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Oscillatory Solutions of Soliton Equations; Phase Transitions and Variational Problems

by

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OSCILLATORY SOLUTIONS OF SOLITON EQUATIONS

PHASE TRANSITIONS AND VARIATIONAL PROBLEMS

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ABSTRACT: Consider the Cauchy problem for the two following soliton equations: Korteweg-de-Vries with a small dispersion parameter and the nonlinear Schrödinger equation in 1+1 dimensions with cubic nonlinearity in the so-called semiclassical limit. It is known that at a certain "caustic" there is a "phase transition" in the solution of each of these equations. Oscillations of high frequency appear, so strong limits no longer exist. Soliton theory enables us to provide detailed information for the solutions as the small parameter goes to zero, prove the existence of weak limits, and describe completely the different oscillatory regions in terms of a variational problem of "electrostatic type". In the case of the focusing NLS this is a non-convex problem.

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MOTIVATION. THE EQUATIONS

A general goal: the understanding of high frequency oscillations appearing in the solutions of nonlinear dispersive equations with a SMALL DISPERSION PARAMETER. A general theory is lacking. For completely integrable equations however, complete and rigorous results are possible.

Archetypal examples of completely integrable equations are the KdV equation and the nonlinear Schrödinger equation. Here, we will only consider the Cauchy problems:

1. The KdV equation [LL]

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \qquad u(x,0) = u_0(x), \tag{1}$$

where $\epsilon > 0$. Eventually we will take $\epsilon \to 0$.

The formal limit of (1) as $\epsilon \to 0$ is

$$u_t - 6uu_x = 0, \qquad u(x,0) = u_0(x), \tag{1a}$$

2. The nonlinear Schrödinger equation. In one spatial dimension and for the special cubic nonlinearity this is

$$ih\psi_t + \frac{h^2}{2}\psi_{xx} \pm |\psi|^2\psi = 0, \qquad \psi(x,0) = A(x)exp[iS(x)/h], \tag{2}$$

where $A(x)$ is real and decaying at infinity (say, for simplicity, belongs to the Schwartz class) and the phase $S(x)$ is real and converges (at a similar rate) to (possibly different) constants as $x \to \pm\infty$. Again, eventually we will take $h \to 0$.

The sign - corresponds to the defocusing case [JLM] and the sign + corresponds to the defocusing case [KMM].

More generally, if u is an N-dimensional vector, one writes

$$ih\psi_t + \frac{h^2}{2}\Delta\psi - U'(|\psi|^2)\psi = 0, \qquad \psi(x,0) = A(x)exp[iS(x)/h], \tag{3}$$

where U is a differentiable function and again $A(x)$ is rapidly decaying at infinity and the components of the N-dimensional vector $S(x)$ converge rapidly to constants as $x \to \pm\infty$. The focusing case is when $U" > 0$ and the defocusing case is $U" < 0$.

Letting

$$\rho = |\psi|^2, \qquad (4)$$
$$\mu = \frac{ih}{2}(\psi\bar{\psi}_x - \psi_x\bar{\psi}),$$

we see that (2) is equivalent to

$$\rho_t + \mu_x = 0, \qquad (5)$$
$$\mu_t + (\frac{\mu^2}{\rho} \mp \frac{\rho^2}{2}) = \frac{h^2}{4}(\rho(log\rho)_x x)x,$$

with

$$\rho(x,0) = |A(x)|^2, \qquad (6)$$
$$\mu(x,0) = |A(x)|^2 S_x.$$

The formal limit of (5) as $h \to 0$ is

$$\rho_t + \mu_x = 0, \qquad (5a)$$
$$\mu_t + (\frac{\mu^2}{\rho} \mp \frac{\rho^2}{2}) = 0,$$

under data (6).

Similarly, if we let

$$\rho = |\psi|^2, \qquad (7)$$
$$\mu = \frac{ih}{2}(\psi\bar{\nabla}\psi - \nabla\psi\bar{\psi}),$$

then (3) is equivalent to

$$\rho_t + \nabla \cdot \mu = 0, \qquad (8)$$
$$\mu_t + \nabla \cdot (\frac{\mu \otimes \mu}{\rho}) + \nabla P(\rho) = \frac{h^2}{4}\nabla \cdot [\rho\nabla^2\rho].$$

where the "pressure"

$$P(\rho) = \rho U'(\rho) - U(\rho) \qquad (9)$$

and

$$\rho(x,0) = |A(x)|^2, \tag{10}$$
$$\mu(x,0) = |A(x)|^2 \nabla S.$$

The formal limit of (8) as $h \to 0$ is

$$\rho_t + \nabla \cdot \mu = 0, \tag{8a}$$
$$\mu_t + \nabla \cdot (\frac{\mu \otimes \mu}{\rho}) + \nabla P(\rho) = 0,$$

with data (10).

