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BERNSTEIN TYPE THEOREMS WITH FLAT NORMAL BUNDLE

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Abstract. We prove Bernstein type theorems for minimal n-submanifolds in $\mathbb{R}^{n+p}$ with flat normal bundle. Those are natural generalizations of the corresponding results of Ecker-Huisken and Schoen-Simon-Yau for minimal hypersurfaces.

1. Introduction

From the counter example given by Lawson-Osserman [LO77], we know that minimal submanifolds of higher codimension in Euclidean space are more complicated. Their general Bernstein property has been studied in [HJW81], [FC80], [JX99] and [Wan03], but one is still far from a complete understanding.

From the geometric point of view, the normal bundle might be complicated in higher codimension, which would influence the submanifold properties. Now, we consider the simplest case, i.e. that of flat normal bundles and expect this case might share similar properties with minimal hypersurfaces, as shown in [Ter86], [Ter85], [Ter87] and [HPT88]. In the previous paper [Xin03] we study such a situation. The w-function determined by the generalized Gauss map (see §3) will play an important role. If it is positive, the submanifolds are like stable minimal hypersurfaces. They have only one end. We also obtained an adequate generalization of the Bernstein theorem for a minimal graph with controlled growth due to Ecker-Huisken [EH90].

Now, we find the result in [Xin03] can be improved and there is no dimension limitation any more. This is Theorem 1. The key point is Lemma 1 which gives us a better estimate for the second fundamental form in the case of higher codimension and flat normal bundle. From such an estimate and the stability inequality (2.7), curvature estimates, as done by Simon-Schoen-Yau for hypersurfaces [SSY75], are available immediately.

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2. Basic formulas

Let M be a minimal submanifold in Euclidean $(n+p)$-space $\mathbb{R}^{n+p}$ with the second fundamental form B. TM and NM denote the tangent bundle and the normal bundle along M, respectively. There are induced connections on TM and NM. If the curvature of the normal bundle vanishes, then M called a submanifold with flat normal bundle

For $\nu \in \Gamma(NM)$ the shape operator $A^\nu : TM \to TM$ satisfies

$$\langle B_{XY}, \nu\rangle = \langle A^\nu(X), Y\rangle\,,$$

where B can be viewed as a map from $\odot^2 TM$ to NM. There is the trace-Laplace operator ∇^2 acting on any cross-section of a Riemannian vector bundle E over M.

We have (see [Sim68])

$$\nabla^2 B = -\tilde{\mathfrak{B}} - \underline{\mathfrak{B}}. \tag{2.1}$$

We recall the following notations:

$$\tilde{\mathfrak{B}} \overset{def.}{=} B \circ B^t \circ B,$$

where B^t is the conjugate map of B,

$$\underline{\mathfrak{B}}_{X\,Y} \overset{def.}{=} \sum_{\alpha=1}^{p} \left(B_{A^{\nu_\alpha} A^{\nu_\alpha}(X)\,Y} + B_{X\,A^{\nu_\alpha} A^{\nu_\alpha}(Y)} - 2\,B_{A^{\nu_\alpha}(X)\,A^{\nu_\alpha}(Y)}\right),$$

where $(\nu_\alpha)_{\alpha=1,\ldots,p}$ is an orthonormal basis of the normal space. It is obvious that $\underline{\mathfrak{B}}_{X\,Y}$ is symmetric in X and Y, which is a cross-section of the bundle $\mathrm{Hom}(\odot^2 TM, NM)$.

