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by

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MEAN CURVATURE FLOW WITH FLAT NORMAL BUNDLES

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Abstract. We show that flatness of the normal bundle is preserved under the mean curvature flow in $\mathbb{R}^n$ and use this to generalize a classical result for hypersurfaces due to Ecker & Huisken [3] in the case of submanifolds with arbitrary codimension.

1. Introduction

Let us consider immersions

$$F : M^m \to \mathbb{R}^n$$

of an m-dimensional submanifold in $\mathbb{R}^n$ of codimension k. Throughout this paper we shall assume that there is a one-parameter family $F_t = F(\cdot, t)$ of immersions with corresponding images $M_t = F_t(M)$ such that *mean curvature flow*

$$\begin{aligned} \frac{d}{dt}F(x,t) &= H(x,t)\,, \quad x \in M \\ F(x,0) &= F_0(x) \end{aligned} \tag{1}$$

is satisfied for some initial data F_0. Here, $H(x,t)$ is the mean curvature vector of M_t at the point $x \in M$, i.e. H is the trace of the second fundamental form

$$A = \nabla dF.$$

The mean curvature flow has been studied intensively by many authors. Most of the results have been obtained for hypersurfaces. A classical result is due to Ecker and Huisken [3], where they study hypersurfaces in $\mathbb{R}^{n+1}$ that can be represented as entire graphs over a flat plane. Their result says that any polynomial growth rate for the height and the gradient of the initial surface M_0 is preserved during the evolution and that in case of Lipschitz initial data with linear growth, (1) has a

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smooth solution for all times $t > 0$. The growth condition was

$$v = \frac{1}{\langle \nu, \omega \rangle} \leq \text{const},$$

where $\omega \in \mathbb{R}^{n+1}$, $|\omega| = 1$ and ν is some choice of unit normal vector such that $\langle \nu, \omega \rangle > 0$. In addition, they proved that these hypersurfaces approach a *selfsimilar expanding* solution of (1) as $t \to \infty$ provided the initial graph was "straight" at infinity.

In higher codimension, the general picture of mean curvature flow is still incomplete even though some work has been carried out by Chen, Li [1], Wang [13], [14] and one of the authors [9]. E.g. based on the interior estimates for hypersurfaces obtained by Ecker and Huisken in [4], Wang [14] proved that any initial compact submanifold that satisfies a local K-Lipschitz condition admits a smooth solution on some time interval $(0, T), T > 0$.

In the case of graphical mean curvature flow in higher codimension there are some longtime existence and convergence results to flat spaces. In [13] Wang defined a similar expression as the above mentioned quantity v which was essential to get the a-priori estimates needed in the longtime existence and convergence results. However, in that paper the author had to assume a smallness condition on v and the theorem did not apply to arbitrary graphs. In [9] it was shown that Lagrangian graphs with convex potentials admit smooth solutions for all times and that the solutions exponentially converge to flat Lagrangian planes.

So, for some time it was unclear how to generalize the results in [3] in the "best" way to the case of arbitrary codimension. Motivated by a recent paper of the third author [15], we believe that the flatness of the normal bundle is the key ingredient to get the convergence results for arbitrary graphs in higher codimension. Note, that trivially the normal bundle of any hypersurface is flat. So our result is a natural extension of the results by Ecker and Huisken.

There are many examples of submanifolds with flat normal bundles. Hypersurfaces are trivial examples. Other trivial examples are curves in $\mathbb{R}^n$. In addition, any submanifold of codimension 2 in $\mathbb{R}^n$ which also is a hypersurface of the standard sphere S^{n-1} must have a flat normal bundle. For more examples, see [2], [10], [11] and [12], where a theory of isoparametric submanifolds was established in the framework of flat normal bundles.

The organization of the paper is as follows: In section 2 we recall the monotonicity formula and the noncompact maximum principle by Ecker and Huisken and introduce the class of solutions for which our results will apply. Section 3 introduces our notation and recalls the most important structure equations in the geometry of real submanifolds. Some basic evolution equations for the mean curvature flow are derived resp. recalled in section 4. In section 5 we prove that polynomial growth rates are preserved in arbitrary codimension and that these growth estimates can be applied even to non-graphical submanifolds, like cylinders. The core of our article is the proof of Theorem 1, i.e. that flatness of the normal bundle is preserved. This will be done in section 6. In the remainder - based on this fundamental observation - we can proceed basically as in [3], to carry over the results by Ecker and Huisken to the case of arbitrary codimension.

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2. The class of solutions

Throughout this article we shall assume that the solutions of (1) that we consider will belong to the class of solutions for which we can apply Huisken's *monotonicity formula*

Proposition 1. *(Huisken) For a function $f(x,t)$ on M we have*

$$\frac{d}{dt}\int_M f\rho d\mu_t = \int_M \left(\frac{d}{dt}f - \Delta f\right)\rho d\mu_t - \int_M f\rho\left|H + \frac{F^\perp}{2(t_0-t)}\right|^2 d\mu_t, \tag{2}$$

where

$$\rho(y,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}}\, e^{-\frac{|y|^2}{4(t_0-t)}}$$

is the backward heat kernel on $\mathbb{R}^n$ at the origin, and $t_0 > t$.

In particular, we will assume that integration by parts is permitted and all integrals are finite for the submanifolds and functions we will consider in the sequel. E.g., this is the case for those smooth solutions of (1) for which the curvature and its covariant derivatives have at most polynomial growth at infinity since then the faster exponential decay rate of the heat kernel ρ yields finite integrals.

As in [3] we will repeatedly use the following *maximum principle* which is based on the monotonicity formula.

Corollary 1. *(Ecker and Huisken* [3]*) Suppose the function $f = f(x,t)$ satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta\right) f \le \langle \mathbf{a}, \nabla f\rangle$$

for some **a** *which is uniformly bounded on* $M \times [0, t_1]$ *for some* $t_1 > 0$*, then*

$$\sup_{M_t} f \leq \sup_{M_0} f$$

for all $t \in [0, t_1]$.