A natural question is if, or when (for what values of x, t) do the solutions of (1), (5), (8) converge to the solutions of (1a), (5a), (8a) respectively as $\epsilon, h \to 0$. The answer is as follows. Up to the time t_0 at which the formal limit solution encounters a shock or a singularity, i.e. a classical solution ceases to exist, then convergence holds. After that time, this is not in general true. Numerical experiments reveal instead the existence of oscillatory regions in the x, t space, in which ψ, ρ, μ oscillate with amplitude independent of the small parameter ϵ, h and a frequency of order $O(1/\epsilon)$ or $O(1/h)$. It then follows that convergence cannot hold as the small parameter goes to zero. We shall see however that at least in the integrable cases (1), (2) the existence of weak limits can be proved.

For the proof of convergence before the "break" time t_0, see [LL] for (1), [JLM] and [KMM] for (2), [Gr] for (3) in the defocusing case and [G] for (3) and any real analytic U with real analytic data.

COMPLETE INTEGRABILITY

Completely integrable equations always admit a ”Lax pair” representaion: they can be written in the form

$$L_t = [L, B] = LB - BL, \tag{11}$$

where $L, B(L)$ are linear differential operators (in x) on a Hilbert space (e.g. L_2) whose coefficents are polynomials in u, u_x, u_{xx},...... The evolution of L is equivalent to an evolution equation for u.

The existence of a Lax pair does not quite guarantee integrability. It is only the first step. In fact, there is no definition of complete integrability for infinite dimensional systems. There is only a set of attributes-indicators which one associates with complete integrability. However, none of these attributes is sufficient for integrability.

Some integrability indicators for infinitely dimensional Hamiltonian systems are:

(i) Existence of a Lax pair

(ii) Existence of infinitely many conservation laws

(iii) Existence of ”soliton” solutions. Solitons are localized solutions that behave like particles in the sense that when two solitons encounter each other they interact, but retain their shape, amplitude and speed after their interaction.

Complete integrability implies the existence of so-called action-angle variables in finite dimensions (Arnold-Liouville Theorem) but also in infinite dimensions (at least in cases investigated so far).

There are three kinds of problems that have been rigorously solved via integrability machinery: initial value problems, periodic (and quasi-periodic) problems and bounday value problems.

The class that we choose to focus on here is the class of initial value problems with data that decay at infinity or converge to constants at infinity (not very slowly). The appropriate theory is the SCATTERING AND INVERSE SCATTERING theory for equations like (1) and (2), or more precisely the associated Lax operator.

The Lax operator L associated to (1) is the Schrödinger operator

$$L = -\epsilon^2 \frac{d^2}{dx^2} + u(x) \tag{12}$$

while

$$B = 4\epsilon^3 \frac{d^3}{dx^3} - 3\epsilon[u\frac{d}{dx} + \frac{d}{dx}u]. \tag{13}$$

The Lax operators for the NLS equations are the Dirac operators

$$L = \begin{pmatrix} ih\partial_x & \mp i\psi(x) \\ -i\psi^*(x) & -ih\partial_x \end{pmatrix}, \tag{14}$$

where sign "+" corresponds to the defocusing case, while sign "-" corresponds to the focusing case. * denotes complex conjugation. It is important to note that, the operator is self-adjoint in the defocusing case and non-self-adjoint in the focusing case.

The evolution equation (11) is equivalent to the soliton equations (1) or (2) depending on the right chose of the operators L and B.

