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On Improvement of the Iterated Galerkin Solution of the Second Kind Integral Equations

by

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On Improvement of the Iterated Galerkin Solution of the Second Kind Integral Equations

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Abstract

For a second kind integral equation with a kernel which is less smooth along the diagonal, an approximate solution obtained by using a method proposed by the author in an earlier paper, is shown to have a higher rate of convergence than the iterated Galerkin solution. The projection is chosen to be either the orthogonal projection or an interpolatory projection onto a space of piecewise polynomials. The size of the system of equations that needs to be solved, in order to compute the proposed solution, remains the same as in the Galerkin method. The improvement of the proposed solution is illustrated by a numerical example.

Key Words : integral equation, Galerkin method, collocation method

AMS subject classification : 65R20

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1 Introduction

Let

$$u - Tu = f, \tag{1.1}$$

denote a second kind integral equation, where T is a compact linear integral operator defined on a complex Banach space X and f and u belong to X. It is assumed that $(I - T)$ is invertible, so that (1.1) has a unique solution.

As, in general, (1.1) can not be solved exactly, an approximate solution is obtained by replacing the operator T by a finite rank operator. If π_n is a sequence of finite rank projections converging to the Identity operator I pointwise, then in the classical Galerkin method (1.1) is approximated by

$$u_n - \pi_n T \pi_n u_n = \pi_n f.$$

In [14], Sloan proposed an improvement of the Galerkin solution by using iteration techniques. Since then the superconvergence properties of the Sloan iterate have been studied by many authors. (See Chandler [4], Chatelin [5], Chatelin-Lebbar [6], [7], Graham-Joe-Sloan [8], Richter [12], Schock [13], Sloan [15], [16], Sloan-Burn-Datyner [17], Spence-Thomas [18].) In [11] Lin-Zhang-Yan have proposed interpolation post-processing technique as an alternative to the iteration technique.

Recently in Kulkarni [10] a new method based on projections is proposed for approximate solution of (1.1). Under the assumption that the kernel of the integral operator T and the right hand side f are smooth, in [10] it is shown that, while it is necessary to solve a system of equations of the same size as for the Galerkin method, the resulting solution obtained converges faster than the Galerkin and the Sloan solution.

The aim of this paper is to extend the results of [10] to the case when the kernel k fails to be sufficiently differentiable because of discontinuities along the diagonal. In the case of orthogonal projections onto a space of piecewise polynomials, it is shown that the order of convergence in the iterated version of the proposed method is higher than the Sloan solution. In the case of interpolatory projection, it is well known that, in general, Sloan solution does not improve upon the Galerkin solution, but there is an improvement in the case of the interpolatory projection at the Gauss points. In this paper it is shown the proposed solution always improves upon the Galerkin solution. This paper extensively uses results from Atkinson-Potra [2] and Chatelin-Lebbar [7].

The paper has been organised as follows. In Section 2, the method proposed in [10] is recalled along with the relevant results and notation is set. Also, the type of kernel which is considered in this paper is specified and some results from [2] and [7] are cited for the future reference. Precise orders of convergence in the case of the orthogonal projection as well as the interpolatory projection are obtained in Section 3. Section 4 is devoted to numerical results.

2 Method and Notation

Let π_n be a sequence of finite rank projections converging to the Identity operator pointwise. In Kulkarni [10] it is proposed to approximate T by the following finite rank operator

$$T_n^M = \pi_n T \pi_n + \pi_n T (I - \pi_n) + (I - \pi_n) T \pi_n.$$

Then

$$\|T - T_n^M\| = \|(I - \pi_n) T (I - \pi_n)\| \to 0 \text{ as } n \to \infty.$$

The corresponding approximation of (1.1) becomes

$$u_n^M - (\pi_n T \pi_n + \pi_n T (I - \pi_n) + (I - \pi_n) T \pi_n) u_n^M = f, \tag{2.1}$$

while the iterative refinement is defined by

$$\tilde{u}_n^M = T u_n^M + f. \tag{2.2}$$

The following result is quoted from [10].

