacetime, that the spacetime is not even asymptotically flat in the weak sense that the metric tends asymptotically to the Minkowski metric. So far the only rigorous result due to Gibbons and Stewart [6] is that there can be no smooth null infinity unless there is an additional axial Killing vector. Here we show that this is already case under the possibly too strong assumption that the spacetime tends to Minkowski spacetime asymptotically. Considering the Einstein equations in the presence of a helical Killing vector for large values of the radial coordinate, we show by using formal expansions of the metric that the spacetime is not asymptotically flat in the strong

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sense that mass and angular momentum can be defined unless there is an additional axial Killing vector. The Weyl scalars behave asymptotically as $1/r$, but the coefficients of the $1/r$ terms have an oscillatory dependence on $r$. Thus there is no smooth $\mathcal{I}$ and no peeling in accordance with [6]. In asymptotically flat spacetimes for a stationary Killing vector, the Komar integral can be used to determine a conserved quantity via a surface integral calculated at finite radius, for which then the limit of an infinite radius is taken. We show that this limit is not defined under the above assumptions unless there is an additional axial Killing vector.

The paper is organized as follows: In section 2 we use the projection formalism for the vacuum Einstein equations in the presence of a helical Killing vector. We discuss the resulting equations for the example of Minkowski spacetime. In section 3 we use $2+1$-decompositions of the 3-space to study the singularities of the equations, the Killing horizons and the light cylinder. We use a formal expansion of the metric functions in local coordinates adapted to the singularities. In section 4 we study the linearized Einstein equations on a Minkowski background asymptotically in terms of formal expansions of the metric. We discuss the Weyl tensor and the Komar integral. In section 5 we add some concluding remarks.

# 2   Quotient space metrics and Ernst equations

The existence of a Killing vector can be used to establish a simplified version of the field equations by dividing out the group action. These quotient space metrics were first used in [34], see also [35]; here we will follow [37]. We use adapted coordinates in which the Killing vector $\xi$ is given by $\xi = \partial_t$ where $t$ is not necessarily a timelike coordinate. The norm of the Killing vector will be denoted by $f$. The decomposition we are using is not defined at the fixed points of the group action, i.e. the zeros of $f$, and the resulting equations will be singular at the set of zeros of $f$. The behavior of the solutions at these singular points will be discussed in the following section by studying formal expansions of the metric in the vicinity of the fixed points.

In contrast to a standard $3+1$-decomposition, the metric is written in this approach in the form

$$ds^2 = -f(dt + k_a dx^a)(dt + k_b dx^b) + \frac{1}{f} h_{ab} dx^a dx^b; \tag{1}$$

latin indices always take the values $1, 2, 3$ corresponding to the spatial coordinate. The Einstein equations in vacuum imply a Maxwell-type equation for what corresponds to the momentum constraint in a standard $3+1$-decomposition (see [37])

$$\frac{1}{2} D_a(f^2 k^{ab}) = 0, \tag{2}$$

where $k_{ab} = k_{b,a} - k_{a,b}$ ($D_a$ denotes the covariant derivative with respect to $h_{ab}$). Notice that all indices here are raised and lowered with $h_{ab}$. If we define the twist potential $b$ via

($\epsilon^{abc}$ is the tensor density with $\epsilon^{123} = 1/\sqrt{h}$)

$$k^{ab} = \frac{1}{\sqrt{h}f^2}\epsilon^{abc}b_{,c}, \tag{3}$$

where $h$ is the determinant of $h_{ab}$, then equation (2) is identically satisfied. The potentials $f$ and $b$ can be combined to the complex Ernst potential $\mathcal{E} = f + ib$ [38]. The equations for $f$ and the integrability condition for $b$ can then be combined to the generalized complex Ernst equation (the Ernst equation was originally obtained for the stationary axisymmetric case in [38])

$$fD_a D^a \mathcal{E} = D_a \mathcal{E} D^a \mathcal{E}. \tag{4}$$

The 4 constraint equations in the standard $3+1$-decomposition are thus replaced by a single scalar complex equation which is an advantage both for the analytical and the numerical treatment.

The equations for the metric $h_{ab}$ can be written in the form

$$R_{ab} = \frac{1}{2f^2}\Re(\mathcal{E}_{,a}\bar{\mathcal{E}}_{,b}), \tag{5}$$

where $R_{ab}$ is the three-dimensional Ricci tensor corresponding to $h_{ab}$. Equations (5) describe three-dimensional gravitation with some matter model which turns out to be a $SL(2,\mathbb{C})/SO(1,1)$ sigma model, see [39]. It is obvious that zeros of the norm of the Killing vector are singular points of the equations.

To illustrate the above equations in the presence of a helical Killing vector, it is instructive to consider Minkowski spacetime in a rotating frame. In an asymptotically non-rotating frame, the Minkowski metric in standard cylindrical coordinates is in the above formalism for the stationary Killing vector given by $f = 1$, $b = 0$ and $h_{ab} = \text{diag}(1,1,\rho^2)$. In a rotating coordinate system where $\phi' = \phi - \Omega t$ with constant $\Omega$, we get for a helical Killing vector $\xi = \partial_t + \Omega\partial_\phi$ (notice that there is a helical Killing vector in Minkowski spacetime for arbitrary $\Omega$)

$$f' = 1 - \Omega^2\rho^2, \quad b' = -2\Omega z, \tag{6}$$

where we have put a physically irrelevant constant in the definition of the twist potential equal to zero. Since the spatial metric $h_{ab}$ in (1) is rescaled by $f$, we have $h'_{ab} = f'h_{ab}$ except for $h'_{\phi\phi}$ which is invariant under a transformation to a rotating frame. The light cylinder where the rotating observers corresponding to the vector $\xi$ move with the velocity of light is given in this case by $\rho = 1/\Omega$. In the interior of this cylinder, the Killing vector is timelike and $f$ is thus positive, in the exterior it is spacelike. At the cylinder the signature of the metric $h_{ab}$ changes from $+3$ to $-1$. In the four-dimensional picture there is no change in the signature of the metric but $t$ and $\phi$ change roles at the light cylinder, $\phi$ being a timelike coordinate in the exterior of the cylinder.

The Ernst equation takes in non-rotating coordinates the simple form

$$f\Delta\mathcal{E} = (\nabla\mathcal{E})^2, \tag{7}$$

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where $\Delta$ and $\nabla$ are the standard differential operators in cylindrical coordinates. In the rotating frame, the Laplace operator is replaced with the linear operator $\mathcal{L}$ defined by

$$\mathcal{L} := \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz} + (1 - \Omega^2\rho^2)\frac{1}{\rho^2}\partial_{\phi\phi}, \tag{8}$$

which is just the helically reduced flat d'Alembert operator in a rotating frame. In the axisymmetric case (no $\phi$-dependence) $\mathcal{L}$ reduces to the flat Laplace operator. In the non-axisymmetric case solutions to the equation $\mathcal{L}\mathcal{E} = 0$ behave for small $\rho$ like solutions to the Laplace equation and for large $\rho$ like solutions to a hyperbolic equation. Separating the angular dependence in spherical coordinates in a standard way via spherical harmonics, $\mathcal{E} = \sum_{lm} R_{lm}(r)Y_{lm}(\theta, \phi)$, one recognizes that the solutions of $\mathcal{L}\mathcal{E} = 0$ behave close to the origin as $r^l$ like solutions to the Laplace equation and for $r \to \infty$ as $e^{i\Omega r}/r$. One has thus to expect an oscillatory behavior for large $r$. Numerical studies of this type of equations have been carried out in [29, 30, 31] and [4, 5]. In general, Sommerfeld conditions (outgoing wave condition at finite values of $r$) have been used. Existence and uniqueness of solutions to boundary value problems for these equations were studied in [26] using the theory of symmetric positive systems.

