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by

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REGULARITY THEOREMS AND ENERGY IDENTITIES FOR DIRAC-HARMONIC MAPS

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Abstract. We study Dirac-harmonic maps from a Riemann surface to a sphere $\mathbb{S}^n$. We show that a weakly Dirac-harmonic map is in fact smooth, and prove that the energy identity holds during the blow-up process.

1. Introduction

Let M be a compact spin Riemann surface, ΣM the spinor bundle over M and N a compact Riemannian manifold. Let ϕ be a map from M to N, ψ a section of the bundle $\Sigma M \otimes \phi^{-1}TN$. Let $\widetilde{\nabla}$ be the connection induced from those on ΣM and $\phi^{-1}TN$. The Dirac operator $\not{D}$ along the map ϕ is defined by $\not{D}\psi := e_\alpha \cdot \widetilde{\nabla}_{e_\alpha}\psi$, where e_1, e_2 is an orthonormal basis on M. We consider the functional

$$L(\phi, \psi) := \int_M [|d\phi|^2 + \langle \psi, \not{D}\psi \rangle_{\Sigma M \otimes TN}].$$

The critical points (ϕ, ψ) are called Dirac-harmonic maps from M to N (these maps were first introduced in our companion paper [3] were also further background and motivation are provided). When ψ vanishes, we obtain the standard energy functional whose minimizers ϕ are harmonic maps. In other words, here we are generalizing that setting by coupling the map with a spinor field with values in the pull-back tangent bundle. The important point is that this generalization preserves a fundamental property of the energy functional on Riemann surfaces, namely its conformal invariance. In fact, our functional is nothing but the action functional for the non-linear supersymmetric sigma model from quantum field theory, with the only difference that here all fields are real valued instead of having Grassmann coefficients. This brings us back into the framework of the calculus of variations.

Since the construction is geometrically quite natural, one should expect that this class of maps can yield new geometric invariants of N. Before one can address that issue, however, one needs to do the basic analytic work. As a first step, we should derive a compactness theorem. To begin this program, we consider in this paper the case that the target is a sphere $\mathbb{S}^n$, that is, in the terminology of quantum field theory, we consider the $O(n+1)$ sigma model. Suppose that (ϕ_k, ψ_k) is a sequence of Dirac-harmonic maps from M to $\mathbb{S}^n$ with uniformly bounded energy $E(\phi_k, \psi_k) = \int_M (|d\phi_k|^2 + |\psi_k|^4)$, then there is a subsequence which we also denote by (ϕ_k, ψ_k) such that $\phi_k \to \phi$ weakly in $W^{1,2}$ and $\psi_k \to \psi$ weakly in L^4, and outside a finite set of points $S = \{p_1, p_2, \cdots, p_I\}$ which we call the blow-up set, the convergence is strong on compact sets. So (ϕ, ψ) is smooth in $M \setminus S$, and it is a weakly Dirac-harmonic map. We show first in this paper that any weakly Dirac-harmonic map is smooth, and so also the present limit is smooth. At every blow-up

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point p_i, by Sacks-Uhlenbeck's blow-up, one gets a finite number of Dirac-harmonic spheres (σ_i^l, ξ_i^l). The regularity of weakly harmonic maps was proved by Helein [6] and [7]. Another main purpose of this paper is to show the so-called energy identity:

$$\lim_{k\to\infty} E(\phi_k) = E(\phi) + \Sigma_i\Sigma_l E(\sigma_i^l); \qquad \lim_{k\to\infty} E(\psi_k) = E(\psi) + \Sigma_i\Sigma_l E(\xi_i^l).$$

The energy identity for a min-max sequence for the energy was proved by Jost [10], for Palais-Smale sequences with uniformly L^2-bounded tension fields by Ding-Tian [4]. For related results see [13], [18], [14], [15] and [11].

2. Regularity theorems for Dirac-harmonic maps

Let $(M, h_{\alpha\beta})$ be a compact two-dimensional Riemannian manifold with a fixed spin structure, ΣM the spinor bundle. For any $X \in \Gamma(TM)$, $\xi \in \Gamma(\Sigma M)$, denote by $X \cdot \xi$ the Clifford multiplication, which satisfies the following skew-adjointness relation:

$$\langle X\cdot\xi, \eta\rangle_{\Sigma M} = -\langle \xi, X\cdot\eta\rangle_{\Sigma M}$$

for any $X \in \Gamma(TM)$, $\xi, \eta \in \Gamma(\Sigma M)$, where $\langle\cdot,\cdot\rangle_{\Sigma M}$ denotes the metric on ΣM induced by the Riemannian metric $h_{\alpha\beta}$). Choosing a local orthonormal basis $\{e_\alpha, \alpha = 1, 2\}$ on M, the usual Dirac operator is defined as: $\partial\!\!\!/ := e_\alpha \cdot \nabla_{e_\alpha}$, where ∇ stands for the spin connection on ΣM (here and in the sequel, we use the Einstein summation convention). (A good reference for the spin geometry tools used in this paper is [12].)

Let ϕ be a smooth map from M to another compact Riemannian manifold (N, g) of dimension $n \geq 2$. Let $\phi^{-1}TN$ be the pull-back bundle of TN by ϕ and consider the twisted bundle $\Sigma M \otimes \phi^{-1}TN$. On $\Sigma M \otimes \phi^{-1}TN$ there is a metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes TN}$ induced from the metrics on ΣM and $\phi^{-1}TN$. Also we have a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ induced from those on ΣM and $\phi^{-1}TN$. In local coordinates, the section ψ of $\Sigma M \otimes \phi^{-1}TN$ is written as

$$\psi = \psi^j \otimes \partial_{y^j}(\phi),$$

where each ψ^j is a usual spinor on M and $\{\partial_{y^j}\}$ is the natural local basis on N. $\widetilde{\nabla}$ becomes

$$\widetilde{\nabla}\psi = \nabla\psi^i \otimes \partial_{y^i}(\phi) + (\Gamma^i_{jk}\nabla\phi^j)\psi^k \otimes \partial_{y^i}(\phi)$$

where the Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of N.
We define the *Dirac operator along the map* ϕ as

$$\begin{aligned} D\!\!\!\!/\,\psi &:= e_\alpha \cdot \widetilde{\nabla}_{e_\alpha}\psi \\ &= \partial\!\!\!/\psi^i \otimes \partial_{y^i}(\phi) + (\Gamma^i_{jk}\nabla_{e_\alpha}\phi^j)(e_\alpha\cdot\psi^k)\otimes\partial_{y^i}(\phi). \end{aligned}$$

It is easy to verify that $D\!\!\!\!/$ is formally self-adjoint, i.e.,

$$\int_M \langle\psi, D\!\!\!\!/\,\xi\rangle_{\Sigma M\otimes TN} = \int_M \langle D\!\!\!\!/\,\psi, \xi\rangle_{\Sigma M\otimes TN},$$

for all $\psi, \xi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)$, the space of smooth sections of $\Sigma M \otimes \phi^{-1}TN$, where $\langle\psi,\xi\rangle_{\Sigma M\otimes TN} := g_{ij}(\phi)\langle\psi^i,\xi^j\rangle_{\Sigma M}$, for $\psi, \xi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$.
Set

$$\mathcal{X}(M,N) := \{(\phi,\psi)\,|\,\phi \in C^\infty(M,N) \text{ and } \psi \in C^\infty(\Sigma M\otimes\phi^{-1}TN)\}. \tag{2.1}$$

We consider the following functional

$$\begin{aligned} L(\phi,\psi) &:= \int_M (|d\phi|^2 + \langle\psi, \not{D}\psi\rangle_{\Sigma M\otimes TN}) \\ &= \int_M \{g_{ij}(\phi)h^{\alpha\beta}\frac{\partial\phi^i}{\partial x^\alpha}\frac{\partial\phi^j}{\partial x^\beta} + g_{ij}(\phi)\langle\psi^i, \not{D}\psi^j\rangle_{\Sigma M}\}. \end{aligned}$$

By a direct computation, we obtain the Euler-Lagrange equations of L:

$$\not{D}\psi^i = \not{\partial}\psi^i + \Gamma^i_{jk}(\phi)\partial_\alpha\phi^j(e_\alpha\cdot\psi^k) = 0, \quad i = 1,2,\cdots,n, \tag{2.2}$$

$$\tau^m(\phi) - \frac{1}{2}R^m{}_{lij}(\phi)\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle_{\Sigma M} = 0, \quad m = 1,2,\cdots,n, \tag{2.3}$$

where $\tau(\phi)$ is the tension field of the map ϕ, $\nabla\phi^l\cdot\psi^j$ denotes the Clifford multiplication of the vector field $\nabla\phi^l$ with the spinor ψ^j, and $R^m{}_{lij}$ stands for a component of the curvature tensor of the target manifold N. Denote

$$\mathcal{R}(\phi,\psi) := \frac{1}{2}R^m{}_{lij}\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle\partial_{y_m}.$$

We write equations (2.2) (2.3) in global form as

$$\begin{aligned} \not{D}\psi &= 0 \\ \tau(\phi) &= \mathcal{R}(\phi,\psi). \end{aligned} \tag{2.4}$$

Solutions (ϕ,ψ) of (2.2) and (2.3) are called *Dirac-harmonic maps from M to N*. When $\psi = 0$, a solution $(\phi, 0)$ is just a harmonic map. Harmonic maps have been extensively studied. See, for instance, two reports of Eells-Lemaire [5]. When ϕ is a constant map, each component of ψ is a usual harmonic spinor. Harmonic spinors also have been well understood, see for instance [8], [12], [2] and [1]. Dirac-harmonic maps thus are a generalization and combination of harmonic maps and harmonic spinors. Non-trivial examples are given in [3].

Let (N', g') be another Riemannian manifold and $f : N \to N'$ a smooth map. For a map $\phi : M \to N$, we have a map $\phi' = \phi\circ f$ from M to N'. The map f naturally induces a map from $M\otimes\phi^{-1}TN \to M\otimes\phi'^{-1}TN$,which is denoted by f_*. Hence for and $(\phi,\psi)\in\mathcal{X}$ we get $(\phi', f_*\psi)\in\mathcal{X}(M,N')$. $\psi' := f_*\psi$ is a spinor field along the map ϕ'.

