

Max-Planck-Institut  
für Mathematik  
in den Naturwissenschaften  
Leipzig

Adaptive Galerkin Boundary Element Methods  
with Panel Clustering

(revised version: November 2006)

by

*Wolfgang Hackbusch, Boris N. Khoromskij, and Stefan A. Sauter*

Preprint no.: 79

2004





# Adaptive Galerkin boundary element methods with panel clustering

Wolfgang Hackbusch, Boris N. Khoromskij  
Max-Planck-Institute for Mathematics in the Sciences,  
Inselstr. 22-26, D-04103 Leipzig, Germany.  
{wh,bokh}@mis.mpg.de

Stefan Sauter  
Institut für Mathematik, Universität Zürich,  
Winterthurerstr 190,  
CH-8057 Zürich, Switzerland.  
stas@amath.unizh.ch

## Abstract

In this paper, we will propose a boundary element method for solving classical boundary integral equations on complicated surfaces which, possibly, contain a large number of geometric details or even uncertainties in the given data. The (small) size of such details is characterised by a small parameter  $\varepsilon$  and the regularity of the solution is expected to be low in such zones on the surface (which we call the wire-basket zones).

We will propose the construction of an *initial* discretisation for such type of problems. Afterwards standard strategies for boundary element discretisations can be applied such as the  $h$ ,  $p$ , and the adaptive  $hp$ -version in a straightforward way.

For the classical boundary integral equations, we will prove the optimal approximation results of our so-called *wire-basket boundary element method* and discuss the stability aspects. Then, we construct the panel-clustering and  $\mathcal{H}$ -matrix approximations to the corresponding Galerkin BEM stiffness matrix. The method is shown to have an almost linear complexity with respect to the number of degrees of freedom located on the wire basket.

*AMS Subject Classification:* 65F50, 65F30

*Key Words:* hierarchical matrices, panel clustering, boundary element method,  $hp$ -version of BEM

## 1 Introduction

In the paper, we will consider the numerical solution of classical boundary integral equations on (open respectively closed) hypersurfaces  $\Gamma \subset \mathbb{R}^3$ . We will focus on applications where the surface contains complicated geometric details and/or the data is given only up to some uncertainties. Typical applications are, e.g., the computation of the elastic behaviour of a plate or a body containing a large number of small cracks. As an illustrative example we have depicted a plate of duplex steel which contains many cracks caused by fatigue material (see Figure 1). In a neighbourhood of the geometric details, the solution, typically, has low regularity while, at proper distance to these zones, the solution is expected to be smooth.

The goal of this paper is to introduce the wire-basket *boundary element method* for the construction of an *initial* discretisation for such type of problems. Based on this initial discretisation, standard strategies for enriching the boundary element space can be applied straightforwardly. Among them are the  $h$ -version or the  $hp$ -version of the boundary element method (and, certainly, various other methods), see e.g., [5], [13], [28], [19], [20], [22], [23], [30], [8] and the references therein.

However, in some applications a refinement of the initial discretisation is not of interest because the error in the data due to, e.g., measurement errors and uncertainties, then, might dominate the error. In this light, the wire-basket BEM is *not* considered as an alternative to standard  $h$ -version or adaptive  $hp$ -version but as an efficient initial discretisation of boundary integral equations on complicated surfaces.

Our construction of the wire-basket discretisation employs techniques such as mesh grading and variable polynomial degree vectors which are well known in the field of adaptive  $hp$ -BEM. However, the essential

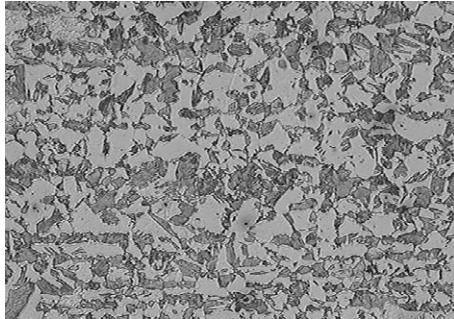


Figure 1: Plate of duplex steel which contains many cracks (reproduced by courtesy of Professor A. Brückner-Foit). For the physical background to this problem we refer to [4].

difference of the wire-basket BEM and the adaptive  $hp$ -BEM is related to the accuracy requirements in the following way. The mesh width of the boundary element mesh for the wire-basket BEM in the local vicinity of the geometric details will be of order  $O(\varepsilon)$  in order to resolve the details of size  $\varepsilon$ . The regularity of the solution is expected to be low in such zones and, hence, low order boundary elements spaces are used there. As a consequence, the error of the Galerkin solution, in general, will be, at least, of size  $C_\kappa \varepsilon^\kappa$ , where  $\kappa$  depends on the (low) regularity of the solution and, typically, satisfies  $\kappa \in ]0, 1]$ . In [25], it is proved for some model problems that under very weak assumptions on the geometry the constant  $C_\kappa$  is moderately bounded independently of the number and size of the geometric details and that  $\kappa \geq 1/2$ .

The idea of the wire-basket BEM is the construction of a boundary element discretisation which contains only very few unknowns and leads to a numerical solution which has accuracy  $O(\varepsilon^\kappa)$  with respect to the energy norm. In contrast, the adaptive  $hp$ -BEM is a method which constructs a sequence of discretisations so that the error converges exponentially.

The moderate accuracy requirements for the wire-basket BEM allows to employ highly non-uniform but *shape-regular* meshes in contrast to the adaptive  $hp$ -BEM where elements with high aspect ratio occur. As a consequence, the evaluation of singular and nearly singular surface integrals for the wire-basket BEM which massively occurs when computing the entries of the system matrix can be performed by simple numerical quadrature (see [27], [9], [28]). In addition, the wire-basket BEM allows the application of the panel-clustering or  $\mathcal{H}$ -matrix techniques (cf. [14, 15, 11]) for the data-sparse representation of the non-local boundary integral operators.

Next, we will specify more formally the class of problems where we see the promising applications of the new boundary element method and introduce the basic notations.

Let  $\Omega$  denote a bounded, three-dimensional Lipschitz domain or the unbounded complement. As a prototype of a homogeneous, linear, elliptic boundary value problem with constant coefficient we choose a Laplace-type equation as our model problem – either with given Dirichlet data or with given Neumann data. This problem can be transformed into a boundary integral equation on the boundary  $\Gamma = \partial\Omega$  by means of the boundary integral equation method.

We are interested especially in problems where the smoothness of the given data and geometry are not uniform on  $\Gamma$ . Instead, we assume that a (thin) zone  $\Gamma^{\text{rough}} \subset \Gamma$  can be described where we expect the solution to have very low regularity while on the remaining part  $\Gamma \setminus \Gamma^{\text{rough}}$  we expect the solution to be analytic. We do not assume that the (one-dimensional) boundary of  $\Gamma^{\text{rough}}$  consists of only few long straight lines, but  $\partial\Gamma^{\text{rough}}$  is a polygonal line with a possibly large number of straight segments. Such problems typically arise, e.g., in crack propagation especially if the data stems from pointwise measurements containing possibly some uncertainties.

The problem class we are interested in can be described by the following assumption.

**Assumption 1.1**  $\Gamma$  is the surface of a Lipschitz polyhedron and can be decomposed into (moderately many) smooth (open) polygonal surface patches  $\Gamma_i$ ,  $1 \leq i \leq q$ , and a remaining part  $\Gamma^{\text{rough}}$  so that the given data, say  $f : \Gamma \rightarrow \mathbb{R}$ , has the property that the restrictions  $f|_{\Gamma_i}$  are analytic for all  $1 \leq i \leq q$ .

We have in mind that the surface measure of  $\Gamma^{\text{rough}}$  is very small compared to the complement  $\Gamma \setminus \Gamma^{\text{rough}}$ .

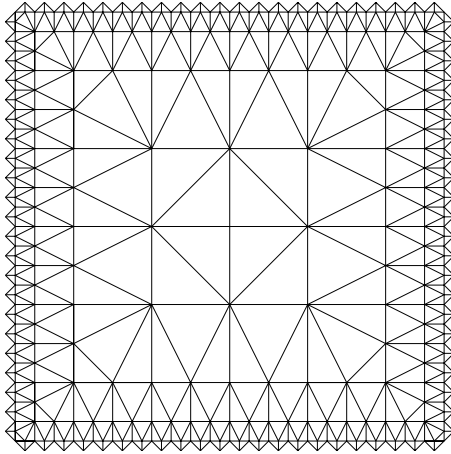


Figure 2: Screen with rough boundary. The mesh is graded geometrically towards the geometric details. Note that the wire-basket BEM does **not** require periodicity of the geometric details.

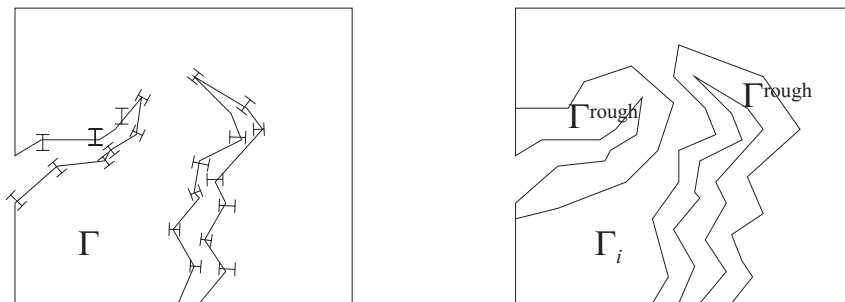


Figure 3: Screen containing two cracks. The error bars indicate that the exact position of the crack is not known due to measurement errors. The zones  $\Gamma^{\text{rough}}$  contain the region, where the solution is expected to have low regularity.

Such kinds of geometric applications motivate the name *wire-basket zone* for the set  $\Gamma^{\text{rough}}$ .

The starting point of the discretisation is the generation of a surface mesh  $\mathcal{G} = \{\tau_1, \dots, \tau_n\}$  consisting of shape regular, triangular panels which are graded geometrically towards the wire-basketed  $\Gamma^{\text{rough}}$ .

We assume that  $\Gamma^{\text{rough}}$  is resolved by  $\mathcal{G}$  in the following sense.

**Assumption 1.2** *There exists a subset  $\mathcal{G}^{\text{rough}} \subset \mathcal{G}$  which is a quasi-uniform and shape regular boundary element mesh for  $\Gamma^{\text{rough}}$ .*

The largest diameter of the elements in  $\mathcal{G}^{\text{rough}}$  is denoted by  $h$ .

$$h := \max_{\tau \in \mathcal{G}^{\text{rough}}} h_\tau \quad \text{with} \quad h_\tau := \text{diam } \tau. \quad (1.1)$$

**Remark 1.3** *Note that  $h$  is **not** considered here as a discretization parameter and convergence is **not** studied as  $h \rightarrow 0$  **but** reflects the size of the geometric details in the domain and the data. If the details are of size  $\varepsilon > 0$  then, typically,  $h \sim \varepsilon$ . We will prove that the wire-basket BEM behaves robustly with respect to the smallness of  $\varepsilon$ , i.e., the discretisation error is of order  $O(\varepsilon^\kappa)$  for some  $\kappa \in ]0, 1]$  uniformly with respect to small  $\varepsilon$ . We further prove that the complexity, under moderate assumptions, is of order (number of geometric details)  $\times \varepsilon^{-1}$  up to logarithmic terms.*

The parts  $\Gamma_i$  of the surface where the solution is assumed to be analytic, is meshed by triangles which are geometrically graded to the wire-basket zones. The precise definition of such meshes will be presented in Definition 2.3 while Figures 2, 4 depict two characteristic examples.

