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TWISTOR SPINORS WITH ZEROS ON COMPACT ORBIFOLDS

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Abstract. We study compact spin orbifolds with finite singularity set carrying twistor-spinors. We show that any non-trivial twistor-spinor admits at most one zero which is singular unless the orbifold is conformally equivalent to a round sphere. We show the sharpness of our results through examples.

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1. Introduction

Twistor spinors occur as solutions of a conformally covariant field equation. Outside their zeros they can be viewed as conformal analogues of parallel spinors or Killing spinors. Using the solution of the Yamabe problem A. Lichnerowicz obtained the following

Theorem 1. (A. Lichnerowicz [Li]) *If a compact Riemannian spin manifold carries a non-trivial twistor spinor with zero then the manifold is conformally equivalent to the standard sphere.*

Non conformally flat Riemannian spin manifolds carrying twistor spinors with zeros were first constructed by W. Kühnel and H.B. Rademacher [KR1] using a conformal completion of the gravitational instanton constructed by T. Eguchi and A. J. Hanson [EH]. One can also use the explicitly given asymptotically Euclidean Ricci flat Kähler metrics with holonomy $\mathrm{SU}(m), m \geq 2$ given by E. Calabi [Ca] to obtain non conformally flat examples in all even dimensions ≥ 4. These examples can be used to construct compact *orbifolds* with finite singularity groups carrying twistor spinors with zeros, cf. Section 5.
In this paper we consider orbifolds having singularities of type $\mathbb{R}^n/\Gamma$ where $\Gamma \subset \mathrm{SO(n)}$ is a finite subgroup of the special orthogonal group acting freely on $\mathbb{R}^n \setminus \{0\}$. We introduce the concept of a spin structure

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on orbifolds in Section 2. In Section 3 we use the connection between manifolds resp. orbifolds carrying twistor spinors with zero and manifolds with parallel spinors with an asymptotically Euclidean end, which was first introduced in [KR2]. We obtain as main result in Section 4

Theorem 2. *Let M be a compact Riemannian spin orbifold with finitely many singularities. If M carries a non-trivial twistor spinor with zero at p then it is the only zero point and p is a singular point unless the orbifold is conformally equivalent to a round sphere. In addition for every point $q \neq p$ the order $\#\Gamma_q$ of the singularity group satisfies*

$$\#\Gamma_q \leq \#\Gamma_p$$

with equality only if the orbifold is a quotient of a round sphere.

This result can be viewed as a singularity result extending Lichnerowicz' Theorem 1. In Section 5 we present examples of compact Riemannian spin orbifolds carrying twistor spinors with zero in even dimensions $2m \geq 4$ and with finite singularity groups $\Gamma \subset \mathrm{SU}(m)$. These metrics are not conformally flat, the space of twistor spinors is 2-dimensional and all twistor spinors have the same zero point.

2. Spin orbifolds

In this section we will recall the basic properties of an orbifold and define a spin structure on it.

Definition 1. *An n-dimensional* orbifold *M is a Hausdorff topological space, together with an atlas of charts (U, f), where U is an open set in M and $f : U \to \mathbb{R}^n/\Gamma_f$ is a homeomorphism, where Γ_f is a finite subgroup of $GL(n, \mathbb{R})$ (acting on $\mathbb{R}^n$), such that the transition functions $g \circ f^{-1} : f(U \cap V) \to g(U \cap V)$ are differentiable in the following sense:*

For any $x \in U \cap V$ there exists a small neighbourhood $W \subset U \cap V$ and a differentiable map (called the lift of the transition function) $F : \pi_f^{-1}(f(W)) \to \pi_g^{-1}(g(W))$ such that $g \circ f^{-1} \circ \pi_f = \pi_g \circ F$, where $\pi_f : \mathbb{R}^n \to \mathbb{R}^n/\Gamma_f$ and $\pi_g : \mathbb{R}^n \to \mathbb{R}^n/\Gamma_g$ are the canonical projections.

Remark. Note that the lift F is not unique unless $\pi_f^{-1}(f(W))$ and $\pi_g^{-1}(g(W))$ are connected, but it is always a local diffeomorphism.

Remark. As any finite subgroup of $GL(n, \mathbb{R})$ is conjugated to one sitting in $O(n)$, we will suppose the groups Γ above are orthogonal. An *orientation* on M is defined, as in the case of a manifold, by the choice of an atlas of charts, such that the Jacobian of the lift of transition functions has positive determinant. This is well-defined if the groups Γ actually lie in $SO(n)$. As we are interested in oriented orbifolds, we will always suppose the groups Γ to lie in $SO(n)$.

The concept of a differentiable map between orbifolds can also be defined using the three steps in the definition above. Of course, the charts and their compositions with the canonical projections $\pi_\Gamma : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ are differentiable.

Definition 2. *The (singularity) group of a point $x \in M$ is the conjugacy class of a minimal (with respect to the inclusion) finite subgroup $\Gamma_x \subset SO(n)$ such that there exists a* minimal *chart (U, f) around x with $\Gamma_f = \Gamma_x$. If $\Gamma_x = \{1\}$ we say that x is a smooth point; otherwise it is called singular.*

If the singularity group of a singular point x acts freely on $\mathbb{R}^n \setminus \{0\}$, then the singularity is *isolated*, i.e., it is surrounded by smooth points. This can only happen in even dimensions.

Remark. A chart (U, f) as in the definition above is called *minimal* because there is no other chart around x having a group Γ' with less elements than Γ_x. Such a chart, composed with the canonical projection from $\mathbb{R}^n$ to $\mathbb{R}^n/\Gamma$, yields a *ramified covering* of a neighbourhood of x by an open set in $\mathbb{R}^n$.

