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Abstract

In this paper we show that the Higgs boson of the (minimal) Standard Model has at most three gauge inequivalent ground states. One of these states is related to ordinary electromagnetism and the other two to electromagnetism within magnetically charged vacua. If space-time is assumed to be rotationally symmetric then the charged electroweak vacua may be identified with Dirac monopoles of magnetic charge $g = \pm 1/2$. This offers a physical interpretation of magnetic monopoles and Dirac's quantization condition of electric charge in terms of the electroweak interaction. Moreover, in the case of the (minimal) Standard Model the three possible gauge inequivalent ground states of the Higgs boson are shown to fully determine the topological structure of the gauge bundle which underlies the electroweak interaction.

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1 Introduction

In this paper we discuss the topological structure that underlies the electroweak interaction as it is described by the (minimal) Standard Model. More specifically, we shall show that the topological structure of the principal $\mathrm{SU}(2) \times \mathrm{U}(1)$ bundle of the electroweak interaction is fully determined by the hypercharge of the Higgs boson.

In [Tolk'03(c)] it has been shown that the symmetry of electromagnetism under charge conjugation is equivalent to the triviality of the electroweak gauge bundle. The aim of this paper is to discuss the topology of the electroweak gauge bundle $\mathcal{P}$ over an arbitrary space-time $(\mathcal{M}, g_{\mathrm{M}})$ when charge conjugation is not taken into account. We will show that the structure of $\mathcal{P}$ is fully determined by Dirac's famous quantization condition of electric charge. It follows that in the case of the (minimal) Standard Model the Higgs boson has at most three gauge inequivalent ground states, one of which corresponds to ordinary electromagnetism and the other two correspond to electromagnetism within a "magnetically charged vacuum". Thus, the latter spoil charge conjugation. The fact that there are two charged vacua corresponds to the $\mathbb{Z}_2-$grading of charge conjugation of ordinary electromagnetism. To prove this, we will first summarize the basic geometrical setup in the next section. In the third section we present the proof of the above statements and discuss some consequences. In the fourth section we propose a specific generalization of the moduli space of all electroweak vacua that has been introduced in loc site and discuss Dirac monopoles from the geometrical viewpoint presented in this paper. We finish with a summary of the results presented.

In the following we summarize the motivation and the physical terminology used in this paper (see also in loc site). This paper is part of a sequence of papers dealing with a globally geometrical analysis of spontaneously broken gauge theories. The discussion is based on a geometrical understanding of the "two shifts" usually performed in order to make perturbation theory on Minkowski space-time $\mathbb{R}^{1,3}$

$$\begin{aligned} d &\mapsto d + A, \\ \mathbf{z}_0 &\mapsto \mathbf{z}_0 + \phi. \end{aligned} \tag{1}$$

For example, in the semi-classical approximation ("tree-level") of the (minimal) Standard Model $A = B + W \in \Omega^1(\mathbb{R}^{1,3}, \mathbb{R}\oplus\mathbb{R}^3)$ denotes the electroweak gauge potential and $\phi \in \Omega^0(\mathbb{R}^{1,3}, \mathbb{C}^2)$ represents the Higgs boson. Moreover, the Higgs boson and the electroweak gauge boson are assumed to represent the following “particle multiplets”

$$\begin{aligned} A &= (A_{\rm elm}, W^\pm, Z^0), \\ \phi &= (\phi_{\rm G}, \phi_{\rm H,phys}). \end{aligned} \tag{2}$$

When “quantized” $A_{\rm elm}$ is identified with the photon and $W^\pm$ and Z^0 with the electroweak vector bosons; $\phi_{\rm G}$ denotes the Goldstone boson and $\phi_{\rm H,phys}$ the physical Higgs boson.

In the unitary gauge the pair $(A, \phi_{\rm H,phys})$ is physically interpreted as a “fluctuation” of a ground state (“semi-classical vacuum”) of the electroweak interaction. From a geometrical perspective such a ground state may be described as a particular “Yang-Mills-Higgs pair” $(d, \mathbf{z}_0)$, where “d” is considered as the exterior covariant derivative with respect to the trivial connection on $\mathbb{C}^2$. The chosen minimum $\mathbf{z}_0 \in \mathbb{C}^2$ of the Higgs potential $V_{\rm H}$ can be regarded as the canonical smooth mapping

$$\begin{aligned} \mathbf{z}_0 : \mathbb{R}^{1,3} &\longrightarrow \mathrm{orbit}(\mathbf{z}_0) \\ x &\mapsto \mathbf{z}_0. \end{aligned} \tag{3}$$

Here, the sub-manifold $\mathrm{orbit}(\mathbf{z}_0) \subset \mathbb{C}^2$ is the orbit of $\mathbf{z}_0$ with respect to a unitary group action of $\mathrm{G} \equiv \mathrm{SU}(2) \times \mathrm{U}(1)$ on $\mathbb{C}^2$.