Let A be the second fundamental form of f, i.e., $A(X,Y) = (\nabla_X df)(Y)$ for any $X, Y \in \Gamma(TN)$. It is well-known that the tension fields of ϕ and ϕ' satisfy the following relation

$$\tau'(\phi') = A(d\phi(e_\alpha), d\phi(e_\alpha)) + df(\tau(\phi)). \tag{2.5}$$

One can also check that the Dirac operators $\not{D}$ and $\not{D}'$ corresponding to ϕ and ϕ' respectively are related by

$$\not{D}'\psi' = f_*(\not{D}\psi) + \mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi), \tag{2.6}$$

where

$$\begin{aligned} \mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi) &:= \phi^i_\alpha e_\alpha\cdot\psi^j\otimes A(\partial_{y^i}, \partial_{y^j}) \\ &= (\nabla\phi^i\cdot\psi^j)\otimes A(\partial_{y^i}, \partial_{y^j}). \end{aligned}$$

When $f : N \to N'$ is an isometric immersion, then $A(\cdot,\cdot)$ is the second fundamental form of the submanifold N in N'. We have

$$\nabla'_X\xi = -P(\xi; X) + \nabla^\perp_X\xi, \qquad \nabla'_X Y = \nabla_X Y + A(X,Y)$$

$\forall X, Y \in \Gamma(TN)$, $\xi \in \Gamma(T^{\perp}N)$, where $T \perp N$ is the normal bundle, ∇ and ∇' are covariant derivatives and $P(\cdot\,;\cdot)$ denotes the shape operator. In this case, for simplicity of notation, we identify ϕ with ϕ' and ψ with ψ'. Using the equation of Gauss, we have

$$\begin{aligned}
& R^{m}{}_{lij}\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle_{\Sigma M} \\
= \;& g^{mk}[\langle A(\partial_{y^k},\partial_{y^i}), A(\partial_{y^l},\partial_{y^j})\rangle_{TN'} - \langle A(\partial_{y^k},\partial_{y^j}), A(\partial_{y^l},\partial_{y^i})\rangle_{TN'} \\
& +R'_{klij}]\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle_{\Sigma M} \\
= \;& 2g^{mk}\langle A(\partial_{y^k},\partial_{y^i}), A(\partial_{y^l},\partial_{y^j})\rangle_{TN'}\langle\psi^i, e_\alpha\cdot\psi^j\rangle_{\Sigma M}\phi^l_\alpha \\
& +g^{mk}R'_{klij}\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle_{\Sigma M} \\
= \;& 2g^{mk}\langle P(A(\partial_{y^l},\partial_{y^j});\partial_{y^i}), \partial_{y_k}\rangle_{TN}\langle\psi^i, e_\alpha\cdot\psi^j\rangle_{\Sigma M}\phi^l_\alpha + R'^{m}{}_{lij}\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle_{\Sigma M},
\end{aligned}$$

where in the last step we used the following relation between the shape operator $P(\cdot\,;\cdot)$ and the second fundamental form $A(\cdot,\cdot)$:

$$\langle P(\xi; X), Y\rangle_{TN} = \langle A(X,Y), \xi\rangle_{TN'}$$

for any $X, Y \in \Gamma(TN)$, $\xi \in \Gamma(T^{\perp}N)$. Set

$$P(\mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi);\psi) := P(A(\partial_{y^l},\partial_{y^j});\partial_{y^i})\langle\psi^i, e_\alpha\cdot\psi^j\rangle_{\Sigma M}\phi^l_\alpha.$$

From the above calculation, we have

$$\mathcal{R}(\phi,\psi) = P(\mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi);\psi) + \mathcal{R}'(\phi,\psi). \tag{2.7}$$

Therefore, using (2.6) and (2.5) and identifying ψ with ψ' and ϕ with ϕ', we can rewrite (2.2) and (2.3) as follows:

$$\not{D}'\psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi), \tag{2.8}$$
$$\tau'(\phi) = A(d\phi(e_\alpha), d\phi(e_\alpha)) + P(\mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi);\psi) + \mathcal{R}'(\phi,\psi). \tag{2.9}$$

In particular, by the Nash-Moser embedding theorem, we embed N into the Euclidean space $N' = \mathbb{R}^K$, and have $\not{D}' = \not{\partial}$ and $\tau' = -\Delta$, where Δ is the (negative) Laplacian. Therefore, we have

$$\not{\partial}\psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi), \tag{2.10}$$
$$-\Delta\phi = A(d\phi, d\phi) + P(\mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\psi);\psi), \tag{2.11}$$

where $\phi : M \to \mathbb{R}^K$ with

$$\phi(x) \in N \tag{2.12}$$

for any $x \in M$ and $\psi = (\psi^1, \psi^2, \cdots, \psi^K)$ with the property that $\psi(x)$ is along the map ϕ, namely,

$$\sum_{i=1}^{K} v_i\psi^i(x) = 0, \qquad \text{for any normal vector } v = (v_1, \cdots, v_K) \text{ at } \phi(x). \tag{2.13}$$

Here $\psi^i \in \Gamma(\Sigma M)$. For any vector $v = (v_1, v_2, \cdots, v_K) \in \mathbb{R}^K$, abusing the notation a little bit, we write $\langle v, \psi\rangle = v_i\psi^i \in \Gamma(\Sigma M)$. And we also write for $\xi \in \Gamma(\Sigma M)$

$$\langle\psi, \xi\rangle := (\langle\psi^1, \xi\rangle_{\Sigma M}, \langle\psi^2, \xi\rangle_{\Sigma M}, \cdots, \langle\psi^K, \xi\rangle_{\Sigma M}) \in \mathbb{R}^K,$$

if there is no confusion.

Set

$$\mathcal{X}^{1,2}_{1,4/3}(M,N) := \{(\phi,\psi) \in W^{1,2} \times W^{1,4/3} \text{ with (2.12) and (2.13) a.e.}\}.$$

For simplicity of notation, we denote $\mathcal{X}^{1,2}_{1,4/3}(M,N)$ by $\mathcal{X}(M,N)$ (thus, we are changing the convention of (2.1)). It is clear that the functional $L(\phi,\psi)$ is well-defined for $(\phi,\psi)\in\mathcal{X}(M,N)$.

Definition. A critical point $(\phi,\psi)\in\mathcal{X}(M,N)$ of the functional L in $\mathcal{X}(M,N)$ is called *a weakly Dirac-harmonic map* from M to N. Equivalently, $(\phi,\psi)\in\mathcal{X}(M,N)$ is a weakly Dirac-harmonic map from M to N if and only if (ϕ,ψ) satisfies

$$\int_M \{\langle\nabla\phi,\nabla\eta\rangle - \langle A(d\phi,d\phi)+P(\mathcal{A}(d\phi(e_\alpha),e_\alpha\cdot\psi);\psi),\eta\rangle\} = 0, \tag{2.14}$$

$$\int_M \{\langle\psi,\partial\!\!\!/\xi\rangle - \langle\mathcal{A}(d\phi(e_\alpha),e_\alpha\cdot\psi),\xi\rangle\} = 0, \tag{2.15}$$

for all $\eta\in C^\infty(M,\mathbb{R}^K)$ and $\xi\in C^\infty(\Sigma M\otimes\mathbb{R}^K)$.

One of our purposes of this paper is to study the regularity of weakly Dirac-harmonic maps. Our main observation is that when the target N is the standard sphere $\mathbb{S}^n$, a weakly Dirac-harmonic map has a special structure like a weakly harmonic map. For weakly harmonic maps, see [7].

Proposition 2.1. *Let M be a Riemann surface with a fixed spin structure and $(\phi,\psi)\in\mathcal{X}(N,M)$ a weakly Dirac-harmonic map from M to $\mathbb{S}^n$. Let D be a simply connected domain of M. Then there exists $M=(M^{ij})\in W^2(D,\mathbb{R}^{n\times n})$ such that*

$$-\Delta\phi = \frac{\partial M}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial M}{\partial y}\frac{\partial\phi}{\partial x}. \tag{2.16}$$

Proof. For $N=\mathbb{S}^n\subset\mathbb{R}^{n+1}$, the equations (2.10) and (2.11) can be respectively written as follows:

$$\partial\!\!\!/\psi^m = -\sum(\nabla\phi^i\cdot\psi^i)\otimes\phi^m, \tag{2.17}$$

$$-\Delta\phi^m = |d\phi|^2\phi + \langle\psi^m\otimes d\phi(e_\alpha),e_\alpha\cdot\psi\rangle_{\Sigma M\otimes\mathbb{R}^K}, \tag{2.18}$$

for $m=1,2,\cdots,n+1$. Set $\phi_\alpha := d\phi(e_\alpha)$.

From (2.18), we have for $m=1,2,\cdots,n+1$ that

(2.19)
$$\begin{aligned}
\Delta\phi^m &= -|d\phi|^2\phi^m + \langle\psi^m\otimes d\phi(e_\alpha),e_\alpha\cdot\psi\rangle_{\Sigma M\otimes\mathbb{R}^K}\\
&= -(\phi^i_\alpha\phi^m - \phi^i\phi^m_\alpha)\phi^i_\alpha + \phi^i_\alpha\langle e_\alpha\cdot\psi^i,\psi^m\rangle_{\Sigma M} \quad (\text{as } 0=\partial_\alpha 1 = 2\phi^i_\alpha\phi^i)\\
&= [\langle\partial_x\cdot\psi^i,\psi^m\rangle_{\Sigma M} - (\phi^i_x\phi^m - \phi^i\phi^m_x)]\phi^i_x\\
&\quad +[\langle\partial_y\cdot\psi^i,\psi^m\rangle_{\Sigma M} - (\phi^i_y\phi^m - \phi^i\phi^m_y)]\phi^i_y\\
&:= A^{mi}\phi^i_x + B^{mi}\phi^i_y.
\end{aligned}$$

We would like to show that there exists some function M^{mi} on D such that

$$A^{mi} = M^{mi}_y, \quad B^{mi} = -M^{mi}_x \tag{2.20}$$

which, by the Frobenius theorem, is equivalent to

$$A^{mi}_x + B^{mi}_y = 0. \tag{2.21}$$

Calculating directly, one derives

$$\begin{aligned} A^{mi}_x + B^{mi}_y &= \langle \partial_x \cdot \psi^i_x, \psi^m\rangle_{\Sigma M} + \langle \partial_x \cdot \psi^i, \psi^m_x\rangle_{\Sigma M} + \langle \partial_y \cdot \psi^i_y, \psi^m\rangle_{\Sigma M} \\ &\quad + \langle \partial_y \cdot \psi^i, \psi^m_y\rangle_{\Sigma M} - (\phi^i_{xx}\phi^m - \phi^i\phi^m_{xx} + \phi^i_{yy}\phi^m - \phi^i\phi^m_{yy}) \\ &= \langle \partial\!\!\!/ \psi^i, \psi^m\rangle_{\Sigma M} - \langle \psi^i, \partial\!\!\!/ \psi^m\rangle_{\Sigma M} - (\Delta\phi^i\phi^m - \Delta\phi^m\phi^i). \end{aligned} \tag{2.22}$$

From equation (2.18), one gets

$$-(\Delta\phi^i\phi^m - \Delta\phi^m\phi^i) = -\phi^k_\alpha\phi^m\langle e_\alpha \cdot \psi^k, \psi^i\rangle_{\Sigma M} + \phi^k_\alpha\phi^i\langle e_\alpha \cdot \psi^k, \psi^m\rangle_{\Sigma M}, \tag{2.23}$$