3. Geometric quantities of immersions

Let $F : M^m \to \mathbb{R}^n$ be an immersion and let k be the codimension of M, i.e. $n = k + m$. We let $(x^i)_{i=1,\ldots,m}$ denote local coordinates on M and we will always use *cartesian* coordinates $(y^\alpha)_{\alpha=1,\ldots,n}$ on $\mathbb{R}^n$. Doubled greek and latin indices are summed form 1 to n resp. from 1 to m. In local coordinates the *differential* dF of F is given by

$$dF = F_i^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i,$$

where $F^\alpha = y^\alpha(F)$ and $F_i^\alpha = \frac{\partial F^\alpha}{\partial x^i}$. The coefficients of the induced metric $g_{ij}\, dx^i \otimes dx^j$ are

$$g_{ij} = \langle F_i, F_j \rangle = g_{\alpha\beta} F_i^\alpha F_j^\beta,$$

where $g_{\alpha\beta} = \delta_{\alpha\beta}$ is the euclidean metric in cartesian coordinates. As usual, the Christoffel symbols are

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

The second fundmental form is defined by

$$A = \nabla dF := A_{ij}^\alpha \frac{\partial}{\partial y^\alpha} \otimes dx^i \otimes dx^j.$$

Here and in the following all canonically induced full connections on bundles over M will be denoted by ∇. Later, we will occasionally also use the connection on the normal bundle which will then be denoted by $\nabla^\perp$. It is easy to check that in cartesian coordinates on $\mathbb{R}^n$ we have

$$A_{ij}^\alpha = F_{ij}^\alpha - \Gamma_{ij}^k F_k^\alpha,$$

where $F_{ij}^\alpha = \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j}$. By definition, A is a section in $F^{-1}T\mathbb{R}^n \otimes T^*M \otimes T^*M$ and it can be easily checked that A is normal, i.e. that

$$A \in \Gamma\left(NM \otimes T^*M \otimes T^*M\right),$$

where NM denotes the normal bundle of M w.r.t. the immersion F. This means that

$$g_{\alpha\beta} A_{ij}^\alpha F_k^\beta = 0, \quad \forall\, i, j, k. \tag{3}$$

In particular, the *mean curvature vector field* $H = H^\alpha \frac{\partial}{\partial y^\alpha}$ with $H^\alpha = g^{ij} A^\alpha_{ij}$ satisfies

$$g_{\alpha\beta} H^\alpha F^\beta_j = 0, \quad \forall\, j.$$

The curvature of the normal bundle is defined locally by $R^{\alpha\beta}{}_{ij} \frac{\partial}{\partial y^\alpha} \otimes \frac{\partial}{\partial y^\beta} \otimes dx^i \otimes dx^j$, where by *Ricci's equation*

$$R^{\alpha\beta}{}_{ij} = A^\alpha_{is} A^{\beta s}_j - A^\alpha_{js} A^{\beta s}_i. \tag{4}$$

The normal bundle is flat if and only if $R^{\alpha\beta}{}_{ij}$ vanishes for any α, β, i and j. In addition we have the *Gauß equations* for the induced curvature tensor on M

$$R_{ijkl} = g_{\alpha\beta}(A^\alpha_{ik} A^\beta_{jl} - A^\alpha_{il} A^\beta_{jk}). \tag{5}$$

Let us also recall the *Codazzi equations*

$$\nabla_i A^\alpha_{jk} = \nabla_j A^\alpha_{ik} + F^\alpha_l R^l{}_{kji}, \tag{6}$$

$$\nabla^k A^\alpha_{jk} = \nabla_j H^\alpha + F^\alpha_l R^l_j. \tag{7}$$

The following rule for interchanging derivatives

$$\nabla_i \nabla_j A^\alpha_{lk} = \nabla_j \nabla_i A^\alpha_{lk} - R^m{}_{lij} A^\alpha_{mk} - R^m{}_{kij} A^\alpha_{lm} \tag{8}$$

and the second Bianchi identity together with the Codazzi equations imply the *Simons' identity*

$$\begin{aligned} \Delta A^\alpha_{lk} &= \nabla_l \nabla_k H^\alpha + R^m_l A^\alpha_{mk} + R^m_k A^\alpha_{ml} - 2A^\alpha_{jm} R_l{}^j{}_k{}^m \\ &\quad + F^\alpha_m (\nabla_l R^m_k + \nabla_k R^m_l - \nabla^m R_{lk}), \end{aligned} \tag{9}$$

where R_{ij} denotes the Ricci curvature of M.

4. Evolution equations

¿From the main evolution equation

$$\frac{d}{dt} F = H$$

we obtain

$$\frac{d}{dt} F^\alpha_i = \nabla_i H^\alpha.$$

Let us define the symmetric tensors

$$a_{ij} := g_{\alpha\beta}H^\alpha A^\beta_{ij}\,, \quad b_{ij} := g_{\alpha\beta}A^\alpha_{ik}A^{\beta k}_j,$$

so that by Gauß' equation the Ricci tensor satisfies

$$R_{ij} = a_{ij} - b_{ij}\,. \tag{10}$$

¿From $g_{\alpha\beta}F^\alpha_i H^\beta = 0$ we derive the evolution equation for the induced metric g_{ij}

$$\frac{d}{dt}g_{ij} = 2g_{\alpha\beta}\nabla_i H^\alpha F^\beta_j = -2g_{\alpha\beta}H^\alpha A^\beta_{ij} = -2a_{ij}. \tag{11}$$

Consequently, the volume form $d\mu$ on M satisfies

$$\frac{d}{dt}\,d\mu = -|H|^2 d\mu. \tag{12}$$

We need to compute the evolution equation for A^α_{ij}. In a first step we get

$$\begin{aligned}\frac{d}{dt}A^a_{ij} &= \frac{d}{dt}F^\alpha_{ij} - F^\alpha_k\frac{d}{dt}\Gamma^k_{ij} - \Gamma^k_{ij}\frac{d}{dt}F^\alpha_k.\\ &= \nabla_i\nabla_j H^\alpha - \frac{d}{dt}\Gamma^k_{ij}F^\alpha_k.\end{aligned}$$

The evolution equation for Γ^α_{ij} is

$$\frac{d}{dt}\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\nabla_i\frac{d}{dt}g_{lj} + \nabla_j\frac{d}{dt}g_{li} - \nabla_l\frac{d}{dt}g_{ij}).$$

The last two equations, the Simons' identity and (10), (11) imply

$$\begin{aligned}\frac{d}{dt}A^\alpha_{ij} &= \Delta A^\alpha_{ij} - R^m_i A^\alpha_{mj} - R^m_j A^\alpha_{mi} + 2A^\alpha_{mn}R^{m\ n}_{\ i\ j}\\ &\quad -F^\alpha_m(\nabla_i b^m_j + \nabla_j b^m_i - \nabla^m b_{ij})\end{aligned} \tag{13}$$