In the stationary axisymmetric case the above equations can be further simplified. It can be shown (see e.g. [40]) that the spatial metric can be written in this case in the form $h_{ab} = \text{diag}(e^{2k}, e^{2k}, \rho^2)$. In this case the Ernst equation decouples from the equations for the metric function $k$ and takes the form (7) for a $\phi$-independent Ernst potential. The metric function $k$ follows for a given Ernst potential in terms of a line integral. In this formalism the Kerr solution for a single black hole with mass $m$ and angular momentum $J = m^2 \sin\varphi$ takes a particularly simple form,

$$\mathcal{E} = \frac{e^{-i\varphi}r_+ + e^{i\varphi}r_- - 2m\cos\varphi}{e^{-i\varphi}r_+ + e^{i\varphi}r_- + 2m\cos\varphi}, \tag{9}$$

where $r_\pm = \sqrt{(z - m\cos\varphi)^2 + \rho^2}$, and where the horizon is given by $\rho = 0$ and $|z| \leq m\cos\varphi$.

Here we consider spacetimes with a single helical Killing vector. We adopt the definition of Friedman et al. [8] that the Killing vector can be written in the form $\xi = \partial_{t'} + \Omega\partial_{\phi'}$ where $\partial_{\phi'}$ is a spacelike vector with circular orbits of length $2\pi$ unless it vanishes. The vector $\partial_{t'}$ is timelike outside the history of some sphere, $\Omega$ is a constant, see [8] for details. This vector generalizes the helical Killing vector of Minkowski spacetime discussed above. It corresponds to the introduction of observers corotating with the binary system. Close to the black holes this vector will be timelike, but it will become null if these observers rotate with the velocity of light which determines the so-called light cylinder.

# 3  $2 + 1$-decomposition, horizons and light cylinder

In this work we are interested in spacetimes with a helical Killing vector that contain binary black holes. Since it was shown in [8] that spatially compact Killing horizons in

this case are event horizons, we are interested in spacetimes with two disconnected Killing horizons of spherical topology.

With this assumption it seems convenient to introduce two systems of spherical coordinates adapted to the horizons in a way that one of them is given by $r = R = const$. We assume that the spacetime can be globally foliated by spheres which is not necessary for the analysis below but for the planned numerical implementation. Since equations (5) describe a model of three-dimensional gravity, it seems natural to use a $2 + 1$ decomposition of the 3-space with respect to the radial coordinate. Let $\mathcal{N}_a = (\mathcal{A}, 0, 0)$ and $\mathcal{N}^a = (1, -\mathcal{B}^\alpha)/\mathcal{A}$ be the unit normal to the $r = const$ surfaces; greek indices take the values 2 and 3. Denoting the metric of the $r = const$ surfaces with $s_{\alpha\beta}$, we can write the metric $h_{ab}$ in the form

$$h_{ab}dx^a dx^b = s_{\alpha\beta}(dx^\alpha + \mathcal{B}^\alpha dr)(dx^\beta + \mathcal{B}^\beta dr) + \mathcal{A}^2 dr^2. \tag{10}$$

The Ricci tensor splits in the standard way (see e.g. [41]) in 3 'evolution' equations which contain second derivatives with respect to $r$,

$$R_{\alpha\beta} = \nabla_{(\alpha}\dot{\mathcal{N}}_{\beta)} + 2\mathcal{K}_{\alpha\gamma}\mathcal{K}^\gamma_\beta - \dot{\mathcal{N}}_\alpha\dot{\mathcal{N}}_\beta + \pounds_\mathcal{N}K_{\alpha\beta} + R^{(2)}_{\alpha\beta} - \mathcal{K}_{\alpha\beta}\mathcal{K} \tag{11}$$

and 3 'constraint' equations below which have at most first derivatives with respect to $r$. Here $\mathcal{K}_{\alpha\beta}$ is the exterior curvature of the $r = const$ surfaces given by

$$\mathcal{K}_{\alpha\beta} = \frac{1}{\mathcal{A}}\left(\nabla_{(\alpha}B_{\beta)} - \frac{1}{2}s_{\alpha\beta,r}\right), \tag{12}$$

where $\nabla_\alpha$ denotes the covariant derivative associated to $s_{\alpha\beta}$. The Lie derivative of $\mathcal{K}_{\alpha\beta}$ in the direction of $\mathcal{N}$ is denoted by $\pounds_\mathcal{N}K_{\alpha\beta}$,

$$\pounds_\mathcal{N}\mathcal{K}_{\alpha\beta} = \mathcal{K}_{\alpha\beta,r}\frac{1}{\mathcal{A}} - \mathcal{K}_{\alpha\beta,\gamma}\frac{\mathcal{B}^\gamma}{\mathcal{A}} - \mathcal{K}_{\alpha\gamma}\left(\frac{\mathcal{B}^\gamma}{\mathcal{A}}\right)_{,\beta} - \mathcal{K}_{\beta\gamma}\left(\frac{\mathcal{B}^\gamma}{\mathcal{A}}\right)_{,\alpha}, \tag{13}$$

and $\dot{\mathcal{N}}_\alpha = -(\ln\mathcal{A})_{,\alpha}$. The two equations corresponding to the momentum constraint in the standard $3 + 1$-decomposition read

$$-R_{a\alpha}\mathcal{N}^a = \nabla_\beta\mathcal{K}^\beta_\alpha - \nabla_\alpha\mathcal{K}^\beta_\beta, \tag{14}$$

the Hamiltonian constraint is given by

$$R_{ab}\mathcal{N}^a\mathcal{N}^b - s^{\alpha\beta}R_{\alpha\beta} = \mathcal{K}^2 - \mathcal{K}_{\alpha\beta}\mathcal{K}^{\alpha\beta} - R^{(2)}. \tag{15}$$

where $\mathcal{K} = \mathcal{K}^\alpha_\alpha$ and $R^{(2)} = s^{\alpha\beta}R^{(2)}_{\alpha\beta}$. In the above equations, greek indices are raised and lowered with $s_{\alpha\beta}$ and its inverse.

For concrete calculations one has to fix a gauge. Since we are considering horizons of spherical topology and an adapted coordinate system, a natural choice of the metric $s_{\alpha\beta}$ would be the standard metric of the 2-sphere. This is in general possible for smooth

metrics which we are interested in here since we are looking for regular horizons. Due to the fact that the spatial metric $h_{ab}$ is rescaled by the norm of the Killing vector which vanishes at the horizon, the metric $h_{ab}$ is expected to vanish there as well. A possible choice would then be a metric conformal to the metric of the 2-sphere,

$$s_{\alpha\beta} = fr^2 \text{diag}(1, \sin^2\theta). \tag{16}$$

This gauge will be convenient in the vicinity of the horizon. As the considerations for Minkowski spacetime in the previous section have shown, this choice is not possible across the light cylinder because of the signature change of the spatial metric there. Since we are interested in setting up equations suited for numerical treatment, we are looking for a system of coordinates which is able to cover the whole spacetime in the exterior of the horizon with a single coordinate patch. This could be possible with coordinates in which the angular part $g_{\alpha\beta}$ of the four-dimensional metric is the standard metric of the 2-sphere as in Schwarzschild coordinates. The latter would imply however the explicit inclusion of the vector $k_a$ in the equations which is contrary to the philosophy of the present approach to work only with its dual, the scalar twist potential $b$.