Theorem 2.1. *For all large n, $I - T_n^M$ is invertible,*

$$\|u - u_n^M\| \quad \leq \quad C \|(I - \pi_n) T (I - \pi_n) u\| \tag{2.3}$$

and

$$\|u - \tilde{u}_n^M\| \leq \|(I - T)^{-1}\| \;\; (\|T (I - \pi_n) T (I - \pi_n) u\| + \|T (I - \pi_n) T (I - \pi_n)\| \;\; \|u - u_n^M\|), \tag{2.4}$$

where C is a constant independent of n.

Throughout this paper C denotes a generic constant independent of n.
The reduction of (2.1) to a linear system of equations is done as follows. (See Kulkarni [10]).
Applying π_n and $(I - \pi_n)$ to equation (2.1) we obtain

$$\pi_n u_n^M - \pi_n T \pi_n u_n^M - \pi_n T (I - \pi_n) u_n^M \quad = \quad \pi_n f, \tag{2.5}$$

$$(I - \pi_n) u_n^M - (I - \pi_n) T \pi_n u_n^M \quad = \quad (I - \pi_n) f. \tag{2.6}$$

Let $w_n^M = \pi_n u_n^M$. The substitution for $(I - \pi_n) u_n^M$ from equation (2.6) in equation (2.5) gives us

$$w_n^M - (\pi_n T \pi_n + \pi_n T (I - \pi_n) T \pi_n) w_n^M = \pi_n f + \pi_n T (I - \pi_n) f, \tag{2.7}$$

which is equivalent to a linear system of equations of size equal to the dimension of the space $\pi_n X$.
We then have

$$u_n^M = w_n^M + (I - \pi_n)Tw_n^M + (I - \pi_n)f \tag{2.8}$$

Let α and γ be integers such that $\alpha \geq \gamma$, $\alpha \geq 0$ and $\gamma \geq -1$. We assume that the kernal k is of the following form.

$$k(s,t) = \begin{cases} k_1(s,t), & 0 \leq s \leq t \leq 1, \\ k_2(s,t), & 0 \leq t \leq s \leq 1, \end{cases}$$

with $k_1 \in C^{\alpha}(\{0 \leq s \leq t \leq 1\})$, $k_2 \in C^{\alpha}(\{0 \leq t \leq s \leq 1\})$. If $\gamma \geq 0$, then it is assumed that $k \in C^{\gamma}([0,1] \times [0,1])$ and if $\gamma = -1$, then the kernel k may have a discontinuity of the first kind along the line $s = t$.
Following Chatelin and Lebbar [7], the class of kernels of the above form is denoted by $\mathcal{C}(\alpha, \gamma)$.
Consider the integral operator

$$(Tx)(s) = \int_0^1 k(s,t)x(t)dt, \quad s \in [0,1], \tag{2.9}$$

where the kernel $k \in \mathcal{C}(\alpha, \gamma)$.
Then the operator $T : L^{\infty}[0,1] \to C[0,1]$ is compact. In fact, the range of T, $R(T)$ is contained in $C^{\gamma_1}[0,1]$, where $\gamma_1 = \min\{\gamma + 1, \alpha\}$.
For $x \in C^k[0,1]$, we define

$$\|x\|_{k,\infty} = \sum_{i=0}^{k} \|x^{(i)}\|_{\infty},$$

where $x^{(i)}$ denotes the ith derivative of x.
For any integer n, let $\Delta^{(n)}$ denote a quasiuniform partition of [0,1]:

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = 1.$$

Let $r \geq 0$ and let $\mathcal{P}_{r,\Delta^{(n)}}$ denote the space of all piecewise polynomials of degree $\leq r$ on each of the subintervals $[t_{i-1}^{(n)}, t_i^{(n)}]$. For the sake of notational simplicity, the index n is dropped. Henceforth we write $\Delta = \Delta^{(n)}$, $\Delta_i = \Delta_i^{(n)} = [t_{i-1}^{(n)}, t_i^{(n)}]$, $t_i = t_i^{(n)}$, $h_i = h_i^{(n)} = t_i^{(n)} - t_{i-1}^{(n)}$, and $h = h^{(n)} = \max_{1 \leq i \leq n} \{h_i^{(n)}\}$.
For $\nu \geq 0$, set

$$C_{\Delta}^{\nu} = \{y \in L^{\infty} : y|\Delta_i \in C^{\nu}(\Delta_i), i = 1, \cdots, n\}.$$

The following result is a particular case of Theorem 4.1 of Atkinson-Potra [2].