In addition

$$\begin{aligned}\frac{d}{dt}H^\alpha &= g^{ij}\frac{d}{dt}A^\alpha_{ij} + 2a^{ij}A^\alpha_{ij}\\ &= g^{ij}\left(\nabla_i\nabla_j H^\alpha - \frac{d}{dt}\Gamma^k_{ij}F^\alpha_k\right) + 2a^{ij}A^\alpha_{ij}\\ &= \Delta H^\alpha - g^{ij}\frac{d}{dt}\Gamma^k_{ij}F^\alpha_k + 2a^{ij}A^\alpha_{ij}\end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}|H|^2 &= 2H_\alpha\frac{d}{dt}H^\alpha\\ &= 2H_\alpha\Delta H^\alpha + 4|a_{ij}|^2,\end{aligned}$$

for $F_k^\alpha H_\alpha = 0$. Thus

$$\frac{d}{dt}|H|^2 = \Delta|H|^2 - 2|\nabla_i H^\alpha|^2 + 4|a_{ij}|^2.$$

Since $H_\alpha F_i^\alpha = 0$ we conclude that

$$\nabla_i H_\beta F_j^\beta F_l^\alpha g^{jl} = -a_i^l F_l^\alpha$$

so that

$$|\nabla_i H^\alpha|^2 = |\nabla_i H^\alpha + a_i^l F_l^\alpha|^2 + |a_{ij}|^2.$$

Hence

$$\frac{d}{dt}|H|^2 = \Delta|H|^2 - 2|\nabla_i H^\alpha + a_i^l F_l^\alpha|^2 + 2|a_{ij}|^2. \tag{14}$$

Let $\nabla^\perp$ denote the normal connection induced from the immersion F. Then $|\nabla^\perp H|^2 = |\nabla_i H^\alpha + a_i^l F_l^\alpha|^2$.

Remark 1. *For a hypersurface with inward unit normal vector ν, scalar mean curvature H and second fundamental tensor h_{ij} we have $A_{ij}^\alpha = h_{ij}\nu^\alpha, H^\alpha = H\nu^\alpha$ and $a_{ij} = Hh_{ij}, \nabla_i H^\alpha = \nabla_i H \nu^\alpha - a_i^l F_l^\alpha$ so that $-2|\nabla_i H^\alpha|^2 = -2|\nabla H|^2 - 2|a_{ij}|^2$ and $|a_{ij}|^2 = H^2|A|^2$.*

Now we compute the evolution equation for $|A|^2$. From (11), (13) and the normality of A we deduce

$$\begin{aligned}
\frac{d}{dt}|A|^2 &= 4a^{ij}b_{ij} + 2A_\alpha^{ij}\frac{d}{dt}A_{ij}^\alpha \\
&= 4a^{ij}b_{ij} + 2A_\alpha^{ij}\left(\Delta A_{ij}^\alpha - R_i^m A_{mj}^\alpha - R_j^m A_{mi}^\alpha + 2A_{mn}^\alpha R^m{}_i{}^n{}_j\right) \\
&= \Delta|A|^2 - 2|\nabla_l A_{ij}^\alpha|^2 + 4|b_{ij}|^2 + 4A_\alpha^{ij}A_{mn}^\alpha R^m{}_i{}^n{}_j. \qquad (15)
\end{aligned}$$

Since

$$4|b_{ij}|^2 + 4A_\alpha^{ij}A_{mn}^\alpha R^m{}_i{}^n{}_j = 2|R^{\alpha\beta}{}_{ij}|^2 + 4|A_n^{\alpha m}A_m^{\beta n}|^2$$

and the tangential part of $\nabla_l A_{ij}^\alpha$ is given by $-A_{ij}^\beta A_{\beta l}^k F_k^\alpha$ we conclude

$$|\nabla_l A_{ij}^\alpha|^2 = |\nabla_l A_{ij}^\alpha + A_{ij}^\beta A_{\beta l}^k F_k^\alpha|^2 + |A_n^{\alpha m}A_m^{\beta n}|^2$$

and finally

$$\frac{d}{dt}|A|^2 = \Delta|A|^2 - 2|\nabla_l A_{ij}^\alpha + A_{ij}^\beta A_{\beta l}^k F_k^\alpha|^2 + 2|A_n^{\alpha m}A_m^{\beta n}|^2 + 2|R^{\alpha\beta}{}_{ij}|^2. \tag{16}$$

Here $|\nabla_l A_{ij}^\alpha + A_{ij}^\beta A_{\beta l}^k F_k^\alpha|^2 = |\nabla^\perp A|^2$.

5. Growth estimates

In this section we will prove that polynomial growth estimates for submanifolds in $\mathbb{R}^n$ are preserved under the mean curvature flow. Note, that in this section we do not require that M_0 or M_t can be written as graphs over some flat m-plane. We will use a flat ℓ-plane merely as a reference submanifold to measure distances. As a consequence, we will also obtain growth estimates for other objects as graphs, e.g. for cylindrical objects as depicted in figure 1.

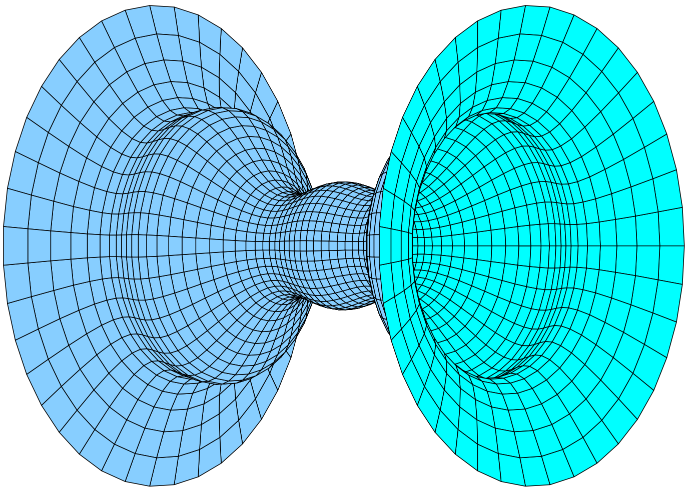

Figure 1. *Growth rates of cylindrical surfaces are preserved. Here, the surface is given by* $(x,\phi) \mapsto (u_1,u_2,u_3) = \big(x,(1+|x|)\cos x\cos\phi,(1+|x|)\cos x\sin\phi\big)$. *Since* $u^2 := u_2^2+u_3^2 = (1+|x|)^2\cos^2 x \le 2(1+x^2)$ *the surface grows linearly over the axis of rotation. We have* $\ell=1$, $m=2$ *and* $n=3$.