Since $B \circ B^t : NM \to NM$ is symmetric, there is a local normal frame field $\{\nu_1, \cdots, \nu_p\}$, such that at a considered point

$$B \circ B^t(\nu_\alpha) = \lambda_\alpha^2 \nu_\alpha.$$

Noting that

$$\begin{aligned}\sum_\alpha \lambda_\alpha^2 &= \left\langle B \circ B^t \nu_\alpha, \nu_\alpha\right\rangle \\ &= \langle A^{\nu_\alpha}, A^{\nu_\alpha}\rangle \\ &= \left\langle B_{e_i\,e_j}, \nu_\alpha\right\rangle \left\langle B_{e_i\,e_j}, \nu_\alpha\right\rangle = |B|^2,\end{aligned}$$

we get

$$\left\langle \tilde{\mathfrak{B}}, B\right\rangle \;=\; \left\langle B \circ B^t \circ B, B\right\rangle = \left\langle B^t \circ B, B^t \circ B\right\rangle$$

$$
\begin{aligned}
&= \left\langle B_{e_i\,e_j}, B_{e_k\,e_l}\right\rangle \left\langle B_{e_i\,e_j}, B_{e_k\,e_l}\right\rangle \\
&= \left\langle B_{e_i\,e_j}, \nu_\alpha\right\rangle \langle B_{e_k\,e_l}, \nu_\alpha\rangle \left\langle B_{e_i\,e_j}, \nu_\beta\right\rangle \langle B_{e_k\,e_l}, \nu_\beta\rangle \\
&= \left\langle e_i \odot e_j, B^t(\nu_\alpha)\right\rangle \left\langle e_k \odot e_l, B^t(\nu_\alpha)\right\rangle \\
&\qquad \left\langle e_i \odot e_j, B^t(\nu_\beta)\right\rangle \left\langle e_k \odot e_l, B^t(\nu_\beta)\right\rangle \\
&= \left\langle B^t(\nu_\alpha), B^t(\nu_\beta)\right\rangle \left\langle B^t(\nu_\alpha), B^t(\nu_\beta)\right\rangle \\
&= \left\langle B \circ B^t(\nu_\alpha), B \circ B^t(\nu_\alpha)\right\rangle \\
&= \sum \lambda_\alpha^4 \le \left(\sum_\alpha \lambda_\alpha^2\right)^2 = |B|^4.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\langle \underline{\mathcal{B}}, B\rangle &= \left\langle \underline{\mathcal{B}}_{e_i\,e_j}, \nu_\alpha\right\rangle \langle B_{e_i\,e_j}, \nu_\alpha\rangle \\
&= \langle [A^{\nu_\beta}, [A^{\nu_\beta}, A^{\nu_\alpha}]]\,(e_i), e_j\rangle \,\langle A^{\nu_\alpha}(e_i), e_j\rangle \\
&= \langle (A^{\nu_\beta}A^{\nu_\beta}A^{\nu_\alpha} - 2\,A^{\nu_\beta}A^{\nu_\alpha}A^{\nu_\beta} + A^{\nu_\alpha}A^{\nu_\beta}A^{\nu_\beta})\,, A^{\nu_\alpha}\rangle\,.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\langle A^{\nu_\alpha}A^{\nu_\beta}A^{\nu_\beta}, A^{\nu_\alpha}\rangle &= \langle A^{\nu_\alpha}A^{\nu_\alpha}A^{\nu_\beta}A^{\nu_\beta}, I\rangle \\
&= \mathrm{trace}(A^{\nu_a}A^{\nu_\alpha}A^{\nu_\beta}A^{\nu_\beta}) \\
&= \langle A^{\nu_\alpha}A^{\nu_a}A^{\nu_\beta}, A^{\nu_\beta}\rangle\,,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\langle \underline{\mathcal{B}}, B\rangle &= \langle A^{\nu_\beta}A^{\nu_\beta}A^{\nu_\alpha} - 2\,A^{\nu_\beta}A^{\nu_\alpha}A^{\nu_\beta} + A^{\nu_\beta}A^{\nu_\beta}A^{\nu_\alpha}, A^{\nu_\alpha}\rangle \\
&= \langle A^{\nu_\beta}A^{\nu_\beta}A^{\nu_\alpha} - A^{\nu_\beta}A^{\nu_\alpha}A^{\nu_\beta}, A^{\nu_\alpha}\rangle \\
&\qquad - \langle A^{\nu_\beta}A^{\nu_\alpha}A^{\nu_\beta} - A^{\nu_\beta}A^{\nu_\beta}A^{\nu_\alpha}, A^{\nu_\alpha}\rangle
\end{aligned}
$$