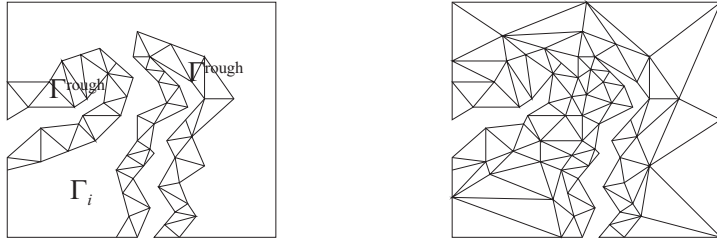


Figure 4: Quasi-uniform mesh in the wire-basket regions and graded mesh in the remaining part of the screen.

The regularity theory for elliptic equations with constant coefficients implies that the solution is analytic on the smooth parts  $\Gamma_i^{\text{smooth}}$  while it might have very low regularity on the portion  $\Gamma^{\text{rough}}$  of the boundary.

We discretise the integral equations by the Galerkin boundary element method where we use high order elements within the parts  $\Gamma_i$ ,  $1 \leq i \leq q$ , and decrease the polynomial degree to lowest order finite elements towards the wire-basked zone  $\Gamma^{\text{rough}}$ . The number of degrees of freedom associated with  $\mathcal{G}^{\text{rough}}$  is proportional to the total number of unknowns. Since we have in mind that the set  $\Gamma^{\text{rough}}$  is very thin, while the parts  $\Gamma_i$  are proper two-dimensional, we say that the discretisation is concentrated to the wire-basket zones.

The (weak) requirement for the discretisation error to be of order  $h^\kappa$  (instead of exponential convergence for the adaptive  $hp$ -BEM) allows the use of panel-clustering to avoid the full system matrix and, in addition, the smoothness requirements on the solution are substantially relaxed compared to the adaptive  $hp$ -BEM. The total computational complexity and memory consumptions depend linearly on the *number of degrees of freedom on the wire-basket zone*. Hence, in the case of smooth data on each part  $\Gamma_i$ , our method can be viewed as an effective reduction of the classical 3D-BEM to the wire-basket of the surface (where all the singularities are located).

The paper is structured as follows. In Section 2, we will introduce the wire-basket boundary element space for the Galerkin BEM. The corresponding approximation theory will be presented in Section 3. The panel-clustering method for the wire-basket BEM will be introduced and analysed in Section 4. The complexity of the method is summarised in Remark 4.9.

## 2 Galerkin discretisation of classical boundary integral equations

### 2.1 Classical boundary integral equations

Throughout this paper,  $\Omega \subset \mathbb{R}^3$  denotes a bounded Lipschitz domain with boundary  $\Gamma$  and normal vector field  $n$  (oriented to the exterior of  $\Omega$ ). We define the Sobolev space  $H^s(\Gamma)$ ,  $s \geq 0$ , in the usual way (see, e.g., [12]). Note that the range of  $s$  for which  $H^s(\Gamma)$  is defined may be limited, depending on the global smoothness of the surface  $\Gamma$ . For  $s < 0$ , the spaces  $H^s(\Gamma)$  are the dual of  $H^{-s}(\Gamma)$ . The norm in  $H^s(\Gamma)$  is denoted by  $\|\cdot\|_s$ .

We will consider the general integral equation

$$(\lambda I + \mathcal{K}) u(x) := \lambda u(x) + \int_{\Gamma} k(x, y) u(y) ds_y = f(x), \quad x \in \Gamma, \quad (2.1)$$

for some given scalar  $\lambda \in \mathbb{R}$  kernel function  $k$  and sufficiently smooth right-hand side  $f$ . The corresponding weak form is

$$\text{find } u \in H^\mu \text{ such that } a(u, v) := ((\lambda I + \mathcal{K})u, v) = (f, v) \quad \text{for all } v \in H^\mu. \quad (2.2)$$

Here  $H^\mu$  denotes the “energy space” for some  $\mu \in \{-1/2, 0, 1/2\}$ . (The bracket  $(\cdot, \cdot)$  denotes the continuous extension of the  $L^2(\Gamma)$  scalar product to the  $H^{-\mu}(\Gamma) \times H^\mu(\Gamma)$  duality pairing.) The operator associated with the bilinear form  $a(\cdot, \cdot)$  is denoted by  $A: H^\mu \rightarrow H^{-\mu}$ , where  $H^{-\mu}$  is the dual of  $H^\mu$ .

Typical examples are: the classical single layer, double layer and hypersingular operators for the operator  $\mathcal{L}_\kappa u := -\Delta u + \kappa^2 u$  for some  $\kappa \geq 0$ . The bilinear form has the following general form

$$a(\cdot, \cdot) = \lambda(\cdot, \cdot) + \hat{a}(\cdot, \cdot), \quad (2.3)$$

where the definition of the integral operators is based on the fundamental solution of the operator  $\mathcal{L}_\kappa$ :

$$S(z) := \frac{e^{-\kappa|z|}}{4\pi|z|}.$$

**Single layer potential:**

$$\lambda := 0, \mu = -1/2, \hat{a}(u, v) := \int_{\Gamma \times \Gamma} S(x-y) v(x) u(y) ds_x ds_y, \quad (2.4a)$$

**Double layer potential:**

$$\lambda := \pm \frac{1}{2}, \mu = 0, \hat{a}(u, v) := \int_{\Gamma \times \Gamma} v(x) u(y) \frac{\partial}{\partial n_y} S(x-y) ds_y ds_x, \quad (2.4b)$$

**Hypersingular operator:**

$$\lambda := 0, \mu = 1/2, \hat{a}(u, v) := \int_{\Gamma} v(x) \frac{\partial}{\partial n_x} \int_{\Gamma} u(y) \frac{\partial}{\partial n_y} S(x-y) ds_y ds_x. \quad (2.4c)$$

**Remark 2.1** For  $\kappa = 0$ , the operator  $\mathcal{L}_\kappa$  is the Laplace operator. In this case, the energy space for the hypersingular operator is the quotient space  $H^{1/2}(\Gamma)/\mathbb{R}$ . To avoid technicalities, we restrict ourselves in this paper, for the hypersingular operator, to  $\kappa > 0$  while the generalisation to  $\kappa = 0$  is straightforward.

## 2.2 Galerkin discretisation

In the standard, conforming Galerkin method we select a subspace  $S \subset H^\mu$  and approximate (2.2) by seeking  $u_S \in S$ , such that

$$a(u_S, v) = (f, v) \quad \text{for all } v \in S. \quad (2.5)$$

In the context of the boundary element method, these subspaces are finite element spaces lifted on the surface  $\Gamma$ .

**Definition 2.2** (a) The master element  $\hat{t} \subseteq \mathbb{R}^2$  is the open triangle with vertices  $(0, 0)^\top$ ,  $(0, 1)^\top$  and  $(1, 1)^\top$ .

(b) A set  $\mathcal{T} = \{t_1, t_2, \dots, t_n\}$  consisting of open and disjoint (possibly curved) triangles in  $\mathbb{R}^3$  such that there is a  $C^k$ -diffeomorphism  $\Psi_t : \hat{t} \rightarrow \bar{t}$  for each  $t \in \mathcal{T}$  is a surface triangulation of  $\Gamma$  if it satisfies

$$\Gamma = \bigcup_{t \in \mathcal{T}} \bar{t}.$$

(c) The triangulation is compatible if the intersection  $\bar{t} \cap \bar{t}' =: e$  of non-identical triangles  $t, t' \in \mathcal{T}$  is either empty, a common vertex, or a common edge and in the case that  $e$  is an edge, there exist affine mappings  $\gamma_t, \gamma_{t'} : [0, 1] \rightarrow \bar{t}$  such that  $\Psi_t \circ \gamma_t = \Psi_{t'} \circ \gamma_{t'}$  and  $\Psi_t \circ \gamma_t : [0, 1] \rightarrow e$  is bijective.

The triangulation for the wire-basket BEM is constructed as follows.

First,  $\mathcal{G}^{\text{rough}}$  is constructed as a set of quasi-uniform and shape regular triangles covering the wire-basket zone  $\Gamma^{\text{rough}}$  (cf. Figure 4).

The connectivity components of the complement  $\Gamma \setminus \Gamma^{\text{rough}}$  define the polygonal subsets  $\Gamma_i$ ,  $1 \leq i \leq q$ . On each component  $\Gamma_i$ , we construct a mesh  $\mathcal{G}_i$  which is concentrated to the (one-dimensional) boundary  $\partial\Gamma_i$  in such a way that the union  $\mathcal{G} := \mathcal{G}^{\text{rough}} \cup (\bigcup_{i=1}^q \mathcal{G}_i)$  is a compatible surface mesh on  $\Gamma$ . The definition of  $\mathcal{G}_i$  below generalises the boundary concentrated meshes introduced in [22, Definition 2.3].

Let  $h_\tau$  denote the diameter of a triangle  $\tau \in \mathcal{G}$  (cf. (1.1)) and recall the notation of  $\mathcal{G}^{\text{rough}}$  as in Assumption 1.2. In the following we will employ frequently the distance function  $\text{dist}_{\text{geo}}(A, B)$  measuring the geodetic distance on  $\Gamma$  of two subset  $A, B \subset \Gamma$ . However, we assume that there are constants  $c, C$  such that geodetic distance is comparable with the three-dimensional Euclidean distance  $\text{dist}(A, B)$

$$c \text{dist}(A, B) \leq \text{dist}_{\text{geo}}(A, B) \leq C \text{dist}(A, B) \quad \forall A, B \subset \Gamma.$$

The estimates in the remaining part of this paper may depend on the constants  $c, C$ .

**Definition 2.3** Let  $\mathcal{G}^{\text{rough}}$  be given with mesh size  $h$  (cf. (1.1)). A compatible, shape-regular mesh  $\mathcal{G} \supset \mathcal{G}^{\text{rough}}$  on  $\Gamma$  is called a mesh concentrated to the wire-basket zones  $\Gamma^{\text{rough}}$ , if there exist  $c_1, c_2 > 0$  such that for all  $\tau \in \mathcal{G} \setminus \mathcal{G}^{\text{rough}}$ :

1. if  $\bar{\tau} \cap \Gamma^{\text{rough}} \neq \emptyset$ , then  $c_1 h \leq h_\tau \leq h$ ,
2. if  $\bar{\tau} \cap \Gamma^{\text{rough}} = \emptyset$ , then  $c_1 \text{dist}_{\text{geo}}(\tau, \Gamma^{\text{rough}}) \leq h_\tau \leq c_2 \text{dist}_{\text{geo}}(\tau, \Gamma^{\text{rough}})$ .

The constant which measures the shape regularity is given by

$$c_{\text{reg}} := \max_{\tau \in \mathcal{G}} h_\tau / \rho_\tau \quad \text{where } \rho_\tau \text{ is the radius of the largest inscribed circle in } \tau.$$

**Remark 2.4** For given  $\delta > 0$ , let

$$\Gamma_\delta^{\text{rough}} := \{x \in \Gamma : \text{dist}_{\text{geo}}(x, \Gamma^{\text{rough}}) < \delta h\} \quad (2.6)$$

be a neighbourhood of  $\Gamma^{\text{rough}}$  and  $\mathcal{G}$  be a mesh on  $\Gamma$  as in Definition 2.3. Then, there exists a constant  $C > 0$  depending only on  $\delta, c_2$  (cf. Definition 2.3) and  $c_{\text{reg}}$  such that all triangles  $\tau \in \mathcal{G}$  with  $\tau \cap \Gamma_\delta^{\text{rough}} \neq \emptyset$  satisfy

$$h_\tau \leq Ch.$$

In order to define  $hp$ -type boundary element spaces on a mesh  $\mathcal{G}$ , we associate a polynomial degree  $p_\tau \in \mathbb{N}$  to each element  $\tau$  and collect them in the polynomial degree vector  $\mathbf{p} := (p_\tau)_{\tau \in \mathcal{G}}$ . The  $hp$ -boundary element space is defined by

$$S_{\mathbf{p}}^k(\mathcal{G}) := \{u \in H^{k+1}(\Gamma) : \forall \tau \in \mathcal{G} : u|_\tau \in \mathbb{P}_{p_\tau}\} \quad \text{for } k = -1, 0 \text{ with } k+1 \geq \mu, \quad (2.7)$$

where  $\mathbb{P}_m$  denotes the space of bivariate polynomials of maximal total degree  $m$ .