Choose an orthonormal basis $e_1,\dots,e_n$ of $\mathbb{R}^n$ and let $1\le\ell\le n$ be an integer (for later purposes we have in mind $\ell=m$ but here in general ℓ and m can be different). For an immersion $F: M^m\to\mathbb{R}^n$ we define n *coordinate functions*

$$u_i := \langle F, e_i\rangle\,,\quad i=1,\dots,n$$

In addition we define

$$x := \sqrt{\sum_{i=1}^{\ell} u_i^2}\,, \quad u := \sqrt{\sum_{i=\ell+1}^{n} u_i^2}$$

such that $|F|^2 = x^2 + u^2$. Since $\left(\frac{d}{dt} - \Delta\right) u_i = 0$ we conclude

$$\frac{d}{dt}|F|^2 = \Delta|F|^2 - 2m, \tag{17}$$

$$\frac{d}{dt}u^2 = \Delta u^2 - 2\sum_{i=\ell+1}^{n} |\nabla u_i|^2, \tag{18}$$

$$\frac{d}{dt}x^2 = \Delta x^2 - 2m + 2\sum_{i=\ell+1}^{n} |\nabla u_i|^2, \tag{19}$$

since

$$\sum_{i=1}^{n} |\nabla u_i|^2 = m.$$

For a constant $c > 0$ to be determined later, we define the function

$$\eta := 1 + ct + x^2.$$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function with $\varphi' \leq 0$, $\varphi'' \geq 0$. We want to compute the evolution equation of $u^2\varphi(\eta)$.

$$\begin{aligned} \frac{d}{dt}\left(u^2\varphi\right) &= \varphi\left(\Delta u^2 - 2\sum_{i=\ell+1}^{n}|\nabla u_i|^2\right) + u^2\varphi'\left(\Delta\eta + 2\sum_{i=\ell+1}^{n}|\nabla u_i|^2 + c - 2m\right) \\ &= \Delta\left(u^2\varphi\right) - 2\varphi'\langle\nabla\eta, \nabla u^2\rangle - \varphi'' u^2|\nabla\eta|^2 - 2(\varphi - u^2\varphi')\sum_{i=\ell+1}^{n}|\nabla u_i|^2 \\ &\quad +(c-2m)u^2\varphi'. \end{aligned}$$

If at some point on M_t we have $u^2 = 0$, then at such a point

$$\frac{d}{dt}\left(u^2\varphi\right) \leq \Delta\left(u^2\varphi\right).$$

We want to prove that this inequality holds at all points on M_t. At those points, where $\varphi' = 0$ we are done as well. So w.l.o.g. we can assume that $u \neq 0$ and $\varphi' \neq 0$. Next observe that

$$|\nabla u^2|^2 \leq 4u^2\sum_{i=\ell+1}^{n}|\nabla u_i|^2$$

implies

$$|\nabla u|^2 \leq \sum_{i=\ell+1}^{n} |\nabla u_i|^2$$

at all points, where $u \neq 0$. We use Schwarz' inequality to estimate

$$-2\varphi'\langle\nabla\eta, \nabla u^2\rangle \leq -2\varphi'(\varepsilon u^2|\nabla\eta|^2 + \frac{1}{\varepsilon}|\nabla u|^2) \leq -2\varepsilon\varphi' u^2|\nabla\eta|^2 - \frac{2}{\varepsilon}\varphi' \sum_{i=\ell+1}^{n} |\nabla u_i|^2,$$

where ε is some positive constant. We choose $\varepsilon = -\frac{\varphi'}{\varphi}$ and get

$$\frac{d}{dt}\left(u^2\varphi\right) \leq \Delta\left(u^2\varphi\right) + \left(\frac{2(\varphi')^2}{\varphi} - \varphi''\right) u^2|\nabla\eta|^2 + (c-2m)u^2\varphi'.$$

As in [3] we have

$$\nabla_l\eta = 2\left\langle F_l, F - \sum_{i=\ell+1}^{n} u_i e_i \right\rangle,$$

and therefore

$$|\nabla\eta|^2 \leq 4|F - \sum_{i=\ell+1}^{n} u_i e_i|^2 = 4x^2 \leq 4\eta,$$

so that always

$$\frac{d}{dt}\left(u^2\varphi\right) \leq \Delta\left(u^2\varphi\right) + u^2\left\{4\eta\left(\frac{2(\varphi')^2}{\varphi} - \varphi''\right) + (c-2m)\varphi'\right\}. \tag{20}$$

Proposition 2. *If for some $c_0 < \infty, p \geq 0$, the inequality*

$$u^2 \leq c_0\left(1 + |F|^2 - u^2\right)^p$$

is satisfied on M_0, then for all $t > 0$,

$$u^2 \leq c_0\left(1 + |F|^2 - u^2 + \big(2m + 4(p-1)\big)t\right)^p.$$

Proof. We choose $\varphi(\eta) = \eta^{-p}$ and $c = 2m + 4(p-1)$. Then $\varphi' = -p\eta^{-p-1}$, $\varphi'' = p(p+1)\eta^{-p-2}$. Inserting this into (20) gives

$$\begin{aligned}\frac{d}{dt}\left(u^2\varphi\right) &\leq \Delta\left(u^2\varphi\right) + \frac{u^2}{\eta}\left(8p^2 - 4p(p+1) - p(c-2m)\right)\\ &= \Delta\left(u^2\varphi\right).\end{aligned}$$

and the result follows from Corollary 1. ∎

6. Preserving flatness of the normal bundle

In this section we will prove that flatness of the normal bundle is preserved. We do not require that M is a graph nor do we assume compactness or completeness. The theorem can be applied to any smooth solution of the mean curvature flow for which M_0 has a flat normal bundle.

Theorem 1. *Let $F : M \times [0,T) \to \mathbb{R}^n$ be a smooth solution of the mean curvature flow and assume that the normal bundle of M_0 is flat. If $|A|^2$ is bounded on each M_t then the normal bundle of M_t is flat as well.*

Note that we do not require that $|A|^2$ is uniformly bounded in t.

Proof. For the proof of this theorem we have to compute the evolution equation of the normal curvature tensor $R^{\alpha\beta}{}_{ij}$. We will show that the squared normal curvature tensor $R^\perp$ satisfies an evolution equation of the form

$$\frac{d}{dt}|R^\perp|^2 = \Delta|R^\perp|^2 - 2|\nabla R^\perp|^2 + A * A * R^\perp * R^\perp, \tag{21}$$

where the last term is a contraction of a term quadratic in A and one which is quadratic in $R^\perp$. Then, by assumption on $|A|^2$, on a compact time interval $[0, t_1]$ we can choose a constant c (depending on t_1) such that

$$\frac{d}{dt}|R^\perp|^2 \le \Delta|R^\perp|^2 - 2|\nabla R^\perp|^2 + c|R^\perp|^2$$

and the function $f := e^{-ct}|R^\perp|^2$ satisfies

$$\frac{d}{dt}f \le \Delta f$$

on $[0, t_1]$. The result then follows from Corollary 1.