Of course, the decomposition (2) and the physical interpretation of $(A, \phi_{\rm H,phys})$ only makes sense if either the topology of space-time $(\mathcal{M}, g_{\rm M})$ or the topology of the underlying gauge bundle $\mathcal{P}$ is assumed to be trivial. But what do we know about the global properties of the respective spaces? And what kind of phenomena can we expect if the respective topologies are non-trivial? Usually, one argues that physics can only make local statements. Consequently, the above given interpretation may be considered as local relations for locally space-time and any (gauge) bundle are topologically trivial. Then, for example, the gauge classes of mappings (3) are known to be classified by $\pi_1(\mathrm{G}/\mathrm{H})$, where the closed sub-group $\mathrm{H} \subset \mathrm{G}$ is isomorphic to the isotropy group of $\mathbf{z}_0$. However, the notion of locality is physically meaningless in the context of gauge theories for the latter do not give rise to a scale (in contrast to gravity). Therefore, the mathematical fact that every bundle is locally trivial has no physical meaning. We are thus forced to consider a bundle as a global geometrical object. This holds true in particular with respect to a gauge bundle $\mathcal{P}$. Likewise, because of the local nature of our experiments it seems more appropriate to determine the topological structure of space-time only by physical reasoning and not by a-priori assumptions.

The assumption that the topology of $(\mathcal{M}, g_{\rm M})$ and of $\mathcal{P}$ is quite general naturally rises the question about the moduli space of gauge classes of semi-classical vacua. This moduli space is related to the topology of space-time and the underlying gauge bundle. One may thus learn something about the global structure of the latter by investigating the moduli space of semi-classical vacua. Notice that the topological structure of space-time and of the gauge bundle also determines the global structure of every associated bundle (up to equivalence). That the moduli space of semi-classical vacua actually provides a good tool to study the possible topolological structure of $\mathcal{P}$ is demonstrated in this paper for the special case of the electroweak interaction. Although restricted to the semi-classical approximation the results presented may also have non-trivial consequences for quantizing the electroweak/electromagnetic interaction on curved space-time. Indeed, our discussion is also intended as a preliminary step towards a geometrical understanding of perturbation theory.

2 The electroweak interaction as a specific Yang-Mills-Higgs gauge theory

For the convenience of the reader we summarize in this section the basic geometrical notions used to geometrically formulate the bosonic sector of the electroweak interaction as a specific Yang-Mills-Higgs (YMH) gauge theory. For the terminology used and the details we refer to [Tolk'03(a)] and [Tolk'03(c)]. A corresponding discussion of fermions can be found in [Tolk'03(b)].

In what follows $(\mathcal{M}, g_{\rm M})$ denotes a smooth orientable (semi-)Riemannian manifold of arbitrary signature and dimension. As a topological space, $\mathcal{M}$ is assumed to be paracompact, connected and Hausdorff. Since we are mainly interested in the case where $(\mathcal{M}, g_{\rm M})$ denotes a four dimensional Lorentzian manifold of signature -2, we call $\mathcal{M}$ "space-time". Likewise, by $\mathcal{P}$ we mean a smooth principal G bundle over space-time $\mathcal{M}$

$$\begin{array}{rccc} \pi_{\rm P}: & {\rm P} & \longrightarrow & \mathcal{M} \\ & p & \mapsto & x, \end{array} \tag{4}$$

with structure group ${\rm G} := {\rm SU}(2)\times{\rm U}(1)$. We call $\mathcal{P}$ the "electroweak gauge bundle". Its topological structure is thought to be given but so-far arbitrary. We will show that the actual bundle structure is fully determined (up to equivalence) by physical reasoning.

We call $(\mathcal{P}, \rho_{\rm H}, V_{\rm H})$ the geometrical data which permit to describe the electroweak interaction as a specific YMH gauge theory. Here,

$$\begin{aligned} \rho_{\rm H} : \mathrm{G} &\longrightarrow \mathrm{GL}(2,\mathbb{C}) \\ g = (g_{(2)}, g_{(1)}) &\mapsto g_{(2)} g_{(1)}^{\rm y} \end{aligned} \tag{5}$$

with $\mathrm{y} \in \mathbb{Q}$ denoting the so-called "hypercharge";

$$\begin{aligned} V_{\rm H} : \mathbb{C}^2 &\longrightarrow \mathbb{R} \\ \mathbf{z} &\mapsto \lambda\, |\mathbf{z}|^4 - \mu^2\, |\mathbf{z}|^2 \end{aligned} \tag{6}$$

is the well-known Higgs potential $(\lambda, \mu > 0)$. Note that the unitary representation $\rho_{\rm H}$ is faithful and the Higgs potential is rotationally symmetric, i. e. $V_{\rm H} = \mathrm{f}_{\rm H} \circ r$ with $r(\mathbf{z}) := |\mathbf{z}|$ the "radial function".

The "Higgs bundle" $\xi_{\rm H}$ and the "Yang-Mills bundle" $\xi_{\rm YM} := \tau^*_{\rm M} \otimes \mathfrak{ad}(\mathcal{P})$ are naturally associated with the geometrical data $(\mathcal{P}, \rho_{\rm H}, V_{\rm H})$. The Higgs bundle is defined by

$$\begin{aligned} \pi_{\rm H} : \mathrm{E}_{\rm H} := \mathrm{P} \times_{\rho_{\rm H}} \mathbb{C}^2 &\longrightarrow \mathcal{M} \\ \mathfrak{z} \equiv [(p, \mathbf{z})] &\mapsto \pi_{\rm P}(p) \end{aligned} \tag{7}$$

and the adjoint bundle $\mathfrak{ad}(\mathcal{P})$ is given by

$$\begin{aligned} \pi_{\rm ad} : \mathrm{ad(P)} := \mathrm{P} \times_{\rm ad} \mathrm{Lie(G)} &\longrightarrow \mathcal{M} \\ \tau \equiv [(p, \mathrm{T})] &\mapsto \pi_{\rm P}(p). \end{aligned} \tag{8}$$

The latter is always considered as a real vector bundle of rank four. Also the Higgs bundle will be mainly regarded as a real vector bundle of rank four. The real vector bundle $\tau^*_{\rm M}$ denotes the cotangent bundle of space-time $\mathcal{M}$.