Example. In dimension 2, the group of a singularity is equal to $\mathbb{Z}_n$, $n \geq 2$, and the "total angle" around such a point is $2\pi/n$ (around a smooth point it is 2π). So here the singularities are always isolated (note that we restricted ourselves to oriented orbifolds). A basic neighbourhood of such a "conical" point is actually homeomorphic to a disk, so every 2-dimensional oriented orbifold is homeomorphic to a manifold. This happens only in dimension 2, in larger dimensions the basic neighbourhood of an isolated singularity is a cone over S^{n-1}/Γ_x, and in general a quotient of a sphere is not homeomorphic to the sphere itself.

If a singularity is not isolated, then the group of a neighbouring point is the isotropy group of the corresponding point in $\mathbb{R}^n$ under the action of Γ_x, so it is a subgroup of Γ_x. The set of singularities of M is then a (not necessarily disjoint) union of submanifolds, each of even codimension (because of the orientability condition). For example, in dimension 3 the singularity set is a disjoint union of circles and open segments.

In this paper we will focus on orbifolds with isolated singularities, but many concepts may be extended to the general setting. For simplicity, from now on we *will assume that the singularity groups $\Gamma \subset SO(n)$ act freely on $\mathbb{R}^n \setminus \{0\}$.*

Fundamental remark: Any object on an orbifold M can be seen, in a neighbourhood of a point x, as a Γ_x-invariant object on a local ramified covering by a smooth manifold (obtained from a minimal chart composed with a canonical projection - we call this *the (minimal) smooth covering of M around x*).

We can now consider tensors on orbifolds: Locally they must come from Γ_x-invariant tensors on the minimal smooth covering around x.

Remark. A vector field on an orbifold must vanish on any isolated singularity and, in general, it must be tangent to the singular set, because in these points x it must be Γ_x-invariant. In general, the tensor fields on M must have particular (Γ_x-invariant) values in the singular points. In particular, a *metric* on an orbifold is locally a Γ_x-invariant metric on the minimal smooth covering around x.

We can carry on most of the differential-geometric constructions on orbifolds: for example, we can consider the Levi-Civita connection, differentiate vector fields, take their Lie bracket etc. All those operations may be performed locally in a chart, and there we will work on the smooth covering with Γ-invariant objects.

We are interested now in putting a spin structure on an orbifold. Before doing that, recall that the Spin bundle is a double covering of the total space of the bundle of orthonormal frames, which is non-trivial on each fiber. Note that, locally around x, the frame "bundle" $SO(M)$ is just the quotient of the frame bundle of the smooth covering under the action (by isometries) of Γ_x. This action is always free, so the frame bundle of an orbifold is a smooth manifold (but no longer a fiber bundle).

We will, however, continue to call $SO(M)$ the *bundle* of orthonormal frames on M, and, in general, we will continue to use the term *bundle* for quotients of (locally trivial) fiber bundles on the local smooth coverings, and for objects which are locally of this type.

Definition 3. *A Spin structure on an orbifold M is given by a two-fold covering of the frame bundle $SO(M)$, which is non-trivial over each fiber SO_x, $\forall x \in M$.*

We can then describe a spin structure on an orbifold as being locally a Γ_x-invariant spin structure on the smooth covering around x, but first we have to be able to lift the action of the group Γ_x of isometries to the Spin bundle of the smooth covering.

Definition 4. *A singularity x is said to be* spin *if there is a lift $G_x \subset Spin(n)$ of $\Gamma_x \subset SO(n)$ which projects isomorphically onto Γ_x via the canonical projection from $Spin(n)$ to $SO(n)$.*

If such a lift exists, it is not necessarily unique; for example, in dimension 2, the group $\mathbb{Z}_n$ can be lifted to $Spin(2)$ if and only if n is odd. In that case there are always 2 such possible lifts.

Remark. There is another, deeper, motivation for the definition above: As any spin structure on M restricts to one on the smooth part $M \setminus S$, it should induce spin structures on the quotients of small spheres around any singular point, i.e., on S^{n-1}/Γ. The condition above is necessary and sufficient for S^{n-1}/Γ to be spin [Fr, p. 47].

The following result shows that is is enough to look at the smooth part of an orbifold to see if it is spin and to determine its spin structure.

Proposition 1. *Let M be an orbifold with isolated singularities, and let S be the set of these singularities. Then M is spin if and only if the manifold $M \setminus S$ is spin. Moreover, the Spin structures on M are in 1-1 correspondence with the spin structures on $M \setminus S$.*

Proof. A spin structure on M can be obtained by gluing together some spin structures on

$$M \setminus \cup_{x\in S} U_x, \text{ and on } \bar{U}_x, \ x \in S$$

along the union of the boundaries of the U_x, $x \in S$. Here U_x is the image of a small ball through the smooth covering around x.

To do the gluing, the spin structures (i.e., the two-fold coverings of the frame bundles) must coincide on the corresponding boundary components (which are diffeomorphic to quotients of spheres). But this if equivalent to saying that the induced spin structures on the boundary are the same.

Of course, $M \setminus S$ is spin if and only if $M \setminus \cup_{x\in S} U_x$ is spin, so what we actually have to show is that any spin structure on $M \setminus \cup_{x\in S} U_x$ can be "filled in", in a unique way, in each U_x.

This is implied by the following fact:

Lemma 1. *Any spin structure on S^{n-1}/Γ can be uniquely filled in on B^n/Γ, where $\Gamma \subset SO(n)$ is the (finite) group of the isolated spin singularity $0 \in B^n/\Gamma$. Here, B^n is the unit ball in $\mathbb{R}^n$ and $n \geq 4$.*

Proof. Let us describe first the spin structures on the orbifold $\mathbb{R}^n/\Gamma$. The orthogonal and spin frame bundles over $\mathbb{R}^n$, denoted by $SO(\mathbb{R}^n)$ and by $Spin(\mathbb{R}^n)$, respectively, are both Lie groups acting by isometries on $\mathbb{R}^n$ (actually $SO(\mathbb{R}^n)$ is the group of Euclidean transformations on $\mathbb{R}^n$), and the canonical projection from the last to the former is a group homomorphism.