It remains to derive the evolution equation for $|R^\perp|^2$. It turns out that the computation is rather complicated, since a number of symmetries of the curvature tensor and the second fundamental form have to be used.

The first observation is

$$|R^\perp|^2 = |R^{\alpha\beta}{}_{ij}|^2 = 2|b_{ij}|^2 - 2c^{\alpha\beta}_{ij}c^{ij}_{\beta\alpha},$$

where

$$c^{\alpha\beta}_{ij} = A^\alpha_{ik}A^{\beta k}_j.$$

$$\begin{aligned}
\frac{d}{dt}b_{ij} &= 2a^{kl}A_{\alpha ik}A^{\alpha}_{lj} + \frac{d}{dt}A^{\alpha}_{kj}A^{k}_{\alpha i} + \frac{d}{dt}A^{\alpha}_{ki}A^{k}_{\alpha j} \\
&= 2a^{kl}A_{\alpha ik}A^{\alpha}_{lj} \\
&\quad + A^{k}_{\alpha j}(\Delta A^{\alpha}_{ik} - R^{m}_{i}A^{\alpha}_{mk} - R^{m}_{k}A^{\alpha}_{mi} + 2A^{\alpha}_{mn}R^{m}{}_{i}{}^{n}{}_{k}) \\
&\quad + A^{k}_{\alpha i}(\Delta A^{\alpha}_{jk} - R^{m}_{j}A^{\alpha}_{mk} - R^{m}_{k}A^{\alpha}_{mj} + 2A^{\alpha}_{mn}R^{m}{}_{j}{}^{n}{}_{k})
\end{aligned}$$

and then

$$\begin{aligned}
\frac{d}{dt}b_{ij} &= 2a^{kl}A_{\alpha ik}A^{\alpha}_{lj} + \Delta b_{ij} \\
&\quad -\nabla^{l}A^{k}_{\alpha j}\nabla_{l}A^{\alpha}_{ik} - R^{m}_{i}b_{jm} - R^{m}_{k}A^{k}_{\alpha j}A^{\alpha}_{mi} + 2A^{k}_{\alpha j}A^{\alpha}_{mn}R^{m}{}_{i}{}^{n}{}_{k} \\
&\quad -\nabla^{l}A^{k}_{\alpha i}\nabla_{l}A^{\alpha}_{jk} - R^{m}_{j}b_{im} - R^{m}_{k}A^{k}_{\alpha i}A^{\alpha}_{mj} + 2A^{k}_{\alpha i}A^{\alpha}_{mn}R^{m}{}_{j}{}^{n}{}_{k}.
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{d}{dt}|b_{ij}|^2 &= 4a^{ik}b_{il}b^{l}_{k} + 2b^{ij}\frac{d}{dt}b_{ij} \\
&= 4a^{ik}b_{il}b^{l}_{k} + 4a^{kl}A^{\alpha}_{ik}A_{\alpha lj}b^{ij} + 2b^{ij}(\Delta b_{ij} - 2\nabla^{l}A^{k}_{\alpha j}\nabla_{l}A^{\alpha}_{ik}) \\
&\quad + 4b^{ij}(-R^{m}_{i}b_{jm} - R^{m}_{k}A^{k}_{\alpha j}A^{\alpha}_{mi} + 2A^{k}_{\alpha j}A^{\alpha}_{mn}R^{m}{}_{i}{}^{n}{}_{k})
\end{aligned}$$

and from $R_{ij} = a_{ij} - b_{ij}$

$$\begin{aligned}
\frac{d}{dt}|b_{ij}|^2 &= 4b^{ij}b^{m}_{i}b_{jm} + 4b^{ij}b^{m}_{k}A^{k}_{\alpha j}A^{\alpha}_{mi} \\
&\quad + \Delta|b_{ij}|^2 - 2|\nabla_{l}b_{ij}|^2 - 4b^{ji}\nabla^{l}A^{k}_{\alpha j}\nabla_{l}A^{\alpha}_{ik} \\
&\quad + 8b^{ij}A^{k}_{\alpha j}A^{\alpha}_{mn}R^{m}{}_{i}{}^{n}{}_{k} \\
&= 4b^{ij}b^{m}_{i}b_{jm} + 4b^{ij}b^{m}_{k}A^{k}_{\alpha j}A^{\alpha}_{mi} \\
&\quad + \Delta|b_{ij}|^2 - 2|\nabla_{l}b_{ij}|^2 - 4b^{ji}\nabla^{l}A^{k}_{\alpha j}\nabla_{l}A^{\alpha}_{ik} \\
&\quad + 8b^{ij}A^{k}_{\alpha j}A^{\alpha mn}(A^{\beta}_{mn}A_{\beta ik} - A^{\beta}_{mk}A_{\beta in}) \\
&= \Delta|b_{ij}|^2 - 2|\nabla_{l}b_{ij}|^2 - 4b^{ji}\nabla^{l}A^{k}_{\alpha j}\nabla_{l}A^{\alpha}_{ik} \\
&\quad + 4b^{ij}b^{m}_{i}b_{jm} + 4b^{ij}b^{m}_{k}A^{k}_{\alpha j}A^{\alpha}_{mi} + 8b^{ij}g_{ln}c^{\alpha\beta}_{ji}c^{nl}_{\alpha\beta} - 8b^{j}_{l}c^{\alpha\beta}_{jk}c^{lk}_{\beta\alpha}
\end{aligned}$$

so that

$$\frac{d}{dt}|b_{ij}|^2 = \Delta|b_{ij}|^2 - 2|\nabla_{l}b_{ij}|^2 - 4b^{ji}\nabla^{l}A^{k}_{\alpha j}\nabla_{l}A^{\alpha}_{ik} + 4\Gamma_1 + 4\Gamma_2 + 8\Gamma_3 - 8\Gamma_4, \tag{22}$$

where

$$\Gamma_1 := b^{ij} b_i^m b_{jm}, \qquad \Gamma_2 := b^{ij} b_k^m A_{\alpha j}^k A_{mi}^\alpha \tag{23}$$
$$\Gamma_3 := b^{ij} g_{ln} c_{ji}^{\alpha\beta} c_{\alpha\beta}^{nl}, \quad \Gamma_4 := b_l^j c_{jk}^{\alpha\beta} c_{\beta\alpha}^{lk}.$$

To continue, we need an expression for $\frac{d}{dt} c_{ij}^{\alpha\beta}$.