COMPARISON WITH A ZERO VISCOSITY PROBLEM

Consider the Burgers problem:

$$u_t - 6uu_x + \epsilon u_{xx} = 0,$$

$$u(x,0) = u_0(x).$$

The Cole-Hopf transformation

$$U_x = u$$

$$U = -2\epsilon logV$$

leads to the heat equation

$$V_t = \epsilon V_{xx}$$

and

(15)
$$u(x,t;\epsilon) = \frac{\int (\frac{x-y}{t}) exp(-\frac{D}{\epsilon}) dy}{\int exp(-\frac{D}{\epsilon}) dy},$$
$$D = \int_{-\infty}^{y} u_0(z) dz + \frac{(x-y)^2}{2t}.$$

Letting $\epsilon \to 0$ and applying the standard asymptotic analysis to exponential integrals we get

(16)
$$lim_{\epsilon \to 0} u(x,t,\epsilon) = \frac{x - y^*}{t},$$

where $y^*(x,t)$ minimizes $D(x,y,t)$.

It is easy to see that, before the shock time, $lim_{\epsilon \to 0} u(x,t,\epsilon)$ solves $u_t - 6uu_x = 0$. In fact, it also provides a nonclassical solution for (almost) all values of x, t. In other words, the Burgers equation regularizes the hyperbolic equation $u_t - 6uu_x = 0$.

We shall see that a variational problem exists also for KdV and NLS ! The zero dispersion limits of (1) and (2) as the small parameters ϵ, h tend to zero can be described via the solution of variational problems of "electrostatic type". However there is no regularization of the formal limit!

SMALL DISPERSION LIMIT OF KdV

We consider equation (1) with data (1a) as $\epsilon \to 0$.

Numerical experiments with small ϵ reveal the coexistence of two regions: there is a 'smooth' (in the sense of non-oscillatory) region and an oscillatory region. The oscillations appear to be regular, of amplitude independent of ϵ and frequency of order $O(1/\epsilon)$. A strong limit does not exist in the oscillatory region. It turns out that a weak limit does exist.

More precisely, [LL] prove that until the shock time for (1a*) the STRONG limit exists, as $\epsilon \to 0$, uniformly in time. After shock time, a weak limit does exist.

RIGOROUS ANALYSIS

For simplicity, we choose negative one-well initial data $u_0(x) = -v(x)$, where $0 < v(x) \leq 1$ and $\int_{\mathbb{R}}(1+x^2)v(x)dx < \infty$.

[LL] prove that as $\epsilon \to 0$, the limit of (1) is described by a so-called pure multi-soliton solution. On the other hand one of the major results of soliton theory is that multisoliton solutions are associated to eigenvalues of the Lax operator. One then needs to study the spectral problem for L , as $\epsilon \to 0$. The study of this prolem goes back to Weyl. The eigenvalues 'fill up' the interval $(-1,0)$ with a certain density which is computable in terms of initial data. In fact both eigenvalues and norming constants (essentially the L_2 norms of the eigenfunctions) can be computed up to an $o(\epsilon)$ error.

The density of the eigenvalues $-\eta_j^2, j = 1, ..., N$ over the interval $(-1,0)$ is $\frac{1}{\pi\epsilon}\phi(-\eta)$, with

$$\phi(\eta) = \int \frac{\eta dx}{(v(x)-\eta^2)^{1/2}},$$

the domain of integration being the interval $[x_-(\eta), x_+(\eta)]$, where $x_- < x_+$ are the two solutions of $v(x) = \eta^2$.

More precisely, the eigenvalues and the associated norming constants are given asymptotically as $\epsilon \to 0$ by

$$\int_{x_-}^{x_+} (v(x) - \eta_j^2)^{1/2} dx = (N - 1/2)\epsilon\pi + o(\epsilon), \tag{16}$$
$$c_j \sim exp[\frac{\eta_j x_+(\eta_j) + \int_{x_+}^{\infty} (\eta_j - (v(x) - \eta_j^2)^{1/2}) dx + 4\eta_j^3 t}{\epsilon}].$$

Now soliton theory provides explicit formulae for multisoliton solutions with specified associated eigenvalues and norming constants. Here is for example a version due to Kay and Moses.

