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Brownian motion of fractal particles: Lévy flights from white noise

by

Kiran M. Kolwankar

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Kiran M. Kolwanker
Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D - 04103 Leipzig, Germany
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We generalise the Langevin equation with Gaussian white noise by replacing the velocity term by a local fractional derivative. The solution of this equation is a Lévy process. We further consider the Brownian motion of a fractal particle, for example, a colloidal aggregate or a biological molecule and argue that it leads to a Lévy flight. This effect can also be described using the local fractional Langevin equation. The implications of this development to other complex data series are discussed.

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Exactly 100 years ago Einstein [1] proposed the theory of Brownian motion which marked the birth of the field of stochastic processes and has worked as a guiding principle in modelling of many fundamental as well as applied phenomena in physical, chemical and biological sciences (see reviews [2, 3] and references therein which document its impact). Present interest in complex high dimensional systems, in wide ranging fields, whose dynamics may not be completely known has given rise to the study of many irregular processes. Experience shows that, in spite of the lack of understanding of the laws governing the phenomena, they can be successfully modelled using stochastic processes. As a result new processes are being introduced [4]. In the past, the Gaussian process used to be the main tool employed to this end. But recently, owing to the frequent occurrence of anomalous diffusions [5] (where $< x^2 > \propto t^\alpha$, $\alpha \neq 1$), two main generalisations of Gaussian process, viz., Lévy processes [6] and fractional Brownian motion (fBm) [7], have turned out to be of importance. These processes respectively relax two important assumptions in the central limit theorem that of finite variance and independence. A process whose second moment diverges but the first moment is finite falls into the domain of attraction the Lévy process of index μ, with $1 < \mu < 2$, where μ characterises the power law tail of the probability distribution function and the one with even the first moment infinite corresponds to the Lévy process with $0 < \mu < 1$. It is now known that many economical time series are better modelled by a truncated Lévy flight [8]. Also in biological physics, modern high speed imaging techniques have uncovered many interesting anomalous diffusive behaviours of large biological molecules [9].

On the other hand, derivatives and integrals of non-integer order [10, 11] have been found to be useful in successfully describing scaling processes. The realm of applications of such a fractional calculus is fast expanding with ever new developments rapidly taking place in the field of statistical and nonlinear physics over the last few years [12–19]. Fractional derivatives were used in the definition of fBm [7, 20] and the fractional Langevin equation has been shown to give rise to fBm [21]. Many researchers have used the diffusion equation involving fractional derivative in space which describes a Lévy process and there are studies wherein the Gaussian white noise in the Langevin equation is replaced by a Lévy noise [22–26]. This models the anomalous behaviour of the environment or the heat bath. But there is no generalisation of the Langevin equation in which one naturally obtains a Lévy process from usual white noise. That is the heat bath is normal but the system is irregular and there is no memory. It has been shown, however, that the Lévy flights can be obtained from continuous time random walks by introduction of an operational time [27].

In this letter we generalise the Langevin equation by incorporating the local fractional derivatives and show that it leads to the Lévy flights from usual white noise. We then consider, as an example, the Brownian motion of rigid irregular particles, a study possibly relevant for aggregates and biological molecules. We argue that the fractal nature gives rise to the Lévy flights. We describe this phenomena using the local fractional Langevin equation. We further point out the importance of this formalism in describing irregular processes which arise in econophysics.

One way of defining the fractional derivative is through the so called Riemann-Liouville fractional derivative [10, 11]. For q, the order of the derivative, between zero and one it is given by:

$$ {}_aD_x^q f(x) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^q} dy. \qquad (1)$$

Clearly this is nonlocal and depends on the lower limit a. It is interesting to note that it is this dependence on the parameter, which led to confusion in different ways of defining a fractional derivative towards the end of 19th century and hence possibly delayed its application, is now becoming important. The value of a is usually dictated by the problem one is investigating. In some cases it is appropriate to put it equal to zero and in some other it is taken to be $\pm\infty$. In the latter case it is called Weyl derivative. Another choice of the lower limit is made in the following definition called the Local fractional derivative (LFD) defined as follows [28, 29]:

$$f^{(q)}(x) = \lim_{x' \to x} {}_{x'}D_x^q (f(x) - f(x')) \quad 0 < q < 1. \qquad (2)$$