Each minimum $\mathbf{z}_0 \in \mathbb{C}^2$ of the Higgs potential induces a specific fiber sub-bundle $\xi_{\rm orb} \subset \xi_{\rm H}$, which we call the "orbit bundle" with respect to $\mathbf{z}_0$:

$$\begin{aligned} \pi_{\rm orb} : \mathcal{O}rbit(\mathbf{z}_0) := \mathrm{P} \times_{\rho_{\rm orb}} \mathrm{orbit}(\mathbf{z}_0) &\longrightarrow \mathcal{M} \\ \mathfrak{z} \equiv [(p, \mathbf{z})] &\mapsto \pi_{\rm P}(p), \end{aligned} \tag{9}$$

with $\rho_{\rm orb} := \rho_{\rm H}|_{{\rm orbit}(\mathbf{z}_0)}$. Note that in the case at hand the Higgs potential has only one orbit of minima. Also, for two different minima the corresponding orbit bundles are equivalent. We therefore refer to (9) as the orbit bundle with respect to the data $(\mathcal{P}, \rho_{\rm H}, V_{\rm H})$. For rotationally symmetric Higgs potentials the orbit bundle can be thought of as a sphere sub-bundle of the Higgs bundle.

Each section $\mathcal{V} \in \Gamma(\xi_{\rm orb})$ is in one-to-one correspondence with a smooth principal H bundle $\mathcal{Q}$ over $\mathcal{M}$

$$\begin{array}{rccc} \pi_{\rm Q}: & {\rm Q} & \longrightarrow & \mathcal{M} \\ & q & \mapsto & x. \end{array} \tag{10}$$

The structure group ${\rm H} \equiv {\rm U}_{\rm elm}(1)$ of $\mathcal{Q}$ is isomorphic to the isotropy group ${\rm I}(\mathbf{z}_0) \subset {\rm G}$ of the minimum $\mathbf{z}_0 \in \mathbb{C}^2$

$${\rm I}(\mathbf{z}_0) \equiv \Big\{ \exp(\theta[{\rm T} + i{\rm y}]) \,|\, {\rm T} = {\rm T}(\mathbf{z}_0) \in {\rm su}(2),\ {\rm tr}[({\rm T} + i{\rm y})^2] = -{\rm n}^2,\ {\rm n} \in \mathbb{N} \Big\}. \tag{11}$$

Moreover, there is an equivariant embedding of principal bundles $\iota : \mathcal{Q} \hookrightarrow \mathcal{P}$, such that $(\iota, \mathcal{Q})$ is an H-reduction of $\mathcal{P}$ (see, [Koba/Nomi'96]). For a discussion of spontaneously broken gauge theories in terms of bundle reductions see, for example, [Trau'80], [Blee'81], [Choq/deWit'89] and [Ster'95]. We call the principal H bundle $\mathcal{Q}$ the "electromagnetic gauge bundle" with respect to the "vacuum section" $\mathcal{V}$. The relation between a section $\mathcal{V}$ of the orbit bundle and the appropriate H-reduction $(\iota, \mathcal{Q})$ of $\mathcal{P}$ is given by $\mathcal{V}(x) = [(\iota(q), \mathbf{z}_0)]|_{q \,\in\, \pi_{\rm Q}^{-1}(x)}$ for all $x \in \mathcal{M}$.

Each $\mathcal{V} \in \Gamma(\xi_{\rm orb})$ gives rise to a distinguished subset of principal connections on $\mathcal{P}$. They are determined by the requirement

$$d_{\rm A}\mathcal{V} = 0, \tag{12}$$

where $\mathcal{A} \in \mathcal{A}(\xi_{\rm H})$ is the corresponding associated connection on $\xi_{\rm H}$. The connections on $\mathcal{P}$ which satisfy the condition (12) are called compatible with the vacuum section $\mathcal{V}$ (resp. with the H-reduction $(\iota, \mathcal{Q})$ of $\mathcal{P}$). These connections have the crucial property that they are flat when restricted to the "physical space-time" $\mathcal{M}_{\rm phys} := \mathcal{V}(\mathcal{M}) \subset {\rm E}_{\rm H}$.

A YMH pair $(\mathcal{A}, \Phi) \in \mathcal{A}(\xi_{\rm H}) \times \Gamma(\xi_{\rm H})$ is called a "vacuum (pair)" iff $\Phi = \mathcal{V}$ is a vacuum section and $\mathcal{A} = \Theta$ is associated with a flat connection on $\mathcal{P}$ which is compatible with $\mathcal{V}$. Each vacuum defines an absolute minimum of the energy functional (if globally defined) that is associated with the well-known YMH functional. The moduli space of gauge classes of vacua is denoted by $\mathfrak{M}_{\rm vac}$. The latter turns out to be non-trivial iff $\mathcal{P}$ is trivial. In this case, $\mathfrak{M}_{\rm vac}$ can be canonically identified with $\mathrm{H}^1_{\rm deR}(\mathcal{M})$. In other words, the moduli space of the electroweak vacua only depends on the topology of space-time. In particular, if $\mathcal{M}$ is simply connected then there is a natural vacuum (pair) $(\Theta_0, \mathcal{V}_0)$ which generates $\mathfrak{M}_{\rm vac}$. This vacuum corresponds to the vacuum usually introduced in perturbation theory via the "shift" (1).