THEOREM. The multisoliton solution of (1) associated to the eigenvalues $-\eta_j^2, j = 1, ..., N$ and the norming constants $c_j, j = 1, ..., N$ is given by

$$u(x, t, \epsilon) = \partial_{xx} log\ det\ (I + G^\epsilon)(x, t),$$

where the N x N matrix G^ϵ has entries

$$g_{ij} = \frac{exp[\frac{(-\eta_i - \eta_j)x}{\epsilon}] c_i c_j}{\eta_i + \eta_j}$$

Plugging (16) into the general multisoliton formula above one derives an asymptotic formula for the zero dispersion limit of KdV. The determinant can be expanded as a sum. Note however that as $\epsilon \to 0$ one has $N = O(1/\epsilon)$ so the number of summands increases. This makes its asymptotic analysis somewhat complicated. In [LL] the sum can be written as a Laplace type integral with respect to a pure point measure which is approximated by a Lebesgue measure in the limit $\epsilon \to 0$. In analogy with the argument we presented for the zero viscosity problem, one can isolate the dominating contribution by solving a maximization problem (see (15)). A rigorous asymptotic analysis produces the following theorem.

THEOREM [LL]. The weak limit of KdV exists and satisfies

$$w - lim_{\epsilon \to 0} u(x, t, \epsilon) = \partial_{xx} Q^*(x, t),$$

$$Q^*(x, t) = min_{\mathbb{A}} Q(\psi; x, t),$$

where $\mathbb{A} = \{\psi \in L^1[0, 1] : 0 \leq \psi \leq \phi\}$ and

$$Q(\psi; x, t) = \frac{4}{\pi} \int_0^1 a(\eta, x, t)\psi(\eta) d\eta - \frac{1}{\pi^2} \int_0^1 \int_0^1 log(\frac{\eta - \mu}{\eta + \mu})^2 \psi(\eta) d\eta \psi(\mu) d\mu,$$
$$a(\eta, x, t) = \eta x - 4\eta^3 t - \eta x_+(\eta) - \int_{x_+(\eta)}^{\infty} (\eta - (u(y) + \eta^2)^{1/2}) dy.$$

For a certain class of regular initial data, the support of the maximizer consists of a finite union of intervals. For example, one has

THEOREM. [DKM] If the data $u_0(x)$ are real analytic, negative, with only one local minimum and such that

$$\int_{\mathbb{R}} (1+x^2)u_0(x)dx < \infty,$$

the support of the maximizer conists of a finite union of, say, M intervals. Naturally M depends on x, t.

In such a case the endpoints of intervals satisfy a set of transcendental equations defined in terms of the inital data $u_0(x)$.

THEOREM. [LL] At $t = 0$, $M = 1$, for all x.

In general the oscillations in the oscillatory region can be described by a deformation of a Riemann surface of genus $M-1$ (whose moduli depend on x, t.) At the "caustic", a curve in the $x-t$ plane, the Riemann surface changes genus. Uniform asymptotic formulae exist [V]; they involve theta functions of dimension $M-1$ associated to the underlying Riemann surface.

We will not provide the details of these asymptotic formulae here. Later in this paper, we discuss the more complicated example of the focusing NLS equation and we will describe in detail the asymptotic formulae as $h \to 0$, together with the appropriate definition of the underlying Riemann surfaces and the associated homology cycles, holomorphic differentials, Abel maps and theta functions.

SEMICLASSICAL LIMIT OF THE NONLINEAR SCHROEDINGER EQUATION

We consider

$$ih\psi_t + \frac{h^2}{2}\psi_{xx} \pm |\psi|^2\psi = 0,$$

in the limit as $h \to 0$, under initial data $A(x)$ that are real analytic, positive, with only one local maximum and decaying rapidly enough at infinity (the phase S is set to 0).

As we have seen, the NLS equation (for any fixed h) is equivalent to a specific perturbation of an Euler system for compressible fluid dynamics.

Indeed, if

$$\rho = |\psi|^2,$$

$$\mu = \frac{ih}{2}(\psi\bar{\psi}_x - \psi_x\bar{\psi}).$$

then

$$\rho_t + \mu_x = 0,$$

$$\mu_t + (\frac{\mu^2}{\rho} \mp \frac{\rho^2}{2}) = \frac{h^2}{4}(\rho(log\rho)_x x)x.$$

with

$$\rho(x,0) = |A(x)|^2,$$

$$\mu(x,0) = 0.$$

In the defocusing case, the unperturbed ($h = 0$) Euler system is hyperbolic. In the focusing case, it is elliptic. Hence the initial value problem is ill-posed in the sense of Hadamard (even though for every fixed h it is well-posed).