Let T be the integral operator defined by (2.9). Then T is a continuous map from C_{Δ}^{α} to C_{Δ}^{α}.
If $x \in C_{\Delta}$, then

$$(Tx)^{(\mu)}(s) = \int_0^1 \frac{\partial^{\mu}}{\partial s^{\mu}} k(s,t)x(t)dt, \quad 0 \leq \mu \leq \gamma_1. \tag{2.10}$$

For $s \notin \Delta$,

$$(Tx)^{(\gamma_1+1)}(s) = \int_0^1 \frac{\partial^{\gamma_1+1}}{\partial s^{\gamma_1+1}} k(s,t)x(t)dt + \frac{\partial^{\gamma_1}}{\partial s^{\gamma_1}} k_1(s,s)x(s) - \frac{\partial^{\gamma_1}}{\partial s^{\gamma_1}} k_2(s,s)x(s). \tag{2.11}$$

Thus

$$\|(Tx)^{(\mu)}\|_\infty \leq C\|x\|_\infty, \quad 0 \leq \mu \leq \gamma_1 + 1. \tag{2.12}$$

3 Orders of Convergence

In this section the main results are proved. In the case of the orthogonal projection onto a space of piecewise polynomials, the orders of convergence of u_n^M and $\tilde{u}_n^M$ are obtained, whereas in the case of the interpolatory projection, the order of convergence of u_n^M is obtained.

3.1 Orthogonal Projection

Let P_n be the restriction to $L^\infty[0,1]$ of the orthogonal projection from $L^2[0,1]$ onto $\mathcal{P}_{r,\Delta}$.
The following result is standard. (See Chatelin-Lebbar [7].) Let

$$\beta = \min\{\alpha, r+1\}.$$

There is a constant C such that for any $x \in C_\Delta^\beta$,

$$\|(I - P_n)x\|_\infty \leq C\|x^{(\beta)}\|_\infty h^\beta. \tag{3.1}$$

Let

$$\beta_1 = \min\{\beta, \gamma+1\} = \min\{\alpha, r+1, \gamma+1\} \text{ and } \beta_2 = \min\{\beta, \gamma+2\} = \min\{\alpha, r+1, \gamma+2\}.$$

Also, if $x \in C_\Delta^{\beta_1}$, then

$$\|(I - P_n)x\|_\infty \leq C\|x^{(\beta_1)}\|_\infty h^{\beta_1}. \tag{3.2}$$

We quote the following estimate from Chatelin-Lebbar [7] for future reference.
Let T be an integral operator with kernel $k \in \mathcal{C}(\alpha, \gamma)$. Then

$$\|T(I - P_n)x\|_\infty \leq C\|x^{(\beta)}\|_\infty h^{\beta+\beta_2}. \tag{3.3}$$

As in the proof of Lemma 9 of [7], it can be shown that

$$\|T(I - P_n)x\|_\infty \leq C\|x^{(\beta_1)}\|_\infty h^{\beta_1+\beta_2}. \tag{3.4}$$

Theorem 3.1. *Let T be an integral operator with the kernel $k \in \mathcal{C}(2\alpha, \gamma)$ and P_n be the orthogonal projection onto $\mathcal{P}_{r,\Delta}$. Assume that the integral equation $u - Tu = f$, $f \in C^{\alpha}_{\Delta}$, is uniquely solvable. Then, for n large enough, (2.1) has a unique solution,*

$$\|u - u_n^M\| \quad = \quad (h^{\beta + \min\{\beta+\beta_1, \gamma+2\}}) \tag{3.5}$$

and

$$\|u - \tilde{u}_n^M\| \quad = \quad O(h^{\beta+\beta_2+\min\{\beta+\beta_1, \gamma+2\}}). \tag{3.6}$$