For any  $\tau$  and any edge  $\gamma_i$  of  $\tau$ ,  $1 \leq i \leq 3$ , we define  $p_{\tau,i}$  as the maximal polynomial degree such that for all elements  $P_{p_{\tau,i}} \in \mathbb{P}_{p_{\tau,i}}$ , the traces onto the edges satisfy  $P_{p_{\tau,i}}|_{\gamma_i} \in S_{\mathbf{p}}^k(\mathcal{G})|_{\gamma_i}$ , i.e.,

$$\left\{ w|_{\gamma_i} : w \in \mathbb{P}_{p_{\tau,i}} \right\} \subset \left\{ u|_{\gamma_i} : u \in S_{\mathbf{p}}^k(\mathcal{G}) \right\}.$$

Then, we set

$$p_\tau^{\max} := \max_{1 \leq i \leq 3} p_{\tau,i} \quad \text{and} \quad p_\tau^{\min} := \min_{1 \leq i \leq 3} p_{\tau,i}.$$

**Assumption 2.5** For  $k = -1, 0$ , let  $S_{\mathbf{p}}^k(\mathcal{G})$  be defined as in (2.7).

- a.  $\mathbf{p} = (p_\tau)_{\tau \in \mathcal{G}}$  is a linear degree vector, i.e., it satisfies

$$c_p + \gamma \log \frac{h_\tau}{h} \leq p_\tau^{\min} \leq p_\tau^{\max} \leq C_p + \gamma \log \frac{h_\tau}{h} \quad (2.8)$$

for some constants  $c_p, C_p \geq 0$  and  $\gamma > 0$  independent of  $h$ .

- b. There exists a constant  $p_{\text{rough}} \in \{0, 1\}$  with  $p_{\text{rough}} > k$  such that, for all  $\tau \in \mathcal{G}^{\text{rough}}$ , there holds  $p_\tau = p_{\text{rough}}$ .

- c. The positive ratio

$$C_{\text{deg}} := \min_{\tau \in \mathcal{G}} \frac{p_\tau^{\min}}{p_\tau}$$

is bounded away from 0 independently of  $h$  and  $\mathbf{p}$ .

In the following we abbreviate  $S_{\mathbf{p}}^k(\mathcal{G})$  by  $S$  if no confusion is possible.



## 2.3 Error estimates

In the case of (2.4a,c) the bilinear form  $a(\cdot, \cdot)$  in (2.3) is  $H^\mu(\Gamma)$ -elliptic

$$|a(u, u)| \geq c \|u\|_\mu^2 \quad \forall u \in H^\mu(\Gamma),$$

and continuous

$$|a(u, v)| \leq C \|u\|_\mu \|v\|_\mu \quad \forall u, v \in H^\mu(\Gamma).$$

Via the Lax-Milgram lemma, continuity and ellipticity imply the unique solvability of the corresponding boundary integral equations and quasi-optimal error estimates of the Galerkin BEM due to C ea's lemma:

$$\|u_S - u\|_\mu \leq c \inf_{v \in S} \|u - v\|_\mu. \quad (2.9)$$

To obtain quantitative error estimate, we will further study the best approximation  $\inf_{v \in S} \|u - v\|_\mu$  under appropriate regularity assumptions.

For the double layer potential (2.4b) we require the coercivity, injectivity and continuity of the bilinear form. In this case, the coercivity is proved for smooth surfaces while the generalisation to other classes of surfaces is still open (cf. [8]) as long as the double layer potential is considered as a mapping from  $L^2(\Gamma) \rightarrow L^2(\Gamma)$ . In contrast, the coercivity of the double layer is proved for the energy space  $H^{1/2}(\Gamma)$  (see [6], [7], [28, Sec. 3.8]).

## 3 Approximation theory

### 3.1 Function spaces

In this section, we will introduce some function spaces and begin with a short outline of their different roles.

The function space for describing the regularity of the solution will be the intersection of two spaces:

- $H^{\mu+\delta}(\Gamma)$  for some  $\delta > 0$ : This space reflects the low global regularity which will be resolved on the wire-basket zone by the fine local mesh width  $h$ .
- $\mathcal{A}_\beta(C, \gamma; \Gamma)$ : Set of functions described in terms of countably normed spaces (cf. (3.1)).

The error estimates will be derived first for the auxiliary (local) function set  $\mathcal{A}_{M,\rho}(\tau)$ , which contains all functions with analytic continuation in certain neighbourhoods of  $\tau$  (cf. Definition 3.3). Then, the error estimates for functions in  $\mathcal{A}_\beta(C, \gamma; \Gamma)$  are derived from those in  $\mathcal{A}_{M,\rho}(\tau)$  by using the alternative characterisation of  $\mathcal{A}_{M,\rho}(\tau)$  (cf. Remark 3.5) and its relation to  $\mathcal{A}_\beta(C, \gamma; \Gamma)$  (cf. (3.1)).

We begin with the definition of the space  $\mathcal{A}_{M,\rho}(\tau)$  which requires several steps. For the interval  $I := (-1, 1)$  and  $\rho > 1$ , the Bernstein's regularity ellipse is given by (cf. [2])

$$\mathcal{E}_\rho := \{z \in \mathbb{C} : |z - 1| + |z + 1| \leq \rho + \rho^{-1}\}.$$

The corresponding semi-axes are  $a = \frac{\rho + \rho^{-1}}{2}$  and  $b = \frac{\rho - \rho^{-1}}{2}$ . Obviously there holds  $a + b = \rho$ .

**Definition 3.1** Let  $I = (-1, 1)$  and  $M > 0$ ,  $\rho > 1$  be given constants.  $\mathcal{A}_{M,\rho}(I)$  is the class of functions  $f \in C^\infty(I)$  having a holomorphic extension to  $\mathcal{E}_\rho(I)$  such that

$$|f(z)| \leq M \quad \forall z \in \mathcal{E}_\rho(I).$$

Next, we introduce the multidimensional analogue of  $\mathcal{A}_{M,\rho}(I)$  on the tensor domain  $I^d := (-1, 1)^d$ . Let  $\mathcal{E}_\rho^{(j)} := I \times \dots \times I \times \mathcal{E}_\rho \times I \times \dots \times I$ .

**Definition 3.2** For given constants  $M > 0$ ,  $\rho > 1$ , the set  $\mathcal{A}_{M,\rho}(I^d)$  consists of all functions  $f \in C^\infty(I^d)$  having holomorphic extensions to  $\mathcal{E}_\rho^{(j)}$ , for all  $1 \leq j \leq d$ , and satisfying

$$\max_{1 \leq j \leq d} \left\{ \sup_{x \in \mathcal{E}_\rho^{(j)}} |f(x)| \right\} \leq M.$$

For triangles  $\tau \in \mathcal{G} \setminus \mathcal{G}^{\text{rough}}$ , let  $B(\tau)$  denote some minimal rectangular bounding box  $B(\tau)$ . Thus, we may fix, for any  $\tau \in \mathcal{G} \setminus \mathcal{G}^{\text{rough}}$ , a bijective affine mapping  $\chi_\tau : I^d \rightarrow B(\tau)$ .

**Definition 3.3** For given constants  $M > 0$ ,  $\rho > 1$ , the set  $\mathcal{A}_{M,\rho}(\tau)$  is the class of functions  $f : \tau \rightarrow \mathbb{R}$  such that the pull back  $f \circ \chi_\tau$  can be extended to a function in  $\mathcal{A}_{M,\rho}(I^d)$ .

We shall deal with functions which locally can be extended to some complex neighbourhoods of the triangles in  $\mathcal{G} \setminus \mathcal{G}^{\text{rough}}$ . To describe this neighbourhood we introduce

$$\mathcal{E}_\rho(\tau) := \chi_\tau \left( \bigcup_{j=1}^d \mathcal{E}_\rho^{(j)} \right) \quad \text{and} \quad \mathcal{E}_\rho(\Gamma) := \bigcup_{\tau \in \mathcal{G} \setminus \mathcal{G}^{\text{rough}}} \mathcal{E}_\rho(\tau).$$

The following assumption concerns the overlap of these complex neighbourhoods.

**Assumption 3.4** There exists a constant  $C_{\text{ol}} > 0$  such that, for all  $x \in \Gamma$ , there holds

$$\#\{\tau \in \mathcal{G} \setminus \mathcal{G}^{\text{rough}} : x \in \mathcal{E}_\rho(\tau)\} \leq C_{\text{ol}}.$$

Note that Assumption 3.4 can be satisfied by a proper choice of  $c_1, c_2$  in Definition 2.3.

Finally, we introduce the set of functions which will be used to describe the regularity of the solution of (2.2). With the distance function  $r = \text{dist}(x, \Gamma^{\text{rough}})$ , and for  $\beta \in [0, 1)$ , we introduce the weighted space  $H_\beta^2(\Gamma)$  as the completion of  $C^\infty(\Gamma)$  under the norm

$$\|u\|_{H_\beta^2(\Gamma)}^2 := |u|_{H^1(\Gamma)}^2 + \|r^\beta \nabla^2 u\|_{L^2(\Gamma)}^2.$$

By  $\mathcal{A}_\beta(C, \gamma; \Gamma)$  we denote the set of functions on  $\Gamma$  that can be described in terms of countably normed spaces

$$\mathcal{A}_\beta(C, \gamma; \Gamma) := \{u \in H_\beta^2(\Gamma) : \|u\|_{H_\beta^2(\Gamma)} \leq C, \|r^{\beta+n} \nabla^{n+2} u\|_{L^2(\Gamma)} \leq C \gamma^n n! \quad \forall n \in \mathbb{N}\}. \quad (3.1)$$

Assume that our solution has a global Sobolev regularity  $u \in H^{\mu+\delta}(\Gamma)$  for some  $\delta > 0$ . For two-dimensional problems, the regularity results imply that the parameter  $\beta \in [0, 1)$  can be specified explicitly by  $\beta = 2 - \mu - \delta$  (cf. [1], [10], [22]). The regularity theory in terms of countably normed spaces  $\mathcal{A}_\beta(C, \gamma; \Gamma)$  is still an open question, hence, in the following, we employ, as a hypothesis, that the choice  $\beta = 2 - \mu - \delta$  is also valid in BEM applications.

The following remark recalls the well-known fact that controlling all higher derivatives of a function implies that it belongs to the class of analytic functions  $\mathcal{A}_{M,\rho}(\tau)$  (see e.g., [3] for the proof in the case  $\tau = I$ ).