$$\begin{aligned}
\frac{d}{dt} c_{ij}^{\alpha\beta} &= 2a^{kl} A_{ik}^\alpha A_{lj}^\beta + \frac{d}{dt} A_{ik}^\alpha A_j^{\beta k} + \frac{d}{dt} A_{kj}^\beta A_i^{\alpha k} \\
&= 2a^{kl} A_{ik}^\alpha A_{lj}^\beta \\
&\quad + A_j^{\beta k} (\Delta A_{ik}^\alpha - R_i^m A_{mk}^\alpha - R_k^m A_{mi}^\alpha + 2A_{mn}^\alpha R^m{}_i{}^n{}_k) - A_j^{\beta k} F_m^\alpha \lambda_{ik}^m \\
&\quad + A_i^{\alpha k} (\Delta A_{jk}^\beta - R_j^m A_{mk}^\beta - R_k^m A_{mj}^\beta + 2A_{mn}^\beta R^m{}_j{}^n{}_k) - A_i^{\alpha k} F_m^\beta \lambda_{jk}^m,
\end{aligned}$$

where

$$\lambda_{ik}^m = \nabla_i b_k^m + \nabla_k b_i^m - \nabla^m b_{ik}.$$

Therefore

$$\begin{aligned}
\frac{d}{dt} c_{ij}^{\alpha\beta} &= 2a^{kl} A_{ik}^\alpha A_{lj}^\beta + \Delta (A_{ik}^\alpha A_j^{\beta k}) \\
&\quad - \nabla^l A_j^{\beta k} \nabla_l A_{ik}^\alpha + A_j^{\beta k} (-R_i^m A_{mk}^\alpha - R_k^m A_{mi}^\alpha + 2A_{mn}^\alpha R^m{}_i{}^n{}_k) - A_j^{\beta k} F_m^\alpha \lambda_{ik}^m \\
&\quad - \nabla^l A_i^{\alpha k} \nabla_l A_{jk}^\beta + A_i^{\alpha k} (-R_j^m A_{mk}^\beta - R_k^m A_{mj}^\beta + 2A_{mn}^\beta R^m{}_j{}^n{}_k) - A_i^{\alpha k} F_m^\beta \lambda_{jk}^m \\
&= \Delta c_{ij}^{\alpha\beta} - 2\nabla^l A_j^{\beta k} \nabla_l A_{ik}^\alpha + 2b^{kl} A_{ik}^\alpha A_{lj}^\beta - R_i^m c_{mj}^{\alpha\beta} - R_j^m c_{im}^{\alpha\beta} \\
&\quad + 2A_{mn}^\alpha A_j^{\beta k} R^m{}_i{}^n{}_k + 2A_i^{\alpha k} A_{mn}^\beta R^m{}_j{}^n{}_k - A_j^{\beta k} F_m^\alpha \lambda_{ik}^m - A_i^{\alpha k} F_m^\beta \lambda_{jk}^m
\end{aligned} \tag{24}$$

Then

$$\begin{aligned}
\frac{d}{dt} \left(c_{ij}^{\alpha\beta} c_{\beta\alpha}^{ij} \right) &= 2 \frac{d}{dt} c_{ij}^{\alpha\beta} c_{\beta\alpha}^{ij} + 4a_l^k c_{kj}^{\alpha\beta} c_{\beta\alpha}^{lj} \\
&= \Delta \left(c_{ij}^{\alpha\beta} c_{\beta\alpha}^{ij} \right) - 2\nabla^l c_{ij}^{\alpha\beta} \nabla_l c_{\beta\alpha}^{ij} - 4c_{\beta\alpha}^{ij} \nabla^l A_{ik}^\alpha \nabla_l A_j^{\beta k} \\
&\quad + 2c_{\beta\alpha}^{ij} \Big\{ 2b^{kl} A_{ik}^\alpha A_{lj}^\beta - R_i^m c_{mj}^{\alpha\beta} - R_j^m c_{im}^{\alpha\beta} + 2A_{mn}^\alpha A_j^{\beta k} R^m{}_i{}^n{}_k \\
&\quad + 2A_i^{\alpha k} A_{mn}^\beta R^m{}_j{}^n{}_k \Big\} + 4a_l^k c_{kj}^{\alpha\beta} c_{\beta\alpha}^{lj} \\
&= \Delta \left(c_{ij}^{\alpha\beta} c_{\beta\alpha}^{ij} \right) - 2\nabla^l c_{ij}^{\alpha\beta} \nabla_l c_{\beta\alpha}^{ij} - 4c_{\beta\alpha}^{ij} \nabla^l A_{ik}^\alpha \nabla_l A_j^{\beta k} \\
&\quad + 8b_l^k c_{kj}^{\alpha\beta} c_{\beta\alpha}^{lj} + 8c_{\beta\alpha}^{ij} A_{mn}^\alpha A_j^{\beta k} R^m{}_i{}^n{}_k \\
&= \Delta \left(c_{ij}^{\alpha\beta} c_{\beta\alpha}^{ij} \right) - 2\nabla^l c_{ij}^{\alpha\beta} \nabla_l c_{\beta\alpha}^{ij} - 4c_{\beta\alpha}^{ij} \nabla^l A_{ik}^\alpha \nabla_l A_j^{\beta k} \\
&\quad + 8\Gamma_4 + 8\Gamma_5 - 8\Gamma_6,
\end{aligned} \tag{25}$$

where

$$\Gamma_5 := c^{\alpha m}_{m\gamma} c^{\beta\gamma}_{jl} c^{lj}_{\beta\alpha}, \quad \Gamma_6 := c^{\alpha\gamma}_{mi} c^{\beta m}_{j\gamma} c^{ij}_{\beta\alpha}. \tag{26}$$

One can also show that

$$R^{\alpha\beta}{}_{ij} R_{\alpha\beta kl} R^{ijkl} = 4(\Gamma_6 - \Gamma_7), \tag{27}$$

where

$$\Gamma_7 := c^{\alpha j}_{i\beta} c^{l\gamma}_{\alpha j} c^{i\beta}_{\gamma l}.$$

Similarly, one has

$$c^{\alpha m}_{m\gamma} R_{\alpha\beta}{}^{ij} R^{\gamma\beta}{}_{ij} = 2(\Gamma_3 - \Gamma_5), \tag{28}$$

$$R^{\alpha\beta}{}_{ij} R_{\alpha}{}^{\gamma i}{}_{l} R_{\beta\gamma}{}^{jl} = \Gamma_1 - 3\Gamma_4 + 3\Gamma_6 - \Gamma_7 \tag{29}$$

and

$$b^{ij} R^{\alpha\beta}{}_{ki} R_{\alpha\beta}{}^{k}{}_{j} = 2(\Gamma_2 - \Gamma_4). \tag{30}$$