$\Gamma \subset SO(n) \subset SO(\mathbb{R}^n)$ is then a subgroup, and so is $G \subset Spin(\mathbb{R}^n)$. As $SO(\mathbb{R}^n/\Gamma) = SO(\mathbb{R}^n)/\Gamma$ (here we notice that Γ acts on $SO(\mathbb{R}^n)$ by left multiplication – hence commutes with the (right) $SO(n)$ action on the fibers, which, in turn, has nothing to do with the Lie group structure of $SO(\mathbb{R}^n)$), we see that the fundamental group of $P := SO(\mathbb{R}^n/\Gamma)$ is the preimage $\tilde{\Gamma} \subset Spin(n)$ of Γ under the fundamental projection $p : Spin(n) \to SO(n)$. So we have the following exact sequence:

$$1 \to \{\pm 1\} \to \pi_1(P) \to \Gamma \to 1.$$

On the other hand, if $\tilde{P}$ is a two-fold covering of P, it is itself covered by $Spin(\mathbb{R}^n)$ which is the universal cover of both P and P', so we have the exact sequence

$$1 \to G' \to \pi_1(P) \to \{\pm 1\} \to 1.$$

$\tilde{P}$ is a spin structure on $\mathbb{R}^n/\Gamma$ if, moreover, the two-fold covering $\tilde{P} \to P$ is non-trivial on each fiber SO_x, $x \in \mathbb{R}^n/\Gamma$. This implies that G' cannot contain the non-trivial element in the $\mathbb{Z}_2$ from the first sequence, so the second sequence is a splitting of the first one.

So $G' = G$ must be a lift of Γ. It is worth mentioning that the double cover $\tilde{P}$ of P is actually determined by the subgroup G of $\tilde{\Gamma}$.

So, there is a 1-1 correspondence between the lifts G of Γ in $Spin(n)$ and the spin structures on $\mathbb{R}^n/\Gamma$, namely

$$G \longmapsto Spin(\mathbb{R}^n)/G,$$

where G acts by left multiplication.

If we remove the singularity (and the corresponding fiber from the frame bundles), the fundamental groups remain the same, because $n > 2$. We can carry on the same argument to conclude (see also [Fr]) that there is a 1-1 correspondence between the lifts G of Γ and the spin structures on $(\mathbb{R}^n \smallsetminus \{0\})/\Gamma$, or, equivalently, the spin structures on S^{n-1}/Γ. $\square$

We can use the Lemma to fill in the spin structures on each U_x, and thus get a spin structure on M starting from one on $M \smallsetminus S$. The converse is obvious. $\square$

Remark. In [Fr] precise algebraic conditions on $\tilde{\Gamma}$ are given for the existence of a lift G of Γ in $Spin(n)$. If $n = 4k+2$, $k \in \mathbb{N}^*$, such a lift is always unique, if it exists [Fr, sec. 2.2].

Example The group $\{\pm 1\} \subset SO(4)$ is the group of a spin singularity: indeed, on $S^3/\{\pm 1\} \simeq \mathbb{R}P^3$ there are exactly 2 (inequivalent) spin structures. On the other hand, the 2 possible lifts of $\{\pm 1\}$ in $Spin(4) \simeq SU(2) \times SU(2)$, which are $\{\pm 1\} \times \{1\}$ and $\{1\} \times \{\pm 1\}$.

From now on, all considered singularities will be isolated and spin, and for all the corresponding groups Γ there will be associated a lift G.

Remark. The action of Γ by isometries on $SO(\mathbb{R}^n)$ commutes with the right action of $SO(n)$ on the fibers of the frame bundle on $\mathbb{R}^n$. The lifted action of G on $Spin(\mathbb{R}^n)$ equally commutes with the right action of $Spin(n)$ on the fibers. It follows that G equally acts on every associated bundle $Spin(\mathbb{R}^n) \times_\rho F$, where $\rho : Spin(n) \times F \to F$ is a C^∞ representation of $Spin(n)$.

In particular, G acts on the spinor bundles $\Sigma^\pm(\mathbb{R}^n)$ of positive, resp. negative Weyl spinors of $\mathbb{R}^n$ (remember that n is even).

Definition 5. *The total spaces of the spinor bundles $\Sigma^\pm(M)$ over an orbifold M are the orbifolds obtained by gluing together the spinor bundles $\Sigma^\pm(M \smallsetminus S)$ with $\Sigma^\pm(U_x) \simeq \Sigma^\pm(\mathbb{R}^n)/G$. A spinor field on an orbifold M is a pair of smooth sections in each of these two bundles, such that the lifts in any local smooth covering around $x \in M$ are smooth (G_x-equivariant) spinors.*

Remark. If the quotient $\Sigma^\pm(\mathbb{R}^n)/G$ is not smooth, then the value of the spinor field in 0 must lie in the singular set, more precisely in the set of fixed points of G in $\Sigma_0^{\pm}$. We will focus later on twistor spinors; they have the property that if the spinor ϕ and the value of $D\phi$ (where D is the Dirac operator) simultaneously vanish in some point, then the twistor spinor is everywhere zero (in a connected manifold), see next section. As ϕ and $D\phi$ are sections in the 2 different spinor bundles, they cannot both vanish at a singularity x, which means first of all that G_x must have non-zero fixed points at least on Σ_0^+ or on Σ_0^-. This implies certain constraints in dimension 4:

Proposition 2. *Let ϕ^+ be a positive and ϕ^- a negative Weyl spinor field on a 4-dimensional orbifold. In a singular point x at least one of them vanishes. Moreover, they both vanish unless there is a complex structure on $\tilde{T}_xM$ such that $\Gamma_x \subset SU(\tilde{T}_xM) \simeq SU(2)$. In that latter case, if $\Gamma_x \neq \{\pm 1\}$, then, for any other local spin structure around x, every spinor field ψ vanishes at x.*

The proof follows from the identity $Spin(4) \simeq SU(2) \times SU(2)$, so any group $G \subset Spin(4)$ having a fixed point in Σ^+ must be totally contained in the second factor and conversely. If this is the case, then any other lift of the corresponding projection $\Gamma \subset SO(4)$ will act nontrivially on Σ^+ (always), and on Σ^- (unless $G = \Gamma = \{\pm 1\}$).