Since $\mathfrak{M}_{\rm vac} = \emptyset$ for non-trivial $\mathcal{P}$, we have to appropriately generalize the notion of the moduli space of electroweak vacua. This will be achieved by use of the fact that $\mathfrak{M}_{\rm vac} \neq \emptyset$ iff electromagnetism is symmetric with respect to charge conjugation. Before we introduce a generalization of $\mathfrak{M}_{\rm vac}$, however, we shall prove in the next section that Dirac's quantization condition of electric charge fully determines the topological structure of the electroweak gauge bundle.

3 The topology of the electroweak gauge bundle

When charge conjugation is taken into account the existence of vacuum sections is equivalent to the triviality of the electroweak gauge bundle. We therefore consider in this section the situation where charge conjugation is spoiled. Physically, this is the case if an absolute magnetic field exists which, for example, is generated by a magnetic monopole. We show that in the case of the Standard Model the Higgs boson may also provide such a state.

Proposition 3.1 *Let $\mathcal{Q}$ be a principal H bundle over a smooth manifold $\mathcal{M}$ and let $\lambda:\ \mathrm{H} \hookrightarrow \mathrm{G}$ be a homomorphism which is also a smooth embedding of Lie groups. Up to isomorphism there is a unique principal G bundle $\mathcal{P}$ over $\mathcal{M}$ together with a smooth embedding $\iota:\ \mathcal{Q} \hookrightarrow \mathcal{P}$ such that $(\iota, \mathcal{Q})$ is an H-reduction of $\mathcal{P}$.*

Proof: For this let $\{(U_i, \varphi_i)\,|\, i \in \Lambda\}$ be a family of local trivializations of the principal H bundle $\mathcal{Q}:\ \pi_{\rm Q}:\ \mathrm{Q} \to \mathcal{M}$. That is, $U_i \subset \mathcal{M}$ is an open subset such that $\mathcal{M} \subset \cup_{i\in\Lambda} U_i$, and

$$\varphi_i:\ \pi_{\rm Q}^{-1}(U_i) \quad \longrightarrow \quad U_i \times \mathrm{H}$$

$$p \quad \mapsto \quad (x := \pi_{\mathrm{Q}}(q), h_i := \varphi_{i,x}(q)) \tag{13}$$

is an equivariant diffeomorphism for all $i \in \Lambda$.

Accordingly, for $U_i \cap U_j \neq \emptyset$ we denote by the mappings

$$\begin{aligned} h_{ij} : U_i \cap U_j \quad &\longrightarrow \quad \mathrm{H} \\ x \quad &\mapsto \quad \varphi_{i,x}(q)\,(\varphi_{j,x}(q))^{-1} \end{aligned} \tag{14}$$

the transition functions with respect to the local trivialization $\{(U_i, \varphi_i)\,|\, i \in \Lambda\}$.

For all $i, j \in \Lambda$ such that $U_i \cap U_j \neq \emptyset$ we define smooth mappings

$$\begin{aligned} g_{ij} : U_i \cap U_j \quad &\longrightarrow \quad \mathrm{G} \\ x \quad &\mapsto \quad \lambda(h_{ij}(x)) \end{aligned} \tag{15}$$

which satisfy the co-cycle condition $g_{ij}(x)g_{jk}(x)g_{ki}(x) = e$ for all $x \in U_i \cap U_j \cap U_k$. Therefore, up to equivalence there is a unique principal G bundle $\mathcal{P} :\ \pi_{\mathrm{P}} :\ \mathrm{P} \to \mathcal{M}$ with local trivialization $\{(U_i, \psi_i)\,|\, i \in \Lambda\}$ such that $g_{ij}(x) = \psi_{i,x}(p)\,(\psi_{j,x}(p))^{-1}$.

We define a smooth family $\{(U_i, \iota_i\,|\, i \in \Lambda)\}$ of mappings

$$\begin{aligned} \iota_i : \pi_{\mathrm{Q}}^{-1}(U_i) \quad &\longrightarrow \quad \pi_{\mathrm{P}}^{-1}(U_i) \\ q \quad &\mapsto \quad \psi_i^{-1}(x, \lambda(\varphi_{i,x}(q)))|_{x\,=\,\pi_{\mathrm{Q}}(q)}. \end{aligned} \tag{16}$$

Since λ is an embedding, these mappings have maximal rank and fulfill $\iota_i(q) = \iota_j(q)$ for all $q \in \pi_{\mathrm{Q}}^{-1}(U_i \cap U_j)$. Therefore, they define a global immersion $\iota :\ \mathrm{Q} \to \mathrm{P}$ which is a homeomorphism onto $\iota(\mathrm{Q})$. Moreover, since λ is a homomorphism, the embedding ι is equivariant and fulfill $\pi_{\mathrm{P}} \circ \iota = \pi_{\mathrm{Q}}$. Consequently, $(\iota, \mathcal{Q})$ is an H-reduction of $\mathcal{P}$ and thus $\mathcal{P}$ a G-extension of $\mathcal{Q}$. Any other G-extension $\mathcal{P}'$ must be equivalent to $\mathcal{P}$, for the structure functions of $\mathcal{P}'$ are equivalent to those of $\mathcal{P}$. □

Therefore, for given data $(\mathcal{Q}, \mathrm{G}, \lambda)$ there is (up to equivalence) a unique G-extension $\mathcal{P}$ of the principal H bundle $\mathcal{Q}$. Since every principal U(1) bundle is characterized by an integer we may apply the Proposition (3.1) to prove our main result.