Let us define

$$G_1 := b^{mk} \nabla^l A^{\alpha}_{ik} \nabla_l A^{i}_{\alpha m}, \quad G_2 := A^{k}_{\beta j} A^{i}_{\alpha m} \nabla^l A^{\alpha}_{ik} \nabla_l A^{\beta}_{mj}$$

and

$$G_3 := c^{mk}_{\alpha\beta} \nabla^l A^{\alpha}_{ik} \nabla_l A^{\beta i}_{m}, \quad G_4 := A^{\beta k}_{j} A^{i}_{\beta m} \nabla^l A^{\alpha}_{ik} \nabla_l A^{jm}_{\alpha}.$$

Since

$$\nabla_l R^{\alpha\beta}{}_{ij} = \nabla_l A^{\alpha}_{ik} A^{\beta k}_{j} + A^{\alpha}_{ik} \nabla_l A^{\beta k}_{j} - \nabla_l A^{\beta}_{ik} A^{\alpha k}_{j} - A^{\beta}_{ik} \nabla_l A^{\alpha k}_{j},$$

we obtain

$$\begin{aligned} |\nabla R^{\alpha\beta}{}_{ij}|^2 &= 4(b^{mk} \nabla A^{\alpha}_{ik} \nabla A^{i}_{\alpha m} + A^{k}_{\beta j} A^{i}_{\alpha m} \nabla A^{\alpha}_{ik} \nabla A^{\beta}_{mj} \\ &\quad - c^{mk}_{\alpha\beta} \nabla A^{\alpha}_{ik} A^{\beta i}_{m} - A^{\beta k}_{j} A^{i}_{\beta m} \nabla A^{\alpha}_{ik} \nabla A^{jm}_{\alpha}) \\ &= 4(G_1 + G_2 - G_3 - G_4). \end{aligned}$$

In addition

$$|\nabla_l b_{ij}|^2 = 2 c^{kl}_{\alpha\beta} \nabla A^{\alpha}_{ik} \nabla A^{\beta i}_{l} + 2 A^{k}_{\alpha j} A^{\beta i}_{l} \nabla A^{\alpha}_{ik} \nabla A^{lj}_{\beta} = 2(G_2 + G_3)$$

and

$$\begin{aligned} \nabla(c^{\alpha\beta}_{ij}) \nabla(c^{ij}_{\beta\alpha}) &= 2 A^{\alpha}_{ik} A^{jl}_{\alpha} \nabla A^{\beta k}_{j} \nabla A^{i}_{\beta l} + 2 c^{\beta l}_{k\alpha} \nabla A^{\alpha k}_{i} \nabla A^{i}_{\beta l} \\ &= 2(G_3 + G_4). \end{aligned}$$

Combining (22), (25) and the last equations we get

$$\begin{aligned}\frac{d}{dt}|R^{\alpha\beta}{}_{ij}|^2 &= \Delta|R^{\alpha\beta}{}_{ij}|^2 - 8(G_2+G_3) - 8G_1 + 8(G_3+G_4) + 8G_3\\ &\quad +8(\Gamma_1+\Gamma_2+2\Gamma_3-2\Gamma_4) - 16(\Gamma_4+\Gamma_5-\Gamma_6)\\ &= \Delta|R^{\alpha\beta}{}_{ij}|^2 - 2|\nabla_l R^{\alpha\beta}{}_{ij}|^2 + 8(\Gamma_1-3\Gamma_4+3\Gamma_6-\Gamma_7)\\ &\quad -8(\Gamma_6-\Gamma_7) + 16(\Gamma_3-\Gamma_5) + 8(\Gamma_2-\Gamma_4)\end{aligned}$$

and in view of the previous equations for $\Gamma_i, i = 1, \dots, 7$, we conclude

$$\begin{aligned}\frac{d}{dt}|R^{\alpha\beta}{}_{ij}|^2 &= \Delta|R^{\alpha\beta}{}_{ij}|^2 - 2|\nabla_l R^{\alpha\beta}{}_{ij}|^2 + 8R^{\alpha\beta}{}_{ij}R_{\alpha}{}^{\gamma i}{}_{l}R_{\beta\gamma}{}^{jl}\\ &\quad -2R^{ijkl}R^{\alpha\beta}{}_{ij}R_{\alpha\beta kl} + 8c^{\alpha m}_{m\gamma}R_{\alpha\beta}{}^{ij}R^{\gamma\beta}{}_{ij} + 4b^{ij}R^{\alpha\beta}{}_{ki}R_{\alpha\beta}{}^{k}{}_{j}\end{aligned} \tag{31}$$

and this equation is of the form given in (21). ∎

Remark 2. *One can replace the gradient term* $\nabla_l R^{\alpha\beta}{}_{ij}$ *by* $\nabla^{\perp}_l R^{\alpha\beta}{}_{ij}$, *where* $\nabla^{\perp}$ *is the normal connection. This simplifies (31) a bit. We leave the details to the reader.*

7. Graphical mean curvature flow with flat normal bundles

In the remaining section we will see that in case of entire graphs with flat normal bundles the computations in the paper by Ecker & Huisken basically carry over unchanged to the case of arbitrary codimension. In the following we will outline the basic steps.

Let us assume that the initial submanifold M_0 can be written as an entire graph over a flat m-plane in $\mathbb{R}^n$ such that the normal bundle of M_0 is flat.

Suppose $\omega \in \Omega^m(\mathbb{R}^n)$ is a parallel m-form with $|\omega| = 1$. ω induces a function w on any immersion $F : M^m \to \mathbb{R}^n$ by

$$F^*\omega =: w d\mu,$$

where $d\mu$ is the induced volume form on M. Since $|\omega| = 1$ we must have

$$-1 \le w(x) \le 1\,, \quad \forall\, x \in M.$$

The angle α defined by $\cos\alpha = w$ measures the angle between the flat plane defined by ω and the tangent planes of the submanifold M. The condition to be a graph then is easily expressed by

$$w > 0.$$

For hypersurfaces, w is also given by the angle between the normal vector ν of M and a fixed normal direction of the reference plane defined by ω. This was considered in [3]. For immersions with higher codimension, a similar w-function was considered in [7]. The evolution equation for $F^*\omega$ was derived earlier in a paper by M.-T. Wang [13] and for the sake of completeness we include the evolution equation in our notation.

$$\frac{d}{dt}\omega_{i_1\cdots i_m} = \Delta\omega_{i_1\cdots i_m} - R\omega_{i_1\cdots i_m} - \sum_{s<j}\omega_{i_1\cdots i_{s-1}\alpha_s i_{s+1}\cdots i_{j-1}\alpha_j i_{j+1}\cdots i_m}R^{\alpha_s\alpha_j}{}_{i_s i_j}, \tag{32}$$

where R denotes the scalar curvature and $\omega_{i_1\cdots i_m} = \omega_{\alpha_1\cdots\alpha_m}F^{\alpha_1}_{i_1}\cdots F^{\alpha_m}_{i_m}$.