3. Zeros of twistor spinors and conformal inversion

Let us first recall the twistor equation on a Riemannian spin manifold of dimension n : We call a spinor field ϕ a *twistor spinor* if the following *twistor equation* holds for all tangent vectors X :

$$\nabla_X \phi + \frac{1}{n} X \cdot D\phi = 0 \,. \tag{1}$$

Here ∇ is the *spin-connection* on the spinor bundle $\Sigma(M)$, the dot "$\cdot$" denotes the Clifford multiplication and D the Dirac operator. This definition extends to the orbifold case: around a singularity x the spinor field ϕ and tangent vectors X must then have G_x-equivariant liftings. The spin-connection $\nabla_X\phi$ and the Dirac operator $D\phi = \sum_{i=1}^n e_i \cdot \nabla_i \phi$ of a G_x-equivariant spinor field are again G_x-equivariant.

If ϕ is a twistor spinor then one computes for the derivative of the Dirac operator $D\phi$:

$$\nabla_X D\phi = \frac{n}{2} L(X) \cdot \phi \,. \tag{2}$$

Here L is the $(1,1)-$*Schouten tensor* defined by

$$L(X) = \frac{1}{n-2}\left(\frac{s}{2(n-1)} X - \mathrm{Ric}(X)\right)$$

with the *Ricci tensor* Ric and the scalar curvature s. One can view Equation (1) and Equation (2) as a parallelism condition: We define on the double spinor bundle $E = \Sigma(M) \oplus \Sigma(M)$ the connection

$$\nabla^E_X = \begin{pmatrix} \nabla_X & \frac{1}{n} X \cdot \\ -\frac{n}{2} L(X) \cdot & \nabla_X \end{pmatrix},$$

i.e.,

$$\nabla^E_X(\phi, \psi) = \left(\nabla_X \phi + \frac{1}{n} X \cdot \psi, -\frac{n}{2} L(X) \cdot \phi + \nabla_X \psi \right).$$

Then we obtain

Lemma 2. [BFGK, ch. 1.4, Thm. 4]. *A twistor spinor ϕ on a Riemannian spin manifold is uniquely determined by a parallel section of the bundle E with connection ∇^E. More precisely, if ϕ is a twistor spinor, then $(\phi, D\phi)$ is a parallel section of (E, ∇^E) and if (ϕ, ψ) is a parallel section of (E, ∇^E) then ϕ is a twistor spinor and $\psi = D\phi$.*

Corollary 1. *Any zero P of a twistor spinor ψ is isolated, more precisely $\nabla_X \psi(P) \neq 0$ for any non-zero vector $X \in T_P M$.*

Proof. If both ψ and $D\psi$ vanish at P, then ψ is identically zero (as unique parallel section in E with this initial data). Therefore $\nabla_X \psi = -\frac{1}{n} X \cdot D\psi$ is non-zero if $X \neq 0$, as Clifford product of a non-zero vector with a non-zero spinor $D\psi$. ☐

We use the following notation for the open Euclidean ball of radius $R : B_R := \{x \in \mathbb{R}^n \mid \|x\| < R\}$, then $\overline{B}_R := \{x \in \mathbb{R}^n \mid \|x\| \leq R\}$ is the corresponding closed Euclidean ball.

Definition 6. (a). *Let Γ be a finite subgroup of $SO(n)$ that acts freely on $\mathbb{R}^n \setminus \{0\}$ and let U be an open subset of a Riemannian manifold. Then the open subset $U \subset M$ carries an* Asymptotically Locally Euclidean (ALE) coordinate system $y = (y_1, \ldots, y_n)$ *of order* (τ, μ) *if the following conditions are satisfied: There is for some $R > 0$ a diffeomorphism $y \in \left(\mathbb{R}^n - \overline{B}_R\right)/\Gamma \mapsto \phi(y) \in U$ such that the metric coefficients $g_{ij}(y) = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$ with respect to the coordinates $y = (y_1, \ldots, y_n)$ and its derivatives*

$$\partial_{i_1 \ldots i_k} g_{ij} = \frac{\partial^k}{\partial y_{i_1} \cdots \partial y_{i_k}} g_{ij}$$

have the following asymptotic behaviour for $\rho = \|y\| = \sqrt{\sum_1^n y_i^2} \to \infty$:

$$g_{ij} - \delta_{ij} = O(\rho^{-\tau}); \partial_{i_1 \ldots i_k} g_{ij} = O\left(\rho^{-\tau-k}\right)$$

for all $k = 1, 2, \ldots, \mu$. If the group $\Gamma = \{1\}$ is trivial, then the coordinate system is called Asymptotically Euclidean.

(b). *We call a non-compact Riemannian manifold $\overline{M}$ of dimension n* Asymptotically Locally Euclidean *or short* ALE *of order (τ, μ) if there is a compact subset M_0 such that the complement $M - M_0$ carries an asymptotically Euclidean coordinate system of order (τ, μ).*

Remark 1. There is a close relation between manifolds carrying twistor spinors with zeros and non-compact manifolds with parallel spinors with an end carrying an ALE coordinate system, more precisely: Let (M, g) be a Riemannian spin manifold with a twistor spinor ϕ having a zero p. It is isolated and the length $\|\phi\|$ behaves like a distance function in the neighbourhood of p (see Corollary 1). Given normal coordinates $x = (x_1, \ldots, x_n) \in B_\epsilon \subset \mathbb{R}^n$ in a neighbourhood U of p we define *inverted normal coordinates* $y = (y_1, \ldots, y_n) \in \mathbb{R}^n - B_{\epsilon^{-1}}, y = x/\|x\|^2$. Then the conformally equivalent metric $(U - \{p\}, \overline{g} = g/\|\phi\|^4)$ carries a parallel spinor and an asymptotically Euclidean coordinate system of order $(3, 2)$. It is locally irreducible unless it is flat. This is shown in [KR4, Theorem 1.2].

