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Uniform rectifiability from mean curvature bounds

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Preprint no.: 85 2004

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Introduction:
In the following we give an explicit estimate on the modulus of continuity of the tangent plane of a surface whose mean curvature is controlled.

The control is of the form e. g.

$$H_{\mathcal{F}} = \vec{v}|_{\mathcal{F}} + [f^{+} - f^{-}]n_{\mathcal{F}}$$

with $\int_{\mathcal{F}} |v|^2 + \int | \bigtriangledown f| < K$. Here $\mathcal{F}$ denotes the surface, $n_{\mathcal{F}}$ its normal $H_{\mathcal{F}}$ its mean curvature.

Controlling the modulus of continuity of the tangent plane means that sequences of surfaces with uniform bounds of the above type will converge to rectifiable surfaces with multiplicity – so called integer rectifiable varifolds.

The method of proof is very close to Allard's original rectifiability proof, making the estimates explicit though. It carries over verbatim to varifolds with lower density bounds. This and an expanded version of the proof will appear elsewhere. Here we stay with surfaces, the only tools from geometric measure theory used are the Vitali covering theorem and the monotonicity formula.

A motivation for the result is a proof that implicit time discretization leads to solutions of mean curvature flow satisfying Brakke's energy decay condition. Also that will appear elsewhere.

The result:
Let $\mathcal{F}$ be a d-dimensional surface in $\Omega \subset \mathbb{R}^n$ without boundary in Ω. Suppose that its mean curvature H satisfies:

$$\int_{\mathcal{F}} H n_{\mathcal{F}} \xi = \int_{\Omega} (\xi v + Tr(AD\xi)) d\mu \text{ for all } \xi \in C_0^1(\Omega, \mathbb{R}^n)$$

where $Hn_{\mathcal{F}}$ denotes the mean curvature vector, μ is a Radon measure and v and A satisfy estimates

$$\varrho^{-d-1} \int_{B_\varrho(x)} |A| d\mu + \varrho^{-d} \int_{B_\varrho(x)} |v| d\mu \leq \partial_\varrho F(\varrho, sup_{\varrho < R < dist(x, \partial\Omega)} R^{-d} \int_{B_R(x)} d\bar{\mu})$$

$\bar{\mu}$ is another Radon measure, $F \geq 0$, $F(0,\cdot) = 0$, $\partial_\varrho F \geq 0$ and with $g(L) = \inf\{R^{-d} + F(R,L) | R \geq 0\} \lim_{L\to\infty} L^{-1} g(L) = 0$.

Then for every $M > 0$, $\hat{\Omega} \subset\subset \Omega$, there exists an exceptional set $K_M \subset \hat{\Omega}$ with $\int_{K_M \cap \mathcal{F}} 1 \leq \omega(M) \to_{M\to\infty} 0$. ω depends only on $F, \int_{\mathcal{F}} 1, \int d\bar{\mu}$, such that for $y_1, y_2 \in \mathcal{F} \backslash K_M$, P_1, P_2 the orthogonal projections on the tangent planes at y_1, y_2

$$|P_1 - P_2| \leq M |\ln |y_1 - y_2||^{-\frac{1}{4}}.$$

Remark:

If $\int H n_{\mathcal{F}} \xi = \int_{\mathcal{F}} v \xi + \int_{\mathcal{F}} (f^+ - f^-) n_{\mathcal{F}} \xi$, $\mathcal{F} = \partial\hat{\Omega} \cap \Omega$,

$$\int_{\mathcal{F}} (|v| + 1) + \int_{\Omega} (|\bigtriangledown f^+| + |\bigtriangledown f^-|) \leq K$$

then $\mathcal{F}$ satisfies the condition. In this case

$$\mu = H^{n-1}|_{\mathcal{F}} + H^n, \ \bar{\mu} = (|v| + 1) H^{n-1}|_{\mathcal{F}} + [|\bigtriangledown f^+| + |\bigtriangledown f^-|] H^n,$$

$A = f^+ \chi_{\tilde{\Omega}} + (1 - \chi_{\tilde{\Omega}}) f^-$ and by the Poincaré estimate

$$\varrho^{-n} \int_{B_\varrho(x)} |A| \leq R^{-n} \int_{B_R(x)} A + \ln\left(\frac{R}{\varrho}\right) \sup_{\varrho < r < R} r^{1-n} \int_{B_r(x)} \left(|\bigtriangledown f^+| + |\bigtriangledown f^-|\right)$$

i. e. $F(\varrho, L) = \varrho(\ln \frac{R}{\varrho} + 1) L$.

Sketch of the proof:

We start with the monotonicity formula for $u(\varrho, x) = \varrho^{-d} \int_{\mathcal{F}} \varphi\left(\frac{(x-y)}{\varrho}\right) dy$ where $\varphi \geq 0, \varphi' \leq 0$, $\varphi(s) = 1$ for $0 \leq s \leq \frac{1}{2}$ and $\varphi(s) = 0$ for $s \geq 1$:

$$\text{(M)} \qquad \begin{aligned} \partial_\varrho u(\varrho, x) &= -\varrho^{d-1} \int_{\mathcal{F}} \varphi'\left(\frac{|x-y|}{\varrho}\right) \frac{|x-y|}{\varrho} \left|(Id - P_y) \frac{x-y}{|x-y|}\right|^2 dy \\ &\quad + \int_{\mathcal{F}} H n_{\mathcal{F}} \frac{x-y}{\varrho^{d+1}} \varphi\left(\frac{|x-y|}{\varrho}\right) dy \end{aligned}$$

Here $H n_{\mathcal{F}}$ is the mean curvature vector, P_y the projection on the tangent space to $\mathcal{F}$ at y.