Proof. Since $f \in C^{\alpha}_{\Delta}$, it follows from Theorem 3 of Chatelin-Lebbar [7] that $u \in C^{\alpha}_{\Delta}$.
Since $P_n u \in C^{\infty}_{\Delta}$, it follows that $u - P_n u \in C^{\alpha}_{\Delta}$. As T is a continuous map from C^{α}_{Δ} to C^{α}_{Δ}, $T(I - P_n)u \in C^{\alpha}_{\Delta}$.
Applying (3.2) we obtain

$$\|(I - P_n)T(I - P_n)u\|_\infty \leq C\|(T(I - P_n)u)^{(\beta_1)}\|_\infty h^{\beta_1}. \tag{3.7}$$

It follows from (2.10) that

$$(T(I - P_n)u)^{(\beta_1)}(s) = \int_0^1 \frac{\partial^{\beta_1}}{\partial s^{\beta_1}} k(s,t)(I - P_n)u(t)dt.$$

Since the kernel $\ell(s,t) = \frac{\partial^{\beta_1}}{\partial s^{\beta_1}} k(s,t) \in \mathcal{C}(\alpha, \gamma - \beta_1)$, by (3.3) we get

$$\|(T(I - P_n)u)^{(\beta_1)}\|_\infty \quad \leq \quad C\|u^{(\beta)}\|_\infty h^{\beta+\min\{\beta, \gamma-\beta_1+2\}}. \tag{3.8}$$

The estimate (3.5) follows from (2.3), (3.7) and (3.8).
Also, by (3.4) and (3.8),

$$\begin{aligned} \|T(I - P_n)T(I - P_n)u\|_\infty \quad &\leq \quad C\|(T(I - P_n)u)^{(\beta_1)}\|_\infty h^{\beta_1+\beta_2} \\ &\leq \quad C\|u^{(\beta)}\|_\infty h^{\beta+\beta_2+\min\{\beta+\beta_1, \gamma+2\}}. \end{aligned} \tag{3.9}$$

By using (2.12) and (3.4), we obtain

$$\begin{aligned} \|T(I - P_n)Tu\|_\infty \quad &\leq \quad C\|(Tu)^{(\beta_1)}\|_\infty h^{\beta_1+\beta_2} \\ &\leq \quad C\|u\|_\infty h^{\beta_1+\beta_2}. \end{aligned}$$

Hence

$$\|T(I - P_n)T\| = O(h^{\beta_1+\beta_2}). \tag{3.10}$$

As a consequence, using (3.5) and the fact that $\|P_n\| \leq C$, a constant independent of n, we have

$$\|T(I - P_n)T(I - P_n)\| \;\; \|u - u_n^M\| = O(h^{\beta+\beta_1+\beta_2+\min\{\beta+\beta_1, \gamma+2\}}). \tag{3.11}$$

Combining (2.4), (3.9) and (3.11), we obtain the estimate (3.6). □

Remark 3.2. The Galerkin and the iterated Galerkin solutions satisfy respectively the following two equations.

$$
\begin{aligned}
u_n^G - P_n T P_n u_n^G &= P_n f, \\
u_n^S - T P_n u_n^S &= f.
\end{aligned}
$$

We quote the following results from Chatelin-Lebbar [7] for comparison.
If the kernel $k \in \mathcal{C}(\alpha, \gamma)$, then

$$\|u - u_n^G\| = O(h^{\beta}), \tag{3.12}$$

$$\|u - u_n^S\| = O(h^{\beta+\beta_2}). \tag{3.13}$$

Let $\alpha \geq 1$. Then it is clear from the above estimates that u_n^S converges to u faster than u_n^G. Also, from the estimates (3.5), (3.6) and (3.13), we see that $\tilde{u}_n^M$ converges to u faster than u_n^S and u_n^M.

If $\gamma \geq 0$ and $\beta < \gamma + 2$, then $\beta_2 = \beta < \min\{\beta + \beta_1, \gamma + 2\}$. Hence u_n^M converges to u faster than u_n^S. If $\beta \geq \gamma + 2$, then the rate of convergence of both u_n^S and u_n^M to u is $\beta + \gamma + 2$.