And from equation (2.17), one obtains

$$\langle \partial\!\!\!/ \psi^i, \psi^m\rangle_{\Sigma M} = -\phi^k_\alpha\phi^i\langle e_\alpha \cdot \psi^k, \psi^m\rangle_{\Sigma M}.$$

Hence

$$\langle \partial\!\!\!/ \psi^i, \psi^m\rangle_{\Sigma M} - \langle \psi^i, \partial\!\!\!/ \psi^m\rangle_{\Sigma M} = \phi^k_\alpha\phi^m\langle e_\alpha \cdot \psi^k, \psi^i\rangle_{\Sigma M} - \phi^k_\alpha\phi^i\langle e_\alpha \cdot \psi^k, \psi^m\rangle_{\Sigma M}. \tag{2.24}$$

Putting (2.23) and (2.24) into (2.22) we have (2.21). Hence we prove the Proposition. □

Theorem 2.2. *Let M be a Riemann surface with a fixed spin structure. Suppose that $(\phi, \psi) \in \mathcal{X}(N, M)$ is a weakly Dirac-harmonic map from M to $\mathbb{S}^n$. Then $\phi \in C^0$. And hence (ψ, ψ) is smooth.*

Proof. From Proposition 2.1 and Wente's well-known lemma ([17]), we know that ϕ^m is continuous, $m = 1, 2, \cdots, n+1$, namely, $\phi \in C^0(M, \mathbb{S}^n)$. By Theorem 2.3 below, we have that ϕ and ψ are smooth. □

Theorem 2.3. *Let $(\phi, \psi) : (D, \delta_{\alpha\beta}) \to (N^n, g_{ij})$ be a weakly Dirac-harmonic map. If ϕ is continuous, then (ϕ, ψ) is smooth.*

To prove this theorem, we first establish two lemmas (Lemma 2.4 and Lemma 2.5 below which are similar to Lemma 8.6.1 and Lemma 8.6.2 in [9]). Since ϕ is in $C^0(D, N)$, we can choose local coordinates $\{y_i\}$ on N such that $\Gamma^i_{jk}(\phi(0)) = 0$. In these coordinates, the equations for ϕ and ψ can be written as

$$\Delta\phi^m = -\Gamma^m_{ij}(\phi)\phi^i_\alpha\phi^j_\alpha + \frac{1}{2}R^m{}_{jkl}(\phi)\langle \psi^k, \nabla\phi^j \cdot \psi^l\rangle_{\Sigma M}, \tag{2.25}$$

$$\partial\!\!\!/ \psi^m = -\Gamma^m_{ij}(\phi)\nabla\phi^i \cdot \psi^j. \tag{2.26}$$

This intrinsic version of our equations is well-defined since $\phi \in C^0$.

Lemma 2.4. *Let (ϕ, ψ) be a weak solution of (2.25) and (2.26). If $\phi \in C^0 \cap W^{1,2}(D, N)$, then for any $\varepsilon > 0$, there is a $\rho > 0$ such that*

$$\int_{D(x_1,\rho)} |\nabla\phi|^2\eta^2(x) \le \varepsilon \int_{D(x_1,\rho)} |\nabla\eta|^2 + C\varepsilon\Big(\int_{D(x_1,\rho)} |\psi|^4\eta^4\Big)^{\frac{1}{2}}, \tag{2.27}$$

where $D(x_1, \rho) \subset D$, $\eta \in W^{1,2}_0(D(x_1, \rho), \mathbb{R})$, C is a positive constant independent of ε, ρ, ϕ and ψ.

Proof. Denote

$$G^m(x, \phi, d\phi) := \Gamma^m_{ij}(\phi)\phi^i_\alpha\phi^j_\alpha - \frac{1}{2}R^m{}_{jkl}(\phi)\langle \psi^k, \nabla\phi^j \cdot \psi^l\rangle_{\Sigma M}, \quad G = (G^1, G^2, \cdots, G^n),$$

then

$$|G_x| \leq C(|d\phi|^3 + |\nabla\psi||\psi||d\phi|),$$
$$|G_\phi| \leq C(|d\phi|^2 + |\psi|^2|d\phi|),$$
$$|G_{d\phi}| \leq C(|d\phi| + |\psi|^2).$$

The weak form of equation (2.25) is:

$$\int_D \nabla_\alpha\phi^i\nabla_\alpha\zeta^i = \int_D G^i(x,\phi,d\phi)\zeta^i, \tag{2.28}$$

for any $\zeta \in W^{1,2} \cap L^\infty(D,\mathbb{R}^K)$. Now we choose $\zeta(x) = (\phi(x) - \phi(x_1))\eta^2(x)$, then

$$\int_{D(x_1,\rho)} \nabla_\alpha\phi^i\nabla_\alpha\zeta^i = \int_{D(x_1,\rho)} |d\phi|^2\eta^2 + 2\int_{D(x_1,\rho)} \eta(\phi^i(x) - \phi^i(x_1))\nabla_\alpha\phi\nabla_\alpha\eta. \tag{2.29}$$

We have

$$\begin{aligned}
\int_{D(x_1,\rho)} G^i(x,\phi,d\phi)\zeta^i \ &= \ \int_{D(x_1,\rho)} [\Gamma^i_{jk}(\phi)\phi^j_\alpha\phi^k_\alpha(\phi^i(x) - \phi^i(x_1))\eta^2(x)] \\
&\quad -\frac{1}{2}\int_{D(x_1,\rho)} [R^i_{jkl}(\phi)\langle\psi^k, \nabla\phi^j\cdot\psi^l\rangle_{\Sigma M}(\phi^i(x) - \phi^i(x_1))\eta^2(x)] \\
&\leq \ C_N\varepsilon_1 \mathrm{Sup}_{D(x_1,\rho)}|\phi(x) - \phi(x_1)| \int_{D(x_1,\rho)} |d\phi|^2\eta^2 \\
&\quad +C_N \mathrm{Sup}_{D(x_1,\rho)}|\phi(x) - \phi(x_1)| \int_{D(x_1,\rho)} |d\phi||\psi|^2\eta^2 \\
&\leq \ C_N\varepsilon_1 \mathrm{Sup}_{D(x_1,\rho)}|\phi(x) - \phi(x_1)| \int_{D(x_1,\rho)} |d\phi|^2\eta^2 \\
&\quad +C_N \mathrm{Sup}_{D(x_1,\rho)}|\phi(x) - \phi(x_1)|(\int_{D(x_1,\rho)} |d\phi|^2)^{\frac{1}{2}} \\
&\quad \times(\int_{D(x_1,\rho)} |\psi|^4\eta^4)^{\frac{1}{2}},
\end{aligned} \tag{2.30}$$

where $\varepsilon_1 > 0$ is a given small number. On the other hand,

$$\begin{aligned}
2\int_{D(x_1,\rho)} \eta(\phi^i(x) - \phi^i(x_1))\nabla_\alpha\phi\nabla_\alpha\eta \ &\leq \ C_N \mathrm{Sup}_{D(x_1,\rho)}|\phi(x) - \phi(x_1)| \int_{D(x_1,\rho)} |d\phi||\nabla\eta|\eta \\
&\leq \ \frac{1}{2}\int_{D(x_1,\rho)} |d\phi|^2\eta^2 + 8\mathrm{Sup}_{D(x_1,\rho)}|\phi(x) - \phi(x_1)|^2 \\
&\quad \times\int_{D(x_1,\rho)} |\nabla\eta|^2.
\end{aligned} \tag{2.31}$$

Substituting (2.29), (2.30) and (2.31) into (2.28), and choosing ρ small enough then yields (2.27). □

Lemma 2.5. *If $\phi \in C^0 \cap W^{1,4} \cap W^{3,2}(D(x_0,R),N)$ and (ϕ,ψ) is a weak solution of (2.25), (2.26), then for R sufficiently small, we have*

$$\|\nabla^2\phi\|_{L^2(D(x_0,R/2))} + \|d\phi\|^2_{L^4(D(x_0,R/2))} \leq C_1\|d\phi\|_{L^2(D(x_0,R))}, \tag{2.32}$$

where $C_1 > 0$ is a constant depending on $|\phi|_{C^0(D,N)}$ and R.

Proof. We may assume $x_0 = 0 \in D$. Given $\varepsilon' > 0$ small, since ϕ is continuous, we can choose R small enough such that $|\phi(x) - \phi(0)| < \varepsilon'$ for all $x \in D(x_0, R)$. For simplicity, we denote $B := D(x_0, R)$. From equation (2.26) we have

$$|\partial\!\!\!/\psi| \le |\Gamma^i_{jk}(\phi(x)) - \Gamma^i_{jk}(\phi(0)||d\phi^j||\psi^k| \le C_N|\phi(x) - \phi(0)||d\phi||\psi|,$$

hence

$$|\partial\!\!\!/\psi| \le C_N\varepsilon'|d\phi||\psi| \tag{2.33}$$

Noting that $\partial\!\!\!/(\psi\eta) = \eta\partial\!\!\!/\psi + \nabla\eta\cdot\psi$, we have

$$\begin{aligned}\|\partial\!\!\!/(\psi\eta)\|_{L^{4/3}(B)} &\le \|\eta\partial\!\!\!/\psi\|_{L^{4/3}(B)} + \|\nabla\eta\cdot\psi\|_{L^{4/3}(B)}\\ &\le C_N\varepsilon'\||d\phi||\psi|\eta\|_{L^{4/3}(B)} + \||\nabla\eta||\psi|\|_{L^{4/3}(B)}\\ &\le C_N\varepsilon'\|d\phi\|_{L^2(B)}\||\psi|\eta\|_{L^4(B)} + \||\nabla\eta||\psi|\|_{L^{4/3}(B)}.\end{aligned} \tag{2.34}$$

By the elliptic estimates for the first order equation, we have

$$\|\nabla(\psi\eta)\|_{L^{4/3}(B)} + \|\psi\eta\|_{L^4(B)} \le C_R\|\partial\!\!\!/(\psi\eta)\|_{L^{4/3}(B)}. \tag{2.35}$$

A proof was given in Lemma 4.8 in [3]. By choosing R small enough such that $C_N\varepsilon'\|d\phi\|_{L^2(B)} < \frac{1}{2}$, we obtain from (2.34) and (2.35) that

$$\|\nabla(\psi\eta)\|_{L^{4/3}(B)} + \|\psi\eta\|_{L^4(B)} \le C_R\||\nabla\eta||\psi|\|_{L^{4/3}(B)}, \tag{2.36}$$

from which we easily derive that

$$\||\nabla\psi|\eta\|_{L^{4/3}(B)} + \|\psi\eta\|_{L^4(B)} \le C_R\||\nabla\eta||\psi|\|_{L^{4/3}(B)}. \tag{2.37}$$

For any $\zeta \in W^{1,2}_0(B, \mathbb{R}^K)$,

$$\int_B \nabla\phi\nabla\zeta = -\int_B \Delta\phi\zeta = \int_B G\zeta, \tag{2.38}$$