**Remark 3.5** Assume that a function  $u : I \rightarrow \mathbb{R}$  satisfies for some  $C_u, \gamma_u \geq 0$

$$\left\| \frac{\partial^n u}{\partial x^n} \right\|_{L^\infty(I)} \leq C_u \gamma_u^n n! \quad \text{for all } n \in \mathbb{N}_0. \quad (3.2)$$

Then  $u \in \mathcal{A}_{M,\rho}(I)$  holds with  $\rho = 1 + \gamma_u^{-1} > 1$ ,  $M = C \cdot C_u$ . Similarly, if we control higher order gradients of a function  $u : B(\tau) \rightarrow \mathbb{R}$ , then  $u \in \mathcal{A}_{M,\rho}(\tau)$ .

### 3.2 Local polynomial approximation on $\tau \in \mathcal{G} \setminus \mathcal{G}^{\text{rough}}$

Due to classical results on the best polynomial approximation we know that for any  $f \in \mathcal{A}_{M,\rho}(I)$ , there holds

$$\inf_{v \in \mathbb{P}_N(I)} \|f - v\|_{C^0(I)} \leq M \rho^{-N}, \quad (3.3)$$

where  $\mathbb{P}_N(I)$  is the set of polynomials of degree  $N$  on  $I$ . Moreover, we have

$$\|f - I_N f\|_{C^0(I)} \leq cM (\log N) \rho^{-N}, \quad (3.4)$$

where  $I_N$  is the polynomial interpolation operator at the  $N + 1$  Chebyshev nodes on  $I$  (see, e.g., [31]) and  $c$  does not depend on  $f$ . The corresponding result for the  $W_\infty^1$ -norm reads as: For each  $1 < \rho_1 < \rho$ ,

$$\|f - I_N f\|_{W_\infty^1(I)} \leq CMN (\log N) \rho_1^{-N}. \quad (3.5)$$

Note that without loss of generality one can choose  $\rho = \rho_1$  in (3.4).

For multivariate functions  $f = f(x_1, \dots, x_d) : \mathbb{R}^d \rightarrow \mathbb{R}$ , we use the tensor product interpolant

$$\mathbf{I}_N f = I_N^1 \dots I_N^d f \in P_N[I_1^d],$$

where  $I_N^i f$  denotes the interpolation polynomial with respect to the variables  $x_i \in I_i := [-1, 1]$ ,  $i = 1, \dots, d$ , at the Chebyshev nodes.

**Proposition 3.6** *Let  $M > 0$  and  $\rho > 1$  be given. For all  $f \in \mathcal{A}_{M,\rho}(I^d)$  and  $N > 1$  the estimate*

$$\|f - \mathbf{I}_N f\|_{C^0(I^d)} \leq c M (\log^d N) \rho^{-N} \quad (3.6)$$

holds. Moreover, we have

$$\|f - \mathbf{I}_N f\|_{W_\infty^1(I^d)} \leq c M N^d (\log^d N) \rho^{-N}. \quad (3.7)$$

*Proof.* The proof of (3.6) is based on a multiple use of the triangle inequality in combination with the familiar estimate to the Lebesgue constant,

$$\|I_N\|_{L^\infty(I) \rightarrow L^\infty(I)} \leq c \log N$$

(see [17] for more details). The second statement is a consequence of (3.5). ■

Lemmata 3.7 and 3.8 below allow us to prove the optimal approximation results in  $L^2$ - and  $H^1$ -norms by the wire-basket  $hp$ -FEM for functions in  $\mathcal{A}_\beta(C, \gamma; \Gamma)$ . Then the result in the  $H^\mu$ -norm for  $\mu \in [0, 1]$ , follows by interpolation. To recover the almost optimal approximation order in  $H^\mu$ -norm with  $\mu < 0$ , we need some modification of the approximation space.

Our arguments here are similar to those from [22, 21], where the approximation theory in the  $H^1$ -norm was derived. We apply the explicit construction of the interpolation operator from [24] and provide the corresponding error analysis based on Proposition 3.6. Let  $I_{p_i,i}$  be the standard interpolation operator along the  $i$ th edge  $\gamma_i$  of the unit triangle  $\hat{\tau}$  at the  $p_i + 1$  Chebyshev nodes (properly scaled to the  $i$ th edge of  $\hat{\tau}$ ).

**Lemma 3.7** *Let  $u \in \mathcal{A}_{M,\rho}(\hat{\tau})$  for some  $M > 0$  and  $\rho > 1$ . For each  $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{N}^3$ , there exists a linear interpolation operator  $\pi_{\mathbf{p}} : C(\hat{\tau}) \rightarrow \mathbb{P}_{\mathbf{p}}(\hat{\tau})$  with  $p := \max_{1 \leq i \leq 3} p_i$  such that*

$$(\pi_{\mathbf{p}} u)|_{\gamma_i} = I_{p_i,i}(u|_{\gamma_i}), \quad i = 1, 2, 3.$$

Let  $C_0 := \min_{1 \leq i \leq 3} p_i/p$ . Then there exists  $b$  with  $0 < C_0 \lesssim b$  such that

$$\|u - \pi_{\mathbf{p}} u\|_{L^\infty(\hat{\tau})} \leq c M p^{3/2} (1 + \log p)^{5/2} \rho^{-bp}. \quad (3.8)$$

Moreover,

$$\|\nabla(u - \pi_{\mathbf{p}} u)\|_{L^2(\hat{\tau})} \leq c M p^4 (1 + \log^2 p) \rho^{-bp}. \quad (3.9)$$

*Proof.* We apply the interpolation operator  $\pi_{\mathbf{p}}$  constructed in [24]. Estimate (3.8) is proved by combining the bound

$$\|u - \pi_{\mathbf{p}} u\|_{L^\infty(\hat{\tau})} \leq c p^{3/2} (1 + \log p)^{1/2} \inf_{v \in \mathbb{P}_{\mathbf{p}}(\hat{\tau})} \|u - v\|_{L^\infty(\hat{\tau})}$$

(see [24, Th. 6.2.6] for the case  $p_1 = p_2 = p_3 = p$ ) with estimate (3.6) for  $N = C_0 p$ . The modification to the case  $p_1 < p_2 \leq p_3 = p$  is rather straightforward because, then,  $p_1 = C_0 p$  and  $b$  in (3.8) and (3.9) may depend on  $C_0$ .

Estimate (3.9) is more involved. We start from the bound

$$\|\nabla(u - \pi_{\mathbf{p}} u)\|_{L^2(\hat{\tau})} \leq \inf_{v \in \mathbb{P}_{\mathbf{p}}(\hat{\tau})} \{\|\nabla(u - v)\|_{L^2(\hat{\tau})} + C p^2 \|u - v\|_{L^2(\hat{\tau})}\}, \quad (3.10)$$

which is a simple modification of [24], (6.2.17). We then choose the element  $v = v_0$  as the Chebyshev interpolant of the extension of  $u$  to  $I^2$ . Thus, this function realises (cf. (3.7) with  $d = 2$ )

$$\|\nabla(u - v_0)\|_{L^2(\hat{\tau})} \leq c M p^2 (1 + \log^2 p) \rho^{-bp}.$$

Since  $v_0$  interpolates  $u$  at some point  $x_0 \in \hat{\tau}$ , i.e.,  $\min_{x \in \hat{\tau}} |(u - v_0)(x)| = 0$ , finally, we treat the simultaneous approximation (3.10) by using the Poincaré-Friedrichs inequality

$$\|u - v_0\|_{L^2(\hat{\tau})} \leq c \|\nabla(u - v_0)\|_{L^2(\hat{\tau})}$$

which completes the proof.  $\blacksquare$

Now we give some auxiliary approximation results with respect to the  $H^m$ -norm with  $m \in [0, 1]$ . Let  $I_{p_i, \gamma_i}$  denote the Gauss-Lobatto interpolant of degree  $p_i$  on the edge  $\gamma_i$  of the reference element  $\hat{\tau}$ . We apply the explicit construction of the interpolation operator in [24] and provide the corresponding error analysis based on Proposition 3.6.

The following interpolation error estimate on  $\tau \in \mathcal{G}$  is a consequence of Lemma 3.7. We employ the convention that if  $\tau$  and  $\mathbf{p} = (p_1, p_2, p_3)$  appear in the same context, then,  $p_i$  is defined as the maximal polynomial degree of the trace functions  $u|_{\gamma_i} \in \mathbb{P}_{p_i}$ ,  $1 \leq i \leq 3$ , for  $u \in S_{\mathbf{p}}^k(\tau)$ , where  $\gamma_i$  denotes the  $i$ th edge of  $\tau$ . For any  $\tau \in \mathcal{G}$ , we define the lifted interpolation operator  $\pi_\tau$  by

$$\pi_\tau f = \left( \pi_{\mathbf{p}} \hat{f} \right) \circ \chi_\tau^{-1} \quad \text{with} \quad \hat{f} = f \circ \chi_\tau \quad \text{and} \quad \chi_\tau \text{ as in Definition 3.3.}$$

For  $u \in \mathcal{A}_\beta(C_u, \gamma_u; \Gamma)$  and for  $\tau \in \mathcal{G}$ , we define (cf. [22])

$$C_\tau := \sqrt{\sum_{n=0}^{\infty} \frac{1}{(2\gamma_u)^{2n} (n!)^2} \|r^{n+\beta} \nabla^{n+2} u\|_{L^2(\tau)}^2}. \quad (3.11)$$

We recall that the polynomial degree on the mesh  $\mathcal{G}^{\text{rough}}$  equals  $p_{\text{rough}} \in \{0, 1\}$  (cf. Assumption 2.5).

**Lemma 3.8** *For each  $\Gamma_i$ ,  $1 \leq i \leq q$ , and  $m \in \{0, 1\}$ , let  $u \in H^{m+\delta} \cap \mathcal{A}_\beta(C, \gamma; \Gamma_i)$ . Then, for all  $\tau \in \mathcal{G} \setminus \mathcal{G}^{\text{rough}}$  there holds*

$$\|u - \pi_\tau u\|_{H^m(\tau)} \lesssim C_\tau h_\tau^{2-m-\beta} P_0(p) \rho^{-bp_\tau}, \quad (3.12)$$

where  $P_0$  is a polynomial of fixed degree that does not depend on  $u$ . For elements  $\tau$  with  $\tau \subset \Gamma_\delta^{\text{rough}}$  (cf. (2.6)) the following estimate is valid

$$\|u - \pi_\tau u\|_{H^m(\tau)} \lesssim h^{\min\{\delta, p_{\text{rough}}+1-m\}} \|u\|_{H^{m+\delta}(\tau)}. \quad (3.13)$$

*Proof.* The estimate (3.12) for  $m = 0$  follows from (3.8), applying the pull-back  $\pi_\tau u \circ \chi_\tau$  of approximation  $\pi_\tau$  (cf. Lemma 3.7) to each triangle  $\tau \in \mathcal{G} \setminus \mathcal{G}^{\text{rough}}$ . First, we estimate the constant  $M$  for the pull-back  $\hat{u} = u \circ \chi_\tau$ . To that end, we note that

$$\|r^{n+\beta} \nabla^{n+2} u\|_{L^2(\tau)}^2 \leq C_\tau (2\gamma_u)^n n! \quad \forall \tau \in \mathcal{G} \quad (3.14)$$

(cf. (3.11)) and, moreover,  $\sum_{\tau \in \mathcal{G}} C_\tau^2 \leq \frac{4}{3} C_u^2$  with  $C_\tau$  as in (3.11). Using (3.14), one can see that  $\hat{u}$  satisfies

$$\|\nabla^{n+2} \hat{u}\|_{L^2(\hat{\tau})} \lesssim C h_\tau^{n+1} \|\nabla^{n+2} u\|_{L^2(\tau)} \quad (3.15)$$

$$\begin{aligned} &\leq C h_\tau^{1-\beta} \|r^{n+\beta} \nabla^{n+2} u\|_{L^2(\tau)} \\ &\leq C_\tau h_\tau^{1-\beta} (2\gamma_u)^n n! \end{aligned} \quad (3.16)$$