$$\begin{aligned}
&= \langle A^{\nu_\beta}A^{\nu_\alpha} - A^{\nu_\alpha}A^{\nu_\beta}, A^{\nu_\beta}A^{\nu_\alpha}\rangle \\
&\quad - \langle A^{\nu_\alpha}A^{\nu_\beta} - A^{\nu_\beta}A^{\nu_\alpha}, A^{\nu_\beta}A^{\nu_\alpha}\rangle \\
&= \langle A^{\nu_\alpha}A^{\nu_\beta} - A^{\nu_\beta}A^{\nu_\alpha}, A^{\nu_\alpha}A^{\nu_\beta}\rangle \\
&\quad - \langle A^{\nu_\alpha}A^{\nu_\beta} - A^{\nu_\beta}A^{\nu_\alpha}, A^{\nu_\beta}A^{\nu_\alpha}\rangle \\
&= \sum_{\alpha\neq\beta} |[A^{\nu_\alpha}, A^{\nu_\beta}]|^2 = 0,
\end{aligned}$$

where the last equation holds by Ricci's equation in the case of a flat normal bundle. Thus, we have

$$\left\langle \nabla^2 B, B\right\rangle \geq -|B|^4. \tag{2.2}$$

It follows that

$$\Delta|B|^2 \geq -2|B|^4 + 2|\nabla B|^2. \tag{2.3}$$

In order to use the formula (2.3), we need an estimate of $|\nabla B|^2$ in terms of $|\nabla |B||^2$. Schoen-Simon-Yau [SSY75] did such an estimate for codimension $p = 1$. For any p with flat normal bundle their technique is also applicable and we have from [Xin03]

$$|\nabla B|^2 - |\nabla |B||^2 \geq \frac{2}{n}|\nabla |B||^2. \tag{2.4}$$

This is our desired Kato-type inequality. From (2.3) and (2.4) we have

Lemma 1. *Let $M \to \mathbb{R}^{n+p}$ be a minimal n-submanifold with flat normal bundle. Its second fundamental form satisfies*

$$|B|\Delta|B| \geq -|B|^4 + \frac{2}{n}|\nabla |B||^2. \tag{2.5}$$

For two simple unit n-vectors

$$A = a_1 \wedge \cdots \wedge a_n, \quad B = b_1 \wedge \cdots \wedge b_n,$$

their inner product is defined by

$$\langle A, B\rangle = \det\left(\langle a_i, b_j\rangle\right).$$

Choose an orthonormal frame field $\{e_i,\, e_\alpha\}$ along M such that $e_i \in TM$ and $e_\alpha \in NM$. Fix a unit simple n-vector $A = a_1 \wedge \cdots \wedge a_n$ in $\mathbb{R}^{n+p}$ and define a function w on M by

$$w = \langle e_1 \wedge \cdots \wedge e_n, a_1 \wedge \cdots \wedge a_n \rangle = \det\left(\langle e_i, a_j \rangle\right).$$

By a direct computation we derived a basic equation for w.

Lemma 2. ([Xin03]) *Let M be an n-submanifold in $\mathbb{R}^{n+p}$ with parallel mean curvature and flat normal bundle. Then the above defined w-function satisfies*

$$\Delta w = -|B|^2\, w. \tag{2.6}$$

This equation is a generalization of the basic equation in constant mean curvature hypersurfaces in Euclidean space.

We assume that the w-function is positive everywhere on M. Then, we have a stability inequality by a theorem in [FCS80]. We state a lemma as follows:

Lemma 3. ([Xin03]) *Let M be a complete n-submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle and parallel mean curvature. If the w-function is positive everywhere on M, then*

$$\int_M |\nabla \phi|^2 * 1 \geq \int_M |B|^2 \phi^2 * 1, \tag{2.7}$$

for any function ϕ with compact support $D \subset M$.