Choosing $\zeta = \nabla_\gamma(\xi^2\nabla_\gamma\phi)$, where $\xi \in C^\infty \cap W^{1,2}_0(B, \mathbb{R})$ is to be determined later, we get

$$\begin{aligned}\int_B \nabla_\gamma(\nabla_\beta\phi)\nabla_\beta(\xi^2\nabla_\gamma\phi) &= -\int_B \nabla_\beta\phi\nabla_\beta(\nabla_\gamma(\xi^2\nabla_\gamma\phi))\\ &= -\int_B G\nabla_\gamma(\xi^2\nabla_\gamma\phi)\\ &= \int_B \nabla_\gamma G\,\nabla_\gamma\phi\,\xi^2.\end{aligned} \tag{2.39}$$

Note that

$$\begin{aligned}\nabla_\gamma(\nabla_\beta\phi)\nabla_\beta(\xi^2\nabla_\gamma\phi) &= |\nabla_\gamma\nabla_\beta\phi|^2\xi^2 + (\nabla_\gamma\nabla_\beta\phi\nabla_\gamma\phi)\nabla_\beta\xi^2\\ &\ge |\nabla^2\phi|^2\xi^2 - 2|\nabla^2\phi||d\phi||\xi\nabla\xi|,\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}|\nabla_\gamma G\nabla_\gamma\phi| \le\ & C_N(|d\phi|^4 + |\psi||\nabla\psi||d\phi|^2 + |d\phi|^3|\psi|^2\\ &+|\nabla^2\phi||d\phi|^2 + |\nabla^2\phi||d\phi||\psi|^2).\end{aligned} \tag{2.41}$$

Substituting (2.40) and (2.41) into (2.39) yields

$$\begin{aligned}\int_B |\nabla^2\phi|^2\xi^2 &\le C_N\int_B |\nabla^2\phi||d\phi||\xi\nabla\xi| + C_N\int_B |\nabla^2\phi||d\phi|^2\xi^2 \\ &\quad + C_N\int_B |d\phi|^4\xi^2 + C_N\int_B |\psi||\nabla\psi||d\phi|^2\xi^2 \\ &\quad + C_N\int_B |d\phi|^3|\psi|^2\xi^2 + C_N\int_B |\nabla^2\phi||d\phi||\psi|^2\xi^2 \\ &:= I + II + III + IV + V + VI. \end{aligned}\tag{2.42}$$

For $\varepsilon_1 > 0$ small, we have

$$I \le C_N\varepsilon_1\int_B |\nabla^2\phi|^2\xi^2 + \frac{C_N}{\varepsilon_1}\int_B |d\phi|^2|\nabla\xi|^2,\tag{2.43}$$

$$II \le C_N\varepsilon_1\int_B |\nabla^2\phi|^2\xi^2 + \frac{C_N}{\varepsilon_1}\int_B |d\phi|^4\xi^2.\tag{2.44}$$

Choosing $\eta = |d\phi|\xi$ in (2.37), we obtain

$$\||\nabla\psi||d\phi|\xi\|_{L^{\frac{4}{3}}(B)} + \|\psi|d\phi|\xi\|_{L^4(B)} \le C\|\nabla(|d\phi|\xi)|\psi|\|_{L^{\frac{4}{3}}(B)}.\tag{2.45}$$

Because

$$\begin{aligned}\|\nabla(|d\phi|\xi)|\psi|\|^2_{L^{\frac{4}{3}}(B)} &\le 2\||\nabla^2\phi|\psi\xi\|^2_{L^{\frac{4}{3}}(B)} + 2\||d\phi||\nabla\xi||\psi|\|^2_{L^{\frac{4}{3}}(B)} \\ &\le 2(\int_B |\nabla^2\phi|^2\xi^2(\int_B |\psi|^4)^{\frac{1}{2}} + 2(\int_B |d\phi|^2|\nabla\xi|^2)(\int_B |\psi|^4)^{\frac{1}{2}} \\ &= 2(\int_B |\psi|^4)^{\frac{1}{2}}(\int_B |\nabla^2\phi|^2\xi^2 + \int_B |d\phi|^2|\nabla\xi|^2),\end{aligned}\tag{2.46}$$

choose $R > 0$ small enough such that

$$C^2\max\{2C_N, C_N^2\}\int_{D(x_0,R)} |\psi|^4 < \frac{1}{8},\tag{2.47}$$

we get

$$\begin{aligned} IV &\le C_N\||\nabla\psi||d\phi|\xi\|_{L^{\frac{4}{3}}(B)}\|\psi|d\phi|\xi\|_{L^4(B)} \\ &\le C_N C^2\|\nabla(|d\phi|\xi)|\psi|\|^2_{L^{\frac{4}{3}}(B)} \quad (\text{ by } (2.45)) \\ &\le 2C_N C^2(\int_B |\psi|^4)^{\frac{1}{2}}(\int_B |\nabla^2\phi|^2\xi^2 + \int_B |d\phi|^2|\nabla\xi|^2) \quad (\text{by}(2.46)) \\ &< \frac{1}{8}\int_B |\nabla^2\phi|^2\xi^2 + \frac{1}{8}\int_B |d\phi|^2|\nabla\xi|^2. \end{aligned}\tag{2.48}$$

Similarly, we can estimate the terms V and VI as follows.

$$
\begin{aligned}
V &\le C_N\varepsilon_1\int_B |d\phi|^2\xi^2|\psi|^4 + \frac{C_N}{\varepsilon_1}\int_B |d\phi|^4\xi^2 \\
&\le C_N\varepsilon_1\Big(\int_B |\psi|^4\Big)^{\frac{1}{2}}\Big(\int_B |\psi|^4|d\phi|^4\xi^4\Big)^{\frac{1}{2}} + \frac{C_N}{\varepsilon_1}\int_B |d\phi|^4\xi^2 \\
&\le C_N\varepsilon_1\Big(\int_B |\psi|^4\Big)^{\frac{1}{2}}C^2|\nabla(|d\phi|\xi)|\psi||^2_{L^{\frac{4}{3}}(B)} + \frac{C_N}{\varepsilon_1}\int_B |d\phi|^4\xi^2 \quad (\text{by}(2.45)) \\
&\le 2C_NC^2\varepsilon_1\Big(\int_B |\psi|^4\Big)\Big(\int_B |\nabla^2\phi|^2\xi^2 + \int_B |d\phi|^2|\nabla\xi|^2\Big) + \frac{C_N}{\varepsilon_1}\int_B |d\phi|^4\xi^2 \\
&< \frac{1}{8}\int_B |\nabla^2\phi|^2\xi^2 + \frac{1}{8}\int_B |d\phi|^2|\nabla\xi|^2 + \frac{C_N}{\varepsilon_1}\int_B |d\phi|^4\xi^2, \qquad (2.49)
\end{aligned}
$$

and

$$
\begin{aligned}
VI &= C_N\int_B (|\nabla^2\phi|\xi)(|\psi|^2|d\phi|\xi) \\
&\le \frac{1}{2}\int_B |\nabla^2\phi|^2\xi^2 + \frac{1}{2}C_N^2\int_B |\psi|^4|d\phi|^2\xi^2 \\
&\le \frac{1}{2}\int_B |\nabla^2\phi|^2\xi^2 + \frac{1}{2}C_N^2\Big(\int_B |\psi|^4\Big)^{\frac{1}{2}}\Big(\int_B |\psi|^4|d\phi|^4\xi^4\Big)^{\frac{1}{2}} \\
&\le \frac{1}{2}\int_B |\nabla^2\phi|^2\xi^2 + \frac{1}{2}C_N^2\Big(\int_B |\psi|^4\Big)^{\frac{1}{2}}C^2|\nabla(|d\phi|\xi)|\psi||^2_{L^{\frac{4}{3}}(B)} \\
&\le \frac{1}{2}\int_B |\nabla^2\phi|^2\xi^2 + C_N^2C^2\Big(\int_B |\psi|^4\Big)\Big(\int_B |\nabla^2\phi|^2\xi^2 + \int_B |d\phi|^2|\nabla\xi|^2\Big) \\
&< \frac{1}{2}\int_B |\nabla^2\phi|^2\xi^2 + \frac{1}{8}\int_B |\nabla^2\phi|^2\xi^2 + \frac{1}{8}\int_B |d\phi|^2|\nabla\xi|^2. \qquad (2.50)
\end{aligned}
$$

Putting (2.43), (2.44), (2.48), (2.49) and (2.50) into (2.42) gives

$$
\int_{D(x_0,R)} |\nabla^2\phi|^2\xi^2 \le C\Big(\int_{D(x_0,R)} |d\phi|^2|\nabla\xi|^2 + \int_{D(x_0,R)} |d\phi|^4\xi^2\Big). \tag{2.51}
$$

Now for $\varepsilon > 0$, let $\rho > 0$ be as in Lemma 2.4, and with $D(x_1,\rho) \subset D(x_0,R)$, choose a cut-off function $\xi \in C_0^\infty(D(x_1,\rho))$, $0 \le \xi \le 1$ such that

$$
\xi \equiv 1 \quad \text{in} \quad D(x_1,\frac{\rho}{2}); \quad |\nabla\xi| \le \frac{4}{\rho} \quad \text{in} \quad D(x_1,\rho),
$$

denote $B_\rho := D(x_1,\rho)$ for simplicity, one derives

$$
\begin{aligned}
\int_{B_\rho} |d\phi|^4\xi^2 &= \int_{B_\rho} |d\phi|^2(|d\phi|\xi)^2 \\
&\le \varepsilon\int_{B_\rho} |\nabla(|d\phi|\xi)|^2 + C\varepsilon\Big(\int_{B_\rho} |\psi|^4|d\phi|^4\xi^4\Big)^{\frac{1}{2}} \quad (\text{by} \quad \text{Lemma2.4}) \\
&\le \varepsilon\int_{B_\rho} |\nabla^2\phi|^2\xi^2 + \varepsilon\int_{B_\rho} |d\phi|^2|\nabla\xi|^2 + C\varepsilon\Big(\int_{B_\rho} |\psi|^4|d\phi|^4\xi^4\Big)^{\frac{1}{2}},
\end{aligned}
$$

it follows from (2.45) that

$$
\||\psi||d\phi|\xi\|_{L^4(B_\rho)} \le C\|\nabla(|d\phi|\xi)|\psi|\|_{L^{\frac{4}{3}}(B_\rho)},
$$

then, by an argument similar to the one used in the proof of (2.46), we can get

$$\||\psi||d\phi|\xi\|^2_{L^4(B_\rho)} \le 2C^2(\int_{B_\rho}|\psi|^4)^{\frac{1}{2}}(\int_{B_\rho}|\nabla^2\phi|^2\xi^2+\int_{B_\rho}|d\phi|^2|\nabla\xi|^2),$$

thus,

$$\begin{aligned}\int_{B_\rho}|d\phi|^4\xi^2 &\le \varepsilon\int_{B_\rho}|\nabla^2\phi|^2\xi^2+\varepsilon\int_{B_\rho}|d\phi|^2|\nabla\xi|^2\\ &\quad +C'\varepsilon(\int_{B_\rho}|\nabla^2\phi|^2\xi^2+\int_{B_\rho}|d\phi|^2|\nabla\xi|^2)\\ &= C''\varepsilon(\int_{B_\rho}|\nabla^2\phi|^2\xi^2+\int_{B_\rho}|d\phi|^2|\nabla\xi|^2).\end{aligned}\tag{2.52}$$