On the other hand one can use a metric with parallel spinor having an end with an ALE coordinate system to produce examples of twistor spinors with zeros:

Lemma 3. *If on the open subset U diffeomorphic to $\left(\mathbb{R}^n \setminus \overline{B}_R\right)/\Gamma$ of a smooth, i.e. C^∞ Riemannian manifold (M, g) there is an ALE-coordinate system y with radius function $\rho = \|y\| = \sqrt{\sum_1^n y_i^2}$ of order (τ, μ) with $\mu \geq \tau - 1 \geq 2$, then the conformally equivalent metric $\overline{g} = \rho^{-4} g$ extends as a $C^{\tau-1}$ metric to the one-point completion $U \cup p_\infty$ diffeomorphic to B_R/Γ.*

Proof. We denote by $y = (y_1, \ldots, y_n), \rho > R$ asymptotically Euclidean coordinates and denote

$$g_{ij}(y) = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \delta_{ij} + h_{ij}(y).$$

Then $h_{ij} = O(\rho^{-\tau}), \frac{\partial^k}{\partial y_{i_1} \cdots y_{i_k}} h_{ij}(y) = O(\rho^{-\tau-k})$ for all $k = 1, 2, \ldots, \mu$. We use the *inversion* $z = \rho^{-2} y$ and obtain with the formula $\frac{\partial}{\partial z_i} = \rho^{-2} \frac{\partial}{\partial y_i} - 2\frac{z_i}{\rho^2} \sum_{k=1}^n z_k \frac{\partial}{\partial y_k}$ for the coefficients $\overline{g}_{ij}$ of the conformally equivalent metric $\overline{g}$ with respect to the inverted coordinates $z = (z_1, \ldots, z_n)$; $z_i = \rho^{-2} y_i$:

$$\begin{aligned} \overline{g}_{ij}(z) &= \hat{g}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = \delta_{ij} + h_{ij} - \frac{2}{\rho^2}\left(z_i \sum_k z_k h_{kj} + z_j \sum_l z_l h_{il}\right) \\ &\quad + \frac{4}{\rho^4} z_i z_j \sum_{k,l} z_k z_l h_{kl}\,. \end{aligned} \tag{3}$$

It follows that $\overline{g}_{ij}(z) = \delta_{ij} + O(r^\tau)$ with $r = \|z\| = \sqrt{\sum_1^n z_i^2} = \rho^{-1}$ and $\frac{\partial^k}{\partial z_{i_1} \cdots \partial z_{i_k}} \overline{g_{ij}(z)} = O\left(r^{\tau-k}\right)$. Hence we obtain that the function $\overline{g}_{ij}(z)$ extends to $z = 0$ as a $(\tau - 1)$-times continuously differentiable function. ∎

The following result establishes a *conformal completion* of a Ricci-flat *ALE* Kähler metric which will provide us with examples of conformal orbifolds admitting twistor spinors with zero.

Theorem 3. *Let* Γ *be a subgroup of* $SO(n), n = 2m$ *acting freely on* $\mathbb{R}^n \setminus \{0\}$ *and let* (M, g) *be an ALE Riemannian spin manifold* (M, g) *(with a* C^∞*-metric) of order* (τ, τ) *(i.e. asymptotically Euclidean to* $\mathbb{R}^n/\Gamma$ *at infinity) with* $\tau \geq 2$ *and holonomy group* SU(m). *Then there is a one point conformal completion* $N = M \cup \{p_\infty\}$ *of* (M, g) *to a compact Riemannian spin orbifold with singular point* p_∞ *whose singularity group is* Γ. *The metric* $\overline{g}$ *on* $M = N - \{p_\infty\}$ *is* C^∞*-smooth and conformally equivalent to* g *on* $M = N - \{p_\infty\}$ *and it is a* $C^{\tau-1}$ *metric on* N.
Then there is a spin structure on the orbifold N *with a two-dimensional space of twistor spinors, and all nontrivial twistor spinors have exactly one zero point, which is the singularity point* p_∞.

Proof. Since the holonomy group is $\mathrm{SU}(m)$ one can conclude that the manifold M is spin and has a preferred spin structure for which the space of parallel spinors is two-dimensional, cf. [Jo, Corollary 3.6.3]. It follows from Lemma 3 that the conformally equivalent metric $\rho^{-4}g$ can be extended to the orbifold $N = M \cup \{p_\infty\}$. If U is a sufficiently small neighbourhood of p_∞ then there is a Riemannian covering $(V \setminus \{p\}, \overline{g})$ of $(U \setminus \{p\}, \overline{g})$ with Riemannian covering group Γ which is diffeomorphic to $B_1^n \setminus \{0\}$ carrying a twistor spinor ϕ invariant under Γ. It follows from the behaviour of twistor spinors under conformal changes that its norm with respect to the Hermitian metric on the spinor bundle of $(V \setminus \{p\}, \overline{g})$ is given by $\|\phi\| = r$, i.e. the spinor field ϕ can be continuously extended to U by setting $\phi(p_\infty) = 0$. As pointed out above a twistor spinor ϕ is in one-to-one correspondence with a parallel section of the bundle E with connection ∇^E. Then it follows from the next Lemma 4 that it extends to a continuously differentiable section, hence it is a Γ-invariant twistor spinor on $(V, \overline{g})$. Therefore the orbifold $(N, \overline{g})$ carries a twistor spinor with zero in p. □

In the proof of Lemma 3 we used the following general result to extend the twistor spinor into the singularity:

Lemma 4. *Let* $E \to M$ *be a* $\mathcal{C}^1$ *vector bundle equipped with a continuous linear connection (i.e., in a* $\mathcal{C}^1$ *trivialization map, the coefficients of the connection form are continuous forms on* M*, with values in the set of linear endomorphisms of the fiber), and let* σ *be a parallel section over* $M \setminus \{p\}$. *Then* σ *can be uniquely extended to a* $\mathcal{C}^1$ *global section (hence parallel).*

Proof. By replacing M, if necessary, with a smaller neighbourhood of p, we can suppose the bundle is trivial, so $E = M \times F$, and the connection form is a continuous 1-form ω on M, with values in $End(F)$. Consider some Riemannian metric on M and an Euclidean metric on F. Without loss of generality we can assume that $\omega_p = 0$, and (by restricting it, if necessary, to a smaller neighbourhood around p),

$$|\omega_x(X) \cdot V| \leq \varepsilon |X||V|, \tag{4}$$

where the dot means matrix multiplication, and the norms of $X \in T_xM$, $V \in F$ are computed using the above chosen Riemannian, resp. Euclidean Metric on M and F.

We consider a parallel lift of a $\mathcal{C}^1$ loop c in M, passing through x, and contained in a compact ball around x. It is a curve $t \mapsto (c(t), \gamma(t))$ in $M \times F$. This lift is characterized by the following linear ODE:

$$\gamma'(t) = -\omega_{c(t)}(c'(t)) \cdot \gamma(t)$$

Suppose $|c'| \equiv 1$, so c is an arc length parameterization. By taking the scalar product in F with $\gamma(t)$, and using (4), we obtain

$$|(|\gamma(t)|^2)'| \leq 2\varepsilon |\gamma(t)|^2,$$

hence

$$|\gamma(t)| \leq e^{\varepsilon t} |\gamma(0)|,$$

and

$$|\gamma'(t)| \leq \varepsilon e^{\varepsilon t} |\gamma(0)|. \tag{5}$$

This gives us a bound for the length of γ, depending continuously on the length of c, and linearly on the initial data $\gamma(0)$. The first conclusion is that the lift is defined for all times.

Let $\sigma : M \smallsetminus \{p\} \to F$ be a parallel section outside p. Let c be an arbitrary $\mathcal{C}^1$ curve on M, such that $c(0) = p$ and $c'(0) \neq 0$. Starting from $c(t_0) \neq p$, for some small time t_0, we consider the lift $t \mapsto (c(t), \sigma(c(t)))$, which is horizontal, because σ is parallel. But we can define this lift for all times, including 0. We obtain a curve γ, depending on c, and a point $(p, \gamma(0)) = (p, A)$ in the fiber over p.

We want to show that this point does not depend on the choice of the curve c. Suppose that, for another curve $\tilde{c}$, equally arc length parametrized, we obtain a lift $\tilde{\gamma}$, such that $\tilde{\gamma}(0) = B \neq A$.

Using c for the beginning, $-\tilde{c}$ for the end (i.e., we run $\tilde{c}$ in the reverse sense), and connecting them smoothly, we can get, for any $n \in \mathbb{N}^*$, a sequence of $\mathcal{C}^1$ arc length parametrized loops c_n in M, of length $l_n \leq 1/n$, such that $c_n(0) = c_n(l_n) = p$.

The corresponding lifts γ_n through σ are well defined over $M \smallsetminus \{p\}$, and because of the argument above, $\gamma_n(0) = \gamma(0) = A$ (the curves c and $c(n)$ coincide around 0), and $\gamma_n(l_n) = \tilde{\gamma}(0)$ (the curves $-\tilde{c}$ and c_n coincide around 0, resp. l_n).

From (5), the length of γ_n is smaller than $|A|(e^{\varepsilon/n} - 1)$, therefore the distance between A and B must be smaller that this expression.

So $A = B$.

This allows us to define $\sigma(p) := A$ and get a continuous global section. Because the lifts through σ of $\mathcal{C}^1$ curves through p are parallel lifts through A, we easily obtain the differentiability σ in p, and actually that σ is $\mathcal{C}^1$.

It is also, by construction, parallel. □

4. Main results

This section contains the proof of Theorem 2.

Let $u := \langle \phi, \phi \rangle$, then the orbifold $\overline{M} = (M - Z_\phi, \overline{g} = u^{-2} g)$ is a Ricci flat Riemannian orbifold carrying a parallel spinor. This follows from the analogue of Remark 1 for orbifolds. If $Z_\phi = \{p_1, \ldots, p_m\}$ then $\overline{M}$ has m ends, at which the metric in inverted normal coordinates is asymptotically locally Euclidean (i.e. asymptotic to $\mathbb{R}^n / \Gamma_{p_j}$).

As in the Riemannian case we first show that if $m > 1$ there is a geodesic line in $\overline{M}$. This geodesic line is obtained as a limit of segments connecting points approaching two different ends (see, for example, [Pe], Ch. 9). We notice that in the process no segment (i.e., minimizing geodesic) can pass close to an isolated singular point (otherwise it wouldn't be minimizing).

So the singularities play no role in the process, which carries over exactly as in the smooth case and we get a geodesic line (which avoids singularities, as well).

On the other hand, there is an orbifold generalization of the splitting Theorem of Cheeger and Gromoll:

Proposition 3. ([BoZ, Theorem 1]) *If $\bar{M}$ is a complete Riemannian orbifold of dimension n with non-negative Ricci curvature carrying a geodesic line, then $\bar{M}$ is isometric to the product $\mathbb{R} \times N$ of the real line with a Riemannian orbifold N of nonnegative Ricci curvature.*

If we therefore have more than one end, there should be such a splitting. If N is not a manifold, then we straightforward obtain a contradiction since the singular set on M is discrete. In the other case $\bar{M}$ is a complete Ricci-flat Riemannian manifold, hence from the splitting theorem of Cheeger and Gromoll it cannot have more than one end.