(cf. [22]). Now Remark 3.5 implies that  $\hat{u} \in \mathcal{A}_{M, \rho}(\hat{\tau})$  with  $M \lesssim C_\tau h_\tau^{1-\beta}$  and  $\rho = 1 + (2\gamma_u)^{-1}$ . Then we have

$$\begin{aligned} \|u - \pi_\tau u\|_{L^2(\tau)} &\lesssim h_\tau \|\hat{u} - \pi_\tau \hat{u}\|_{L^2(\hat{\tau})} \\ &\lesssim h_\tau P_0(p) \rho^{-bp_\tau} \max_{x \in \mathcal{E}_\rho(\hat{\tau})} |\hat{u}| \\ &\lesssim C_\tau h_\tau^{2-\beta} P_0(p) \rho^{-bp_\tau}. \end{aligned}$$

Using (3.9), the case  $m = 1$  can be proven similarly. For elements lying in a  $\delta$ -neighbourhood of  $\Gamma^{\text{rough}}$  (cf. (2.6)), we apply standard finite element error estimation to obtain

$$\|u - \pi_\tau u\|_{H^m(\tau)} \lesssim h^{\min\{\delta, p_{\text{rough}}+1-m\}} \|u\|_{H^{m+\delta}(\tau)}. \quad \blacksquare$$

### 3.3 Approximation of $H^{\mu+\delta}(\Gamma) \cap \mathcal{A}_\beta(C, \gamma; \Gamma)$ -functions by $S_{\mathbf{p}}^k(\mathcal{G})$ for $\mu \geq 0$

In this Section, we will prove optimal approximation results by our  $hp$ -BEM for functions in  $H^{\mu+\delta}(\Gamma) \cap \mathcal{A}_\beta(C, \gamma; \Gamma)$  for certain range of parameters  $\mu, \delta, \beta$ . The regularity results for the Sobolev space  $H^{\mu+\delta}(\Gamma)$  on the Lipschitz surfaces are well presented in the literature on BEM. Concerning the conditions which guarantee certain regularity in countably normed spaces we refer to [1], [10], [19], [26] and literature therein.

We will need an assumption concerning the geometric structure of the wire-basket.

**Assumption 3.9** *There exists a constant  $\omega_0 > 0$  independent of  $h$  such that for all  $\omega > \omega_0$  and  $\beta > \omega h$ , the measure of the subregions*

$$F(\omega, \beta) := \{x \in \Gamma : \omega h \leq \text{dist}_{\text{geo}}(x, \Gamma^{\text{rough}}) \leq \beta\}$$

satisfies

$$|F(\omega, \beta)| \lesssim \beta - \omega h,$$

where the constant being hidden in the “ $\lesssim$ ”-estimate may depend on the length of the boundary  $\partial\Gamma^{\text{rough}}$  but not on  $h$ .

Next, we introduce on each component  $\Gamma_i$ ,  $1 \leq i \leq q$ , a layer-type structure in the triangulation  $\mathcal{G}$ . Let  $L$  be the largest integer such that  $(\text{diam } \Gamma) 2^{-L} > \omega_0 h$  with  $\omega_0$  as in Assumption 3.9. For  $0 \leq \ell < L$ , we define the subgrids  $\mathcal{G}_\ell$  by

$$\mathcal{G}_\ell := \{\tau \in \mathcal{G} : (\text{diam } \Gamma) 2^{-\ell-1} \leq \text{dist}_{\text{geo}}(\tau, \Gamma^{\text{rough}}) < (\text{diam } \Gamma) 2^{-\ell}\} \quad \text{and} \quad \mathcal{G}_L := \mathcal{G} \setminus \left( \bigcup_{\ell=0}^{L-1} \mathcal{G}_\ell \right) \quad (3.17)$$

and the subregions  $\gamma_\ell$  by

$$\gamma_\ell := \{x \in \Gamma : (\text{diam } \Gamma) 2^{-\ell-1} \leq \text{dist}_{\text{geo}}(x, \Gamma^{\text{rough}}) \leq (\text{diam } \Gamma) 2^{-\ell}\}. \quad (3.18)$$

In the following lemma, the symbol  $O(\dots)$  means the two-sided estimate.

**Lemma 3.10** *Let  $\mathcal{G}$  be a mesh as in Definition 2.3 and derive the distribution of the polynomial degrees according to (2.8). Then, for all  $0 \leq \ell \leq L$  and all  $\tau \in \mathcal{G}_\ell$  we have*

$$h_\tau = O(2^{-\ell}), \quad p_\tau \leq C(1 + L - \ell), \quad N_\ell := \sum_{\tau \in \mathcal{G}_\ell} 1 \leq C2^\ell \quad (3.19)$$

and, for  $\ell < L$ ,

$$\text{dist}_{\text{geo}}(\tau, \Gamma^{\text{rough}}) = O(2^{-\ell}). \quad (3.20)$$

*Proof.* Let  $\tau \in \mathcal{G}_\ell$ . The estimate  $h_\tau = O(2^{-\ell})$  follows by combining Definition 2.3 and (3.17). The estimate  $p_\tau = O(1 + \gamma(L - \ell))$  is a consequence of the previous one, (2.8), and Assumption 2.5.c, while (3.20) follows from (3.17). It remains to estimate the number of elements in  $\mathcal{G}_\ell$ . All triangles  $\tau \in \mathcal{G}_\ell$  satisfy  $h_\tau = O(2^{-\ell})$  and the shape regularity of the mesh (cf. Definition 2.3) implies the estimate  $|\tau| = O(2^{-2\ell})$  for the area of  $\tau$ . From Assumption 3.9 we conclude (cf. (3.18))

$$\gamma_\ell = F\left(\left(\text{diam } \Gamma\right) \frac{2^{-\ell-1}}{h}, \left(\text{diam } \Gamma\right) 2^{-\ell}\right)$$

and for the area we obtain

$$|\gamma_\ell| \leq C2^{-\ell}.$$

Comparing this area with the area  $|\tau|$  leads to the estimate for  $N_\ell$ . ■

**Theorem 3.11** *Assume  $u \in H^s(\Gamma) \cap \mathcal{A}_\beta(C_u, \gamma_u; \Gamma)$  for some  $s \geq \mu$ . Let  $b > 0$  be as in (3.8),  $\mathcal{G}$  be a geometric mesh with mesh size  $h$  (cf. (1.1)) and let  $\mathbf{p} = \{p_\tau\}$  be a linear degree vector on  $\mathcal{G}$  with slope  $\gamma > 0$  provided that  $2 - \beta - b\gamma < 0$ . Let Assumptions 3.4 and 3.9 be satisfied. For  $k = -1, 0$ , let  $S_{\mathbf{p}}^k(\mathcal{G})$  be defined as in (2.7).*

*Then for each  $m \in [0, \min\{s, k + 1\}]$ , there exists a constant  $C > 0$  depending on  $u$  and  $\beta$  such that*

$$\inf\{\|u - v\|_{H^m(\Gamma)} : v \in S_{\mathbf{p}}^k(\mathcal{G})\} \leq Ch^{\min\{s, p_{\text{rough}} + 1, 2 - \beta\} - m}. \quad (3.21)$$

*Proof.* First, we will consider the case  $m = 0$ . Based on Lemma 3.8, we explicitly construct an element  $\pi u \in S_{\mathbf{p}}^k(\mathcal{G})$  providing an optimal approximation property in the  $L^2(\Gamma)$ -norm. For any  $\tau \in \mathcal{G}_L$  there holds  $p_\tau = p_{\text{rough}}$ . The combination of Remark 2.4 and (3.13) yields

$$\|u - I_\tau u\|_{L^2(\tau)} \leq ch^{\min\{s, p_{\text{rough}}+1\}} \|u\|_{H^s(\tau)}, \quad (3.22)$$

where  $I_\tau$  is the linear interpolant in the case of continuous boundary elements while it is the  $L^2(\tau)$ -orthogonal projection for discontinuous boundary elements. For  $\ell = 0, 1, \dots, L-1$  and  $\tau \in \mathcal{G}_\ell$ , we apply the results of Lemma 3.8 elementwise to obtain

$$\begin{aligned} \sum_{\ell=0}^{L-1} \sum_{\tau \in \mathcal{G}_\ell} \|u - \pi_\tau u\|_{L^2(\tau)}^2 &\lesssim \sum_{\ell=0}^{L-1} \rho^{-2b\gamma(L-\ell)} \sum_{\tau \in \mathcal{G}_\ell} h_\tau^{2(2-\beta)} C_\tau^2 \\ &\lesssim h^{2(2-\beta)} \sum_{\ell=0}^{L-1} 2^{2(2-\beta)(L-\ell)-2b\gamma(L-\ell)} C_u^2 \\ &\lesssim h^{2(2-\beta)} C_u^2, \end{aligned} \quad (3.23)$$

taking into account that  $2 - \beta - b\gamma < 0$ , by assumption. In the estimate above we made use of the *finite overlapping property* between the regularity ellipses  $\mathcal{E}_\rho(\tau)$  for different  $\tau \in \mathcal{G}_\ell$  (cf. Assumption 3.4). The combination of (3.23) and (3.22) completes our proof in the case  $m = 0$ .

For  $k = 0$ , we have to consider the case  $m = 1$  which can be treated similarly (cf. [22]). The estimate for the intermediate indices follows by interpolation.  $\blacksquare$

### 3.4 Approximation in Sobolev norm $H^{-\mu}$ , $\mu > 0$

Now we discuss the approximation properties of functions  $u \in H^{\mu+\delta}(\Gamma) \cap \mathcal{A}_\beta(C, \gamma; \Gamma)$  with respect to the Sobolev norm with negative indices  $H^{-\mu}$ ,  $\mu > 0$ . Let  $v_N \in S_{\mathbf{p}}^k(\mathcal{G})$  be the best approximation to  $u$  in the  $L^2$ -norm. Then, with an arbitrary  $z_N \in S_{\mathbf{p}}^k(\mathcal{G})$ , there holds

$$\|u - v_N\|_{H^{-\mu}} = \sup_{z \in H^\mu \setminus \{0\}} \frac{(u - v_N, z - z_N)}{\|z\|_{H^\mu}},$$

and we readily obtain

$$\|u - v_N\|_{H^{-\mu}} \leq C \|u - v_N\|_0 \sup_{z \in H^\mu \setminus \{0\}} \inf_{z_N \in S_{\mathbf{p}}^k(\mathcal{G})} \frac{\|z - z_N\|_0}{\|z\|_{H^\mu}}. \quad (3.24)$$

Since we approximate on large panels with high order polynomials we cannot gain from the term

$$\inf_{z_N \in S_{\mathbf{p}}^k(\mathcal{G})} \|z - z_N\|_0 \leq C \|z\|_{H^\mu}$$

since  $z \in H^\mu$ , in general, is not smooth enough to make use of the high-order polynomial degrees on the elements with proper distance to the wire basket. To improve the approximation properties in (3.24) we modify the approximation space as follows. Fix the surface patch  $\Gamma_i$ . Choose a coarse mesh parameter  $H = \sqrt{h}$  and introduce the domain  $\Gamma_H \subset \subset \Gamma_i$  by

$$\Gamma_H := \{x \in \Gamma_i : \text{dist}(x, \partial\Gamma_i) \geq H\}.$$

Let us modify the original triangulation in such a way that it remains a quasi-uniform mesh of the size  $H$  in the domain  $\Gamma_H$  (cf. Fig. 5 with  $N_\Gamma = 4 \cdot 2^L$ ). Correspondingly to the new triangulation, we introduce the modified space  $S_{H, \mathbf{p}}^k(\mathcal{G})$ , where the polynomial degrees on the elements in  $\Gamma_H$  are all chosen as a constant corresponding to that for the elements of the original space  $S_{\mathbf{p}}^k(\mathcal{G})$  on level  $L_{\text{unif}} \approx L/2$ , thus having the diameter  $O(H) = O(\sqrt{h})$ .