Now, for real analytic initial data, the Cauchy-Kovalewski ensures the existence and uniqueness of an analytic solution, at least for small time. But the elliptic problem is unstable in the sense that small perturbations of the initial data can produce exponentially large perturbations of the solution (even for small times).

Again, the natural question is: to what extent does the unperturbed Euler system describe the semiclassical limit of the NLS equations?

Once more, the answer is: until the time where the unperturbed Euler system develops a singularity, after which no classical solution is possible.

Once more there is caustic curve in the x, t plane which separates the non-oscillatory and the oscillatory regions. One has oscillations of bounded amplitude and high $O(1/h)$ frequency. Strong limits exist only before the caustics, but weak limits exist always.

INTEGRABILITY

The Lax operator at time zero is

$$L = \begin{pmatrix} ih\partial_x & iA(x) \\ \pm iA(x) & -ih\partial_x \end{pmatrix}.$$

It easily follows from (11) that L undergoes an isospectral deformation, i.e. the eigenvalues of L are constant in time. Furthermore, it is not hard to see that the norming constants associated with the eigenvalues λ_j undergo a very simple evolution

$$c_j(t) = c_j(0)exp[i\lambda_j^2 t/h].$$

As a result we only need to focus on the direct scattering problem at time $t = 0$. Of course, the inverse scattering problem is a different story.

In the defocusing case L is self-adjoint and the situation is very similar to KdV.

In the focusing case L is not self-adjoint. The spectrum is not necessarily real. There exist non-real eigenvalues. As for KdV, reflection coefficient is exponentially small (away from zero), so one has pure soliton solutions (asymptotically as $h \to 0$). But the Lax-Levermore method asymptotic analysis does not work here. Quite different methods are needed for the analysis (involving deformations of so-called Riemann-Hilbert factorization problems; see [KMM] for a rigorous detailed presentation). Accordingly, the associated variational problem is more complicated. It is a maxmin problem involving contours allowed to move anywhere in the complex plane (not just on the real line).

The eigenvalues of the Lax operator accumulate in the imaginary segment $[-iA_{max}, iA_{max}]$ with the density

(17) $$\rho^0(\lambda) = \int_{x_-(\lambda)}^{x_+(\lambda)} \frac{\lambda dx}{\pi(A^2(x) + \lambda^2)^{1/2}},$$

where $x_- < x_+$ are the two solutions of $A^2(x) + \lambda^2 = 0$.

VARIATIONAL PROBLEM FOR FOCUSING NLS [KR]

We will assume here, for simplicity, that the density ρ^0 defined in (17) admits a nice holomorphic extension in the upper complex plane.

Let $\mathbb{K} = \{z : Imz > 0\} \setminus \{z : Rez = 0, 0 < Imz \leq A\}$. We define $\mathbb{F}$ to be the set of all "continua" F in $\bar{\mathbb{K}}$ (i.e. connected compact sets) containing the distinguished points $0_+, 0_-$. Define

$$\phi(z) = \int log \frac{|z-\eta^*|}{|z-\eta|} \rho^0(\eta) d\eta - Re[i\pi \int_z^{iA} \rho^0(\eta) d\eta + 2i(zx + z^2 t)],$$

Next, let $\mathbb{M}$ be the set of all positive Borel measures on $\mathbb{F}$, such that both the (free) energy

$$E(\mu) = \int \int log \frac{|x-y^*|}{|x-y|} d\mu(x) d\mu(y), \ \mu \in \mathbb{M}$$

and $\int \phi d\mu$ are finite. Define the "weighted" energy

$$E_\phi(\mu) = E(\mu) + 2 \int \phi d\mu,$$

for any $\mu \in \mathbb{M}$.

Now, given any $F \in \mathbb{F}$, the equilibrium measure λ^F supported in F is defined by

$$E_\phi(\lambda^F) = min_{\mu \in M(F)} E_\phi(\mu),$$

where $M(F)$ is the set of measures in $\mathbb{M}$ which are supported in F.

We want to maximize $E_\phi(\lambda^F)$ over continua.