A first observation is

Lemma 1. *Suppose $F : M^m \times [0,T) \to \mathbb{R}^n$ is a smooth solution of the mean curvature flow as in Theorem 1. Then w defined as above satisfies*

$$\frac{d}{dt}w = \Delta w + w|A|^2. \tag{33}$$

Proof. We know that the normal bundle will be flat for all t. Hence

$$\frac{d}{dt}F^*\omega = \Delta F^*\omega - RF^*\omega$$

and the conclusion follows from $\frac{d}{dt}d\mu = -|H|^2 d\mu$ and $R = |H|^2 - |A|^2$. ■

The second observation is

Lemma 2. *Suppose $F : M^m \times [0,T) \to \mathbb{R}^n$ is as in Lemma 1. Then*

$$\begin{aligned} \frac{d}{dt}|A|^2 &= \Delta|A|^2 - 2|\nabla^\perp A|^2 + 2|A^{\alpha m}_n A^{\beta n}_m|^2 \\ &\le \Delta|A|^2 - 2|\nabla^\perp A|^2 + 2|A|^4 \end{aligned} \tag{34}$$

Proof. The Lemma follows from (16), Theorem 1 and the estimate

$$|A^{\alpha m}_n A^{\beta n}_m|^2 \le |A|^4.$$

■

Hence, in the case of a flat normal bundle the differential inequality for $|A|^2$ is the same as in the case of codimension one. Using these observations, we can follow [3] closely to consider the mean curvature flow with flat normal bundles.

Now let us assume the following *linear growth* condition

$$v := \frac{1}{w} \le c_1 < \infty \tag{35}$$

holds everywhere on M_0 for some constant c_1. (33) implies that if M_0 satisfies (35) then M_t also satisfies (35) with the same constant.

Lemma 3. *The term* $|A|^2v^2$ *satisfies*

$$\left(\frac{d}{dt}-\Delta\right)|A|^2v^2 \le -2v^{-1}\langle\nabla v, \nabla(|A|^2v^2)\rangle. \tag{36}$$

Proof. To prove the Lemma, we need the following Kato type inequality

$$|\nabla^\perp A|^2 \ge \frac{m+2}{m}|\nabla|A||^2, \tag{37}$$

which was proved in [15] for immersions with flat normal bundle. Now the proof follows from the proof of Lemma 4.1 in [3]. ∎

Note that when the codimension is one, (37) was proved in [8]. A direct consequence of Lemma 3 and Corollary 1 is

Corollary 2. *If* M_t *is a smooth solution of (1) with bounded* v *and bounded* $|A|^2$ *on each* M_t*, then there is the following estimate*

$$\sup_{M_t}|A|^2v^2 \le \sup_{M_0}|A|^2v^2.$$

One can also get a differential inequality for higher derivatives of the second fundmental form

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\left(t^{l+1}|(\nabla^\perp)^lA|^2\right) \quad &\le \quad -2t^{l+1}|(\nabla^\perp)^{l+1}A|^2+(l+1)t^l|(\nabla^\perp)^lA|^2\\ &\quad +C(n,l)t^{l+1}\sum_{i+j+k=l}|(\nabla^\perp)^iA||(\nabla^\perp)^jA||(\nabla^\perp)^lA||(\nabla^\perp)^lA|,\end{aligned} \tag{38}$$

for any integer $l \ge 0$, where $C(n,l)$ is a constant depending only on n and l. Therefore, the higher order estimates in [3] can also be obtained in our case. Furthermore we get

Proposition 3. *Let* M_t *be a smooth solution of (1) with flat normal bundle. Then for each* $m \ge 0$ *there is a consant* $C(m)$ *such that*

$$t^{m+1}|(\nabla^\perp)^mA|^2 \le C(m). \tag{39}$$

Rescaling as in [3] (see also [6]), we define

$$\widetilde{F}(s) = \frac{1}{\sqrt{2t+1}}F(t),$$

where s is given by $s = \frac{1}{2}\log(2t+1)$. In the new time variable s, we have a normalized equation

$$\frac{d}{ds}\widetilde{F} = \widetilde{H} - \widetilde{F}. \tag{40}$$

From Proposition 2, Corollary 2 and Proposition 3 we have estimates for the rescaled immersion $\widetilde{F}$

$$\begin{aligned}\tilde{u}^2(\widetilde{F}, s) &\le \tilde{c}_0(1 + \widetilde{x}^2(\widetilde{F}, s))\\ \widetilde{v}(\widetilde{F}, s) &\le c_1\\ |\widetilde{A}|^2(\widetilde{F}, s) &\le c_2,\end{aligned}$$

where $\tilde{u}^2 := |\widetilde{F}^\perp|^2$, $\tilde{x}^2 = |\widetilde{F}^T|^2$ and $\tilde{c}_0$, c_1, c_2 are some constants depending only on the initial immersion F_0.

Eventually, since the term $|a_{ij}|^2$ appearing in the evolution equation of the squared mean curvature (14) can be estimated by

$$|a_{ij}|^2 \le |H|^2|A|^2,$$

the computations can be carried out in the same way to derive

Theorem 2. *Suppose M_0 is an entire graph with bounded curvature over some $\mathbb{R}^m$ in $\mathbb{R}^n$ satisfying the linear growth condition (35) and that M_0 has a flat normal bundle. Then the mean curvature flow admits a smooth solution for all $t > 0$ with uniformly bounded curvature quantities. If in addition we assume that*

$$u^2 \le c_3(1 + |F|^2)^{1-\delta} \tag{41}$$

holds on M_0 for some constant $c_3 < \infty$ and $\delta > 0$, then the solution $\widetilde{M}_s$ of the normalized mean curvature flow (40) converges for $s \to \infty$ to a limiting surface $\widetilde{M}_\infty$ satisfying the equation for selfsimilarly expanding solutions

$$F^\perp = H. \tag{42}$$

Remark 3. *We note that Proposition 4.5 in* [3]*, the spatial decay behaviour, can also easily be done in the same way. We leave the details to the reader.*

Remark 4. *We improved the previous results of the third author. The dimension limitation in* [15] *can be removed. That will appear in another paper.*

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