On the other hand, by (2.51), we have

$$\int_{B_\rho}|\nabla^2\phi|^2\xi^2 \le C(\int_{B_\rho}|d\phi|^2|\nabla\xi|^2+\int_{B_\rho}|d\phi|^4\xi^2),\tag{2.53}$$

substituting (2.52) into (2.53) yields

$$\int_{B_\rho}|\nabla^2\phi|^2\xi^2 \le C\int_{B_\rho}|d\phi|^2|\nabla\xi|^2,$$

hence,

$$\int_{D(x_1,\frac{\rho}{2})}|\nabla^2\phi|^2 \le \frac{C}{\rho^2}\int_{D(x_1,\rho)}|d\phi|^2.\tag{2.54}$$

Covering $D(x_0,\frac{R}{2})$ with $\{D(x_1,\frac{\rho}{2})\}$ and using (2.54) we obtain (2.32). □

Now we are in the position to give the

Proof of Theorem 2.3. First, we show that $\phi \in W^{2,2}\cap W^{1,4}(D(x_0,\frac{R}{2}),N)$. This can be done just by replacing weak derivatives by difference quotients in the proof of Lemma 2.5. Denote

$$\Delta_i^h\phi(x) := \frac{\phi(x+hE_i)-\phi(x)}{h}, \quad (\text{if} \quad \operatorname{dist}(x,\partial D) > |h|),$$

where $(E_1,E_2,\cdots,E_K)$ is an orthonarmal basis of $\mathbb{R}^K$, $h\in\mathbb{R}$. $\Delta^h := (\Delta_1^h,\Delta_2^h,\cdots,\Delta_K^h)$. Let $\zeta := \Delta_\gamma^{-h}(\xi^2\Delta_\gamma^h\phi)$, then, similar to (2.51), we have

$$\int_{D(x_0,R)}|\nabla(\Delta^h\phi)|^2\xi^2 \le C\int_{D(x_0,R)}|\Delta^h\phi|^2|\nabla\xi|^2+C\int_{D(x_0,R)}|\nabla\phi|^2|\Delta^h\phi|^2\xi^2.\tag{2.55}$$

Since (cf. [9], p.382)

$$C\int_{D(x_0,R)}|\Delta^h\phi|^2|\nabla\xi|^2 \le C\int_{D(x_0,R)}|\nabla\phi|^2|\nabla\xi|^2,$$

applying Lemma 2.4 to the right hand side, we obtain the following estimate analogous to (2.54):

$$\int_{D(x_1,\frac{\rho}{2})}|\nabla(\Delta^h\phi)|^2\xi^2 \le \frac{C}{\rho^2}\int_{D(x_1,\rho)}|d\phi|^2,$$

from which it follows that the weak derivative $\nabla^2\phi$ exists and (2.32) still holds true with ρ sufficiently small and $C_1>0$ which depends on $|\phi|_{C^0(D,N)}$ and ρ.

Next, since $\phi \in W^{2,2}$, we have that $\phi \in W^{1,p}$ for any $p > 0$, thus, the right hand side of (2.26) is in $L^p (p > 2)$, so $\psi \in C^{0,\gamma}$ for some $\gamma > 0$. By the elliptic estimates for the equation (2.25), we have $\phi \in W^{2,p}$ for any $p > 2$, thus $\phi \in C^{1,\gamma}$. By the elliptic estimates for the equation (2.26), we have $\psi \in C^{1,\gamma}$. Then the standard arguments yield that both ϕ and ψ are smooth. This completes the proof. □

3. Energy identities

First, using the elliptic estimates, we can establish the following vanishing theorem which will be used later in obtaining the energy identities, and in which we see that the $W^{1,2}$-norm of ϕ and L^4-norm of ψ play an important role in the analytic properties of ϕ and ψ. We note that these two norms are conformally invariant.

Theorem 3.1. *Let $(M^2, h_{\alpha\beta})$ be a compact Riemann surface with a fixed spin structure, and (N^n, g_{ij}) be a compact Riemannian manifold. There is a constant $\varepsilon_0 > 0$ small enough such that if (ϕ, ψ) is a smooth solution of (2.2) and (2.3) satisfying*

$$\int_M (|d\phi|^2 + |\psi|^4) < \varepsilon_0, \tag{3.5}$$

then ϕ is constant and consequently $\partial\!\!\!/ \psi^i \equiv 0$, $i = 1, 2, \cdots, n$.

Proof. In the sequel, we write $\|\cdot\|_{D,k,p}$ for the $L^{k,p}$-norm on the domain D, and if there is no confusion, we may drop the subscript D. Embed N into some $\mathbb{R}^K$ isometrically, then from the ϕ-equation (2.11) we have

$$|\Delta\phi| \le \|A\|_\infty |d\phi|^2 + C_N |d\phi||\psi|^2,$$

where $C_N > 0$ is a constant depending only on N, $\|A\|_\infty := \max_N |A|$. It follows from the above inequality that

$$\begin{aligned}
\|\Delta\phi\|_{0,\frac{4}{3}} &\le \|A\|_\infty \||d\phi|^2\|_{0,\frac{4}{3}} + C_N \||d\phi||\psi|^2\|_{0,\frac{4}{3}} \\
&\le C(\|d\phi\|_{0,2}^2 + \|\psi\|_{0,4}^2)\|d\phi\|_{0,4} \\
&\le C(\|d\phi\|_{0,2}^2 + \|\psi\|_{0,4}^2)\|d\phi\|_{1,4/3}.
\end{aligned}$$

If $\int_M (|d\phi|^2 + |\psi|^4) < \varepsilon_0$ for small $\varepsilon_0 > 0$, then $\phi \equiv const$. From equation (2.10), we have $\partial\!\!\!/ \psi^i \equiv 0$, $i = 1, 2, \cdots, n$. □

Now we prove the small energy regularity. Since the problem is local, we assume that M is flat.

Theorem 3.2 ($\varepsilon-$**regularity theorem**)**.** *There is an $\varepsilon_0 > 0$ such that if $(\phi, \psi) : (D, \delta_{\alpha\beta}) \to (N, g_{ij})$ is a C^∞ Dirac-harmonic map satisfying*

$$\int_D (|d\phi|^2 + |\psi|^4) < \varepsilon_0, \tag{3.6}$$

then

$$\|d\phi\|_{\widetilde{D},1,p} \le C(\widetilde{D}, p)\|d\phi\|_{D,0,2}, \tag{3.7}$$

$$\|\nabla\psi\|_{\widetilde{D},1,p} \le C(\widetilde{D}, p)\|\psi\|_{D,0,4}, \tag{3.8}$$

$$\|\nabla\psi\|_{C^0(\widetilde{D})} \le C(\widetilde{D})\|\psi\|_{D,0,4}, \quad \|\psi\|_{C^0(\widetilde{D})} \le C(\widetilde{D})\|\psi\|_{D,0,4} \tag{3.9}$$

$\forall \widetilde{D} \subset D, p > 1$, where $C(\widetilde{D}, p) > 1$ is a constant depending only on $\widetilde{D}$ and p.

To prove this theorem, we first estimate $|d\phi|$.

Lemma 3.3 *There is an $\varepsilon_0 > 0$ such that if $(\phi, \psi) : (D, \delta_{\alpha\beta}) \to (N, g_{ij})$ is a C^∞ Dirac-harmonic map satisfying (3.6), then*

$$\|\phi\|_{D^1,1,4} \leq C(D^1)\sqrt{\varepsilon_0}, \quad \forall D^1 \quad \text{with} \quad \overline{D^1} \subset D, \tag{3.10}$$

where $C(D^1) > 0$ is a constant depending only on D^1.

Proof. Choose a cut-off function $\eta : 0 \leq \eta \leq 1$, with $\eta|_{D^1} \equiv 1$ and $\text{Supp}\eta \subset D$. By (2.11) we have

$$\begin{aligned} |\Delta(\eta\phi)| &\leq C(|\phi| + |d\phi|) + \|A\|_\infty |d\phi|(|d(\eta\phi)| + |\phi d\eta|) + |\eta\alpha| \\ &\leq \|A\|_\infty |d\phi||d(\eta\phi)| + C(|\phi| + |d\phi|) + |\eta\alpha|, \end{aligned}$$

where $\alpha := P(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)$, thus, for any $p > 1$,

$$\|\Delta(\eta\phi)\|_{0,p} \leq \|A\|_\infty \||d\phi||d(\eta\phi)|\|_{0,p} + C\|\phi\|_{1,p} + \|\eta\alpha\|_{0,p}. \tag{3.11}$$

Let $p = \frac{4}{3}$, and without loss of generality we assume $\int_D \phi = 0$ so that $\|\phi\|_{1,p} \leq C'\|d\phi\|_{0,p}$, then

$$\|A\|_\infty \||d\phi||d(\eta\phi)|\|_{0,\frac{4}{3}} \leq \|A\|_\infty \|\eta\phi\|_{1,4}\|d\phi\|_{0,2},$$

from this and (3.11) we have

$$\|\eta\phi\|_{2,\frac{4}{3}} \leq C(\|A\|_\infty \|\eta\phi\|_{1,4}\|d\phi\|_{0,2} + \|d\phi\|_{0,\frac{4}{3}} + \|\eta\alpha\|_{0,\frac{4}{3}}).$$

Bythe Sobolev inequality, $\|\eta\phi\|_{1,4} \leq C'\|\eta\phi\|_{2,\frac{4}{3}}$, so,

$$(C'^{-1} - C\|A\|_\infty \|d\phi\|_{0,2})\|\eta\phi\|_{1,4} \leq C(\|d\phi\|_{0,\frac{4}{3}} + \|\eta\alpha\|_{0,\frac{4}{3}}), \tag{3.12}$$

moreover,

$$\begin{aligned} \|\eta\alpha\|_{0,\frac{4}{3}} &\leq C_N \||\psi|^2|\eta d\phi|\|_{0,\frac{4}{3}} \\ &= C_N \||\psi|^2|d(\eta\phi) - \phi d\eta|\|_{0,\frac{4}{3}} \\ &\leq C\||\psi|^2|d(\eta\phi)|\|_{0,\frac{4}{3}} + C\||\psi|^2|\phi||d\eta|\|_{0,\frac{4}{3}} \\ &\leq C\|\psi\|_{0,4}^2 \|\eta\phi\|_{1,4} + C\|\psi\|_{0,\frac{4}{3}}^2 \\ &\leq C\|\psi\|_{0,4}^2 \|\eta\phi\|_{1,4} + C\|\psi\|_{0,4}^2, \end{aligned}$$

putting this into (3.12) we get:

$$\|\eta\phi\|_{1,4} \leq C(\|d\phi\|_{0,\frac{4}{3}} + \sqrt{\varepsilon_0}\|\eta\phi\|_{1,4} + \|\psi\|_{0,4}^2),$$

which yields

$$\|\eta\phi\|_{1,4} \leq C(\|d\phi\|_{0,\frac{4}{3}} + \|\psi\|_{0,4}^2) < 2\sqrt{\varepsilon_0}C.$$

□

Next we estimate ψ.