"THEOREM". (See [KR] for a precise version.) Under certain conditions on the initial data a maximizing continuum C of the equilibrium measure exists; in other words there exists a $C \in \mathbb{F}$ such that

$$E_\phi(\lambda^C) = max_{C \in \mathbb{F}} min_{\mu \in M(F)} E_\phi(\mu). \tag{18}$$

Furthermore, the continuum C is in fact a piecewise smooth contour; λ^C is absolutely continuous with respect to the Lebesgue measure and its support consists of a finite union of $M(x,t)$ analytic arcs. (Under more relaxed conditions the support of C consists of an infinite (generically finite) union of analytic arcs).

THEOREM [KMM] At t=0, $M(x,0) = 1$.

ASYMPTOTICS. RIEMANN SURFACES AND THETA FUNCTIONS.

Let $M(x,t)$ be the number of components of a maximizer C of the equilibrium measure. The support of the equilibrium measure (as it turns out) can be written as the union of arcs $I_0 = [0, \lambda_0], I_1 = [\lambda_1, \lambda_2], ..., I_M = [\lambda_{2M-1}, \lambda_{2M}]$. Uniqueness of C cannot be guaranteed. However, one can a posteriori conclude the uniqueness of $M(x,t)$ and the points λ_j.

We will actually "double up" our picture by considering the whole plane and the extended contour $C \cup C^*$ (C and its image under reflection with respect to the real line). The "bands" will then be $I_0 \cup I_0^*, I_J, I_j^*, j = 1, ..., M$. The connected components of the set $C \cup C^* \setminus \cup_j (I_j \cup I_j^*)$ are the so-called "gaps", for example the gap Γ_1 joins λ_0 to λ_1, etc.

Define the signed measure $\rho(\eta)d\eta$ via

$$\rho(\eta)d\eta = \lambda^C(d\eta), \ Im\eta > 0,$$
$$\rho(\eta)d\eta = -\lambda^C(d\eta^*), \ Im\eta < 0.$$

Let

$$g(\lambda) = \frac{1}{2}\int_{C\cup C^*} log(\lambda - \eta)\rho(\eta)d\eta.$$

The choice of the logarithm branch is irrelevant here.

For $\lambda \in C$, define the functions

$$\theta(\lambda) := i(g_+(\lambda) - g_-(\lambda)),$$
$$\phi(\lambda) := \int_0^{iA} log(\lambda - \eta)\rho^0(\eta)\, d\eta + \int_{-iA}^0 log(\lambda - \eta)\rho^0(\eta^*)^*\, d\eta$$
$$+2i\lambda x + 2i\lambda^2 t + i\pi \int_\lambda^{iA} \rho^0(\eta)\, d\eta - g_+(\lambda) - g_-(\lambda).$$

It follows from the Euler-Lagrange conditions for the variational problem that the function ϕ is constant (say α_j) in each band I_j while θ is constant (say θ_j) in each gap Γ_j.

The asymptotics of the solution of (1)-(2) as $h \to 0$ can be given by the next Theorem.

THEOREM [KMM]. Let x_0, t_0 be given. The solution $\psi(x,t)$ of (1)-(2) is asymptotically described (locally) as a slowly modulated M-phase wavetrain. Setting $x = x_0 + h\hat{x}$ and $t = t_0 + h\hat{t}$, so that x_0, t_0 are "slow" variables while $\hat{x}, \hat{t}$ are "fast" variables, and letting $G = M - 1$, there exist parameters

$a, U = (U_0, U_1, ..., U_G)^T$, $k = (k_0, k_1, ..., k_G)^T$, $w = (w_0, w_1, ..., w_G)^T$, $Y = (Y_0, Y_1, ..., Y_G)^T$, $Z = (Z_0, Z_1, ..., Z_G)^T$ depending on the slow variables x_0 and t_0 (but not $\hat{x}, \hat{t}$) such that

$$\begin{aligned} \psi(x,t) = \psi(x_0 + h\hat{x}, t_0 + h\hat{t}) \sim A(X_0, t_0) e^{iU_0(x_0,t_0)/h} e^{i(k_0(x_0,t_0)\hat{x} - w_0(x_0,t_0)\hat{t})} \\ \cdot \frac{\Theta(Y(x_0,t_0) + iU(x_0,t_0)/h + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}{\Theta(Z(x_0,t_0) + iU(x_0,t_0)/h + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}. \end{aligned} \tag{19}$$

All parameters can be defined in terms of an underlying Riemann surface X of genus G. The moduli of X are given by $\lambda_j,\ j = 0, ..., G$ and their complex conjugates $\lambda_j^*,\ j = 0, ..., G$. (Recall that they are uniquely defined.) The moduli of X vary slowly with x, t, i.e. they depend on x_0, t_0 but not $\hat{x}, \hat{t}$. For the exact formulae for the parameters as well as the definition of the theta functions we present the following construction.