By standard mesh refinement techniques it is easy to subdivide the triangles  $\tau$  in  $\mathcal{G}$  with  $\text{diam}(\tau) > \sqrt{h}$  (see Fig. 5, where we have  $N_\Gamma = 4 \cdot 2^L$ ). The polynomial degree vector for the refined mesh is chosen according to (2.8) by replacing the ratio  $h_\tau/h$  by  $\min\{h_\tau, \sqrt{h}\}/h$  and the definition of  $S_{H, \mathbf{p}}^k(\mathcal{G})$  would correspond to (2.7) by using the new degree vector and the new mesh. One can easily verify the following properties:

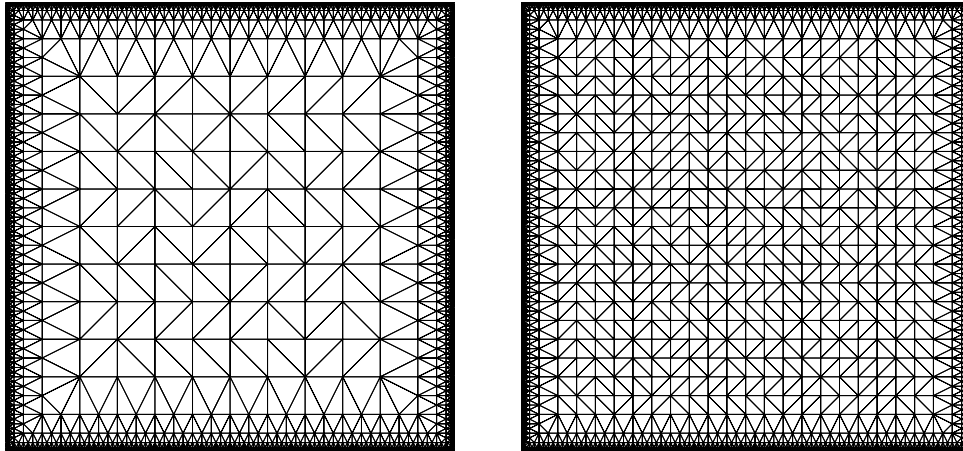


Figure 5: Modified BCM:  $L = 8, L_{unif} = 3$  (left),  $L = 8, L_{unif} = 4$  (right).

- The number of unknowns for the new method is of the same order as for  $S_{\mathbf{p}}^k(\mathcal{G})$ . In fact, from Lemma 3.10, one derives that the number of unknowns for the original mesh and for the original polynomial degree vector is  $O(h^{-1}(\log h^{-1})^3)$ , while the number of unknowns for the refined mesh and modified degree vector is  $O(h^{-1}(\log h^{-1})^3 + h^{-1}(\log h^{-1})^2)$ .
- The approximation property holds:

$$\inf_{z_N \in S_{H,\mathbf{p}}^k(\mathcal{G})} \|z - z_N\|_0 \leq C H^\mu \|z\|_{H^\mu(\Gamma)}. \quad (3.25)$$

Now we arrive at the following result.

**Corollary 3.12** *Let the assumptions of Theorem 3.11 be valid. Then for  $\mu = 1/2$ , there exists a constant  $C > 0$  depending on  $u$  and  $\beta$  such that*

$$\inf\{\|u - v\|_{H^{-\mu}(\Gamma)} : v \in S_{H,\mathbf{p}}^k(\mathcal{G})\} \leq C h^{\mu/2 + \min\{s, \text{prouph} + 1, 2 - \beta\}} (\|u\|_{H^s(\Gamma)} + C_u). \quad (3.26)$$

Due to this corollary we see a reduced gap of  $h^{\mu/2}$  between the optimal result and (3.26), which certainly improves (3.24), where  $\inf_{z_N \in S_{\mathbf{p}}^k(\mathcal{G})} \frac{\|z - z_N\|_0}{\|z\|_{H^\mu}} \leq C$  (cf. 3.25).

## 4 Hierarchical clustering in wire-basket BEM

In this section, we construct and analyse an hierarchical matrix approximation to the exact stiffness matrix  $\mathbf{A}$ , corresponding to the Galerkin BEM (cf. (2.5)). In Subsection 4.1 we recall the basic concepts of panel-clustering and  $\mathcal{H}$ -matrices, which are applied verbatim to the wire-basket BEM and introduce the basic notations. For a more detailed introduction to panel-clustering and  $\mathcal{H}$ -matrices we refer, e.g., to [14], [18], [28].

In Subsection 4.2, the corresponding error analysis is developed and the complexity estimates are derived. These results are new and differ from the standard estimates since (a) highly non-uniform meshes occur in wire-basket BEM and (b) the polynomial degrees are given by a *variable* degree vector.

### 4.1 Construction of the hierarchical clustering

We restrict ourselves here to the Galerkin boundary element discretisation of boundary integral equations while collocation or Nyström discretisations can be considered as well.

Note that the representation of the term  $\lambda(u, v)$  in (2.3) with respect to the nodal basis  $(b_i)_{i \in \Theta}$  leads to a sparse matrix. (The number of non-zero entries of this matrix is bounded from above by

$$C \sum_{\ell=0}^L \sum_{\tau \in \mathcal{G}_\ell} p_\tau^4 \leq C \sum_{\ell=0}^L \sum_{\tau \in \mathcal{G}_\ell} (1 + L - \ell)^4 \leq C \sum_{\ell=0}^L (1 + L - \ell)^4 2^\ell \leq C 2^L \sum_{\ell=0}^{\infty} (1 + \ell)^4 2^{-\ell} \leq CN,$$

where  $N$  is the number of degrees of freedom located in  $\Gamma^{\text{rough}}$ ). Hence, we concentrate on the sparse representation of the bilinear form  $\hat{a}(\cdot, \cdot)$  (cf. (2.4)). The matrix corresponding to the bilinear form  $\hat{a}$  is given by

$$\mathbf{A} = (\hat{a}(b_j, b_i))_{i,j \in \Theta} = \int_{\Gamma} b_i(x) \int_{\Gamma} k(x, y) b_j(y) ds_y ds_x \quad (4.1)$$

and, due to the non-localness of the kernel function, is fully populated. In (4.1),  $k$  is the kernel function corresponding to one of the forms in (2.4).

If the dimension of  $S$  is very large, iterative solvers should be employed for the solution of the arising linear system. Such solvers require a matrix-vector multiplication as a basic operation while the knowledge of all matrix entries of  $\mathbf{A}$ , typically, is not needed explicitly. The idea of the panel-clustering algorithm is to represent the bilinear form  $\hat{a}(\cdot, \cdot)$  in an alternative way so that a matrix-vector multiplication can be performed approximately. Here, we will generalise the panel-clustering method to the  $hp$ -discretisation in our applications.

We start with the description of the general idea. Let  $\Theta$  denote the index set of unknowns. The data-sparse representation of integral operators starts with the definition of the clusters and a cluster tree.

**Definition 4.1 (Cluster)** *A cluster is a non-empty subset of  $\Theta$ . The support and the diameter of a cluster  $c$  are given by*

$$\Gamma_c := \text{supp} \sum_{i \in c} b_i \quad \text{and} \quad \text{diam } c := \text{diam } \Gamma_c,$$

where  $b_i$  are the basis functions from (4.1). The cluster-box  $Q_c$  is the minimal axis-parallel box which contains  $\Gamma_c$  and the cluster centre  $Z_c$  is the centre of mass of  $Q_c$ . The distance of two clusters  $c, s$  is given by

$$\text{dist}(c, s) := \text{dist}(\Gamma_c, \Gamma_s).$$

For a finite element function  $u = \sum_{i \in \Theta} u_i b_i$ , its restriction to  $c$  is denoted by

$$u_c := \sum_{i \in c} u_i b_i.$$

**Definition 4.2 (Cluster Tree)** *A cluster tree  $\mathbb{T}$  is a tree<sup>1</sup> whose vertices (called ‘‘clusters’’) are certain subsets of  $\Theta$ . These are required to satisfy the following properties:*

- (i)  $\Theta$  is the root of  $\mathbb{T}$ .
- (ii)  $\mathcal{L}(\mathbb{T}) = \{\{i\} : i \in \Theta\}$ , where  $\mathcal{L}(\mathbb{T})$  denotes the set of leaves of  $\mathbb{T}$ .
- (iii) If  $\sigma \in \mathbb{T}$  is not a leaf, there is a set of vertices of  $\mathbb{T}$  (denoted  $\text{sons}(\sigma)$ ) such that  $\sigma$  is the disjoint union:  $\sigma = \cup_{\sigma' \in \text{sons}(\sigma)} \sigma'$ .

There are standard procedures for constructing cluster trees (see for example [11, Example 2.1]). Once  $\mathbb{T}$  has been constructed, a second tree,  $\mathbb{T}_2$ , whose vertices are pairs of clusters may be constructed with the following properties:

**Definition 4.3**  $\mathbb{T}_2$  is uniquely defined by

- (i)  $(\Theta, \Theta) \in \mathbb{T}_2$  is the root of  $\mathbb{T}_2$ ,
- (ii) For  $\mathbf{b} = (\sigma', \sigma'') \in \mathbb{T}_2$ , the set of sons is defined as follows:

$$\text{sons}(\mathbf{b}) := \begin{cases} \text{sons}(\sigma') \times \text{sons}(\sigma'') & \text{if } \sigma', \sigma'' \in \mathbb{T} \setminus \mathcal{L}(\mathbb{T}), \\ \{\sigma'\} \times \text{sons}(\sigma'') & \text{if } \mathbf{b} \in \mathcal{L}(\mathbb{T}) \times \mathbb{T} \setminus \mathcal{L}(\mathbb{T}), \\ \text{sons}(\sigma') \times \{\sigma''\} & \text{if } \mathbf{b} \in \mathbb{T} \setminus \mathcal{L}(\mathbb{T}) \times \mathcal{L}(\mathbb{T}), \\ \emptyset & \text{if } \mathbf{b} \in \mathcal{L}(\mathbb{T}) \times \mathcal{L}(\mathbb{T}). \end{cases}$$

<sup>1</sup>Usually a tree is a graph  $(V, E)$  with vertices  $V$  and edges  $E$  having a certain structure. Here the structure will be given by the sons of the vertices (defined below), while  $V$  is identified with  $\mathbb{T}$ .



The key point in the hierarchical clustering algorithm is to select pairs of clusters  $(\sigma', \sigma'') \in \mathbb{T}_2$  and to approximate the corresponding integrals by replacing the kernel  $k$  of the integral operator by a suitable separable expansion. This cannot be done on all pairs of clusters, but only on pairs which are sufficiently far apart relative to their diameters. This leads to the definition of an admissible pair of clusters:

**Definition 4.4 (Admissible Pair)** For  $\eta > 0$ , a pair  $(\sigma', \sigma'') \in \mathbb{T}_2$  is called  $\eta$ -admissible if

$$\max\{\text{diam } \sigma', \text{diam } \sigma''\} \leq 2\eta \text{ dist}(\sigma', \sigma''). \quad (4.2)$$

Using the concept of admissibility, the integration domain  $\Gamma \times \Gamma$  in (4.1) is split into a nearfield and a farfield, characterised by the subsets  $P_{\text{far}}$  (“farfield”) and  $P_{\text{near}}$  (“nearfield”) of  $\mathbb{T}_2$ , defined as follows.