Proposition 3.2 *Let $(\mathcal{P}, \rho_{\rm H}, V_{\rm H})$ be the geometrical data which specifies the electroweak interaction of the (minimal) Standard Model as a YMH gauge theory. Up to equivalence the topological structure of the electroweak gauge bundle is fully determined by the hypercharge of the Higgs boson. Moreover, for a given hypercharge* $\mathrm{y} > 0$ *there are* $2|\mathrm{n}| + 1$ *gauge inequivalent sections of the orbit bundle where*

$$\mathrm{y} = \sqrt{\tfrac{1}{2}\left(\mathrm{n}^2 - \tfrac{1}{2}\right)}, \qquad \mathrm{n} \in \mathbb{Z}^*. \tag{17}$$

Proof: To prove the statement we make use of the fact that every principal U(1) bundle is characterized by its Chern number $\mathrm{n} \in \mathbb{Z}$, where n=0 corresponds to the trivial bundle (see [Free/Uhle'84]). For a given hypercharge $\mathrm{y} \in \mathbb{Q}$, the definition (11) of the isotropy group of a minimum $\mathbf{z}_0 \in \mathbb{C}^2$ of the Higgs potential $V_{\rm H}$ determines an integer $\mathrm{n} \in \mathbb{Z}^* \equiv \mathbb{Z}\backslash\{0\}$ which is unique modulo $\mathbb{Z}_2$. For example, one may assume that $\mathbf{z}_0 = (0, 0, 0, \mathrm{r}_0)$ with $\mathrm{r}_0 := \sqrt{\mu^2/2\lambda}$ and $\mathrm{T}(\mathbf{z}_0) = i\tau_3/2 \in \mathrm{su}(2)$ to prove the relation (17). In the case where $\mathrm{y} < 0$, we may replace y by $-|\mathrm{y}|$ in the definition of the hypercharge. So, we may assume $\mathrm{y} > 0$ without loss of generality. The relation (17) is Dirac's quantization condition of electric charge in terms of the hypercharge of the Higgs boson. Modulo $\mathbb{Z}_2$, this condition fixes a specific principal $\mathrm{U}_{\rm elm}(1)$ bundle $\mathcal{Q}$ over $\mathcal{M}$, where $h \in \mathrm{U}_{\rm elm}(1)$ is given by $h = \det\exp(\theta[\mathrm{T} + i\mathrm{y}])$. Accordingly, the embedding λ reads

$$\begin{aligned} \lambda : \mathrm{U}_{\rm elm}(1) &\hookrightarrow \mathrm{SU}(2) \times \mathrm{U}(1) \\ h = \exp(in\theta) &\mapsto (\exp(\theta\mathrm{T}), \exp(i\theta\mathrm{y})). \end{aligned} \tag{18}$$

Therefore, the structure of the electroweak gauge bundle $\mathcal{P}$ is fully determined by the hypercharge of the Higgs boson. Moreover, if the case n=0 is taken into account one obtains $2|\mathrm{n}| + 1$ $\mathrm{U}_{\rm elm}(1)-$reductions of the principal $\mathrm{SU}(2) \times \mathrm{U}(1)$ bundle $\mathcal{P}$.

□

The case n=0 is special in several respects and has been thoroughly discussed in [Tolk'03(c)]. For instance, as already mentioned, the corresponding electromagnetic gauge bundle equals the trivial principal $\mathrm{U}_{\rm elm}(1)$ bundle independently from the topology of space-time

$$\begin{aligned} \mathrm{pr}_1 : \mathcal{M} \times \mathrm{U}_{\rm elm}(1) &\longrightarrow \mathcal{M} \\ q = (x, h) &\mapsto x. \end{aligned} \tag{19}$$

This electromagnetic gauge bundle is the only one which possesses a flat connection. Moreover, it has a natural flat connection that is induced by the Maurer-Cartan form on $\mathrm{U}_{\mathrm{elm}}(1)$. Since the electroweak gauge bundle is trivial too, the corresponding embedding $\iota:\ \mathcal{Q}\hookrightarrow\mathcal{P}$ is given by

$$\begin{array}{rcl} \iota:\ \mathcal{M}\times\mathrm{U}_{\mathrm{elm}}(1) & \hookrightarrow & \mathcal{M}\times(\mathrm{SU}(2)\times\mathrm{U}(1)) \\ (x,h) & \mapsto & (x,\lambda(h)). \end{array} \tag{20}$$

Accordingly, the vacuum section $\mathcal{V}_0$ reads

$$\begin{array}{rcl} \mathcal{V}_0:\ \mathcal{M} & \longrightarrow & \mathcal{M}\times\mathrm{orbit}(\mathbf{z}_0) \\ x & \mapsto & (x,\mathbf{z}_0). \end{array} \tag{21}$$

Indeed, it has been shown that, for the trivial electroweak gauge bundle, any other vacuum section $\mathcal{V}$ must be gauge equivalent to the canonical section $\mathcal{V}_0$. It follows that $\mathfrak{M}_{\mathrm{vac}}\simeq\mathrm{H}^1_{\mathrm{deR}}(\mathcal{M})$, which turns out to be equivalent to the existence of charge conjugation. Therefore, the case n=0 corresponds to ordinary electromagnetism generated by the ground state $\mathcal{V}_0$ of the Higgs boson.