The Riemann surface X is constructed by cutting two copies of the complex sphere along the slits $I_0 \cup I_0^*, I_j, I_j^*, j = 1...., G$, and pasting the "top" copy to the "bottom" copy along these very slits.

We define the homology cycles $a_j, b_j,\ j = 1, ..., G$ as follows. Cycle a_1 goes around the slit $I_0 \cup I_0^*$ joining λ_0 to λ_0^*, remaining on the top sheet, oriented counterclockwise, a_2 goes through the slits I_{-1} and I_1 starting from the top sheet, also oriented counterclockwise, a_3 goes around the slits $I_{-1}, I_0 \cup I_0^*, I_1$ remaining on the top sheet, oriented counterclockwise, etc. Cycle b_1 goes through I_0 and I_1 oriented counterclockwise, cycle b_2 goes through I_{-1} and I_1, also oriented counterclockwise, cycle b_3 goes through I_{-1} and I_2, and around the slits $I_{-1}, I_0 \cup I_0^*, I_1$, oriented counterclockwise, etc.

On X there is a complex G-dimensional linear space of holomorphic differentials, with basis elements $\nu_k(P)$ for $k = 1, \ldots, G$ that can be written in the form

$$\nu_k(P) = \frac{\sum_{j=0}^{G-1} c_{kj}\lambda(P)^j}{R_X(P)}\, d\lambda(P)\,,$$

where $R_X(P)$ is a "lifting" of the function $R(\lambda)$ from the cut plane to X: if P is on the first sheet of X then $R_X(P) = R(\lambda(P))$ and if P is on the second sheet of X then $R_X(P) = -R(\lambda(P))$. The coefficients c_{kj} are uniquely determined by the constraint that the differentials satisfy the normalization conditions:

$$\oint_{a_j} \nu_k(P) = 2\pi i \delta_{jk}.$$

From the normalized differentials, one defines a $G \times G$ matrix H (the period matrix) by the formula:

$$H_{jk} = \oint_{b_j} \nu_k(P).$$

It is a consequence of the standard theory of Riemann surfaces that H is a symmetric matrix whose real part is negative definite.

In particular, we can define the theta function

$$\Theta(w) := \sum_{n \in \mathbb{Z}^G} \exp(\frac{1}{2} n^T H n + n^T w),$$

where H is the period matrix associated to X. Since the real part of H is negative definite, the series converges.

We arbitrarily fix a base point P_0 on X. The Abel map $A : X \to Jac(X)$ is then defined componentwise as follows:

$$A_k(P; P_0) := \int_{P_0}^{P} \nu_k(P'),\ k = 1, \ldots, G,$$

where P' is an integration variable.

A particularly important element of the Jacobian is the Riemann constant vector K which is defined, modulo the lattice Λ, componentwise by

$$K_k := \pi i + \frac{H_{kk}}{2} - \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq k}}^{G} \oint_{a_j} \left(\nu_j(P) \int_{P_0}^{P} \nu_k(P') \right),$$

where the index k varies between 1 and G.

Next, we will need to define a certain meromorphic differential on X. Let $\Omega(P)$ be holomorphic away from the points ∞_1 and ∞_2, where it has the behavior

$$\Omega(P) = dp(\lambda(P)) + \left(\frac{d\lambda(P)}{\lambda(P)^2}\right),\ P \to \infty_1,$$
$$\Omega(P) = -dp(\lambda(P)) + O\left(\frac{d\lambda(P)}{\lambda(P)^2}\right),\ P \to \infty_2,$$

and made unique by the normalization conditions

$$\oint_{a_j} \Omega(P) = 0, j = 1, \dots, G.$$

Here p is a polynomial, defined as follows.