Therefore we choose here a generalized Weyl gauge which we call 'quasi-isotropic'. We write

$$s_{\alpha\beta} = \text{diag}(r^2\mathcal{A}^2, \mathcal{C}). \tag{17}$$

A possible choice for $\mathcal{C}$ is $\mathcal{C} = r^2\sin^2\theta(1 - R^2/r^2)^2$. This corresponds to standard Weyl coordinates in which the horizon of a Kerr black hole is a sphere of radius $R$. These coordinates are related to the coordinates (9) via $\rho = (r - R^2/r)\sin\theta$ and $z = (r + R^2/r)\cos\theta$. The quasi-isotropic gauge thus reduces to the standard Weyl coordinates in the axisymmetric case. It can be used in principle throughout the light cylinder, but it remains to be shown whether it can be used globally. Due to the divergence structure of the Ernst equation (4) which can be written free of covariant derivatives in the form

$$f(h^{ab}\sqrt{h}\mathcal{E}_{,a})_{,b} = h^{ab}\sqrt{h}\mathcal{E}_{,a}\mathcal{E}_{,b}, \tag{18}$$

the equation has in this gauge the standard terms of the Laplace operator for the $rr$ and $\theta\theta$ derivatives. This helps in the numerical treatment of the equations since standardized differential operators can be numerically inverted. In this gauge we have the non-vanishing Christoffel symbols corresponding to $s_{\alpha\beta}$

$$\Gamma^2_{22} = (\ln\mathcal{A})_{,\theta}, \Gamma^2_{23} = (\ln\mathcal{A})_{,\phi}, \Gamma^2_{33} = -\frac{1}{\mathcal{A}^2}(1 - R^2/r^2)^2 \sin\theta\cos\theta, \tag{19}$$

and

$$\Gamma^3_{22} = -\frac{\mathcal{A}\mathcal{A}_{,\phi}}{(1 - R^2/r^2)^2\sin^2\theta}, \Gamma^3_{23} = \cot\theta. \tag{20}$$

The components of the Ricci tensor read $R^{(2)}_{23} = 0$ and

$$-R^{(2)}_{22} = -\frac{\mathcal{A}^2}{(1 - R^2/r^2)^2\sin^2\theta} R^{(2)}_{33} = \frac{\mathcal{A}\mathcal{A}_{,\phi\phi}}{(1 - R^2/r^2)^2\sin^2\theta} - 1 - \cot\theta(\ln\mathcal{A})_{,\theta}. \tag{21}$$

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The equations for $R_{ab}$ can be treated as in the case of a $3 + 1$-decomposition: if the constraints are satisfied for some value of $r$, this will be the case for solutions to the evolution equations for all values of $r$. Since the horizon is a singularity for the equations, one has to give boundary conditions there which are compatible with the constraints and the evolution equations. With these boundary conditions one has to solve the Ernst equation which corresponds to two real equations and the three evolution equations $(11)$[1]. Thus one has to solve in total 5 equations as in the case of the IWM problem in [22]. The difference is here that the equations are not elliptic in the exterior of the light cylinder in contrast to the IWM equations and that the spacetime will not be asymptotically flat as discussed in the following section.

To study the behavior of the metric at the horizon, we use formal power series in the local coordinate $y = r - R$. As in the case of ordinary differential equations, we adopt for a function $F(r, \theta, \phi)$ the ansatz

$$F(r, \theta, \phi) = y^{n_F} \sum_{j=0}^{\infty} F_j(\theta, \phi) y^j, \tag{22}$$

but here with coefficients $F_j(\theta, \phi)$ depending on $\theta$ and $\phi$. The question is whether there are formal solutions to the Einstein equations of this form with vanishing norm $f$ of the Killing vector for $y = 0$ which are more general than the Kerr solution for a single black hole. Here we are only interested in providing formal solutions intended for the use in the numerical treatment. Therefore we do neither discuss the convergence of the series nor global questions. We get (again we ignore a physically irrelevant constant in the definition of $b$)

**Proposition 3.1.** *In the gauge (17), the equations (4) and (5) have formal solutions of the form (22) with*

$$f = f_0(\theta, \phi)y^2 + f_1(\theta, \phi)y^3 + f_2(\theta, \phi)y^4 + \ldots, \quad b = b_0(\theta, \phi)y^4 + b_1(\theta, \phi)y^5 + b_2(\theta, \phi)y^6 + \ldots, \tag{23}$$

*and*

$$\mathcal{A} = \mathcal{A}_0(\theta, \phi)y + \mathcal{A}_1(\theta, \phi)y^2 + \mathcal{A}_2(\theta, \phi)y^3 + \ldots, \quad \mathcal{B}_2 = \Theta_0(\theta, \phi)y^3 + \ldots, \quad \mathcal{B}_3 = \Phi_0(\theta, \phi)y^3 + \ldots. \tag{24}$$

*The functions $f_0(\theta, \phi)$ and $b_0(\theta, \phi)$ are free functions of $\theta$ and $\phi$ with $f_{0,\phi}(0, \phi) = 0$. All other coefficient functions in the expansions (23) and (24) can be expressed in dependence of $f_0(\theta, \phi)$ and $b_0(\theta, \phi)$, the leading order terms being*

$$\mathcal{A}_0(\theta, \phi) = \kappa f_0(\theta, \phi), \quad \frac{\mathcal{A}_1}{\mathcal{A}_0} = -\frac{3}{2R}, \quad f_1 = -\frac{f_0}{R}, \quad b_1 = -\frac{2b_0}{R}, \tag{25}$$

*where the constant $\kappa$ is given by $\kappa = 2/(R f_0(0, 0))$.*

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[1] As in [42], a fully constraint approach can be used alternatively in the sense that only the constraint equations instead of the evolution equations are solved.

The constant $\kappa$ in the relation between $\mathcal{A}_0$ and $f_0$ indicates a freedom in the choice of $f_0(0,0)$. This freedom is due to the fact that a scale in the norm of the Killing vector is not fixed, after multiplication with some constant, $\xi$ is still a Killing vector.

*Proof:*

With (23) and (24), we get for the exterior curvature (12) by using (19) and (20)

$$
\begin{aligned}
\mathcal{K}_{22} &= -\mathcal{A}_0(R^2 + 3Ry + 2y^2) - \mathcal{A}_1(2R^2y + 5Ry^2) - 3R^2\mathcal{A}_2 y^2 \\
&\quad + (\Theta_{0,\theta} - (\ln \mathcal{A}_0)_{,\theta}\Theta_0)\frac{y^2}{\mathcal{A}_0} + \frac{R^2 \mathcal{A}_{0,\phi}\Phi_0}{4\sin^2\theta}y^2 + 0(y^3), \\
\mathcal{K}_{23} &= \left(\frac{1}{2}\Theta_{0,\phi} - (\ln \mathcal{A}_0)_{,\phi}\Theta_0\right)\frac{y^2}{\mathcal{A}_0} + \left(\frac{1}{2}\Phi_{0,\theta} - \cot\theta\,\Phi_0\right)\frac{y^2}{\mathcal{A}_0} + 0(y^3), \\
\mathcal{K}_{33} &= -\frac{4}{\mathcal{A}_0}\sin^2\theta\left(1 - \frac{3}{2R}y + \frac{5y^2}{2R^2} - \frac{\mathcal{A}_1}{\mathcal{A}_0}y\left(1 - \frac{3y}{2R}\right) + \frac{\mathcal{A}_1^2}{\mathcal{A}_0^2}y^2 - \frac{\mathcal{A}_2}{\mathcal{A}_0}y^2\right) \\
&\quad + \Phi_{0,\phi}\frac{y^2}{\mathcal{A}_0} + \frac{4\sin\theta\cos\theta}{R^2\mathcal{A}_0^3}\Theta_0 y^2 + 0(y^3). \tag{26}
\end{aligned}
$$

It is straight forward to check that the Hamiltonian constraint (15) is satisfied to leading order (which is $1/y^4$). The momentum constraint (14) leads in lowest order to

$$
(\ln \mathcal{A}_0)_{,\alpha} = (\ln f_0)_{,\alpha}. \tag{27}
$$

Thus we have $\mathcal{A}_0 = \kappa f_0$ with $\kappa = const.$

To ensure a regular axis ($\theta = 0$) in spherical coordinates, the axis must be 'elementary flat', i.e. small circles around the axis must have an invariant circumference of $2\pi$ times the invariant radius in the limit of vanishing radius. This means for $\rho = r\sin\theta$, $z = r\cos\theta$

$$
\lim_{\rho \to 0}\int_0^{2\pi}\sqrt{g_{\phi\phi}}d\phi = 2\pi \lim_{\rho \to 0}\int_0^{\rho}\sqrt{g_{\rho\rho}(\rho', z, \phi)}d\rho'. \tag{28}
$$