Since for $\mathrm{n}\neq 0$ the corresponding electromagnetic gauge bundles possess no flat connection, we call the appropriate ground states of the Higgs boson "magnetically charged". They spoil the symmetry of ordinary electromagnetism under charge conjugation like a magnetic monopole. However, the $\mathbb{Z}_2-$symmetry of charge conjugation is hidden in the two-to-one relation (17) between the Chern number and the hypercharge of the Higgs boson. Physically this means that it is possible to absolutely distinguish between positive and negative electrically charged particles if we know the gauge class of the ground states of the Higgs boson. We stress that for $\mathrm{n}\neq 0$ the electroweak gauge bundle $\mathcal{P}$ is non-trivial and possesses no flat connection.

When the Gell-Mann-Nishijima relation between the hypercharges and the electric charges of the fermions is taken into account, the hypercharge of the Higgs boson yields y=1/2 (c. f., for instance, [Ait/Hey'82], [Nach'89] or [Wein'01]). Thus, in the case of the (minimal) Standard Model the Higgs boson has exactly three gauge inequivalent ground states, which are parameterized by the Chern numbers $\mathrm{n}=0,\pm1$. These ground states correspond to the lowest non-vanishing electric charge a particle may have.

Let $\mathcal{O}\in\Gamma(\xi_{\mathrm{H}})$ be the zero section and $\Gamma^*(\xi_{\mathrm{H}})\equiv\Gamma(\xi_{\mathrm{H}})\backslash\{\mathcal{O}\}$. Then, the Proposition (3.2) shows that the Higgs bundle ξ_{H} has at least $2|\mathrm{n}|+1$ non-vanishing sections.

Therefore, $\Gamma^*(\xi_{\rm H}) \neq \emptyset$, and for each $\Phi \in \Gamma^*(\xi_{\rm H})$ there is a unique $\mathrm{n} \in \mathbb{Z}$ and a non-vanishing function $\varphi \in \mathcal{C}^\infty(\mathcal{M})$ such that

$$\begin{aligned} \Phi : \mathcal{M} &\longrightarrow \mathrm{E}_{\rm H} \\ x &\mapsto \varphi(x)\mathcal{V}(x). \end{aligned} \tag{22}$$

Consequently, also for $\mathrm{n} \neq 0$ every non-vanishing section of the Higgs bundle is fully determined by its length. This is but a geometrical variant of what is usually refered to as “unitary gauge”. In fact, despite the chosen terminology the unitary gauge is not a choice of gauge (which may not exist globally). Instead, it refers to the moduli space of ground states of the Higgs boson.

4 Magnetically charged vacua and monopoles

As discussed in the last section, ordinary electromagnetism corresponds to the n=0 ground states of the Higgs boson. Moreover, the moduli space of electroweak vacua is related to the topology of space-time via the isomorphism $\mathfrak{M}_{\rm vac} \simeq \mathrm{H}^1_{\rm deR}(\mathcal{M})$.

As we have already mentioned, in the case of $\mathrm{n} \neq 0$ the electroweak gauge bundle possesses no flat connection and thus $\mathfrak{M}_{\rm vac} = \emptyset$. One may therefore ask for an appropriate generalization of $\mathfrak{M}_{\rm vac}$. For this we call a solution $F_{\rm mag} \in \Omega^2(\mathcal{M})$ of the Maxwell equations a “Dirac-Higgs (DH) monopole”, provided it satisfies the following conditions: a) $[F_{\rm mag}]/2\pi \in \mathrm{H}^2_{\rm deR}(\mathcal{M})$ is integral; b) the Chern number n of the isomorphism class of principal U(1) bundles defined by $F_{\rm mag}$ is either zero or related to the hypercharge of the Higgs boson via Dirac's quantization condition (17); c) for each $x \in \mathcal{M}$ there is a geodesic normal coordinate system (U, φ) such that $i_{\partial_t} F_{\rm mag} = 0$, where ∂_t is the (local) time-like vector field that is induced by (U, φ). In particular, for n=0 it is assumed that condition c) holds true for every geodesic normal coordinate system. We let $\mathfrak{M}_{\rm mag}$ be the moduli space of all gauge classes of YMH pairs $(\mathcal{A}, \mathcal{V}) \in \mathcal{A}(\xi_{\rm H}) \times \Gamma(\xi_{\rm H})$ such that $\mathcal{A}$ is associated with a connection on $\mathcal{P}$ which is compatible with the vacuum section $\mathcal{V}$ and whose curvature corresponds to $F_{\rm mag}$.

The condition c) physically means that there is always an inertial reference system (U, φ) such that with respect to this system $F_{\rm mag}$ is purely magnetic: $\iota_t^* F_{\rm mag} = B_t \in \Omega^2(\Sigma_t)$ with $\iota_t : \Sigma_t \hookrightarrow U \subset \mathcal{M}$ is defined by the local space-like hyper-surface

$t = \text{const.}$. Moreover, for n=0 we have $F_{\text{mag}} = 0$.

For $\mathrm{n} \neq 0$ we call $\mathfrak{M}_{\text{mag}}$ the "(magnetically) charged sector" of the moduli space of the electroweak vacua. Accordingly, for n=0 we call $\mathfrak{M}_{\text{mag}} = \mathfrak{M}_{\text{vac}}$ the "(magnetically) uncharged sector" of the electroweak vacua. Note that the charged sector of $\mathfrak{M}_{\text{mag}}$ also depends on the geometry of space-time[1].