Thus, if $r + 1 \leq \alpha$ and $r \leq \gamma$, then $\beta = \beta_1 = \beta_2 = r + 1$ and from the estimates (3.12), (3.13), (3.5) and (3.6) we get

$$
\begin{aligned}
\|u - u_n^G\| &= O(h^{r+1}), && (3.14)\\
\|u - u_n^S\| &= O(h^{2r+2}), && (3.15)\\
\|u - u_n^M\| &= O(h^{\min\{3r+3, r+\gamma+3\}}), && (3.16)\\
\|u - \tilde{u}_n^M\| &= O(h^{\min\{4r+4, 2r+\gamma+4\}}). && (3.17)
\end{aligned}
$$

If $r + 1 \leq \alpha$ and $r > \gamma$, then $\beta = r + 1$, $\beta_1 = \gamma + 1$ and $\beta_2 = \gamma + 2$. Then

$$
\begin{aligned}
\|u - u_n^G\| &= O(h^{r+1}), && (3.18)\\
\|u - u_n^S\| &= O(h^{r+\gamma+3}), && (3.19)\\
\|u - u_n^M\| &= O(h^{r+\gamma+3}), && (3.20)\\
\|u - \tilde{u}_n^M\| &= O(h^{r+2\gamma+5\}}). && (3.21)
\end{aligned}
$$

If the kernal k is only continuous, then $\gamma = 0$. Then, for the piecewise constant functions, that is, $r = 0$, from (3.14) - (3.17) we have

$$\begin{aligned}
\|u - u_n^G\| &= O(h), \\
\|u - u_n^S\| &= O(h^2), \\
\|u - u_n^M\| &= O(h^3), \\
\|u - \tilde{u}_n^M\| &= O(h^4),
\end{aligned}$$

whereas for piecewise linear functions, that is, $r = 1$, from (3.18) - (3.21) we obtain

$$\begin{aligned}
\|u - u_n^G\| &= O(h^2), \\
\|u - u_n^S\| &= O(h^4), \\
\|u - u_n^M\| &= O(h^4), \\
\|u - \tilde{u}_n^M\| &= O(h^6).
\end{aligned}$$

3.2 Interpolatory Projection

For $i = 1, \cdots, n$, let

$$\tau_{i1} < \tau_{i2} < \cdots < \tau_{i(r+1)}$$

be $r + 1$ distinct points in $[t_{i-1}, t_i]$ and let

$$A = \{\tau_{ij},\ i = 1, \cdots, n,\ j = 1, \cdots, r + 1\}$$

be the set of the collocation points.

Let

$$Q_n : C_\Delta[0, 1] \to \mathcal{P}_{r,\Delta}$$

be the interpolatory projection defined as follows.

$$Q_n x \in \mathcal{P}_{r,\Delta},\ (Q_n x)(\tau_{ij}) = x(\tau_{ij}),\ \ 1 \leq i \leq n,\ 1 \leq j \leq r + 1.$$

If $t_{i-1} = \tau_{i1},\ \tau_{i(r+1)} = t_i,\ 1 \leq i \leq n$, then $Q_n x \in C[0, 1]$.

We quote the following estimate from Chatelin-Lebbar [7].

There is a constant C such that for any $x \in C_\Delta^\alpha$,

$$\|(I - Q_n)x\|_\infty \leq C\|x^{(\beta)}\|_\infty h^\beta. \tag{3.22}$$

Similarly, if $x \in C_\Delta^{\beta_1}$, then

$$\|(I - Q_n)x\|_\infty \leq C\|x^{(\beta_1)}\|_\infty h^{\beta_1}. \tag{3.23}$$

Let

$$\beta_3 = \min\{\alpha, 2r+2, r+\gamma+3\},\ \beta_4 = \min\{\alpha, 2r+2, r+\gamma+2\}.$$

If the kernel $k \in \mathcal{C}(\alpha, \gamma)$ with $\alpha \geq r+1$ and the collocation points are the Gauss points, then

$$\|T(I-Q_n)x\|_\infty \quad \leq \quad C\|x\|_{\beta_3,\infty} h^{\beta_3}. \tag{3.24}$$