With the above relations this implies ($k_a$ is bounded at the horizon)

$$
r\sin\theta\left(1 - \frac{R^2}{r^2}\right)\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\sqrt{f(r,0,\phi)}}d\phi = \frac{\mathcal{A}(r,0,\phi)}{\sqrt{f(r,0,\phi)}}. \tag{29}
$$

Expanding this relation in $y$ and using (23), (24) and (27), we get in lowest order of $y$ the condition that $f_0(0,\phi)$ must be independent of $\phi$ and thus be a constant. This constant is related to $\kappa$ via

$$
\kappa = \frac{2}{Rf_0(0,0)}. \tag{30}
$$

The Ernst equation (4) is satisfied in leading order for arbitrary $f_0(\theta,\phi)$ and $b_0(\theta,\phi)$, in the next higher order the real part implies

$$
f_1 + \frac{f_0}{R} = 0, \tag{31}
$$

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whereas the imaginary part leads to

$$-8b_0 f_1 + \frac{2}{R} b_0 f_0 + 5b_1 f_0 = 0. \tag{32}$$

This determines $f_1$ and $b_1$ in dependence of $f_0$ and $b_0$. In order $y^n$, the leading terms in the real part of the Ernst equation are

$$(n-2)^2 f_0 f_{n-2} \tag{33}$$

and

$$n(n-4) f_0 b_{n-4} + 8(2-n) b_0 f_{n-2} \tag{34}$$

for the imaginary part. The Ernst equation can thus be used to all orders determine $f_{n-2}$ and $b_{n-4}$ in dependence of quantities of lower order.

With (26) we get for the Lie derivative of the exterior curvature (13)

$$
\begin{aligned}
\pounds_{\mathcal{N}} \mathcal{K}_{22} &= -\frac{1}{y}\left(3R + 2R^2 \frac{\mathcal{A}_1}{\mathcal{A}_0}\right) - 4 - 7R\frac{\mathcal{A}_1}{\mathcal{A}_0} + 2R^2 \frac{\mathcal{A}_1^2}{\mathcal{A}_0^2} - 6R^2 \frac{\mathcal{A}_2}{\mathcal{A}_0} \\
&\quad + \frac{4\Theta_{0,\theta}}{\mathcal{A}_0^2} - 7(\ln \mathcal{A}_0)_{,\theta}\frac{\Theta_0}{\mathcal{A}_0^2} + \frac{3R^2(\ln \mathcal{A}_0)_{,\phi}\Phi_0}{4\sin^2\theta} + 0(y) \\
\pounds_{\mathcal{N}} \mathcal{K}_{23} &= \frac{1}{\mathcal{A}_0^2}\left(2\Theta_{0,\phi} - 5(\ln \mathcal{A}_0)_{,\phi}\Theta_0 + 2\Phi_{0,\theta} - 4\cot\theta\Phi_0 - (\ln \mathcal{A}_0)_{,\theta}\Phi_0\right) + 0(y) \\
\pounds_{\mathcal{N}} \mathcal{K}_{33} &= \frac{4\sin^2\theta}{\mathcal{A}_0^2 y}\left(\frac{3}{2R} + \frac{\mathcal{A}_1}{\mathcal{A}_0}\right) - \frac{4\sin^2\theta}{\mathcal{A}_0^2}\left(\frac{5}{R^2} + \frac{9\mathcal{A}_1}{2R\mathcal{A}_0} + \frac{3\mathcal{A}_1^2}{\mathcal{A}_0^2} - \frac{2\mathcal{A}_2}{\mathcal{A}_0}\right) \\
&\quad + 16\sin\theta\cos\theta\frac{\Theta_0}{R^2\mathcal{A}_0^4} - 4\sin^2\theta(\ln \mathcal{A}_0)_{,\theta}\frac{\Theta_0}{R^2\mathcal{A}_0^4} + \frac{4\Phi_{0,\phi}}{\mathcal{A}_0^2} - 3(\ln \mathcal{A}_0)_{,\phi}\frac{\Phi_0}{\mathcal{A}_0^2} + 0(y)
\end{aligned}
\tag{35}
$$

Since there are second order derivatives with respect to $r$ in the the evolution equations as in the Ernst equation, higher order terms in the expansion of the metric functions will appear here before they do in the constraints. Therefore there are no further conditions on the lowest order terms. In order $1/y$ the equation for $R_{22}$ (there are no contributions from the Ernst potential in this order) leads to

$$\frac{R^2}{y}\left(\frac{3}{2R} + \frac{\mathcal{A}_1}{\mathcal{A}_0}\right) = 0. \tag{36}$$

This implies that there are no terms of order $y$ in the exterior curvature (26) and no terms of order $1/y$ in the Lie-derivatives (35).

The leading terms in order $y^{n-3}$ in the evolution equations are for $n > 2$ in $R_{22}$

$$
-R^2(1-n)^2 \frac{\mathcal{A}_{n-1}}{\mathcal{A}_0} + \frac{1}{\mathcal{A}_0^2}\left(n\Theta_{n-3,\theta} - (n+3)(\ln \mathcal{A}_0)_{,\theta}\Theta_{n-3} + \cot\theta\Theta_{n-3}\right)
$$

$$
+ \frac{R^2}{4\sin^2\theta}\left(\Phi_{n-3,\phi} + (n-1)(\ln \mathcal{A}_0)_{,\phi}\Phi_{n-3}\right). \tag{37}
$$

Similarly we get for $R_{23}$

$$\frac{1}{2\mathcal{A}_0^2}\left((n-1)\Theta_{n-3,\phi} - 2n(\ln\mathcal{A}_0)_{,\phi}\Theta_{n-3} + (n-1)\Phi_{n-3,\theta} - 2(n-1)\cot\theta\Phi_{n-3} - 2(\ln\mathcal{A}_0)_{,\theta}\Phi_{n-3}\right),\tag{38}$$

and for $R_{33}$

$$\frac{4\sin^2\theta}{R^2\mathcal{A}_0^4}\left(\Theta_{n-3,\theta} - 2(\ln\mathcal{A}_0)_{,\theta}\Theta_{n-3} + n\cot\theta\Theta_{n-3}\right)$$
$$+\frac{1}{\mathcal{A}_0^2}\left(n\Phi_{n-3,\phi} - 2(\ln\mathcal{A}_0)_{,\phi}\Phi_{n-3}\right).\tag{39}$$

It is straight forward to solve the equations 38) and (39) for $\Theta_{n-3}$ and $\Phi_{n-3}$. Function $\mathcal{A}_{n-1}$ then follows from equation (37). The Ernst equations (33) and (34) determine consequently $f_{n-2}$ and $b_{n-4}$. We have thus shown that the evolution equations and the Ernst equation can be solved in this way to all orders. This completes the proof.

The fact that $\mathcal{A}_{n-1}$ does not appear in the equations (38) and (39) implies that $\Theta$ and $\Phi$ can be chosen to vanish for $\phi$-independent $f_0$ and $b_0$. This leads as expected to the Kerr solution. Note that $f_{n-2}$, $b_{n-4}$ and $\mathcal{A}_{n-1}$ are determined algebraically by the above equations. Just to determine $\Theta_{n-1}$ and $\Phi_{n-1}$, one has to integrate which leads to free integration functions. The latter are related to the fact that the used gauge conditions do not fix the gauge completely. There are transformations of the form

$$r' = r + y^3 P(\theta,\phi) + \dots, \quad \theta' = \theta + y^2 S(\theta,\phi) + \dots, \quad \phi' = \phi + y^2 T(\theta,\phi) + \dots \tag{40}$$

for non-trivial $P$, $S$ and $T$ which do not change the gauge. Since $h'_{r\theta} = h_{r\theta} + h_{\theta\theta}2yS$ and similarly for $h_{r\phi}$, there are gauge modes in $\Theta$ and $\Phi$ which show up in the form of free integration functions.