To present an example which demonstrates the non-triviality of the magnetically charged sector of $\mathfrak{M}_{\text{mag}}$ we consider the exterior Schwarzschild space-time $(\mathcal{M}, g_{\text{M}})$. Here, $\mathcal{M} \simeq \mathbb{R} \times [r_0, \infty[\times \mathrm{S}^2$ with $r_0 \in \mathbb{R}_+$ the Schwarzschild radius. Consequently, $\mathcal{M} \approx \mathrm{S}^2$ where the latter is a spacelike sub-manifold of $(\mathcal{M}, g_{\text{M}})$. Since the pull-back of g_{M} to this sub-manifold equals the Riemannian standard metric on $\mathrm{S}^2 \subset \mathbb{R}^3$ it is straightforward to check that

$$F_{\text{mag}} = \tfrac{\mathrm{n}}{2} \sin\vartheta \, d\vartheta \wedge d\varphi \tag{23}$$

defines a Dirac-Higgs monopole provided $\mathrm{n} \neq 0$ satisfies Dirac's quantization condition (17). The corresponding electromagnetic gauge bundle $(\mathcal{Q}, \iota)$ over $(\mathcal{M}, g_{\text{M}})$ is equivalent to the (generalized) Hopf bundle (see, for instance, [Trau'80], [Trau'84] and [Nab'00] for a very readable approach)

$$\pi_{\mathrm{n}} : \mathrm{S}^3/\mathbb{Z}_{\mathrm{n}} \longrightarrow \mathrm{S}^2, \tag{24}$$

where $\mathbb{Z}_{\mathrm{n}} := \{e^{2k\pi\, i/\mathrm{n}} \mid k = 0, 1, \ldots, \mathrm{n}-1\} \subset \mathrm{U}(1)$ and π_{n} generalizes the famous Hopf map π_1 between spheres (c. f. [Hopf'31]; For a recent discussion of the various physical meanings of the Hopf fibration see [Urb'03]).

In classical electrodynamics the DH monopole field (23) is regarded as being created by a massive magnetically charged pointlike particle ("Dirac monopole") moving in Minkowski space-time $\mathbb{R}^{1,3}$. In this context the monopole field $F_{\text{mag}} \in \Omega^2(\mathcal{M})$ is known as the Dirac monopole field on $\mathcal{M} := \mathbb{R}^{1,3} \backslash \Gamma$, where $\Gamma \subset \mathcal{M}$ denotes the worldline of the Dirac monopole, see [Dir'31]. However, in the context of the Standard Model the physical interpretation of (23) is different. Notice that in either case $(g_{\text{M}}, F_{\text{mag}})$ is not a solution of the combined Einstein-Maxwell equations. Also notice that the DH monopole has finite energy since the Schwarzschild radius r_0 acts like an ultra-violet cut-off. Indeed, the energy-momentum current $\tau \in \Gamma(\mathrm{End}(\tau_{\text{M}}))$ of the DH monopole

[1] We would like to thank G. Naber for an appropriate hint.

field (23) reads $\tau = \frac{\mathrm{n}^2}{2r^4}\,\mathrm{Id}_{\mathrm{TM}}$. Hence, the "vacuum energy" Λ is given by the magnetical analogue of the electrostatic energy of an electrically charged sphere

$$\Lambda = 4\pi\,\frac{\mathrm{g}^2}{r_0}, \tag{25}$$

where $\mathrm{g} \equiv \mathrm{n}/2$ is the magnetic charge of the DH monopole. Notice that the vacuum energy either is zero or uniquely determined by the Schwarzschild radius and the hypercharge of the Higgs boson, i.e by the topology of space-time and the electroweak gauge bundle. In particular, in the case of the Standard Model one has $\Lambda \in \{0, \pi/r_0\}$.

The given example of a DH monopole also demonstrates that the electroweak gauge bundle $\mathcal{P}$ is nontrivial in general. However, to determine the structure of the moduli space $\mathfrak{M}_{\mathrm{mag}}$ of electroweak vacua for more general space-times $(\mathcal{M}, g_{\mathrm{M}})$ is certainly a major challenge. The geometry of (static) monopoles (in Minkowski space-time) is thoroughly discussed, for example, in [Atiy'79] and [Atiy/Hit'88] (see also, in [Trau'77]). However, the point here is to not regard magnetic monopoles as individual "classical (point-like) particles" in space-time but instead to consider monopoles as specific ground states of the Higgs boson whose realizations depend on both the topology and the geometry of space-time. Indeed, the notion of a world line $\Gamma \subset \mathcal{M}$ is a purely classical concept which seems to make no sense within (quantum) field theory. Also, Dirac's famous quantization condition $\mathrm{qn} \in \mathbb{Z}$ of electric charge $\mathrm{q} \in \mathbb{Q}$ (again, when measured in appropriate units, see again [Dir'31]) holds true only if the appropriate monopole bundle (which is characterized by $\mathrm{n} \in \mathbb{Z}^*$) is identified with the electromagnetic gauge bundle. However, this is consistent with the Standard Model only if the monopole bundle is regarded as a specific $\mathrm{U}_{\mathrm{elm}}(1)-$reduction of the electroweak gauge bundle $\mathcal{P}$. That is, the monopole is identified with a specific gauge class of ground states of the Higgs boson which are not gauge equivalent to those considered in perturbation theory.