3. The Geometric Meaning of $w > 0$

For an n-dimensional oriented submanifold M in Euclidean space $\mathbb{R}^{n+p}$ we have the generalized Gauss map. By the parallel translation in the ambient Euclidean space, the tangent space T_xM at each point $x \in M$ is moved to the origin of $\mathbb{R}^{n+p}$ to obtain an n-subspace in $\mathbb{R}^{n+p}$, namely, a point of the Grassmannian manifold $\gamma(x) \in \mathbf{G}_{n,p}$. Thus, we define a generalized Gauss map $\gamma : M \to \mathbf{G}_{n,p}$.

For defining the w-function we need to fix a unit n-vector $A = a_1 \wedge \cdots \wedge a_n$. Without loss of generality, we assume that A is defined by the first n-axes. If the tangent vectors to M are

$$e_1 = (e_{11}, e_{12}, \cdots, e_{1n}, \cdots, e_{1n+p})$$

$$\cdots\cdots\cdots\cdots$$

$$e_n = (e_{n1}, e_{n2}, \cdots, e_{nn}, \cdots, e_{nn+p}),$$

then

$$w = \det \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix}.$$

If $w \neq 0$ on M, then the tangent n-plane is also spanned by n vectors

$$e_i' = a_i + z_{i\alpha} e_{n+\alpha},$$

where $(z_{i\alpha})$ is an $n \times p$ matrix. Thus, if w never vanishes on M, the image under the generalized Gauss map for a submanifold M lies in one local coordinate neighborhood in the Grassmannian manifold.

For the fixed n-planes A and $\gamma(T_x M)$, considered as two points in the Grassmannian manifold $\mathbf{G}_{n,p}$, we can define Jordan angles between them. Those are

$$\theta_\alpha = \cos^{-1}(\lambda_\alpha),$$

where λ_α^2 are eigenvalues of the symmetric matrix $W^T W$, where $W = (\langle e_i, a_j \rangle)$. In the case of $w \neq 0$, the Jordan's angles between A and $\gamma(T_x M)$ lie in $[0, \frac{\pi}{2})$. We also know the relation between w and those angles [JX99]

$$w = \prod_{\alpha=1}^{n} \cos\theta_\alpha.$$

4. Bernstein Type Theorems for Controlled Growth

In the case when w is positive, we set $v = \frac{1}{w}$. We have a Bernstein type result as follows.

Theorem 1. *Let M be a minimal n-submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle. If M has polynomial volume growth and v has growth*

$$v = O\left(R^\mu\right),$$

where $0 \le \mu < 1$ and R is the Euclidean distance from any point in M. Then M has to be an affine linear subspace.

Proof. From (2.6) we obtain

$$\Delta v = v\,|B|^2 + \frac{2}{v}|\nabla v|^2. \tag{4.1}$$

From (2.5) and (4.1) we obtain for any real q and s

$$\Delta\left(v^q|B|^s\right) \geq q\,(q+1)\,v^{q-2}|B|^s\,|\nabla v|^2 + s\,\left(s - \frac{n-2}{n}\right)\,v^q|B|^{s-2}|\nabla|B||^2$$

$$+\,(q-s)\;v^q|B|^{s+2} + 2\,q\,s\,v^{q-1}|B|^{s-1}\,\langle\nabla v, \nabla|B|\rangle\,. \tag{4.2}$$

By using the Cauchy inequality with $\varepsilon > 0$

$$v^{q-1}|B|^{s-1}\,\langle\nabla v, \nabla|B|\rangle \leq \frac{1}{2}\left(\varepsilon^{-1}v^{q-2}|B|^s|\nabla v|^2 + \varepsilon v^q|B|^{s-2}|\nabla|B||^2\right)$$

which is substituted in (4.2) we have

$$\Delta\left(v^q|B|^s\right) \geq q\,(q+1-\varepsilon^{-1}s)\,v^{q-2}|B|^s|\nabla v|^2$$

$$+s\left(s - \frac{n-2}{n} - \varepsilon\,q\right)\,v^q|B|^{s-2}|\nabla|B||^2 + (q-s)\;v^q|B|^{s+2} \tag{4.3}$$