*Remark* 3.1. Due to the homogeneity of the Ernst equation in the Ernst potential, the functions $f_0(\theta,\phi)$ and $b_0(\theta,\phi)$ are not determined in the above expansions in the vicinity of a horizon. This gives hope that there might be a second regular horizon of spherical topology in the spacetime for a suitable choice of these functions. Whereas the behavior of the Ernst potential with respect to $y$ is the same as in the case of a single Kerr black hole, the functions $f_0$ and $b_0$ may be different.

To treat the light cylinder we use a similar approach as for the horizon. By the definition of the Killing vector of [8] we are using here, the light cylinder will have cylindrical topology. We assume that the spacetime can be foliated by cylindrical surfaces, and use cylindrical coordinates in which the light cylinder is given by $\rho = \rho_0 = const$. In an abuse of notation we use the same symbols for the $2+1$-decomposition as used for the spherical case,

$$h_{ab}dx^a dx^b = s_{\alpha\beta}(dx^\alpha + \mathcal{B}^\alpha d\rho)(dx^\beta + \mathcal{B}^\beta d\rho) + \mathcal{A}^2 dr^2.\tag{41}$$

where $dx^2 = dz$. We use again the quasi-isotropic gauge which reads in this case

$$s_{\alpha\beta} = \text{diag}(\mathcal{A}^2, \mathcal{C}).\tag{42}$$

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The choice $\mathcal{C} = \rho^2$ is possible near the cylinder.

We assume that $f$ has a zero of first order in $v = \rho - \rho_0$ at the cylinder since the Killing vector is supposed to change from timelike to spacelike there. Again we consider formal expansions of the metric function of the form

$$F(\rho, z, \phi) = v^{n_F} \sum_{j=0}^{\infty} F_j(z, \phi) v^j, \tag{43}$$

We get

**Proposition 3.2.** *In the gauge (42) the equations (4) and (5) have formal power series solutions of the form (43) with*

$$
\begin{aligned}
f &= f_0(z)v + f_1(z, \phi)v^2 + \dots, \quad b = b_0(z) + b_2(z, \phi)v^2 + \dots, \\
\mathcal{A} &= \mathcal{A}_0(z)\sqrt{v} + \mathcal{A}_1(z, \phi)v^{\frac{3}{2}} + \dots \quad \mathcal{B}_2 = Z_0(z, \phi)v^2 + \dots, \quad \mathcal{B}_3 = \Phi_0(z, \phi)v. \quad (44)
\end{aligned}
$$

*The functions $b_0(z)$, $\mathcal{A}_0(z)$ and $f_2(z, \phi)$ are free functions of $z$ and $z$, $\phi$ respectively. All other coefficient functions in the expansion (44) can be expressed in dependence of $b_0$, $\mathcal{A}_0$ and $f_2$, the leading term being*

$$b_{0,z}^2 = f_0^2. \tag{45}$$

Proof:
The formulas for the $2 + 1$-decomposition in section 2 apply with the trivial change that $r$ has to be replaced by $\rho$. The chosen gauge is also very similar to the one used for the horizon, the only difference being the factor $r^2$ in the expression for $s_{22}$. Therefore we will not give explicit formulas for the Christoffel symbols and $R_{\alpha\beta}^{(2)}$ here. For the exterior curvature (12), we get with (44)

$$
\begin{aligned}
\mathcal{K}_{22} &= -\frac{\mathcal{A}_0}{2\sqrt{v}} - \frac{3}{2}\mathcal{A}_1\sqrt{v} + 0(v^{\frac{3}{2}}), \\
\mathcal{K}_{23} &= \frac{\sqrt{v}}{2\mathcal{A}_0}\Phi_{0,z} + 0(v^{\frac{3}{2}}), \\
\mathcal{K}_{33} &= -\frac{\rho}{\mathcal{A}_0\sqrt{v}}\left(1 - \frac{\mathcal{A}_1}{\mathcal{A}_0}v\right) + \frac{\sqrt{v}}{\mathcal{A}_0}\Phi_{0,\phi} + 0(v^{\frac{3}{2}}).
\end{aligned} \tag{46}
$$

This implies for the Lie derivative of the exterior curvature

$$
\begin{aligned}
\pounds_{\mathcal{N}}\mathcal{K}_{22} &= \frac{1}{4v^2} - \frac{\mathcal{A}_1}{\mathcal{A}_0 v} + 0(v^0), \\
\pounds_{\mathcal{N}}\mathcal{K}_{23} &= \frac{1}{4\mathcal{A}_0^2 v}\Phi_{0,z} + 0(v^0), \\
\pounds_{\mathcal{N}}\mathcal{K}_{33} &= \frac{\rho_0}{2\mathcal{A}_0^2 v^2} - \frac{1}{2\mathcal{A}_0^2 v} + \frac{\Phi_{0,\phi}}{2\mathcal{A}_0^2 v} + 0(v^0).
\end{aligned} \tag{47}
$$

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In lowest order, the equation for $R_{22}$ then yields (45), whereas the relations for $R_{23}$ and $R_{33}$ can only be satisfied in this case for

$$b_{0,\phi} = 0. \tag{48}$$

The $z$-component of the momentum constraint implies

$$2b_{0,z}b_2 + f_{0,z}f_0 - \frac{Z_0}{\mathcal{A}_0^2}b_{0,z}^2 = 0, \tag{49}$$

which determines $b_2$, whereas the $\phi$-component of the momentum constraint gives

$$\mathcal{A}_{0,\phi} = 0. \tag{50}$$

With condition (45) the Hamiltonian constraint is satisfied to leading order.

The real part of the Ernst equation gives no additional condition in leading order, in order $v$ it leads to

$$f_1 = \frac{f_0}{2\rho_0}. \tag{51}$$

The imaginary part gives with (45) in leading order (50). Consequently $b_2$ and $f_1$ are determined in this order.

The equation for $R_{22}$ reads with (11) in order $1/v$

$$\frac{\mathcal{A}_1}{\mathcal{A}_0} = \frac{1}{\rho_0}. \tag{52}$$

In the same order $R_{23}$ is identically satisfied. $R_{33}$ takes the form

$$0 = \frac{2\mathcal{A}_1}{\mathcal{A}_0} + \frac{1}{\rho_0}\Phi_{0,\phi}. \tag{53}$$

This fixes $\Phi_0$ and $\mathcal{A}_1$. Thus the functions $b_0(z)$ and $\mathcal{A}_0(z)$ are not determined by the above equations. In addition $Z_0$ is not yet fixed.