In [Tolk'03(c)] it has been shown that, with respect to any electroweak vacuum, the Yang-Mills bundle decomposes as

$$\xi_{\mathrm{YM}} = \xi_{\mathrm{elm}} \oplus \xi_{\mathrm{Z}^0} \oplus \xi_{\mathrm{W}^\pm}. \tag{26}$$

Here, respectively, ξ_{elm} and ξ_{Z^0} are trivial line bundles which geometrically represent an (asymptotically free) photon and an electrically neutral massive weak vector boson. In contrast, $\xi_{\mathrm{W}^\pm} := \tau^*_{\mathrm{M}} \otimes \xi_{\mathrm{W}}$, with $\xi_{\mathrm{W}} \subset \mathfrak{ad}(\mathcal{P})$ denoting a rank two vector bundle, simultaneously represents both of the electrically charged and massive weak vector bosons

$W^{\pm}$ iff electromagnetism is symmetric with respect to charge conjugation. Of course, one may expect that only electrically charged particles permit a physical distinction between the magnetically charged and uncharged ground states of the Higgs boson. With respect to a magnetically charged vacuum, the (massive) vector bosons W^{+} and W^{-} are no longer charge conjugate to each other due to their electromagnetic interaction with the corresponding DH monopole. Geometrically, this is expressed by the non-triviality of $\xi_{\rm W}$ and that the asymptotically free states of the $W^{\pm}-$bosons have to satisfy the field equation (see also eq. 35 in [Tolk'03(c)])

$$\delta_{\rm A} d_{\rm A} W^{\pm} + {\rm m}_{\rm W}^2 W^{\pm} = 0. \tag{27}$$

Here, respectively, $d_{\rm A}$ and $\delta_{\rm A}$ is the exterior covariant derivative and its formal adjoint with respect to a DH monopole connection $\mathcal{A}$ and $W^{\pm} \in \Gamma(\xi_{W^{\pm}})$ is (the electrically charged part of) a smooth "fluctuation" of $\mathcal{A}$. Moreover, ${\rm m}_{\rm W}^2 \in \mathbb{R}_{+}$ is a non-vanishing eigenvalue of the Yang-Mills mass matrix[2] $\mathcal{V}^{*}{\rm M}_{\rm YM}^2 \in \Gamma({\rm End}(\xi_{\rm YM}))$ with $[(\mathcal{A},\mathcal{V})] \in \mathfrak{M}_{\rm mag}$.

However, in the case of a magnetically charged vacuum the $\mathbb{Z}_2-$symmetry of charge conjugation is restored in the two-to-one correspondence between the charge of the vacuum and the hypercharge of the Higgs boson. Of course, it is interesting to also ask for appropriate physical effects which permit to distinguish between the gauge inequivalent ground states of the Higgs boson. Though we do not want to discuss this question here, we would like to stress again that such an interaction can only occur for a topologically non-trivial space-time. Moreover, such an interaction also depends on the space-time geometry, i. e. on the gravitational field. Consequently, appropriate physical effects caused by the interaction of the electrically charged weak vector bosons (resp. fermions) with the electroweak vacuum may provide the possibility to gain some insight into the topology and the geometry of space-time.

5 Conclusion

We have shown that the topological structure of the electroweak gauge bundle either is trivial or fully determined by the hypercharge of the Higgs boson. This is a geometrical variant of Dirac's quantization condition within the realm of the electroweak interaction. For this it is crucial, however, that in the case of the (minimal) Standard

[2]For the terminology used please see loc site.

Model “charge comes with mass”. Indeed, it is a remarkable fact that no massless electrically charged particles are known to exist in nature. In general, the moduli space of the electroweak vacua consists of a magnetically charged and an uncharged sector. The uncharged sector corresponds to ordinary electromagnetism. It is determined by the assumption that the electrically charged massive weak vector bosons are charge conjugate to each other. The uncharged sector is fixed by the topology of space-time via the first de Rham cohomology group of $\mathcal{M}$. In contrast, a necessary condition for the existence of the charged sector (i. e. the existence of DH monopoles) is the non-triviality of the second de Rham cohomology group of space-time. Moreover, a charged ground state of the Higgs boson can only exist if it also fits with the geometry of space-time.

In the case of the (minimal) Standard Model the physical Higgs boson geometrically appears as a fluctuation of any of the three gauge inequivalent ground states of the Higgs boson which are characterized by the Chern numbers $\mathrm{n} = 0, \pm 1$. These ground states correspond to the lowest possible non-vanishing electric charge a particle may assume. However, the magnetically charged ground states $\mathrm{n} = \pm 1$ can be realized only if space-time possesses a non-trivial topology as, for example, in the case of a rotationally symmetric space-time. In this case, the corresponding electromagnetic gauge bundles are equivalent to Hopf fibrations. The appropriate DH monopoles generalize the well-known Dirac monopoles of magnetic charge $\mathrm{g} = \pm 1/2$ to the electroweak interaction within the (minimal) Standard Model. This example may also motivate the terminology of “magnetically charged electroweak vacua”. In general, these topologically non-trivial ground states of the Higgs boson yield electric charge quantization analogous to ordinary Dirac monopoles. Clearly, whether these ground states of the Higgs boson can be actually realized for a general space-time manifold $(\mathcal{M}, g_{\mathrm{M}})$ needs a more thorough analysis of the space of solutions $F_{\mathrm{mag}} \in \Omega^2(\mathcal{M})$ of the corresponding Maxwell equations. Moreover, like in the usual discussion of magnetic monopoles also the definition of DH monopoles does not refer to some field equation of gravity. Of course, this seems unsatisfying from a physical viewpoint. However, to mathematically discuss the structure of the correspondingly enlarged moduli space of electroweak vacua is obviously even more challenging. In any case, the geometrical viewpoint presented here with respect to the electroweak interaction of the (minimal) Standard Model suggests that the mechanism of spontaneous symmetry breaking may provide a better understanding of the link between topology and geometry.

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