By the Vitali covering theorem we can estimate the mean curvature term outside a set $K_1(L)$ of the form $K_1(L) = UB_{\varrho_i}(x_i)$ with $\sum \varrho_i^d \leq L^{-1} \int d\bar{\mu}$ by:

$$\int_{\mathcal{F}} H n_{\mathcal{F}} \frac{x-y}{\varrho^{d+1}} \varphi\left(\frac{|x-y|}{\varrho}\right) \leq c\partial_\varrho F(\varrho, L)$$

More precisely if $x \notin K_1(L)$ or $x = x_i,\ \varrho \geq \varrho_i$

$$\partial_\varrho\left(u(\varrho, x) + cF(\varrho, L)\right) \geq \varrho^{-d-1} \int_{\mathcal{F}} |\varphi'| \left(\frac{|x-y|}{\varrho}\right) \frac{|x-y|}{\varrho} \left|(Id - P_y)\frac{x-y}{|x-y|}\right|^2 dy$$

The trick is to integrate this in equality with respect to x and ϱ to get for $R < dist(\hat{\Omega}, \partial\Omega)$

$$(\mathrm{M_i}) \quad \left(R^{-d} \int_{\mathcal{F}} 1\right) + c_1 F(R, L) \geq c_2 \int_{B_R(y)\cap\mathcal{F}\setminus K_1(L)} \int_{\hat{\Omega}\cap\mathcal{F}} |x-y|^{-d} \left|(Id - P_y)\frac{x-y}{|x-y|}\right|^2$$

Interchanging the order of integration outside a set $K_2(L_1)$ of measure proportional to $\frac{1}{L_1}$

$$(\mathrm{M'_{L_1}}) \qquad \int_{B_R(y)\cap\mathcal{F}\setminus K_1(L)} |x-y|^{-d} \left|(Id - P_y)\frac{x-y}{|x-y|}\right|^2 \leq L_1 \text{ for } y \in \hat{\Omega}\cap\mathcal{F}\setminus K_2(L_1)$$

now by another use of Vitali's covering theorem and the estimate

$$\int_{\mathcal{F}\cap B_{\alpha\varrho_i}(x)} 1 \leq \alpha^d \varrho_i^d g(L) \int_{\mathcal{F}} 1\ .$$

The set

$$K_3(\varepsilon, L) = \left\{y^2 | \text{ there exists } \rho > 0 \int_{B_\varrho(y)\cap K_1(L)\cap\mathcal{F}} 1 \geq \varepsilon \int_{B_{8\varrho}(y)\cap\mathcal{F}} 1\right\}$$

can be estimated

$$\int_{\mathcal{F}\cap K_3(\varepsilon,L)} \leq (c + \frac{1}{\varepsilon}) g(L) \left(\int_{\mathcal{F}} 1\right)\left(\int d\bar{\mu}\right)$$

Now using the transversality estimate:

If $|P_{y_1} - P_{y_2}| \geq \gamma$ there exists a $d-1$ dimensional space such for the projection π onto this subspace

$$|(Id - \pi)x|^2 \leq \frac{4}{\gamma^4}\left[|(Id - P_{y_1})x|^2 + |(Id - P_{y_2})x|^2\right]$$

and adding (M'_{L_1}) for y_1 and y_2 one gets with $y = \frac{1}{2}(y_1 + y_2),\ \rho = |y_1 - y_2|$

$$\sum_{l=0}^{\ln(R/\rho)} (2^l\rho)^{-d} \int_{B_{2^l\rho}(y)\cap\mathcal{F}\setminus K_1(L)} \left|(Id-\pi)\frac{y-x}{|y-x|}\right|^2 \leq \frac{c(L_1+g(L))}{\gamma^4}$$

Split $B_{2^l\rho}(y)$ into $B_{2^l\rho}(y)\cap\{(Id-\pi)\frac{y-x}{|y-x|}|^2 < \delta\}$ and the rest. Using the monotonicity formula in $B_{2^l\rho}(y)\cap\mathcal{F}\setminus K_1(L)$ again,

$$\sum_{l=0}^{\ln R/\rho} \left((2^l\rho)^{-d}\left(\int_{\mathcal{F}\cap(B_{2^l\rho}(y)\setminus K_1(L)} 1 - c\delta \int_{\mathcal{F}\cap B_{2^l\rho}(y)}\right) - c\delta F(2^{l+1}\rho, L) + c\delta F(2^l\rho\delta, L)\right) \leq c\frac{L_1+g(L)}{\gamma^4\delta^2}$$

If $y_1 \notin K_3(\varepsilon, L)$ then

$$\sum_{l=0}^{\ln R/\varrho} \left((2^l\rho)^{-d}\left(\int_{\mathcal{F}\cap B_{2^l\rho}} 1 - (\varepsilon + c\delta)\int_{\mathcal{F}\cap B_2^{l+3}\rho} 1\right) - F(2^{l+1}\rho, L) + F(2^l\rho\delta, L)\right) = 0(\gamma^{-4})$$

Choosing ε, δ appropriately we get a contradiction if $\gamma^4 = 0(\frac{1}{|\ln\rho|})$

This proves the theorem.

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