First, let us introduce the function $R(\lambda)$ defined by

$$R(\lambda)^2 = \prod_{k=0}^{G} (\lambda - \lambda_k)(\lambda - \lambda_k^*),$$

choosing the particular branch that is cut along the bands I_k^+ and I_k^- and satisfies

$$\lim_{\lambda\to\infty} \frac{R(\lambda)}{\lambda^{G+1}} = -1,$$

This defines a real function, i.e. one that satisfies $R(\lambda^*) = R(\lambda)^*$. At the bands, we have $R_+(\lambda) = -R_-(\lambda)$, while $R(\lambda)$ is analytic in the gaps. Next, let's introduce the function $k(\lambda)$ defined by

$$k(\lambda) = \frac{1}{2\pi i}\sum_{n=1}^{G/2} \theta_n \int_{\Gamma_n^+\cup\Gamma_n^-} \frac{d\eta}{(\lambda-\eta)R(\eta)} + \frac{1}{2\pi i}\sum_{n=0}^{G/2} \int_{I_n^+\cup I_n^-} \frac{\alpha_n\ d\eta}{(\lambda-\eta)R_+(\eta)}\,.$$

Finally let

$$H(\lambda) = k(\lambda)R(\lambda).$$

The function k satisfies the jump relations

$$k_+(\lambda) - k_-(\lambda) = -\frac{\theta_n}{R(\lambda)},\ \lambda \in \Gamma_n^+ \cup \Gamma_n^-$$
$$k_+(\lambda) - k_-(\lambda) = -\frac{\alpha_n}{R_+(\lambda)},\ \lambda \in I_n^+ \cup I_n^-,$$

and is otherwise analytic. It blows up like $(\lambda-\lambda_n)^{-1/2}$ near each endpoint, has continuous boundary values in between the endpoints, and vanishes like $1/\lambda$ for large λ. It is the only such solution of the jump relations. The factor of $R(\lambda)$ renormalizes the singularities at the endpoints, so that, as desired, the boundary values of $H(\lambda)$ are bounded continuous functions. Near infinity, there is the asymptotic expansion:

$$
\begin{aligned}
H(\lambda) = H_G\lambda^G + H_{G-1}\lambda^{G-1} + \cdots + H_1\lambda + H_0 + O(\lambda^{-1})\\
= p(\lambda) + O(\lambda^{-1}),
\end{aligned}
$$

where all coefficients H_j of the polynomial $p(\lambda)$ can be found explicitly by expanding $R(\lambda)$ and the Cauchy integral $k(\lambda)$ for large λ. It is easy to see from the reality of θ_j and α_j that $p(\lambda)$ is a polynomial with real coefficients.

Thus the polynomial $p(\lambda)$ is defined and hence the meromorphic differential $\Omega(P)$ is defined.

Let the vector $U \in \mathbb{C}^G$ be defined componentwise by

$$U_j := \oint_{b_j} \Omega(P).$$

Note that $\Omega(P)$ has no residues.

Let the vectors V_1, V_2 be defined componentwise by

$$
\begin{aligned}
V_{1,k} &= (A_k(\lambda_{1+}^*) + A_k(\lambda_{2+}) + A_k(\lambda_{3+}^*) + \cdots + A_k(\lambda_{G+})) + A_k(\infty) + \pi i + \frac{H_{kk}}{2},\\
V_{2,k} &= (A_k(\lambda_{1+}^*) + A_k(\lambda_{2+}) + A_k(\lambda_{3+}^*) + \cdots + A_k(\lambda_{G+})) - A_k(\infty) + \pi i + \frac{H_{kk}}{2},
\end{aligned}
$$

where $k = 1, ..., G$, and the $+$ index means that the integral for A is to be taken on the first sheet of X, with base point λ_+^0.

Finally, let

$$a = \frac{\Theta(Z)}{\Theta(Y)} \sum_{k=0}^{G} (-1)^k \Im(\lambda_k),$$

$$k_n = \partial_x U_n,\ w_n = -\partial_t U_n,\ n = 0, \ldots, G,$$

where

$$Y = -A(\infty) - V_1,\ Z = A(\infty) - V_1,$$

and $U_0 = -(\theta_1 + \alpha_0)$ where θ_1 is the (constant in λ) value of the function θ in the gap Γ_1 and α_0 is the (constant) value of the function ϕ in the band I_0.

Now, the parameters appearing in formula (19) are completely described.

We simply note here that the U_i and hence the k_i and w_i are real. We also note that the denominator in (19) never vanishes (for any $x_0, t_0, \hat{x}, \hat{t}$).

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