So there is only one end. The inequality in the conclusion of the Theorem is proved using growth estimates for the volume of balls (more precisely, an extension to orbifolds of Bishop's Theorem):

Proposition 4. ([Bo, Proposition 20]) *Let $\bar{M}$ be a complete Riemannian orbifold with singular set Σ and non-negative Ricci curvature* $\mathrm{Ric} \geq 0$. *Then for every point $p \in \bar{M}$ the function*

$$r \in (0, \infty) \mapsto \psi(r) := \frac{\mathrm{vol} B(p, r)}{\omega_n r^n}$$

is non-increasing. Here $\mathrm{vol}B(p,r)$ *is the volume of the geodesic ball* $B(p,r)$ *of radius* r *around* p *and* ω_n *is the volume of a unit ball in Euclidean space* $\mathbb{R}^n$*. Furthermore*

$$\lim_{r\to 0}\psi(r) = \frac{1}{\#\Gamma_p}$$

where Γ_p *is the isotropy subgroup of the point* p*. Moreover, if, for some* $r>0$, $\psi(r)=\frac{1}{\#\Gamma_p}$*, then* $B(p,r)$ *is isometric to the quotient* B^n_r/Γ_p *of the ball of radius* r *in* $\mathbb{R}^n$ *with* Γ_p*.*

The proof of [KR4, Lemma 2.3] carries on to the orbifold case and gives the following volume estimation of large balls in an ALE orbifold:

Lemma 5. *Let* $(\bar{M},g)$ *be an ALE Riemannian orbifold (with corresponding group* Γ*) of order* (τ,μ)*, with* $\tau>1,\mu\geq 0$*. Then the function* ψ *(representing the relative volume of a ball in* $\bar{M}$ *w.r.t. the Euclidean space) satisfies:*

$$\lim_{r\to\infty}\psi(r) = \frac{1}{\#\Gamma}.$$

Choose a point $p\in\bar{M}$ (if p is a smooth point, then $\Gamma_p=\{1\}$). Since the function $\psi(p;r)=\frac{\mathrm{vol}B(p,r)}{\omega_n r^n}$ is non-increasing it follows from the previous Lemma and Proposition 4 that

$$\#\Gamma_p \leq \#\Gamma,$$

where Γ is the group of the (unique) ALE end (identified to the singularity group of the zero of the twistor spinor).

The equality occurs, according to Proposition 4, if and only if $\bar{M}$ is isometric to $\mathbb{R}^n/\Gamma_p$, therefore $\Gamma\simeq\Gamma_p$ and M is conformally equivalent to S^n/Γ (where Γ acts by isometries and has two mutually opposed fixed points).

Remark. We can show that the vanishing locus $\{p_1,\dots,p_k\}$ of a twistor spinor on a compact orbifold contains only singular points without using Proposition 3. Indeed, from the volume estimates given in Proposition 4 we obtain

$$1\geq\frac{1}{\#\Gamma_{p_1}}+\cdots+\frac{1}{\#\Gamma_{p_k}},$$

so if any p_i is smooth, it must be unique and we get equality in the volume estimate, so the metric is flat. This simplifies the proof of the Lichnerowicz' Theorem 1, as contained in [KR4].

Note that the above inequality does not exclude the existence of multiple *singular* zeros of a twistor spinor, so the use of Proposition 3 is necessary.

Corollary 2. *Let* (M,g) *be a compact Riemannian spin orbifold with finite singularity set carrying a twistor spinor with non-empty zero set.*

Then either M is a quotient of S^n or it is a bad *(or:* non-trivial*) Riemannian orbifold.*

Proof. If $M = N/\Gamma$ for a compact Riemannian spin manifold N then also N carries a twistor spinor with non-empty zero set. Hence it is a round sphere.

We can actually say a little more about the sphere quotients that can occur: they have exactly 2 singular points:

Proposition 5. *Any compact, conformally flat orbifold M carrying a twistor spinor with zero is conformally equivalent to S^n/Γ, where $\Gamma \subset SO(n)$ acts on $\mathbb{R}^n \subset \mathbb{R}^{n+1} \supset S^n$.*

Proof. We keep the notations as above. $\bar{M}$ is a flat, complete orbifold with singularity set $\bar{\Sigma}$. $\bar{M} \setminus \bar{\Sigma}$ is a manifold, therefore it has a universal cover Q. Now, in a neighbourhood of a singular point $P \in \bar{\Sigma}$, $\pi : Q \to \bar{M} \setminus \bar{\Sigma}$ looks like a disjoint union of connected coverings of $\mathbb{R}^n \setminus \{0\}/\Gamma_P$, where Γ_P is the singularity group of P. Each of these connected components V_i of $\pi^{-1}(U)$ can be completed by adding a point P_i to V_i. Furthermore, $\bar{Q} := Q \cup \{P_i \mid P \in \bar{\Sigma}\}$ can be given the structure of a flat orbifold, and $\pi : \bar{Q} \to \bar{M}$ is then a ramified covering of orbifolds.

But Q is simply connected and flat, therefore any vector field can be globally extended to a parallel one. But V_i is isometric to some quotient of a ball, and its tangent space is spanned by parallel vector fields if and only if it is a ball itself. So all V_i's are isometric to balls and $V_i \cap \{P_i\}$ are smooth balls. So $\bar{\mathbb{Q}}$ is a smooth manifold, flat and complete, thus $\bar{\mathbb{Q}} \simeq \mathbb{R}^n$. So it has, like $\bar{M}$, one end, therefore the covering π is finite (the end of $\bar{M}$ has finite fundamental group).

So $\bar{M} \simeq \mathbb{R}^n/G$, where $G \subset \mathrm{Isom}(R^n)$ is finite and has only isolated fixed points. As each $g \in G$ may have at most one fixed point, the set Σ of all fixed points, and $\bar{\Sigma} \subset \pi(\Sigma)$.