Lemma 3.4. *There is an $\varepsilon_0 > 0$ such that if $(\phi, \psi) : (D, \delta_{\alpha\beta}) \to (N, g_{ij})$ is a C^∞ Dirac-harmonic map satisfying (3.6), then*

$$\|\psi\|_{D^2,0,q} \leq C(D^2)\|\psi\|_{D,0,4}, \quad \forall q > 1, \quad \forall D^2 \quad \text{with} \quad \overline{D^2} \subset D, \tag{3.13}$$

where $C(D^2) > 0$ is a constant depending only on D^2.

Proof. Choose a cut-off function $\eta : 0 \le \eta \le 1$, with $\eta|_{D^2} \equiv 1$ and Supp$\eta \subset D$. For any $i = 1, 2, \cdots, n$, ψ^i is an ordinary spinor field, and $\xi^i := \eta\psi^i$ has compact support in D, so, by the well-known Lichnerowitz's formula, we have

$$\partial\!\!\!/^2 \xi^i = -\Delta\xi^i + \frac{1}{4}R\xi^i = -\Delta\xi^i$$

because the scalar curvature $R \equiv 0$ on D. Integrating this yields

$$\begin{aligned}\int_D |\nabla\xi^i|^2 &= \int_D |\partial\!\!\!/\xi^i|^2 \\ &= \int_D |\partial\!\!\!/(\eta\psi^i)|^2 \\ &= \int_D |\nabla\eta\cdot\psi^i + \eta\partial\!\!\!/\psi^i|^2 \\ &\le C(\int_D |\psi^i|^2 + \int_D |\partial\!\!\!/\psi^i|^2) \\ &\le C(\int_D |\psi^i|^2 + C\int_D |d\phi|^2|\psi|^2),\end{aligned}$$

hence,

$$\begin{aligned}\|\nabla\xi^i\|_{D,0,2} &\le C(\|\psi^i\|_{D,0,2} + \|d\phi\|^2_{D,0,4}\|\psi\|^2_{D,0,4}) \\ &\le C\|\psi\|_{D,0,4}(1 + \|d\phi\|_{D,0,4}) \\ &\le C'\|\psi\|_{D,0,4},\end{aligned}$$

from which it follows that

$$\|\psi\|_{D^2,0,q} < C\|\psi\|_{D,0,4}.$$

□

Lemma 3.5. *There is an $\varepsilon_0 > 0$ such that if $(\phi, \psi) : (D, \delta_{\alpha\beta}) \to (N, g_{ij})$ is a C^∞ Dirac-harmonic map satisfying (3.6), then*

$$\|d\phi\|_{D^2,0,4} \le C(D^2)\|d\phi\|_{D,0,2}, \quad \forall D^2 \quad \text{with} \quad \overline{D^2} \subset D, \tag{3.14}$$

where $C(D^2) > 0$ is a constant depending only on D^2.

Proof. Choose a cut-off function $\eta : 0 \le \eta \le 1$, with $\eta|_{D^2} \equiv 1$ and Supp$\eta \subset D$. By (3.12), we have

$$\begin{aligned}\|\eta\phi\|_{D,1,4} &\le C(\|d\phi\|_{D,0,\frac{4}{3}} + \|\eta\alpha\|_{D,0,\frac{4}{3}}) \\ &\le C(\|d\phi\|_{D,0,2} + \|\eta\alpha\|_{D,0,\frac{4}{3}}),\end{aligned}$$

and it is clear that

$$\begin{aligned}\|\alpha\|_{D,0,\frac{4}{3}} &\le C\||\psi|^2|d\phi|\|_{D,0,\frac{4}{3}} \\ &\le C(\int_D |\psi|^8)^{\frac{1}{4}}(\int_D |d\phi|^2)^{\frac{1}{2}} \\ &\le C\|\psi\|_{D,0,4}\|d\phi\|_{D,0,2},\end{aligned}$$

therefore,

$$\|\eta\phi\|_{D,1,4} \le C\|d\phi\|_{D,0,2}.$$

□

Proof of Theorem 3.2. Choose $\widetilde{D} \subset D^3 \subset D^2 \subset D^1 \subset D$. Also choose a cut-off function $\eta : 0 \leq \eta \leq 1$, with $\eta|_{D^3} \equiv 1$ and $\text{Supp}\eta \subset D^2$. By (3.11) on D^2 for $p = 2$ (we temporarily assume $\int_{D^2} \phi = 0$):

$$\begin{aligned} \|\eta\phi\|_{D^2,2,2} &\leq C[\|A\|_\infty \|d(\eta\phi)\|_{D^2,0,4}\|d\phi\|_{D^2,0,4} + \|\phi\|_{D^2,1,2} + \||\psi|^2|d\phi|\|_{D^2,0,2}] \\ &\leq C(\|A\|_\infty \|\eta\phi\|_{D^2,1,4}\|d\phi\|_{D^2,0,4} + \|\phi\|_{D^2,1,2} + \|\psi\|^2_{D^2,0,8}\|d\phi\|_{D^2,0,4}). \end{aligned}$$

Since

$$\|\eta\phi\|_{D^2,1,4} \leq C(D^2)\|\eta\phi\|_{D^2,2,\frac{4}{3}} \leq C\|\eta\phi\|_{D^2,2,2},$$

using this and (3.10) we otain

$$\|A\|_\infty \|\eta\phi\|_{D^2,1,4}\|d\phi\|_{D^2,0,4} \leq C'\sqrt{\varepsilon_0}\|A\|_\infty \|\eta\phi\|_{D^2,2,2},$$

which yields

$$\begin{aligned} (1 - CC'\sqrt{\varepsilon_0}\|A\|_\infty)\|\eta\phi\|_{D^2,2,2} &\leq C(\|\phi\|_{D^2,1,2} + \|\psi\|^2_{D^2,0,8}\|d\phi\|_{D^2,0,4}) \\ &\leq C\|\phi\|_{D^2,1,4}, \end{aligned}$$

therefore,

$$\|\phi\|_{D^2,2,2} \leq C\|\phi\|_{D^2,1,4} \leq \|d\phi\|_{D^2,0,4}.$$

By the Sobolev inequality, we have

$$\|d\phi\|_{D^3,0,p} < C\|d\phi\|_{D^2,0,4}, \quad \forall p > 1. \tag{3.15}$$

This also holds for ϕ without $\int_{D^2} \phi = 0$.

Now for $\widetilde{D} \subset D^3$, as above, we choose a cut-off function $\eta : 0 \leq \eta \leq 1$, with $\eta|_{\widetilde{D}} \equiv 1$ and $\text{Supp}\eta \subset D^3$. By (3.11) on D^3 for any $p > 1$ (we again temporarily assume $\int_{D^3} \phi = 0$) we have:

$$\|\eta\phi\|_{D^3,2,p} \leq C[\|A\|_\infty \||d\phi||d(\eta\phi)|\|_{D^3,0,p} + \|\phi\|_{D^3,1,p} + \||\psi|^2|d\phi|\|_{D^3,0,p}].$$

Using (3.15), we obtain:

$$\||d\phi||d(\eta\phi)|\|_{D^3,0,p} \leq (\int_{D^3} |d\phi|^{2p})^{\frac{1}{p}} \leq C\|d\phi\|^2_{D^2,0,4} \leq C\|d\phi\|_{D^2,0,4}$$

and

$$\begin{aligned} \||\psi|^2|d\phi|\|_{D^3,0,p} &= (\int_{D^3} |\psi|^{2p}|d\phi|^p)^{\frac{1}{p}} \\ &\leq (\|\psi\|_{D^3,0,4p})^2\|d\phi\|_{D^3,0,2p} \\ &\leq C\|d\phi\|_{D^2,0,4}, \end{aligned}$$

we conclude that

$$\|\eta\phi\|_{D^3,2,p} \leq C\|d\phi\|_{D^2,0,4},$$

from which and (3.14) it follows that

$$\|\phi\|_{\widetilde{D},2,p} \leq C\|d\phi\|_{D^2,0,4} \leq C\|d\phi\|_{D,0,2}.$$

Clearly, this also holds for ϕ without $\int_{D^3} \phi = 0$. Now let us give the estimates for ψ. By Lemmas 3.4 and 3.5, for $D^1 \subset D$, the following estimates hold

$$\|\psi\|_{D^1,0,p} \leq C\|\psi\|_{D,0,4}, \tag{3.16}$$

$$\|d\phi\|_{D^1,1,p} \leq C\|d\phi\|_{D,0,2}. \tag{3.17}$$

On the other hand, for $D^1 \subset D$, by an argument similar to the one used in the proof of Lemma 3.4, we have

$$\|\nabla\psi^i\|_{D^1,0,2} \leq C\|\psi\|_{D,0,4}. \tag{3.18}$$

Computing directly, one gets

$$\not{D}^2\psi^i = -(\widetilde{\nabla}_{e_\alpha}\widetilde{\nabla}_{e_\alpha}\psi)^i + \frac{1}{2}R^i_{kpj}\nabla\phi^p\cdot\nabla\phi^j\cdot\psi^k,$$

from which we have

$$\begin{aligned}\Delta\psi^i + (\Gamma^i_{jk,p} + \Gamma^i_{pm}\Gamma^m_{jk})\phi^p_\alpha\phi^j_\alpha\psi^k - \frac{1}{2}R^i_{kpj}\nabla\phi^p\cdot\nabla\phi^j\cdot\psi^k \\ +\Gamma^i_{jk}\phi^j_{\alpha\alpha}\psi^k + 2\Gamma^i_{jk}\phi^j_\alpha\nabla_{e_\alpha}\psi^k = 0.\end{aligned} \tag{3.19}$$

For any $\eta \in C^\infty(D,R)$ with $0 \leq \eta \leq 1$, we have

$$|\Delta(\eta\psi)| \leq C(|\psi| + |\nabla\psi| + |d\phi|^2|\psi| + |\nabla^2\phi||\psi| + |d\phi||\nabla\psi|), \tag{3.20}$$

on $D^2 \subset D^1$. Choose a cut-off function $\eta : 0 \leq \eta \leq 1$, with $\eta|_{D^2} \equiv 1$ and $\mathrm{Supp}\eta \subset D^1$. For any $p > 1$ we have

$$\begin{aligned}\|\eta\psi\|_{D^1,2,p} \leq\ & C_p[\|\psi\|_{D^1,0,p} + \|\nabla\psi\|_{D^1,0,p} + \||d\phi|^2|\psi|\|_{D^1,0,p} \\ & +\||\nabla^2\phi||\psi|\|_{D^1,0,p} + \||d\phi||\nabla\psi|\|_{D^1,0,p}].\end{aligned} \tag{3.21}$$

Let $p = \frac{4}{3}$, by (3.16), (3.17) and (3.18), we obtain

$$\begin{aligned}\|\psi\|_{D^2,2,\frac{4}{3}} \leq\ & C[\|\psi\|_{D^1,0,\frac{4}{3}} + \|\nabla\psi\|_{D^1,0,\frac{4}{3}} + \|d\phi\|^2_{D^1,0,4}\|\psi\|_{D^1,0,4} \\ & +\|\nabla^2\phi\|_{D^1,0,2}\|\psi\|_{D^1,0,4} + \|d\phi\|_{D^1,0,4}\|\nabla\psi\|_{D^1,0,2}] \\ \leq\ & C\|\psi\|_{D,0,4},\end{aligned}$$

from which it follows that

$$\|\psi\|_{D^2,1,4} \leq C\|\psi\|_{D,0,4}, \tag{3.22}$$

and consequently,

$$\|\psi\|_{C^0(D^2)} \leq C\|\psi\|_{D,0,4},$$

This proves the second inequality in (3.9).