First set  $P_{\text{near}} = \emptyset = P_{\text{far}}$ , and then initiate a call **divide**( $\Gamma, \Gamma$ ) of the following recursive procedure:

```

procedure divide( $\sigma', \sigma''$ );
begin  if ( $\sigma', \sigma''$ ) is  $\eta$ -admissible then  $P_{\text{far}} := P_{\text{far}} \cup \{(\sigma', \sigma'')\}$ 
        else  if ( $\sigma', \sigma''$ ) is a leaf then  $P_{\text{near}} := P_{\text{near}} \cup \{(\sigma', \sigma'')\}$ 
        else  for all ( $c', c''$ )  $\in$  sons( $\sigma', \sigma''$ ) do divide( $c', c''$ )
end;

```

As a result of this call,  $P := P_{\text{near}} \cup P_{\text{far}}$  describes a non-overlapping partitioning of  $\Theta \times \Theta$  in the sense that  $\cup\{\sigma' \times \sigma'' : (\sigma', \sigma'') \in P\} = \Theta \times \Theta$  and all contributions  $\sigma' \times \sigma''$  have empty intersection.

In this light, the part of the bilinear form associated with the integral operator,

$$\hat{a}(u, v) = (\mathcal{K}u, v)_{L^2(\Gamma)} = \int_{\Gamma} v(x) \int_{\Gamma} k(x, y) u(y) ds_y ds_x, \quad (4.3)$$

can be written in the form

$$\hat{a}(u, v) = \sum_{\mathbf{b}=(\sigma, s) \in P} \sum_{(i, j) \in \mathbf{b}} v_i u_j \int_{\Gamma_{\sigma} \times \Gamma_s} b_i(x) k(x, y) b_j(y) ds_y ds_x.$$

The goal is to approximate the kernel functions on admissible blocks  $\Gamma_{\sigma} \times \Gamma_s$  by a separable expansion with respect to appropriate function systems  $\Phi_c^{\nu} : \Gamma_c \rightarrow \mathbb{R}$  and  $\Psi_c^{\mu} : \Gamma_c \rightarrow \mathbb{R}$ , for all  $c \in \mathbb{T}$  and  $\nu \in I_m$ . Here  $I_m$  denotes an index set and  $m$  the approximation order. In typical examples, the function systems could be the tensorised three-dimensional Lagrange basis functions restricted to the surface patches  $\Gamma_c$ . In this case,  $I_m$  is the index set

$$I_m := \{\nu \in \mathbb{N}_0^3 : 0 \leq \nu_i \leq m \text{ for all } 1 \leq i \leq 3\}.$$

Let  $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$ . For  $x \in \Gamma_{\sigma}$ ,  $y \in \Gamma_s$ , we use a separable approximation  $k_{\mathbf{b}}(x, y) \approx k(x, y)$  of the form:

$$k_{\mathbf{b}}(x, y) := \sum_{\nu \in I_m, \mu \in I_m} \mathbf{k}_{\mathbf{b}}^{(\nu, \mu)} \Phi_{\sigma}^{(\nu)}(x) \Psi_s^{(\mu)}(y). \quad (4.4)$$

For kernel functions which are related to linear elliptic PDEs of second order with constant coefficients one can prove (cf. [28] and references therein) the exponential convergence estimate

$$|k(x, y) - k_{\mathbf{b}}(x, y)| \leq C_1 \frac{(\eta')^m}{\text{dist}(\sigma, s)^{\kappa}} \quad (4.5)$$

for all  $x \in \Gamma_{\sigma}, y \in \Gamma_s$  and  $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$ , where  $\eta' = C_2 \eta$  for some constant  $C_2$  and  $\eta$  as in Definition 4.4 can be chosen such that  $\eta' < 1$ . The number  $\kappa > 0$  is the blow-up rate of the kernel

$$|k(x, y)| \leq C_3 |x - y|^{-\kappa}, \quad x, y \in \Gamma, x \neq y. \quad (4.6)$$

Note that the constants  $C_1$  and  $C_2$  are independent of the clusters. In the following, we assume that (4.5) holds.

The panel-clustering approximation of the bilinear form  $\hat{a}$  in (2.2) acting on the finite-dimensional space  $S \times S$  is given by

$$\hat{a}_{\text{pc}}(u, v) = \sum_{\mathbf{b}=(\sigma, s) \in P} \sum_{(i, j) \in \mathbf{b}} v_i u_j \int_{\Gamma_{\sigma} \times \Gamma_s} b_i(x) \mathbf{k}_{\mathbf{b}}(x, y) b_j(y) ds_y ds_x \quad (4.7)$$

and

$$k_{\mathbf{b}}(\mathbf{x}, \mathbf{y}) := \begin{cases} k(\mathbf{x}, \mathbf{y}) & x \in \Gamma_{\sigma}, y \in \Gamma_s \text{ with } \mathbf{b} = (\sigma, s) \in P_{\text{near}}, \\ k_{\mathbf{b}}(\mathbf{x}, \mathbf{y}) & x \in \Gamma_{\sigma}, y \in \Gamma_s \text{ with } \mathbf{b} = (\sigma, s) \in P_{\text{far}}. \end{cases} \quad (4.8)$$

The algorithmic realisation of (4.7) is as in the standard panel-clustering algorithm (cf. [28]). We skip the details here and proceed with the error analysis.

## 4.2 Error analysis

The replacement of the kernel function in (4.3) by the panel-clustering approximation  $k_{\mathbf{p}}$  leads to the perturbed bilinear form  $\hat{a}_{\text{pc}}$ . We will employ Strang's lemma to prove stability and consistency of the perturbed Galerkin method. First, we will formulate suitable assumptions on the geometry of  $\Gamma$ , on the finite element mesh  $\mathcal{G}$ , and on the construction of the cluster tree.

Let  $Q_{\Gamma}$  denote the minimal axis-parallel 3D-box containing  $\Gamma$ . By subdividing  $Q_{\Gamma}$  step-by-step into eight congruent sub-boxes results an (infinite, virtual) octree  $\mathcal{Q}$ . The set  $\mathcal{Q}_{\ell}$  contains all boxes in  $\mathcal{Q}$  which have depth  $\ell$  in  $\mathcal{Q}$ . We assume conventionally that the boxes in  $\mathcal{Q}_{\ell}$  are pairwise disjoint, i.e., any face, edge and vertex belongs to only one box, and the union of the boxes is in  $Q_{\Gamma}$ .

We identify the cluster tree with the set of clusters

$$\mathbb{T} := \{c \subset \Theta \mid \exists Q \in \mathcal{Q} : Z_{\{i\}} \subset Q \text{ for all } i \in c\}$$

and the set sons( $c$ ) of a cluster  $c$  is the minimal subset of  $\mathbb{T}$  so that its union is  $c$ . If such a set does not exist, then,  $c$  is called a leaf  $c \in \mathcal{L}(\mathbb{T})$ . The subsets  $\mathbb{T}_{\ell} \subset \mathbb{T}$  contain all clusters with depth  $\ell$  in the cluster tree.

### Assumption 4.5

1. There exists a constant  $C_4$  such that

$$\forall c \in \mathbb{T} : C_4^{-1} (\text{diam } c)^2 \leq |\Gamma_c| \leq C_4 (\text{diam } c)^2. \quad (4.9)$$

2. There exists a constant  $C_5$  independent of  $\ell$  such that for all  $\ell$

$$\forall c \in \mathbb{T}_{\ell} : C_5^{-1} 2^{-\ell} \leq |\Gamma_c| \leq C_5 2^{-\ell}. \quad (4.10)$$

3. There exists  $C_6$  such that

$$C_6^{-1} h \leq 2^{-L} \leq C_6 h,$$

where  $L$  denotes the minimal constant such that  $\mathbb{T}_{\ell} = \emptyset$  for all  $\ell > L$ .

The next assumption concerns the partitioning  $P$  of  $\Theta \times \Theta$ . For  $c \in \mathbb{T}$ , let

$$P_{\text{far}}(c) := \{s \in \mathbb{T} : (c, s) \in P_{\text{far}}\} \quad \text{and} \quad U_c := \bigcup_{s \in P_{\text{far}}(c)} \Gamma_s.$$

The following assumption expresses the fact that the triangles in  $\mathcal{G}$  are shape regular and, for sufficiently small control parameter  $\eta = O(1)$  in (4.2), the set  $P_{\text{far}}(c)$  is quasi-uniform.

### Assumption 4.6

1. There exists a constant  $C_7$  such that

$$\forall c \in \mathbb{T} : |U_c| \leq C_7 (\text{diam } c)^2. \quad (4.11)$$

2. There exists a constant  $C_8$  independent of  $\ell$  such that

$$\forall x \in \Gamma : \#\{c \in \mathbb{T}_{\ell} : x \in U_c\} \leq C_8. \quad (4.12)$$

Assumption 4.5 and 4.6 allow to estimate the perturbation in the bilinear form.

**Lemma 4.7** *Let Assumption 4.5 and 4.6 be satisfied. For the kernel approximation we assume (4.5). Then,*

$$\forall u, v \in S : |\hat{a}(u, v) - \hat{a}_{\text{pc}}(u, v)| \leq C(h) (\eta')^m \|v\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)}.$$

The function  $C(h)$  is given by

$$C(h) := C_9 \begin{cases} 1 & \kappa < 2, \\ |\log h| & \kappa = 2, \\ h^{-1} & \kappa = 3, \end{cases} \quad (4.13)$$

where  $C_9$  is positive and depends continuously on  $C_1, \eta, \kappa, C_4, C_5, C_6, C_7, C_8$ .

*Proof.* For all  $u, v \in S$ , we have

$$\begin{aligned}
|\hat{a}(u, v) - \hat{a}_{\text{pc}}(u, v)| &\stackrel{(4.5)}{\leq} C_1(\eta')^m \sum_{(\sigma, s) \in P_{\text{far}}} \int_{\Gamma_\sigma} \int_{\Gamma_s} \frac{|v_\sigma(x)| |u_s(y)|}{\text{dist}(\sigma, s)^\kappa} ds_y ds_x \\
&\leq C_1(\eta')^m \sum_{\ell=0}^L \sum_{\sigma \in \mathbb{T}_\ell} \sum_{s \in P_{\text{far}}(\sigma)} \|v_\sigma\|_{L^1(\Gamma_\sigma)} \|u_s\|_{L^1(\Gamma_s)} \text{dist}(\sigma, s)^{-\kappa} \\
&\stackrel{(4.2), (4.9), (4.11)}{\leq} C_1(\eta')^m (2\eta)^\kappa \sqrt{C_4 C_7} \sum_{\ell=0}^L \sum_{\sigma \in \mathbb{T}_\ell} (\text{diam } \sigma)^{2-\kappa} \|v_\sigma\|_{L^2(\Gamma_\sigma)} \|u_s\|_{L^2(U_\sigma)} \\
&\leq \tilde{C}(\eta')^m \sum_{\ell=0}^L 2^{\ell(\kappa-2)} \sum_{\sigma \in \mathbb{T}_\ell} \|v_\sigma\|_{L^2(\Gamma_\sigma)} \|u_s\|_{L^2(U_\sigma)} \\
&\leq \tilde{C} C_8 (\eta')^m \|v\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)} \sum_{\ell=0}^L 2^{\ell(\kappa-2)}.
\end{aligned}$$

where  $\tilde{C}$  is positive depending continuously only on  $C_1, \eta, \kappa, C_4, C_5$ , and  $C_7$ .