In higher orders of the expansion, the reasoning will be similar since the general structure of the equations is the same: the Ernst equation in order $v^{n-1}$ with $n > 1$ contains the leading terms $n(n-3)f_0f_{n-1}$ and

$$n(n-3)b_n + 2b_{0,z}\frac{Z_{n-2}}{\mathcal{A}_0^2} \tag{54}$$

in the real and the imaginary part respectively. The evolution equations in order $v^{n-2}$ contain the leading terms

$$-\frac{b_{n,z}}{b_{0,z}} - \left(n^2 - n - \frac{1}{4}\right)\frac{\mathcal{A}_n}{\mathcal{A}_0} + \frac{1}{\mathcal{A}_0^2}\left(\left(n - \frac{1}{2}\right)Z_{n-2,z} - (n-2)(\ln \mathcal{A}_0)_{,z}Z_{n-2}\right), \tag{55}$$

$$-\mathcal{A}_0^2 \frac{b_{n,\phi}}{b_{0,z}} + nZ_{n-1,\phi} + \left(n - \frac{1}{2}\right)\Phi_{n-1,z}, \tag{56}$$

and

$$(2n-1)\frac{\mathcal{A}_n}{\mathcal{A}_0} + \frac{n}{2\rho_0}\Phi_{n-1,\phi} - \frac{1}{\mathcal{A}_0^2}Z_{n-2,z}. \tag{57}$$

Thus one can determine $f_{n-1}$, $b_n$, $\mathcal{A}_n$, $Z_{n-1}$ and $\Phi_{n-1}$ from the above equations unless $n = 3$. In this case the real part of the Ernst equation determines the function $Z_0$ which was still free, the function $f_2$ remains undetermined. The equations in higher order fix all expansion functions except $\mathcal{A}_0$, $b_0$ and $f_2$. Free functions in $Z$ and $\Phi$ occurring after integration are again related to residual gauge freedoms as was the case in the vicinity of the horizon. Thus the series (44) provide a formal solution in the vicinity of the light cylinder. This completes the proof.

*Remark* 3.2. It should be possible to apply Fuchsian methods as in [33] to prove existence of the above solutions near the horizon and the light cylinder for some non-vanishing radius of convergence. However this would not answer the decisive question whether there can be two smooth horizons and a smooth light cylinder in the spacetime. Therefore we will not apply these methods here.

# 4   Asymptotic behavior

The formal solutions in terms of a series in the vicinity of the two horizons and the light cylinder in the previous section obviously do not imply global existence of a solution describing a spacetime with a helical Killing vector and two regular Killing horizons. The radius of convergence of these series is unknown. Therefore it is also not possible to make precise statements on the asymptotic behavior of the metric. As shown in [6] such spacetimes cannot be asymptotically flat in the strong sense that they have a smooth null infinity. However this does not exclude the possibility that the spacetime is weakly asymptotically flat in the sense that the spacetime tends to the Minkowski spacetime. This is what we will show in this section though the assumption of an asymptotic Minkowski metric might be too strong as indicated by the work of [36]. However the used techniques will also be applicable for even weaker asymptotic conditions.

Since the coordinate system we were using in the previous sections is asymptotically rotating (see the considerations for Minkowski spacetime in section 2), the Ernst potential is expected to have the kinematic terms (6). These terms will lead to technical difficulties if one wants to consider an expansion of the metric functions in powers of $1/r$. Therefore we will consider in this section asymptotically non-rotating coordinates $(t', r, \theta, \phi')$ where $\phi' = \phi + \Omega t$, $t' = t$. The metric will be studied via the linearized Einstein equations on a Minkowski background. Due to the helical symmetry the metric functions depend on $t'$ and $\phi'$ only via the combination $x = \phi' - \Omega t'$.

We assume the metric to be of the form $g_{AB} = \eta_{AB} + \delta_{AB}$ (capital indices take the values 0,1,2,3) for $r \to \infty$ where $\eta_{AB} = \text{diag}(-1, 1, r^2, r^2 \sin^2\theta)$ is the Minkowski metric

in spherical coordinates, and where $\delta_{AB}$ gives the deviation from Minkowski spacetime for large $r$. In cartesian coordinates $\delta_{AB}$ is assumed to be of order $1/r$. An analysis of the equations as in section 2 indicates however that the terms of order $1/r$ will have an oscillatory dependence in $r$ (we exclude here possible logarithmic terms in the metric function). We will therefore consider a formal expansion of the metric functions of the form

$$F(r, \theta, \phi) = \sum_{j=0}^{\infty} \frac{F_j(r, \theta, x)}{r^{n_F + j}}, \tag{58}$$

where the $r$-dependence of the $F_j$ is to be understood to be purely oscillatory. In spherical coordinates we expect for the algebraic dependence on $r$ that $\delta_{00}$, $\delta_{01}$ and $\delta_{11}$ are of order $1/r$, that $\delta_{02}$, $\delta_{03}$, $\delta_{12}$ and $\delta_{13}$ are of order $r^0$, and that $\delta_{22}$, $\delta_{23}$ and $\delta_{33}$ are of order $r$. We note that this ansatz for $\delta_{03}$ allows for a so-called Newman-Unti-Tamburini parameter which corresponds to a magnetic monopole in electrodynamics. It is unclear whether such terms have to be expected in the present context, and the used methods are not suited to answer this question.

To fix the gauge freedom we consider coordinate transformations of the form below which do not change $\eta_{AB}$ to leading order. The wanted gauge transformations can be put into the form

$$
\begin{aligned}
\tilde{t} &= t + \frac{T(r, \theta, x) + \alpha x}{r}, \\
\tilde{r} &= r + \beta + \frac{R(r, \theta, x)}{r}, \\
\tilde{\theta} &= \theta + \frac{Q(r, \theta, x) + \gamma x}{r^2}, \\
\tilde{\phi} &= \phi + \frac{P(r, \theta, x)}{r^2},
\end{aligned}
\tag{59}
$$

where the $r$- and $x$-dependence of the potentials $T$, $R$, $Q$ and $P$ is understood as before to be purely oscillatory, and where $\alpha$, $\beta$ and $\gamma$ are functions of $\theta$ only. This implies for the transformed Minkowski metric in leading order

$$
\tilde{g}_{00} = -1 + \frac{2\Omega}{r}(T_{,x} + \alpha), \quad \tilde{g}_{01} = -\frac{1}{r}(T_{,r} + \Omega R_{,x}), \quad \tilde{g}_{02} = -\Omega(Q_{,x} + \gamma),
$$

$$
\tilde{g}_{03} = -\Omega \sin^2 \theta P_{,x}, \quad \tilde{g}_{11} = 1 + \frac{2}{r}R_{,r}, \quad \tilde{g}_{12} = Q_{,r}, \tag{60}
$$

$$
\tilde{g}_{13} = \sin^2 \theta P_{,r}, \quad \tilde{g}_{22} = r^2\left(1 + \frac{2\beta}{r}\right), \quad \tilde{g}_{23} = Q_{,x} + \sin^2 \theta P_{,\theta}, \quad \tilde{g}_{33} = r^2 \sin^2 \theta \left(1 + \frac{2\beta}{r}\right).
$$

For metrics obtained by solving the linearized Einstein equations on a Minkowski background, this implies that the functions $T$ to $P$ in (60) can be used to establish a certain gauge. By an appropriate choice of $P$, we can choose $\tilde{g}_{23}$ to vanish under the made assumptions. By considering higher orders of this expansion, this should be possible to all orders. Similarly by choosing $R$ and $\beta$ we obtain $\tilde{g}_{22} = r^2 \tilde{g}_{11}$. The additional freedom

can be used to have a vanishing $\delta_{00}$ and $g_{02}$. We assume that the gauge potentials $T$, $R$, $Q$ and $P$ are of the form $T = \sum_{m \in \mathbb{Z}} (T_m^+(\theta) e^{im(x+\Omega r)} + T_m^-(\theta) e^{im(x-\Omega r)})$. Therefore we had to add the functions $\alpha$, $\beta$ and $\gamma$ depending only on $\theta$ to make sure that terms in the $\delta_{AB}$ which are constant with respect to $x$ can be compensated. This would lead, however, to terms proportional to $x$ in $\delta_{00}$ and $\delta_{02}$. Therefore we allow for a purely $\theta$-dependent $\delta_{00}$ and $\delta_{02}$ which will not enter the linearized field equations to retain periodic potentials in $x$ and $r$. Dropping the tilde, we thus choose the gauge (which is not exactly equivalent one used in the previous section, but close to it)

$$g_{00} = -\left(1 + \frac{f_0(\theta)}{r}\right), \quad g_{01} = \frac{c}{r}, \quad g_{03} = a, \quad g_{33} = r^2 \sin^2\theta \left(1 + \frac{F}{r}\right), \quad (61)$$

and

$$g_{02} = h_0(\theta), \quad g_{11} = \frac{1}{r^2} g_{22} = 1 + \frac{A}{r}, \quad g_{12} = \frac{\Theta}{r}, \quad g_{13} = \Phi. \quad (62)$$