Now for $D^3 \subset D^2$, choose a cut-off function $\eta : 0 \leq \eta \leq 1$, with $\eta|_{D^3} \equiv 1$ and $\mathrm{Supp}\eta \subset D^2$. By (3.21) on D^2 for $p = 2$ we have:

$$\begin{aligned}\|\eta\psi\|_{D^2,2,2} \leq\ & C[\|\psi\|_{D^2,0,2} + \|\nabla\psi\|_{D^2,0,2} + \||d\phi|^2|\psi|\|_{D^2,0,2} \\ & +\||\nabla^2\phi||\psi|\|_{D^2,0,2} + \||d\phi||\nabla\psi|\|_{D^2,0,2}] \\ \leq\ & C[\|\psi\|_{D^2,0,4} + \|\psi\|_{D,0,4} + \|d\phi\|^2_{D^2,0,8}\|\psi\|_{D^2,0,4} \\ & +\|\nabla^2\phi\|_{D^2,0,4}\|\psi\|_{D^2,0,4} + \|d\phi\|_{D^2,0,4}\|\nabla\psi\|_{D^2,0,4}] \\ \leq\ & C\|\psi\|_{D,0,4},\end{aligned}$$

consequently,

$$\|\psi\|_{D^3,1,p} \leq C\|\psi\|_{D,0,4}, \quad \forall p > 1. \tag{3.23}$$

We again use (3.21) on D^3 for a cut-off function $\eta : 0 \leq \eta \leq 1$, with $\eta|_{\widetilde{D}} \equiv 1$ and $\mathrm{Supp}\eta \subset D^3$. By (3.16) (3.17) (3.18) and (3.23) we have:

$$\|\eta\psi\|_{D^3,2,p} \leq C\|\psi\|_{D,0,4}, \quad \forall p > 1.$$

that is,

$$\|\nabla\psi\|_{\widetilde{D},1,p} \le C(\widetilde{D},p)\|\psi\|_{D,0,4}.$$

We therefore obtain

$$\|\nabla\psi\|_{C^0(\widetilde{D})} \le C(\widetilde{D})\|\psi\|_{D,0,4}.$$

This proves the first inequality in (3.9). □

Theorem 3.6 (Energy identities). *Let $\{(\phi_k,\psi_k) : M \to \mathbb{S}^n\}$ be a sequence of smooth Dirac-harmonic maps with uniform bounded energy:*

$$E(\phi_k,\psi_k) \le \Lambda < +\infty,$$

and assume that $\{(\phi_k,\psi_k)\}$ weakly converges to a Dirac-harmonic map (ϕ,ψ) in $W^{1,2}(M,\mathbb{S}^n) \times L^4(\Sigma M \otimes \mathbb{R}^{n+1})$, then the blow-up set

$$S := \cap_{r>0}\{x \in M | \mathrm{liminf}_{k\to\infty} \int_{Dx,r)} (|d\phi_k|^2 + |\psi_k|^4) \ge \varepsilon_0\}$$

is a finite set of points $\{p_1,p_2,\cdots,p_I\}$, and there are a subsequence, still denoted by $\{(\phi_k,\psi_k)\}$, and a finite set of Dirac-harmonic maps $(\sigma_i^l,\xi_i^l) : \mathbb{S}^2 \to \mathbb{S}^n, i = 1,2,\cdots,I$; $l = 1,2,\cdots,L_i$ such that

$$\mathrm{lim}_{k\to\infty}E(\phi_k) = E(\phi) + \Sigma_{i=1}^I\Sigma_{l=1}^{L_i}E(\sigma_i^l), \tag{3.24}$$

$$\mathrm{lim}_{k\to\infty}E(\psi_k) = E(\psi) + \Sigma_{i=1}^I\Sigma_{l=1}^{L_i}E(\xi_i^l), \tag{3.25}$$

where ε_0 is as in Theorem 3.2, $E(\phi_k) := \int_M |d\phi_k|^2$, $E(\psi_k) := \int_M |\psi_k|^4, E(\phi_k,\psi_k) := \int_M(|d\phi_k|^2 + |\psi_k|^4)$.

Proof. Since $E(\phi_k,\psi_k) \le \Lambda < +\infty$, the blow-up set S must be finite. So we can find small disks D_{δ_i} for each p_i such that $D_{\delta_i} \cap D_{\delta_j} = \emptyset$ for $i \ne j$, $i,j = 1,2,\cdots,I$. On $M \setminus \cup_{i=1}^I D_{\delta_i}$, $\{(\phi_k,\psi_k)\}$ strongly converges to (ϕ,ψ) in $W^{1,2}\times L^4$, so equivalently we need to prove that there are Dirac-harmonic spheres $(\sigma_i^l,\xi_i^l) : \mathbb{S}^2 \to \mathbb{S}^n, i = 1,2,\cdots,I$; $l = 1,2,\cdots,L_i$ such that

$$\Sigma_{i=1}^I\mathrm{lim}_{\delta_i\to 0}\mathrm{lim}_{k\to\infty}E(\phi_k;D_{\delta_i}) = \Sigma_{i=1}^I\Sigma_{l=1}^{L_i}E(\sigma_i^l), \tag{3.26}$$

$$\Sigma_{i=1}^I\mathrm{lim}_{\delta_i\to 0}\mathrm{lim}_{k\to\infty}E(\psi_k;D_{\delta_i}) = \Sigma_{i=1}^I\Sigma_{l=1}^{L_i}E(\xi_i^l). \tag{3.27}$$

In fact, we will prove for each $i = 1,2,\cdots,I$ that there exist Dirac-harmonic spheres $(\sigma_i^l,\xi_i^l) : \mathbb{S}^2 \to \mathbb{S}^n$, $l = 1,2,\cdots,L_i$ such that

$$\mathrm{lim}_{\delta\to 0}\mathrm{lim}_{k\to\infty}E(\phi_k;D_\delta) = \Sigma_{l=1}^L E(\sigma^l), \tag{3.28}$$

$$\mathrm{lim}_{\delta\to 0}\mathrm{lim}_{k\to\infty}E(\psi_k;D_\delta) = \Sigma_{l=1}^L E(\xi^l), \tag{3.29}$$

where, for simplicity, we have dropped the subscript i of D_{δ_i}, L_i etc. and denote p_i by p. Certainly, in each D_δ there is only one blow-up point p.

Let us first prove (3.28) and (3.29) for the case that there is only one bubble at the blow-up point p. Then, we need to prove that there exists Dirac-harmonic sphere $(\sigma^1,\xi^1) : \mathbb{S}^2 \to \mathbb{S}^n$, such that

$$\mathrm{lim}_{\delta\to 0}\mathrm{lim}_{k\to\infty}E(\phi_k;D_\delta) = E(\sigma^1), \tag{3.30}$$

$$\mathrm{lim}_{\delta\to 0}\mathrm{lim}_{k\to\infty}E(\psi_k;D_\delta) = E(\xi^1). \tag{3.31}$$

For each (ϕ_k, ψ_k), we choose λ_k such that

$$\max_{x\in D_\delta(p)} E(\phi_k, \psi_k; D_{\lambda_k}(x)) = \frac{\varepsilon_0}{2},$$

and then choose $x_k \in D_\delta$ such that

$$E(\phi_k, \psi_k; D_{\lambda_k}(x_k)) = \frac{\varepsilon_0}{2}.$$

We may assume that $\lambda_k \to 0$ and $x_k \to p$ as $k \to \infty$. Let

$$\widetilde{\phi}_k(x) := \phi_k(x_k + \lambda_k x), \quad \widetilde{\psi}_k(x) := \lambda_k^{-\frac{1}{2}} \psi_k(x_k + \lambda_k x),$$

then

$$E(\widetilde{\phi}_k, \widetilde{\psi}_k; D_1(0)) = E(\phi_k, \psi_k; D_{\lambda_k}(x_k)) = \frac{\varepsilon_0}{2} < \varepsilon_0,$$

$$E(\widetilde{\phi}_k, \widetilde{\psi}_k; D_K) = E(\phi_k, \psi_k; D_{\lambda_k K}(x_k)) \leq \Lambda,$$

from the $\varepsilon-$regularity theorem Theorem 3.2, we have a subsequence, still denoted by $(\widetilde{\phi}_k, \widetilde{\psi}_k)$, that strongly converges to some $(\widetilde{\phi}, \widetilde{\psi})$ in $W^{1,2}(D_R, N) \times L^4(\Sigma D_R \times \mathbb{R}^K)$ for any $R \geq 1$. Thus, we get a nonconstant Dirac-harmonic map $(\widetilde{\phi}, \widetilde{\psi})$ on $\mathbb{R}^2$, and by stereographic projection, we obtain a nonconstant Dirac-harmonic map (σ^1, ξ^1) on $\mathbb{S}^2 \setminus \{N\}$ with bounded energy. By the regularity theorem, we have a nonconstant Dirac-harmonic map (σ^1, ξ^1) on the whole $\mathbb{S}^2$, this is the first bubble at the blow-up point p.