Theorem 3.3. *Let T be an integral operator with the kernel $k \in \mathcal{C}(\alpha, \gamma)$ and let Q_n be the interpolatory projection onto $\mathcal{P}_{r,\Delta}$ described above. Assume that the integral equation $u - Tu = f,\ f \in C^\alpha_\Delta$, is uniquely solvable. Then*

$$\|u - u_n^M\| \quad = \quad O(h^{\beta+\beta_2}). \tag{3.25}$$

If T is an integral operator with the kernel $k \in \mathcal{C}(2\alpha, \gamma)$ with $\alpha \geq r+1$ and the set A of collocation points consists of Gauss points, then

$$\|u - u_n^M\| \quad = \quad O(h^{\beta_1+\beta_4}.) \tag{3.26}$$

Proof. Since $u \in C^\alpha_\Delta$, by (3.22),

$$\|(I-Q_n)u\|_\infty \leq C\|u^{(\beta)}\|_\infty h^\beta. \tag{3.27}$$

Also, for $x \in C_\Delta$, by (3.23) and by (2.12),

$$\begin{aligned} \|(I-Q_n)Tx\|_\infty &\leq C\|(Tx)^{(\beta_2)}\|_\infty h^{\beta_2} \\ &\leq C\|x\|_\infty h^{\beta_2}. \end{aligned}$$

Hence

$$\|(I-Q_n)T\| \leq Ch^{\beta_2}. \tag{3.28}$$

The estimate (3.25) follows from (2.3), (3.27) and (3.28).

If $\alpha \geq r+1$ and the collocation points are the Gauss points, then on applying (3.23) we obtain

$$\|(I-Q_n)T(I-Q_n)u\|_\infty \leq C\|(T(I-Q_n)u)^{(\beta_1)}\|_\infty h^{\beta_1}. \tag{3.29}$$

As in the proof of Theorem 3.1 it follows from (2.10) that

$$(T(I-Q_n)u)^{(\beta_1)}(s) = \int_0^1 \frac{\partial^{\beta_1}}{\partial s^{\beta_1}} k(s,t)(I-Q_n)u(t)dt.$$

Since by assumption, the kernel $k \in \mathcal{C}(2\alpha, \gamma)$, the kernel $\ell(s,t) = \frac{\partial^{\beta_1}}{\partial s^{\beta_1}} k(s,t) \in \mathcal{C}(\alpha, \gamma - \beta_1)$. Hence by (3.24) we get

$$\|(T(I-Q_n)u)^{(\beta_1)}\|_\infty \quad \leq \quad C\|u\|_{\beta_4,\infty} h^{\beta_4}. \tag{3.30}$$

The estimate (3.26) follows from (2.3), (3.29) and (3.30). □

Remark 3.4. If the interpolation points are not Gauss points, the iteration does not improve the order of convergence. Hence only the order of convergence of u_n^M is given. However, in the case of interpolation at the Gauss points, as is observed in the numerical example, the iteration is expected to improve the order of convergence. It was not possible to obtain an estimate for $\|u - \tilde{u}_n^M\|$ justifying this improvement.

The collocation and the iterated collocation solutions satisfy respectively the following two equations.

$$\begin{aligned} u_n^C - Q_n T Q_n u_n^C &= Q_n f, \\ u_n^S - T Q_n u_n^S &= f. \end{aligned}$$

We quote the following results from Chatelin-Lebbar [7] for comparison.
If the kernel $k \in \mathcal{C}(\alpha, \gamma)$, then

$$\|u - u_n^C\| = O(h^{\beta}). \tag{3.31}$$

In general, u_n^S does not improve upon u_n^C.
A comparison of (3.25) and (3.31) shows that while u_n^S converges to u at the same rate as u_n^C, u_n^M converges faster than u_n^C. If $\alpha \geq r+1$, then while

$$\|u - u_n^C\| = O(h^{r+1}),$$

we have

$$\|u - u_n^M\| = O(h^{r+1+\min\{r+1, \gamma+2\}}).$$

Thus, if we consider the space of piecewise linear continuous functions with respect to a partition Δ of $[0,1]$ and if the collocation points are chosen to be the the partition points $t_i, i = 0, 1, \cdots, n$, then since $r = 1$, we obtain

$$\|u - u_n^C\| = O(h^2),$$

whereas

$$\|u - u_n^M\| = O(h^4).$$

This result is illustrated by a numerical example.