The gauge is fixed up to a free function of $r, \phi$ in $a$, and functions of $\theta$ only in $c$, $\Theta$, and $\Phi$, since we assume periodicity in $r$ and $x$ of the terms $F_n(r, \theta, x)$ in (58). There could be a contribution in order $r^0$ to $\Theta$, but this must be a function of $\theta$ alone as a consequence of the Einstein equations below and the periodicity condition in $r$. We put this function equal to zero here to fix a gauge freedom. This implies in leading order for the inverse metric

$$g^{00} = -\left(1 - \frac{f_0}{r}\right), \quad g^{01} = \frac{c}{r}, \quad g^{03} = \frac{a}{r^2 \sin^2\theta}, \quad g^{33} = \frac{1}{r^2 \sin^2\theta}\left(1 - \frac{F}{r}\right), \quad (63)$$

and

$$g^{02} = \frac{h_0}{r^2}, \quad g^{11} = r^2 g^{22} = 1 - \frac{A}{r}, \quad g^{13} = -\frac{\Phi}{r^2 \sin^2\theta}, \quad g^{12} = -\frac{\Theta}{r^3}. \quad (64)$$

We obtain

**Proposition 4.1.** *The linearized Einstein equations on a Minkowski background for the metric (61), (62) lead to two wave equations for the functions $A$ and $a$,*

$$A_{,rr} - \Omega^2 A_{,\phi\phi} = 0, \quad a_{,rr} - \Omega^2 a_{,\phi\phi} = 0. \quad (65)$$

*The remaining metric potentials follow in terms of quadratures.*

Proof:
With relation

$$R_{abcd} = \frac{1}{2}(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}) \quad (66)$$

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for the linearized Riemann tensor we get for the Ricci tensor in lowest order in $1/r$

$$
\begin{aligned}
2r R_{00} &= -2\Omega c_{,r\phi} - 2\Omega^2 A_{,\phi\phi} - \Omega^2 F_{,\phi\phi} \\
2r R_{01} &= \Omega(A+F)_{,r\phi} \\
2r R_{02} &= c_{,r\theta} - \Omega\Theta_{,r\phi} + \Omega(A+F)_{,\phi\theta} \\
2R_{03} &= -\Omega\Phi_{,r\phi} - a_{,rr} \\
2r R_{11} &= 2\Omega c_{,r\phi} + \Omega^2 A_{,\phi\phi} - (A+F)_{,rr} \\
2r R_{12} &= \Omega c_{,\theta\phi} - F_{,r\theta} + \Omega^2\Theta_{,\phi\phi} \\
2R_{13} &= \Omega a_{,r\phi} + \Omega^2\Phi_{,\phi\phi} \\
\frac{2}{r}R_{22} &= \Omega^2 A_{,\phi\phi} - A_{,rr} \\
2R_{23} &= \Omega a_{,\theta\phi} + \Phi_{,r\theta} \\
\frac{2}{r\sin^2\theta}R_{33} &= \Omega^2 F_{,\phi\phi} - F_{,rr}.
\end{aligned}
\tag{67}
$$

It is a consequence of the equations for $R_{03}$ and $R_{13}$ that

$$
\Omega\Phi_{,\phi} + a_{,r} = G_1(\theta),
\tag{68}
$$

where $G_1$ is a free function of $\theta$ only which is gauge invariant under transformations of the form (59). The equation for $R_{23}$ then implies

$$
\Omega^2 a_{,\phi\phi} - a_{,rr} = G_2(r,\phi),
\tag{69}
$$

where $G_2$ is a free function of $r$ and $\phi$ which reflects a gauge freedom and can be put equal to zero. Equation (69) represents the first of the two wave equations. We write the solution in the form of a Fourier series

$$
a = \sum_{m\in\mathbb{Z}} e^{im\phi}(a_m^+(\theta)e^{i\Omega mr} + a_m^-(\theta)e^{-i\Omega mr}).
\tag{70}
$$

The reality condition for $a$ implies $a_{-m}^\pm = \bar{a}_m^\pm$. Thus we get for $\Phi$

$$
\Phi = -\sum_{m\in\mathbb{Z}} e^{im\phi}(a_m^+(\theta)e^{i\Omega mr} - a_m^-(\theta)e^{-i\Omega mr}) + G_0(\theta).
\tag{71}
$$

The equation for $R_{01}$ implies

$$
A + F = G_3(\theta,\phi) + G_6(\theta,r).
\tag{72}
$$

It is then a consequence of $R_{22}$ ($R_{33}$ is identically satisfied) that

$$
A = \sum_{m\in\mathbb{Z}} e^{im\phi}(A_m^+(\theta)e^{i\Omega mr} + A_m^-(\theta)e^{-i\Omega mr}).
\tag{73}
$$

This gives the second wave equation. Again reality of $A$ implies $A_{-m}^{\pm} = \bar{A}_m^{\pm}$.

Equations $R_{00}$ and $R_{11}$ lead to

$$c = -\frac{\Omega}{2} \int_{r_0}^r A_{,\phi} dr + G_4(\theta). \tag{74}$$

These equations also determine that the right-hand side of (72) is only a function of $\theta$ if the periodicity in $r$ and $\phi$ is taken into account. Equations $R_{02}$ and $R_{12}$ then imply

$$\Theta = -\frac{1}{2} \int_{r_0}^r A_{,\theta} dr + G_5(\theta). \tag{75}$$

This completes the proof.

*Remark* 4.1. If the black holes have equal 'mass', i.e. equal combination of mass and angular momentum which can be defined via the Komar integral below, the spacetime has an additional discrete symmetry, it is invariant in a suitably defined coordinate system under the transformation $\phi \to -\phi$. This implies for (70) and (73) $A_m^+ = \bar{A}_m^-$ and $a_m^+ = \bar{a}_m^-$. In this case no additional boundary conditions at infinity need to be given. A Sommerfeld condition which is typically considered at finite radius would only allow trivial solutions in this example if imposed at infinity.

*Remark* 4.2. In case the functions $a$ and $\Phi$ have leading terms of order $1/r$, i.e. if there is no NUT-parameter, the following equations for the Ricci tensor (67) change

$$\begin{aligned}
2rR_{03} &= c_{,r\phi} - \Omega\Phi_{,r\phi} - a_{,rr} + 2\Omega A_{,\phi\phi}, \\
2rR_{13} &= \Omega a_{,r\phi} + \Omega c_{,\phi\phi} - A_{,r\phi} + \Omega^2 \Phi_{,\phi\phi}, \\
2rR_{23} &= \Omega a_{,\theta\phi} + \Phi_{,r\theta} + \Theta_{,r\phi} - A_{,\theta\phi}.
\end{aligned} \tag{76}$$

These equations again imply wave equation (69) for $a$ and

$$\Phi = -\frac{1}{\Omega} \int_{\phi_0}^{\phi} a_{,r} d\phi + \frac{3}{2} \int_{r_0}^r A_{,\phi} dr + G_7(\theta, r). \tag{77}$$

The ansatz (58) already implies that the ADM mass and additional asymptotic multipoles cannot be defined due to the oscillatory behavior of the metric functions. We will show that it is also not possible to use the Komar integral asymptotically in a standard way to define a conserved quantity. This integral can be used to relate a locally calculated mass to the ADM mass for an asymptotically flat spacetime with a stationary Killing vector. The idea is to evaluate a surface integral at finite radius $R$ and then to take the limit $R \to \infty$. Basically one uses that $\star d\xi$ is an exact differential which means that one can apply Gauss' theorem. We get for an integration over a sphere with $t = const, r = const$

$$\int_S \xi_{[A,B]} g^{0A} g^{1B} \sqrt{-g} d\theta d\phi = 0. \tag{78}$$

To calculate the integral near the horizon we need the inverse of the 4-dimensional metric

$$g^{00} = -\frac{1}{f} + f k_a k^a, \quad g^{0a} = -f k^a, \quad g^{ab} = f h^{ab}, \tag{79}$$

where spatial indices are raised and lowered with $h_{ab}$. In the quasi-isotropic gauge we get with the results of section 3 for the surface integral that only the term

$$\int_S \sqrt{-g} d\theta d\phi g^{00} g^{11} \xi_{[0,1]} \tag{80}$$

contributes. This leads with $\mathcal{A}_0 = \kappa f_0$ to

$$\frac{1}{2} \int_S \sin\theta d\theta d\phi \frac{f_{,r}}{\mathcal{A}} = \frac{4\pi}{\kappa}. \tag{81}$$

For a single black hole the constant $\kappa = 1/(2m)$. The Komar integral is of course only defined up to a scaling of the helical Killing vector, see the remarks in [8].