By choosing $s = q > n-2$ and $\varepsilon = \frac{q}{q+1}$ we have

$$\Delta(v^q|B|^q) \geq 0.$$

We also choose $q = s+1 > n-1$ and $\varepsilon = \frac{q-1}{q+1}$ in (4.3), and then we have

$$\Delta(v^q|B|^{q-1}) \geq v^q|B|^{q+1}. \tag{4.4}$$

Hence we may apply the mean value inequality for any subharmonic function on a minimal submanifold M in $\mathbb{R}^{n+p}$ [CLY84], [Ni01] to $v^q|B|^{q-1}$ which gives

$$v^q|B|^q(o) \leq \frac{C}{R^n}\int_{D_R} v^{2q}|B|^{2q} * 1 \leq \frac{C\,\mathrm{vol}(D_R)^{\frac{1}{2}}}{R^n}\left(\int_{D_R} v^{2q}|B|^{2q} * 1\right)^{\frac{1}{2}}, \tag{4.5}$$

where we assume $o \in M \subset \mathbb{R}^{n+p}$ and C is a constant depending only on n.

Multiplying by $v^q|B|^{q-1}\phi^{2q}$, where ϕ is any smooth function with compact support, in (4.4), then integrating by parts and using the Cauchy inequality, we have

$$\int_M v^{2q}|B|^{2q}\phi^{2q} * 1 \leq \int_M v^q|B|^{q-1}\phi^{2q}\Delta(v^q|B|^{q-1}) * 1$$

$$= -\int_M \left\langle \nabla(v^q|B|^{q-1}\phi^{2q}), \nabla(v^q|B|^{q-1}\right\rangle * 1$$

$$= -\int_M |\nabla(v^q|B|^{q-1})|^2\phi^{2q} * 1$$

$$-2q\int_M \left\langle \phi^{q-1}|B|^{q-1}v^q\nabla\phi, \phi^q\nabla(v^q|B|^{q-1})\right\rangle * 1$$

$$\leq C_1(q)\int_M v^{2q}|B|^{2q-2}\phi^{2q-2}|\nabla\phi|^2 * 1. \tag{4.6}$$

By using Young's inequality

$$ab \leq \frac{\alpha^p a^p}{p} + \frac{\alpha^{-q}b^q}{q}$$

for any positive real numbers p, q, α, a, b with $\frac{1}{p} + \frac{1}{q} = 1$, (4.6) becomes

$$\int_M v^{2q}|B|^{2q}\phi^{2q} * 1 \leq C_2(q)\int_M v^{2q}|\nabla\phi|^{2q} * 1.$$

Choosing ϕ as the standard cut-off function, we obtain

$$\int_{D_R} v^{2q}|B|^{2q} * 1 \leq C_2(q)\, R^{-2q}\int_{D_{2R}} v^{2q} * 1$$

$$\leq C_2(q)\, R^{-2q}\mathrm{vol}(D_{2R})\sup_{D_{2R}} v^{2q}. \tag{4.7}$$

We know that M has polynomial volume growth of order $n+m,\ m \geq 0$. From (4.5) and (4.7) we obtain

$$v^q|B|^q(o) \leq C_3(n)R^{-n-q}R^{n+m}\sup_{D_{2R}} v^q,$$

then

$$v|B|(o) \leq C(n)\, R^{-1+\frac{m}{q}+\mu}.$$

For given $m \geq 0$ and $0 \leq \mu < 1$, we can choose q large enough such that

$$-1 + \frac{m}{q} + \mu < 0.$$

Let R go to infinity, we have $|B|(o) = 0$. Since o is any point in M, we complete the proof. □

In case M is an entire graph defined by p functions on $\mathbb{R}^{n+p}$, v is just the volume element. If

$$v = O\left(R^{\mu}\right),$$

then

$$\mathrm{vol}(D_R) = O(R^{n+\mu}).$$

In this case the assumption that M has polynomial growth is redundant and we have the following result, which is a generalization of the results by Ecker-Huisken and Nitsche [EH90], [Nit89].