Let

$$A(\delta, R, k) := \{x \in \mathbb{R}^2 | \lambda_k R \leq |x - x_k| \leq \delta\},$$

then to prove the assertion of (3.30) and (3.31) is equivalent to proving

$$\lim_{R\to\infty} \lim_{\delta\to 0} \lim_{k\to\infty} E(\phi_k; A(\delta, R, k)) = 0, \tag{3.32}$$

$$\lim_{R\to\infty} \lim_{\delta\to 0} \lim_{k\to\infty} E(\psi_k; A(\delta, R, k)) = 0. \tag{3.33}$$

Now let (r_k, θ_k) be the polar coordinate system centered at x_k, $f : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^2$, $f(t, \theta) = (e^{-t}, \theta)$, where $\mathbb{R} \times \mathbb{S}^1$ is given the metric $g = dt^2 + d\theta^2$, which is conformal to ds^2 on $\mathbb{R}^2$: $(f^{-1})^* g = e^{2t} ds^2$. Denote $\Phi_k := f^*\phi_k$, $\Psi_k := f^*\psi_k$, then $E(\Phi_k, \Psi_k) \leq \Lambda$ since the energy functional is conformally invariant. Denote $T_0 := |\log\delta|$, $P_T := [T_0, T_0 + T] \times \mathbb{S}^1$, $T > 0$, then $(\Phi_k, \Psi_k) \to (f^*\phi, f^*\psi) := (\Phi, \Psi)$ strongly on P_T for any $T > 0$, hence

$$E(\Phi_k, \Psi_k; P_T) \to E(\phi, \psi; D_\delta \setminus D_{\delta e^{-T}}), \quad \text{as} \quad k \to \infty.$$

Given any small $\varepsilon > 0$, since $E(\phi, \psi) \leq \Lambda$, we may choose $\delta > 0$ small enough such that $E(\phi, \psi; D_\delta) < \varepsilon/2$, then for any $T > 0$, there is a $k(T)$ such that when $k \geq k(T)$,

$$E(\Phi_k, \Psi_k; P_T) < \frac{\varepsilon}{2}. \tag{3.34}$$

Similarly, denote $T_k := |\log\lambda_k R|$, $Q_{T,k} := [T_k - T, T_k] \times \mathbb{S}^1$, then

$$E(\Phi_k, \Psi_k; Q_{T,k}) < \frac{\varepsilon}{2}, \quad k \geq k(T). \tag{3.35}$$

Now we prove that there is a $K > 0$ such that if $k \geq K$, then

$$\int_{[t,t+1]\times\mathbb{S}^1} (|d\Phi_k|^2 + |\Psi_k|^4) < \varepsilon, \quad \forall t \in [T_0, T_k - 1]. \tag{3.36}$$

Suppose this is false, then there exists a sequence of $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, and

$$\int_{[t_k,t_k+1]\times\mathbb{S}^1} (|d\Phi_k|^2 + |\Psi_k|^4) \geq \varepsilon, \quad \forall t \in [T_0, T_k - 1],$$

because of (3.34) and (3.35), we know that $t_k - T_0, T_k - t_k \to \infty$, by a translation from t to $t - t_k$, we get some $(\widetilde{\Phi}_k, \widetilde{\Psi}_k)$, and for all k the following holds

$$\int_{[0,1]\times\mathbb{S}^1} (|d\widetilde{\Phi}_k|^2 + |\widetilde{\Psi}_k|^4) \geq \varepsilon. \tag{3.37}$$

By our assumption on (ϕ_k, ψ_k), we may assume that $(\widetilde{\Phi}_k, \widetilde{\Psi}_k)$ weakly converges to $(\Phi_\infty, \Psi_\infty)$ in $W^{1,2}_{loc} \times L^4_{loc}(\mathbb{R}\times\mathbb{S}^1)$. If this is strong convergence on $[0,1]\times\mathbb{S}^1$, then we obtain a nonconstant Dirac-harmonic map $(\Phi_\infty, \Psi_\infty)$ on the whole $\mathbb{R}\times\mathbb{S}^1$, and hence a nonconstant Dirac-harmonic map $(\sigma_\infty, \xi_\infty)$ on $\mathbb{S}^2/\{N, S\}$ with finite energy. Again by the regularity theorem, we have a nonconstant Dirac-harmonic map $(\sigma_\infty, \xi_\infty)$ on $\mathbb{S}^2$, a contradiction to the assumption that $L = 1$. On the other hand, if $(\widetilde{\Phi}_k, \widetilde{\Psi}_k)$ does not strongly converge to $(\Phi_\infty, \Psi_\infty)$ on $[0,1]\times\mathbb{S}^1$, then we may find some point $(t_0, \theta_0) \in [0,1]\times\mathbb{S}^1$ at which $\{(\Phi_k, \Psi_k)\}$ blows up, in this case, we can still get a second nonconstant Dirac-harmonic map $(\sigma_\infty, \xi_\infty)$ on $\mathbb{S}^2$, again contradicting $L = 1$. Therefore, (3.36) holds true.

Now from (3.36) and the $\varepsilon-$regularity theorem, we have

$$||d\Phi_k||_{L^\infty([t,t+1]\times\mathbb{S}^1)} \leq C\sqrt{\varepsilon},$$

where $C > 0$ is a constant independent of t. Back to $\mathbb{R}^2$, we have

$$|d\phi_k|(x) \leq \frac{C\sqrt{\varepsilon}}{r}, \quad r = r(x) = |x|, \quad x \in A(\delta, R, k), \tag{3.38}$$

from which we can conclude that

$$||d\phi_k||_{L^{(2,\infty)}(A(\delta,R,k))} \leq C\sqrt{\varepsilon}, \tag{3.39}$$

where $||\cdot||_{L^{(2,\infty)}(A(\delta,R,k))}$ denotes the norm in $L^{(2,\infty)}$ (cf. [7]).

On the other hand, we know that (for the target $\mathbb{S}^n$):

$$\begin{aligned}\Delta\phi^m &= [(\phi^i_1\phi^m - \phi^i\phi^m_1) - \langle e_1 \cdot \psi^i, \psi^m\rangle_{\Sigma M}\phi^i_1] \\ &\quad +[(\phi^i_2\phi^m - \phi^i\phi^m_2) - \langle e_2 \cdot \psi^i, \psi^m\rangle_{\Sigma M}\phi^i_2]\end{aligned}$$

belongs to the Hardy space $\mathbb{H}^1$, so we have

$$||d\phi_k||_{L^{(2,1)}(A(\delta,R,k))} \leq C, \tag{3.40}$$

where $||\cdot||_{L^{(2,1)}(A(\delta,R,k))}$ denotes the norm in the $L^{(2,1)}$ space. Therefore, by the duality of $L^{(2,1)}$ and $L^{(2,\infty)}$, (3.39) and (3.40) we have

$$\int_{A(\delta,R,k)} |d\phi_k|^2 \leq ||d\phi_k||^2_{L^{(2,\infty)}(A(\delta,R,k))}||d\phi_k||^2_{L^{(2,1)}(A(\delta,R,k))} \leq C\varepsilon,$$

that is, for k, R large enough, δ small enough,

$$E(\phi_k; A(\delta, R, k)) < C\varepsilon,$$

which proves (3.32).

Now we turn to the proof of (3.33). Choose a cut-off function η on $D(x_k, 2\delta)$ as follows:

$$\eta \in C_0^\infty(D_{2\delta} \setminus D_{\lambda_k R/2}); \qquad \eta \equiv 1 \quad \text{in} \quad D_\delta \setminus D_{\lambda_k R}$$
$$|\nabla\eta| \leq C/\delta \quad \text{in} \quad D_{2\delta} \setminus D_\delta; \qquad |\nabla\eta| \leq C/\lambda_k R \quad \text{in} \quad D_{\lambda_k R} \setminus D_{\lambda_k R/2},$$

where we denote $D_\delta := D(x_k, \delta)$ etc. for simplicity. Then from $\partial\!\!\!/(\eta\psi_k) = \eta\partial\!\!\!/\psi_k + \nabla\eta \cdot \psi_k$, we have

$$\begin{aligned}
\|\eta\psi_k\|_{L^4} &\leq C\|\eta\partial\!\!\!/\psi_k + \nabla\eta \cdot \psi_k\|_{L^{\frac{4}{3}}} \\
&\leq C\||\eta||d\phi_k||\psi_k| + |\nabla\eta||\psi_k|\|_{L^{\frac{4}{3}}} \\
&\leq C\|d\phi_k\|_{L^2(A(2\delta,R/2,k))}\|\psi_k\|_{L^4(A(2\delta,R/2,k))} + C[\int_{A(2\delta,R/2,k)} (|\nabla\eta||\psi_k|)^{\frac{4}{3}}]^{\frac{3}{4}} \\
&\leq C\sqrt{\varepsilon} + [\int_{D_{2\delta}\setminus D_\delta} (|\nabla\eta||\psi_k|)^{\frac{4}{3}}]^{\frac{3}{4}} + [\int_{D_{\lambda_k R}\setminus D_{\lambda_k R/2}} (|\nabla\eta||\psi_k|)^{\frac{4}{3}}]^{\frac{3}{4}},
\end{aligned}$$

where in the last line we used (3.32). By the definition of η we have

$$\begin{aligned}
\|\psi_k\|_{L^4(A(\delta,R,k))} &\leq C\sqrt{\varepsilon} + C[\int_{D_{2\delta}\setminus D_\delta} |\psi_k|^4]^{\frac{1}{4}} + C[\int_{D_{\lambda_k R}\setminus D_{\lambda_k R/2}} |\psi_k|^4]^{\frac{1}{4}} \\
&\leq C\sqrt{\varepsilon} + C\varepsilon^{\frac{1}{4}} + C\varepsilon^{\frac{1}{4}},
\end{aligned}$$

where in the last step, we used (3.36). This proves (3.33). Therefore, the energy identities holds true for the case $L = 1$.

For a fixed blow-up point p, the number L of bubbles (σ, ξ) must be finite. This follows easily from the fact that there is a number $C(\mathbb{S}^n) > 0$ such that for all nonconstant Dirac-harmonic maps $(\sigma, \xi) : \mathbb{S}^2 \to \mathbb{S}^n$, we have

$$E(\sigma, \xi; \mathbb{S}^2) \geq C(\mathbb{S}^n).$$

In fact, by Theorem 3.1, there is a constant $\varepsilon_0 > 0$ such that if $E(\sigma, \xi; \mathbb{S}^2) < \varepsilon_0$, then $\sigma \equiv$ *const.* and ψ satisfies the Dirac equation $\partial\!\!\!/\psi = 0$ on $\mathbb{S}^2$ which implies that $\psi \equiv 0$. We therefore have $E(\sigma, \xi; \mathbb{S}^2) \geq \varepsilon_0$ for all nonconstant Dirac-harmonic maps $(\sigma, \xi) : \mathbb{S}^2 \to \mathbb{S}^n$.

The case of $L > 1$ can be proved by induction on the number L, we omit the details, as one may see the argument in [4]. □

Remark. From the proof we see that at each blow-up point p_i $(i = 1, 2, \cdots, I)$, the Dirac-harmonic maps $(\sigma_i^l, \xi_i^l) : \mathbb{S}^2 \to \mathbb{S}^n, l = 1, 2, \cdots, L_i$ in Theorem 3.6 come from the blow-up process at p_i.

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