In the case of collocation at Gauss points, with $\alpha \geq r+1$, we have (Chatelin-Lebbar [7])

$$\|u - u_n^S\| = O(h^{\beta_3}). \tag{3.32}$$

It is seen from (3.31) and (3.32) that u_n^S converges faster as compared to u_n^C.

In this case, if $r \leq \gamma$, then

$$\beta_1 = \beta_2 = r + 1, \ \beta_3 = \beta_4 = 2r + 2.$$

Thus, using (3.32) and (3.26) we get

$$\|u - u_n^S\| \quad = \quad O(h^{2r+2})$$

and

$$\|u - u_n^M\| \quad = \quad O(h^{3r+3}).$$

On the other hand, if $r > \gamma$, then

$$\beta_1 = \gamma + 1, \ \beta_2 = \gamma + 2, \ \beta_3 = r + \gamma + 3, \ \beta_4 = r + \gamma + 2.$$

As a consequence, using (3.32) and (3.26) we get

$$\|u - u_n^S\| \quad = \quad O(h^{r+\gamma+3})$$

and

$$\|u - u_n^M\| \quad = \quad O(h^{r+2\gamma+3}).$$

In this case, u_n^M converges faster than u_n^S if $\gamma > 0$. If $\gamma = 0$, then both u_n^S and u_n^M converge to u at the rate of $r + 3$.

4 Numerical Results

We consider the integral equation

$$u(s) - \int_0^1 k(s,t)u(t)dt = f(s), \quad 0 \leq s \leq 1,$$

where

$$k(s,t) = \begin{cases} s(1-t) & \text{if } 0 \leq s \leq t \leq 1, \\ t(1-s) & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

and the right hand side is so chosen that the exact solution is

$$u(s) = s^{9/2}.$$

Thus $u \in C^4[0,1]$, but $u \notin C^5[0,1]$. Let $\mathcal{P}_{1,\Delta}$ be the space of piecewise linear continuous functions with respect to the uniform partition of [0,1]:

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1$$

and $Q_n : C[0,1] \to \mathcal{P}_{1,\Delta}$ be the interpolatory projection defined by

$$Q_n u(\frac{i}{n}) = u(\frac{i}{n}), \quad i = 0, 1, \cdots, n.$$

Since the dimension of $\mathcal{P}_{1,\Delta}$ is $n+1$, (2.7) is equivalent to a system of linear equations of size $n+1$.

In this example we have $\alpha = \infty$, $\gamma = 0$ and $r = 1$. Thus $\beta = \beta_2 = 2$. Hence by the estimates (3.31) and (3.25), we have

$$\|u - u_n^C\| = O(h^2)$$

and

$$\|u - u_n^M\| = O(h^4).$$

In the computation of the matrices representing $Q_n T Q_n$ and $Q_n T^2 Q_n$, the operator T is replaced by a discrete operator T_m. In the case of $Q_n T Q_n$, the operator T_m is obtained on replacing the integration in T by the composite Simpson quadrature with respect to Δ and is given by

$$T_m x(s) = \frac{1}{6n}\left(k(s,0)x(0) + 2\sum_{i=1}^{n} k(s, \frac{2i-1}{2n})x(\frac{2i-1}{2n}) + 4\sum_{i=1}^{n-1} k(s, \frac{i}{n})x(\frac{i}{n}) + k(s,1)x(1) \right), \ s \in [0,1].$$

On the other hand, since the kernel k is only continuous along the diagonal $s = t$, in the matrix representing $Q_n T^2 Q_n$, the integration is replaced by the composite Simpson quadrature with resepect to an uniform partition of $[0,1]$ with mesh $1/(2n)$. This choice of numerical quadrature retains the expected order of convergence 4 for the approximation u_n^M .

Note that n this case the Sloan solution u_n^S has the same order of convergence as the Galerkin solution u_n^C and $\tilde{u}_n^M$ and u_n^M have the same orders of convergence.

In the following table the error between the exact solution u and the Galerkin solution u_n^C as well as the proposed solution u_n^M at $s = \frac{2}{3}$ is given. Using two successive values of n, the orders of convergence are computed and are denoted by μ_1 and μ_2, respectively. It is seen that u_n^M improves on u_n^C and the observed values of μ_1 and μ_2 match well with the theoretically predicted values.