To check whether the surface integral can be defined for $r \to \infty$, we determine the integral for $r = r_0$ where $r_0 \gg R$ and study whether the limit $r_0 \to \infty$ exists. This could be possible if one uses the periodicity of the functions in $\phi$ in the $\phi$-integration over a complete period at a finite value of $r$. As we will show below, this will not be the case because of the bilinear terms in the integrand. Note that the Killing vector reads in the used coordinates $\xi = \partial_{t'} + \Omega \partial_{\phi'}$. It is readily seen that the integral can only exist if $G_1 = 0$, since the corresponding terms in the integrand are of order $r^2$. If we assume that this is the case, we get for the surface integral

$$\frac{1}{2} \int_S r \sin\theta d\theta d\phi (\Omega^2 \Phi F_{,\phi} + a\Omega(2 + F_{,r})). \tag{82}$$

The integral can only exist if these terms vanish after integration with respect to $\phi$ since the integrand diverges as $r$. Writing the integrands as a Fourier series as we have done in the proof of proposition 4.1, we get after integration with respect to $\phi$ that the integrand is proportional to

$$a_0^+ + a_0^- - i\Omega \sum_{m \in \mathbb{Z}} m(a_m^+ \bar{A}_m^- e^{2i\Omega m r} - a_m^- \bar{A}_m^+ e^{-2i\Omega m r}). \tag{83}$$

This expression can only vanish if $a_m^+ \bar{A}_m^- + \bar{a}_m^- A_m^+ = 0$ for $m > 0$. In the equal mass case, this is only possible if either $a_m$ or $A_m$ vanish. If $a$ and $\Phi$ have leading contributions in order $1/r$ as in remark 4.2, the terms of order $0(r^0)$ in the surface integral are due to (80) and are of the form

$$\sum_{m \in \mathbb{Z}} m(A_m^- \bar{A}_m^+ e^{2i\Omega m r} - A_m^+ \bar{A}_m^- e^{-2i\Omega m r}). \tag{84}$$

The integral can only exist if the terms $A_m^+ A_m^-$ vanish for all $m \neq 0$. Since there cannot be purely 'outgoing' or 'ingoing' waves in the case of a helical Killing vector, this condition will lead to the axisymmetric case. Thus the surface integral cannot be defined

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asymptotically in the presence of a helical Killing vector unless there is in addition an asymptotically axial Killing vector. In this case the integral just gives the expected value $M + \Omega J$, the combination of mass and angular momentum corresponding to a helical Killing vector.

Gibbons and Stewart [6] showed that periodic boundary conditions are incompatible with a smooth null infinity. This is in accordance with the ansatz (58) as can be seen from the following consideration: We define the standard null-tetrad of Minkowski spacetime,

$$ k^a = \frac{1}{\sqrt{2}} \left( \partial_{t'} + \partial_r \right), \quad m^a = \frac{1}{\sqrt{2}} \left( -\partial_{t'} + \partial_r \right), \quad t^a = \frac{1}{\sqrt{2}r} \left( \partial_\theta + \frac{i}{\sin\theta} \partial_{\phi'} \right), \qquad (85) $$

We define the Weyl scalars as in [43] (we can use the above tetrad since we are only considering a linearization on a Minkowski background)

$$
\begin{aligned}
C_1 &= 2 C_{abcd} m^a m^c t^b t^d, \\
C_2 &= -C_{abcd} m^a t^b (k^c m^d + \bar{t}^c t^d), \\
C_3 &= 2 C_{abcd} m^a t^b k^c \bar{t}^d, \\
C_4 &= -C_{abcd} k^a \bar{t}^b (k^c m^d + \bar{t}^c t^d), \\
C_5 &= 2 C_{abcd} k^a k^c \bar{t}^b \bar{t}^d.
\end{aligned}
\qquad (86)
$$

Determining the components of the Riemann tensor for the asymptotic metric of proposition 4.1, we get for the Weyl scalars in leading order

$$
\begin{aligned}
C_1 &= -\frac{1}{2r} (A_{rr} + 2\Omega A_{r\phi} + \Omega^2 A_{\phi\phi}), \\
C_2 &= C_3 = C_4 = 0, \\
C_5 &= -\frac{1}{2r} (A_{rr} - 2\Omega A_{r\phi} + \Omega^2 A_{\phi\phi}).
\end{aligned}
\qquad (87)
$$

Thus the Petrov type is N. The Weyl scalars vanish for $r \to \infty$, but this limit is in general not defined for $rC_i$, $i = 1, \dots, 5$ because of the oscillatory behavior of the metric functions. Thus in accordance with [6], there is no smooth $\mathcal{I}$ and no peeling in this case even if we assume that the metric tends asymptotically to the Minkowski metric.

# 5    Outlook

In the previous sections we have given a set of equations describing binary black hole spacetimes with a helical Killing vector. The equations have regular singularities at the Killing horizons and the light cylinder, and a non-regular singularity at infinity. This leads to a set of five equations which could be useful for a numerical implementation. The equations appear to be well suited for the multi-domain spectral method used in [22], see also [44]. It is straight forward to include regular singularities in the spectral formalism

in the adapted coordinates we used for the analytical discussion, since the formal expansions we were discussing is very close to the philosophy of a spectral expansion. The main difficulty from a numerical point of view seems to be the oscillatory behavior at infinity. Typically a cut-off at some finite radius is used, but it is unclear which boundary conditions have to be used there. A possibility would be to match the solution at some large radius to an analytical solution of the linearized Einstein equations where it is not yet clear on which background the equations can be linearized (the asymptotic form of the solutions is still an open question). The main problem will be in any case the numerical resolution of the oscillatory metric close to infinity.

From a mathematical point of view the most interesting question is whether there exist solutions with two regular Killing horizons in a vacuum spacetime with a helical Killing symmetry. In this paper we have only considered formal expansions of the metric in the vicinity of the singularities. The fact that the solution close to the horizons contains two free functions of the angular variables gives hope that such solutions might exist globally, but this needs to be proven. In case such solutions exist, it would be interesting to obtain the precise asymptotic behavior, whether the metric tends to the Minkowski metric asymptotically, and whether a NUT parameter is needed. Numerical results could give hints on how to answer these mathematical questions.

The physical relevance of the studied model is clearly to obtain fully relativistic values for the ISCO and to get initial data for numerical calculations of the last phase of the binary system. In a real physical situation, the helical symmetry will be only an approximate symmetry. Therefore it would be interesting to study perturbations of a spacetime with an exact helical Killing vector studied here. There have been activities in this direction: in [5], the Killing symmetry holds only in a finite region of space and time, the spacetime is asymptotically matched to a wave-zone. An approximate Killing vector was considered in [45].

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