For the sum in the last estimate, we obtain

$$\sum_{\ell=0}^L 2^{\ell(\kappa-2)} \leq C_{10} \begin{cases} 1 & \kappa < 2, \\ |\log h| & \kappa = 2, \\ h^{-1} & \kappa = 3. \end{cases}$$

In order to prove existence, uniqueness, and quasi-optimal error estimates for the Galerkin discretisation with panel-clustering via Strang's lemma we will show ■

$$\forall u, v \in S : |\hat{a}(u, v) - \hat{a}_{\text{pc}}(u, v)| \leq Ch^\alpha \|v\|_{H^\mu(\Gamma)} \|u\|_{H^\mu(\Gamma)},$$

where  $\alpha = \min\{s, 2 - \beta, p_{\text{rough}} + 1\} - \mu_+$  with  $s, \mu, \beta$  as in Theorem 3.11 and  $\mu_+ := \mu$  for  $\mu \geq 0$  and  $\mu_+ := \mu/2$  for negative  $\mu$ .

**Theorem 4.8** *Let the assumptions of Subsection 2.3 and of Theorem 3.11 be satisfied. Choose*

$$m := \left\lceil (\alpha + 1) \frac{|\log h|}{|\log \eta'|} \right\rceil \quad \text{with} \quad \alpha = \min\{\delta, 2 - \beta, p_{\text{rough}} + 1\} - \mu_+ \tag{4.14}$$

*as the expansion order for the panel-clustering algorithm. Then, the Galerkin discretisation with panel clustering has a unique solution  $\tilde{u}_S \in S$  for sufficiently small  $h$  which satisfies the error estimate*

$$\|u - \tilde{u}_S\|_{H^\mu(\Gamma)} \leq C_u h^\alpha.$$

*Proof.* The definition of  $m$  and  $C(h)$  (cf. (4.14) and (4.13)) leads to the estimate

$$\forall u, v \in S : |\hat{a}(u, v) - \hat{a}_{\text{pc}}(u, v)| \leq C_9 h^\alpha \|v\|_{L^2(\Gamma)} \|u\|_{L^2(\Gamma)}.$$

For  $\mu \geq 0$ , we conclude that

$$\forall u, v \in S : |\hat{a}(u, v) - \hat{a}_{\text{pc}}(u, v)| \leq C_9 h^\alpha \|v\|_{H^\mu(\Gamma)} \|u\|_{H^\mu(\Gamma)}. \tag{4.15}$$

The combination of Céa's lemma 2.9, Strang's lemma with Theorem 3.11 resp. Corollary 3.12 and (4.15) proves the assertion for  $\mu \geq 0$ .

For  $\mu < 0$ , we employ the inverse inequality for shape regular meshes (cf. [29, Thm. 4.76]): For all  $u \in S$  and all  $\tau \in \mathcal{G}$ , we have

$$\|u\|_{H^1(\tau)} \leq C \frac{p_\tau^2}{h_\tau} \|u\|_{L^2(\tau)}.$$

This implies

$$\|u\|_{L^2(\tau)}^2 \leq C \|u\|_{H^1(\tau)} \|u\|_{H^{-1}(\tau)} \leq C \frac{p_\tau^2}{h_\tau} \|u\|_{L^2(\tau)} \|u\|_{H^{-1}(\tau)}$$

and the desired inverse inequality for the  $L^2$ -norm follows. For  $\tau \in \mathcal{G}_\ell$ , there holds (cf. Lemma 3.10)

$$p_\tau^2/h_\tau \leq C(1+L-\ell)^2 2^\ell \leq C2^L \max_{x \geq 0} \frac{(1+x)^2}{2^x} \leq Ch^{-1}.$$

Hence,

$$\forall u \in S : \|u\|_{L^2(\Omega)} \leq Ch^{-1} \|u\|_{H^{-1}(\Omega)}.$$

Interpolating this result with the trivial estimate  $\|u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$  results in  $\|u\|_{L^2(\Omega)} \leq Ch^{-1/2} \|u\|_{H^{-1/2}(\Omega)}$  and

$$\forall u, v \in S : |\hat{a}(u, v) - \hat{a}_{\text{pc}}(u, v)| \leq CC_9 h^{\alpha+1} h^{-1} \|v\|_{H^\mu(\Gamma)} \|u\|_{H^\mu(\Gamma)} \leq Ch^\alpha \|v\|_{H^\mu(\Gamma)} \|u\|_{H^\mu(\Gamma)}.$$

The combination with the Strang Lemma again yields the assertion. ■

**Remark 4.9** *The complexity analysis for the panel clustering and  $\mathcal{H}$ -matrix construction in the presence of geometrically refined grids remains basically the same as for quasi-uniform grids. Recall that in [16] the cardinality and geometry balanced partitionings have been considered in 1D-case as well as in the case of 2D composed grids (with local mesh refinement). The analysis in the more general setting was presented in [11]. Applying the above mentioned constructions we obtain that the storage requirements and complexity of the matrix-vector product are estimated by (cf. [11])*

$$N_{\text{St}}(\mathbb{T}, k) \leq 2\#LC_{sp} \max\{k, n_{\min}\} \#\Theta_p, \quad N_{\mathcal{H}.v} \leq 2N_{\text{St}}(\mathbb{T}, k),$$

where  $C_{sp}$  is the so-called sparsity constant (in our particular case we have  $C_{sp} \approx 25$ ,  $n_{\min} \approx 20$ ). Hence, we arrive at a linear-logarithmic complexity with respect to  $N_{\Gamma_{\text{rough}}}$ . The specific feature of our construction is that we obtain not fully balanced block cluster tree, since some admissible blocks may have small size and thus they are represented by full matrices. However, this does not destroy the almost linear complexity.

## References

- [1] I. Babuška and B. Guo: *Regularity of the solution of elliptic problems with piecewise analytic data Part II. The trace spaces and application to the boundary value problems with nonhomogeneous boundary conditions*. SIAM J. Math. Anal., **20** (1989), 763–781.
- [2] S.N. Bernstein: *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle*. Paris: Gauthier-Villars 1926.
- [3] S. Börm, M. Löhndorf, J-M. Melenk: *Approximation of integral operators by variable-order interpolation*, Numer. Math. **99** (2005), 605-643.
- [4] A. Brücker-Foit, X. Huang, Y. Motoyashiki: *Mesoscopic simulations of damage accumulation under fatigue loading*, ECF 15, Stockholm 2004.
- [5] G. Chen and J. Zhou: *Boundary element methods*. Academic Press, New York, 1992.
- [6] M. Costabel: *Boundary integral operators on Lipschitz domains: Elementary results*. SIAM J. Math. Anal. **19** (1988), 613-626.
- [7] J. Elschner: *The double layer potential operator over polyhedral domains I: solvability in weighted Sobolev spaces*. Appl. Anal. **45** (1992), 117-134.
- [8] J. Elschner: *On the exponential convergence of some boundary element methods for Laplace's equation in non-smooth domains*, in Boundary Value Problems and Integral Equations in Nonsmooth Domains, M. Costabel et al., eds., New York, Dekker, 1995, 69-80.
- [9] S. Erichsen and S. Sauter: *Efficient automatic quadrature in 3-d Galerkin BEM*. Comp. Meth. Appl. Mech. Eng. **157** (1998), 215–224.
- [10] B.Guo and I. Babuška: *Regularity of the solutions for problems on nonsmooth domains in  $\mathbb{R}^3$ , Part 2: Regularity in a neighborhood of edges*, Proc. Royal Soc. Edinb., Sect. **A127** (1997), 517-545.

- [11] L. Grasedyck and W. Hackbusch: *Construction and arithmetics of  $\mathcal{H}$ -matrices*. Computing **70** (2003), 295-334.
- [12] P. Grisvard: *Elliptic problems in non-smooth domains*. Pitman, London, 1985.
- [13] W. Hackbusch: *Integral equations. Theory and numerical treatment*. ISNM 128. Birkhäuser, Basel, 1995.
- [14] W. Hackbusch: *A sparse matrix arithmetic based on  $\mathcal{H}$ -matrices. Part I: Introduction to  $\mathcal{H}$ -matrices*. Computing **62** (1999), 89-108.
- [15] W. Hackbusch and B.N. Khoromskij: *A sparse  $\mathcal{H}$ -matrix arithmetic. Part II: Application to multi-dimensional problems*. Computing **64** (2000), 21-47.
- [16] W. Hackbusch and B.N. Khoromskij:  $\mathcal{H}$ -Matrix approximation on graded meshes. The Mathematics of Finite Elements and Applications X, MAFELAP 1999, J.R. Whiteman (ed), Elsevier, Amsterdam, Chapter 19, 307-316, 2000.
- [17] W. Hackbusch and B.N. Khoromskij: *Towards  $\mathcal{H}$ -matrix approximation of the linear complexity*. Operator Theory: Advances and Applications, vol. 121, Birkhäuser Verlag, 2001, 194-220.
- [18] W. Hackbusch, B. N. Khoromskij, and S. Sauter: *On  $\mathcal{H}^2$ -matrices*. In: Lectures on Applied Mathematics (H.-J. Bungartz, R. Hoppe, C. Zenger, eds.), Springer Verlag, 2000, pp. 9-29.
- [19] N. Heuer and E. P. Stephan: *Boundary integral operators in countably normed spaces*. Math. Nachr. **191** (1998) 123-151.
- [20] N. Heuer, M. Maischak, and E. P. Stephan: *Exponential convergence of the hp-version for the boundary element method on open surfaces*. Numer. Math., **83**(4) (1999) 641-666.
- [21] B.N. Khoromskij: *Hierarchical matrix approximation to Green's function via boundary concentrated FEM*. J. Numer. Math. **11** (2003), 195-223.
- [22] B.N. Khoromskij and J.M. Melenk: *Boundary concentrated finite element methods*. SIAM J. Numer. Anal. **41** (2003), 1-36.
- [23] B.N. Khoromskij and J.M. Melenk: *Efficient direct solver for the boundary concentrated FEM in 2D*. Computing **69** (2002), 91-117.
- [24] J.M. Melenk: *hp-finite element methods for singular perturbations*. Springer Lecture Notes in Mathematics, **1796** (2002).
- [25] S. Nicaise and S.A. Sauter: *Efficient numerical solution of Neumann problems on complicated domains*. Calcolo **43** (2006), 95-120.
- [26] T.v. Petersdorff and E.P. Stephan: *Decomposition in edge and corner singularities for the solution of the Dirichlet problem of the Laplacian in a polyhedron*, Math. Nachr. **149** (1990), 71-104.
- [27] S. A. Sauter: *Über die effiziente Verwendung des Galerkinverfahrens zur Lösung Fredholmscher Integralgleichungen*. PhD thesis, Inst. f. Prakt. Math., Universität Kiel, 1992.
- [28] S. Sauter and C. Schwab: *Randelementmethoden*. B.G. Teubner, 2004.
- [29] C. Schwab: *p- and hp-Finite Element Methods*. Oxford University Press, 1998.
- [30] E.P. Stephan: *Multilevel methods for the h-, p-, and hp-versions of the boundary element method*. J. of Computational and Applied Math. **125** (2000), 503-519.
- [31] E. Tadmor: *The exponential accuracy of Fourier and Chebychev differencing methods*. SIAM J. Numer. Anal. **23** (1986), 1-23.