Let K be the convex envelop of Σ an let $P \in \Sigma$ be one of the vertices of the polyhedron K. Then $\Sigma \cap \bar{H} = \{P\}$ for a certain "exterior" closed halfspace $\bar{H}$. Let $P' \neq P$ be another point in Σ. Then $P'' := \sum_{k=1}^{N_g} g^k(P')$ is another fixed point for g, where $g \in G \setminus \{1\}$ is such that $g(P) = P$ and $g^{N_g} = 1$. But $P'' \in \mathbb{R}^n \setminus \bar{H}$, therefore $P'' \neq P$. In this case g would have two distinct fixed points, so it would leave the whole line connecting them fixed, contradiction with the finiteness of Σ.

So Σ consists of only one point and the conclusion easily follows. $\square$

So the equality case in our Theorem characterizes the conformally flat orbifolds carrying twistor spinors with zero (which actually turn out to be sphere quotients).

5. Examples

Let Γ be a finite subgroup of the group $\mathrm{Sp}(1) = \mathrm{SU}(2)$ acting freely on $\mathbb{H} \setminus \{0\}$, here $\mathbb{H} \cong \mathbb{R}^4$ is the space of quaternions. Then a hyperkähler ALE space (of order $(4, \infty)$) for this group is also called *gravitational instanton* in the physics literature. Outside a compact set the metric is asymptotic to the quotient $\mathbb{H}/\Gamma$ with order $(4, \infty)$ (cf. Definition 6) and in addition also the hyperkähler structure is asymptotic to the Euclidean hyperkähler structure, cf. [Jo, Definition 7.2.1]. A hyperkähler space is in particular Ricci flat, therefore if follows from the rigidity part of the Bishop-Gromov volume comparison theorem [KR4] that the group is non-trivial unless the metric is flat.

Example 1. The first examples of a gravitational instanton are the *Eguchi Hanson spaces* (M_{EH}, g_{EH}), for the subgroup $\Gamma = \mathbb{Z}_2 = \{\pm 1\}$. These can be given explicitly, cf. [EH], [Jo, Example 7.2.2]. The space M_{EH} is the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ at 0, it can be identified with the cotangent bundle $T^*\mathbb{CP}^1 \cong T^*S^2$ of the 1-dimensional complex projective space resp. the 2-sphere. Hence this space is simply-connected and spin. It is shown in [KR1] that one can conformally compactify the Eguchi Hanson space to a compact orbifold (with C^∞-metric) with one singular point p_∞ whose singularity group is $\mathbb{Z}_2$. The existence of this compactification (at least as a C^3-metric) follows also from Theorem 3. Hence we obtain a compact 4-dimensional Riemannian spin orbifold (M, g) with one singular point p_∞ whose singularity group is $\mathbb{Z}_2$ carrying two linearly independent twistor spinors ψ_1, ψ_2. The singularity point is the unique zero point of ψ_1, ψ_2.

Example 2. Gibbons and Hawking generalized the Eguchi Hanson construction and obtained hyperkähler ALE spaces asymptotic to $\mathbb{H}/\mathbb{Z}_k$ for all $k \geq 2$, cf. [GH]. Finally Kronheimer described in [Kr1] and [Kr2] the construction and classification of hyperkähler ALE spaces asymptotic to $\mathbb{H}/\Gamma$ for nontrivial finite subgroup $\Gamma \subset \mathrm{SU}(2)$, see also [Jo, Theorem 7.2.3].

Example 3. On $\mathbb{C}^m \setminus \{0\}$ the group $\mathbb{Z}_m$ generated by complex multiplication with $\zeta = \exp(2\pi\sqrt{-1}/m)$ acts freely. The blow-up X of $\mathbb{C}^m/\mathbb{Z}_m$ at 0 can be identified with a complex line bundle over the $(m-1)$-dimensional complex projective space $\mathbb{CP}^{m-1}$. One can explicitly write down an ALE Kähler metric with holonomy $\mathrm{SU}(m)$, cf. [Ca], [FG] or [Jo, Example 8.2.5]. Using Theorem 3 or the explicit form given in [KR2] one obtains a compact spin orbifold of real dimension $n = 2m$ with one singular point p_∞ and singularity group $\mathbb{Z}_m$.
This orbifold carries a spin structure with a two-dimensional space of twistor spinors whose common zero point is the singular point.

All the examples described so far are defined using a *crepant resolution* of the isolated singularity $0 \in \mathbb{C}^2$. For a definition of a crepant resolution see [Jo, ch. 6.4]. In particular for a crepant resolution the first Chern class vanishes, hence one can show that a resolution of $\mathbb{C}^m/\Gamma$ carries a Ricci flat ALE Kähler metric only if the resolution is crepant, cf. [Jo, Proposition 8.2.1]. In complex dimensions $2, 3$ all resolutions of $\mathbb{C}^m/\Gamma$ are crepant, but in higher dimensions this does no longer hold, for example $\mathbb{C}^4/\{\pm 1\}$ does not carry any crepant resolution, cf. [Jo, Example 6.4.5]. For crepant resolutions of the quotient singularity $\mathbb{C}^m/\Gamma$ there is a solution of the Calabi conjecture, cf. [Jo, Theorem 8.2.3], [Jo, Theorem 8.2.4]: Let Γ be a finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m - \{0\}$ and let X be a crepant resolution of $\mathbb{C}^m/\Gamma$. Then in every Kähler class of ALE Kähler metrics (with order $(2m, \infty)$) there is a unique Ricci-flat Kähler metric with holonomy $\mathrm{SU}(m)$. Then we can conclude from [Jo, Corollary 3.6.3] that this metric is spin and carries a 2-dimensional space of parallel spinors. The resolution is simply-connected, hence there is only one spin structure. Hence starting from a crepant resolution of the quotient sigularity $\mathbb{C}^m/\Gamma$ with an ALE Kähler metric (of order $(2m, \infty)$) one obtains a compact spin orbifold of dimension $n = 2m$ with a C^{2m-1}-Riemannian metric which carries a 2-dimensional space of twistor spinors. All twistor spinors have exactly one zero in the only singularity point.

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