Corollary 1. *Let $M = (x, f(x))$ be a minimal graph given by p functions $f^{\alpha}(x^1, \cdots, x^n)$ with flat normal bundle. If for $0 \leq \mu < 1$*

$$(det(\delta_{ij} + f_i^{\alpha} f_j^{\alpha}))^{\frac{1}{2}} = O(R^{\mu}),$$

where $R^2 = |x|^2 + |f|^2$. Then f^{α} are affine linear functions.

5. Curvature Estimates

Replacing ϕ by $|B|^{1+q}\phi$ in (2.7) gives

$$\int_M |B|^{2q+4}\phi^2 * 1 \leq \int_M \Big[(1+q)^2|B|^{2q}|\nabla|B||^2\phi^2 + |B|^{2q+2}|\nabla\phi|^2$$

$$+2(1+q)\phi|B|^{2q+1}(\nabla\phi)\cdot(\nabla|B|)\Big] * 1. \tag{5.1}$$

Multiplying $\phi^2|B|^{2q}$ with both sides of (2.5) and integrating by parts, we have

$$\frac{2}{n}\int_M \phi^2|B|^{2q}|\nabla|B||^2 * 1 \leq -(1+2q)\int_M \phi^2|B|^{2q}|\nabla|B||^2 * 1$$

$$+\int_M |B|^{4+2q}\phi^2 - 2\int_M \phi|B|^{2q+1}(\nabla\phi)\cdot(\nabla|B|) * 1. \tag{5.2}$$

By adding up both sides of (5.1) and (5.2) we have

$$\left(\frac{2}{n}-q^2\right)\int_M \phi^2|B|^{2q}|\nabla|B||^2 * 1$$

$$\leq \int_M |B|^{2q+2}|\nabla\phi|^2 + 2q\int_M \phi|B|^{2q+1}(\nabla\phi)\cdot(\nabla|B|) * 1. \tag{5.3}$$

Since

$$\begin{aligned}2q\phi|B|^{2q+1}(\nabla\phi)\cdot(\nabla|B|) &\leq 2q\phi|B|^{2q+1}|\nabla\phi||\nabla|B||\\ &\leq \varepsilon q^2\phi^2|B|^{2q}|\nabla|B||^2+\varepsilon^{-1}|B|^{2q+2}|\nabla\phi|^2,\end{aligned}$$

(5.3) becomes

$$\begin{aligned}\left[\frac{2}{n}-(1+\varepsilon)q^2\right]&\int_M \phi^2|B|^{2q}|\nabla|B||^2 * 1\\ &\leq (1+\varepsilon^{-1})\int_M |B|^{2q+2}|\nabla\phi|^2 * 1.\end{aligned}$$

When

$$0 \leq q < \sqrt{\frac{2}{n}}$$

we can choose ε sufficiently small such that

$$\int_M \phi^2|B|^{2q}|\nabla|B||^2 * 1 \leq C_1\int_M |B|^{2q+2}|\nabla\phi|^2 * 1, \tag{5.4}$$

where $C_1 > 0$ is a constant depending on $n,\ q$.

Since

$$2\,\phi|B|^{2q+1}(\nabla\phi)\cdot(\nabla|B|) \leq \phi^2|B|^{2q}|\nabla|B||^2 + |B|^{2q+2}|\nabla\phi|^2,$$

(5.1) becomes

$$\begin{aligned}\int_M |B|^{2q+4}\phi^2 * 1 &\leq (1+q)^2\int_M |B|^{2q}|\nabla|B||^2\phi^2 * 1 + \int_M |B|^{2q+2}|\nabla\phi|^2 * 1\\ &+(1+q)\int_M |B|^{2q}(\nabla|B||^2)\phi^2 * 1 + (1+q)\int_M |B|^{2q+2}|\nabla\phi|^2 * 1\\ &= (1+q)(2+q)\int_M |B|^{2q}|\nabla|B||^2\phi^2 * 1 + (q+2)\int_M |B|^{2q+2}|\phi|^2 * 1\\ &\leq C_2\int_M |B|^{2q+2}|\nabla\phi|^2 * 1,\end{aligned} \tag{5.5}$$

where in the last inequality above we used (5.4). Replacing ϕ by ϕ^{q+2} in (5.5) gives

$$\int_M |B|^{2q+4}\phi^{2q+4} * 1 \le C_3 \int_M |B|^{2q+2}\phi^{2q+2}|\nabla\phi|^2 * 1.$$