Table 5.1: $s = \frac{2}{3}$

n	$\|(u - u_n^C)(s)\|$	μ_1	$\|(u - u_n^M)(s)\|$	μ_2
4	3.9×10^{-2}		2.2×10^{-4}	
8	1.2×10^{-2}	1.80	1.7×10^{-5}	3.70
16	2.6×10^{-3}	2.11	9.4×10^{-7}	4.16
32	6.7×10^{-4}	1.95	6.2×10^{-8}	3.92
64	1.7×10^{-4}	2.03	3.8×10^{-9}	4.04
128	4.2×10^{-5}	1.99	2.4×10^{-10}	3.98
256	1.0×10^{-5}	2.01	1.5×10^{-11}	4.01
512	2.6×10^{-6}	2.00	9.3×10^{-13}	3.99

We next consider $\mathcal{P}_{0,\Delta}$, the space of piecewise constant functions with respect to Δ as the approximating space. Let $Q_n : C_\Delta[0,1] \to \mathcal{P}_{0,\Delta}$ be the interpolatory projection defined by

$$Q_n u(\frac{2i-1}{2n}) = u(\frac{2i-1}{2n}), \quad i = 1, \cdots, n.$$

Thus the collocation points are the Gauss points. $\alpha = \infty$, $\gamma = 0$ and $r = 0$, we have $\beta = \beta_1 = \beta_3 = 1$ and $\beta_4 = 2$. Hence by the estimates (3.31), (3.32) and (3.26), we have

$$\|u - u_n^C\| = O(h),$$

$$\|u - u_n^S\| = O(h^2)$$

and

$$\|u - u_n^M\| = O(h^3).$$

Since the collocation points τ_i are the interior points of $[t_{i-1}, t_i]$, the integration in the evaluation of the matrix corresponding to $Q_n T Q_n$ is replaced by the composite Simpson rule associated with an uniform partition of mesh $\frac{1}{2n}$, whereas in the representation of $Q_n T^2 Q_n$, the composite Simpson rule associated with the uniform partition of mesh $\frac{1}{4n}$ is used.

The computed orders of convergence in the Galerkin, Sloan, New and the iterated New solutions are denoted by μ_1, μ_2, μ_3 and μ_4, respectively and given in the following table along with the error at $s = \frac{2}{3}$.

Table 5.2: $s = \frac{2}{3}$

n	$\lvert(u - u_n^C)(s)\rvert$	μ_1	$\lvert(u - u_n^S)(s)\rvert$	μ_2	$\lvert(u - u_n^M)(s)\rvert$	μ_3	$\lvert(u - \tilde{u}_n^M)(s)\rvert$	μ_4
4	3.9×10^{-2}		1.1×10^{-3}		8.6×10^{-5}		1.5×10^{-5}	
8	2.4×10^{-2}	0.70	3.6×10^{-4}	1.65	1.7×10^{-5}	2.33	6.3×10^{-7}	4.55
16	1.1×10^{-2}	1.15	8.8×10^{-5}	2.03	1.7×10^{-6}	3.37	5.2×10^{-8}	3.62
32	5.8×10^{-3}	0.92	2.3×10^{-5}	1.95	2.3×10^{-7}	2.82	2.9×10^{-9}	4.17
64	2.8×10^{-3}	1.04	5.6×10^{-6}	2.02	2.7×10^{-8}	3.09	1.9×10^{-10}	3.91
128	1.4×10^{-3}	0.98	1.4×10^{-6}	1.99	3.6×10^{-9}	2.96	1.1×10^{-11}	4.60
256	7.1×10^{-4}	1.01	3.5×10^{-7}	2.00	4.4×10^{-10}	3.02	7.3×10^{-13}	4.68
512	3.5×10^{-4}	1.00	8.8×10^{-8}	2.00	5.5×10^{-11}	2.99	4.5×10^{-14}	4.32

We see that the computed orders of convergence given in the above table match with these predicted values. The order of convergence for $\tilde{u}_n^M$ is about 4. It is seen from the above table that u_n^M improves upon u_n^C and u_n^S, whereas $\tilde{u}_n^M$ improves upon u_n^M.

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