By using Young's inequality, we have

$$|B|^{2q+2}\phi^{2q+2}|\nabla\phi|^2 \le \varepsilon|B|^{2q+4}\phi^{2q+4} + C_4|\nabla\phi|^{2q+4},$$

Therefore

$$\int_M |B|^{2q+4}\phi^{2q+4} * 1 \le C \int_M |\nabla\phi|^{2q+4} * 1. \tag{5.6}$$

Replacing ϕ by ϕ^{q+1} in (5.5) gives

$$\int_M |B|^{2q+4}\phi^{2q+2} * 1 \le C_3' \int_M |B|^{2q+2}\phi^{2q}|\nabla\phi|^2 * 1. \tag{5.7}$$

Choosing

$$q' = \frac{2}{q+1},\ t = q+1,\ s = \frac{q+1}{q}$$

and using Young's inequality again

$$\begin{aligned}|B|^{2q+2}\phi^{2q}|\nabla\phi|^2 &= |B|^{2q+2-q'}\phi^{2q}|B|^{q'}|\nabla\phi|^2\\ &\le \varepsilon(|B|^{2q+2-q'}\phi^{2q})^s + C_3'\,|B|^{q't}|\phi|^{2t}\\ &= \varepsilon|B|^{2q+4}\phi^{2q+2} + C_3'|B|^2|\phi|^{2q+2}.\end{aligned}$$

Thus, (5.7) becomes

$$\int_M |B|^{2q+4}\phi^{2q+2} * 1 \le C' \int_M |\phi|^{2q+2}|B|^2 * 1. \tag{5.8}$$

The above inequalities (5.6) and (5.8) enable us to prove the following results, which is a generalization of the results by Schoen-Simon-Yau [SSY75].

Theorem 2. *Let M be a minimal submanifold in $\mathbb{R}^{n+p}$ of codimension p with flat normal bundle and positive w. If for $q \in \left[0, \sqrt{\frac{2}{n}}\right)$,*

$$\lim_{R\to\infty} R^{-(2q+4)} vol(D_R) = 0,$$

then M is flat.

Proof. Choose a cut-off function ϕ to be

$$\phi = \begin{cases} 1 & \text{in} \quad D_R, \\ 0 & \text{outside of } D_{2R}. \end{cases}$$

with $\phi \geq 0$ and $|\nabla\phi| \leq \frac{C_0}{R}$ almost everywhere. From (5.6) we have

$$\int_{D_R} |B|^{2q+4} * 1 \leq \int_{D_{2R}} |B|^{2q+4}\phi^{2q+4} * 1$$
$$\leq C \int_{D_{2R}} |\nabla\phi|^{2q+4} * 1 \leq C\frac{C_0^{2q+4}}{R^{2q+4}}\mathrm{vol}D_{2R} = \frac{A\ \mathrm{vol}D_R}{R^{2q+4}},$$

where A is constant. Letting $R \to \infty$ gives

$$\int_M |B|^{2q+4} * 1 = 0.$$

□

Remark 1. *We know that an n- dimensional minimal submanifold in Euclidean space has at least polynomial growth of order n* [Xin03]. *Theorem 2 is interesting only for $n \leq 5$.*

From (5.8) it is easy to obtain the following result.

Theorem 3. *Let M be a minimal n-submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle and positive w-function. If M has finite total curvature, then M is totally geodesic.*

Using the similar method we can generalize a result in [HOS82] to higher codimension. We state the result without proof.

Theorem 4. *Let M be a complete surface in $\mathbb{R}^{2+p}$ with parallel mean curvature and flat normal bundle. If the w-function is positive everywhere on M, then M has to be a plane.*

Remark 2. *After this work was finished and after authors announced these results in* [SWX04] *(Remark 4), M.-T. Wang* [Wan04] *also submitted Bernstein theorems for minimal submanifolds with flat normal bundles to the Archiv.*

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