where we introduce the notation that the superscript in the brackets on the LHS denotes the LFD of that order. This definition naturally appears in the local fractional Taylor expansion [28] giving it a geometrical interpretation. It should be noted that the extra limit in the definition of the LFD makes it very different from other definitions of fractional derivatives and some of its properties are very different from the non-local versions of fractional derivatives (see [30–34] for more mathematical properties and some applications of the LFD). One important difference is that though it reduces to the usual derivative when $q = 1$, the LFD is not an analytic function of q for a given function. For example, if the function is smooth then the LFD of any order q less than one is zero. In fact, in general for any continuous nondifferentiable function there exists a critical order of differentiability between zero and one below which all the derivatives are zero [51] and above which they do not exist. This critical order of differentiability is equivalent to the local Hölder exponent or the local power law exponent [29]. and for this formalism to yield meaningful results we should work at the order which is equal to this exponent. So it can be said that the LFD is a nonanalytic extension of the usual derivative. This means that the LFD signals an emergence of a new calculus with its new rules to which one should get accustomed. This is much in the same way as in the case of, for example, the Ito calculus. It should be noted that the LFD characterises the local scaling whereas the nonlocal fractional derivatives are useful to study asymptotic scaling. Owing to these complimentary roles played by these two versions of the definitions it can not be ruled out that in some applications a combination of the two is indispensable. The limit in eqn (2) which is akin to the limit in the renormalization group transformation, in fact, bestows the LFD a physical interpretation. It can be used to relate and study the dynamics of renormalised quantities. Carpinteri and Cornetti [33] used it to relate renormalised stress and strain when the stress is concentrated on a singular set.

The next step is to consider the equations involving the LFD. The simplest such equation is

$$f^{(q)}(x) = g(x) \qquad (3)$$

where $g(x)$ is a known function and $f(x)$ an unknown. Using the local fractional Taylor expansion [28] its solution can be written as a generalised Riemann sum giving

$$f(x) = \int g(x) d^q x = \lim_{N \to \infty} \sum_{i=0}^{N-1} g(x_i^*) \frac{(x_{i+1} - x_i)^q}{\Gamma(q+1)} \qquad (4)$$

where $g(x_i^*)$ is an appropriately chosen point in the interval $[x_i, x_{i+1}]$. Such integrals should be useful for integrating physical quantities over a fractal boundary, for example, a current passing through an interface. The order of the integral, in this case, of course being equal to the Hölder exponent of the fractal curve. For $q < 1$, if the function $g(x)$ is, say, nonnegative in some interval then the above sum grows in the limit [52]. Yet, two classes of functions $g(x)$ can be identified which will yield a nontrivial function $f(x)$ as a solution. The first class corresponds to the functions which have fractal support [35]. Then if $q = \alpha$ the dimension of the support of the function the sum above converges since only a few terms contribute to the sum giving rise to a finite solution. This function, called the "devil's staircase", changes only on the points of the support of $g(x)$ and is constant everywhere else. We denote this solution by $P_g(x)$. The second class, which is a main focus of this work, consists of rapidly oscillating functions which oscillate around zero in any small interval. These oscillations then result in cancellations again giving rise to a finite solution. A realisation of the white noise is one example in this class of functions. This immediately prompts us to consider a generalisation of the Langevin equation which involves LFD and $g(x)$ is chosen as white noise.

So we consider a generalisation of the Langevin equation [36] in the high friction limit where one neglects the acceleration term and replaces the first derivative term by the LFD to arrive at

$$x^{(\alpha)}(t) = \zeta(t), \qquad (5)$$

where $< \zeta(t) >= 0$ and $< \zeta(t)\zeta(t') >= \delta(t - t')$ the Dirac delta function. The solution of the above equation follows from Eq. (4). Heuristically it can be seen that

$$< x(t) > \; = \; t^\alpha \lim_{N \to \infty} \frac{N^{-\alpha}}{\Gamma(\alpha+1)} N^{1/2} \qquad (6)$$

and therefore the average is zero if $\alpha > 1/2$ and it does not exist if $\alpha < 1/2$. Now we consider the second moment.

$$< x(t_1)x(t_2) > \; = \; \int_0^{t_1} \int_0^{t_2} \delta(t_1' - t_2') d^\alpha t_1' d^\alpha t_2' \qquad (7)$$

which exists only when $\alpha = 1$. In order to see this systematically we generalise the concept of the delta function. The usual Dirac delta function is defined as the derivative of the Heaviside step function ($\theta(x) = 0$ for $x < 0$ and 1 for $x > 0$); $\delta(x) = \theta'(x)$. Here two things can be generalised, first the derivative can be replaced by the LFD of order γ and second, the Heaviside function can be replaced by a scaling function $\bar{\theta}(x) = x^\beta$ for $x > 0$ and 0 for $x < 0$ and $0 < \beta < 1$. As a result, our generalised delta function is defined as $\delta(x;\gamma,\beta) = \bar{\theta}^{(\gamma)}(x - y)|_{x=y}$. It follows from this definition that $\int \delta(x;\gamma,\beta) d^\alpha x = 0$ for $\alpha > \gamma - \beta$ and ∞ when $\alpha < \gamma - \beta$. Now $\delta(x) = \delta(x;1,0)$ leading to the above conclusion. This shows that when $0 < \alpha 1/2$ both the first and the second moments diverge whereas when $1/2 < \alpha < 1$ only the second moment diverges. This implies that the above process is in the domain of attraction of a Lévy process of index 2α for $\alpha < 1$. This brings forth some interesting perspective. The generalisation of the Langevin equation using non-local fractional operators gives rise to the fractional Brownian motion whereas the one using the local fractional operator results in the Lévy process.

As an application of this formalism we consider the Brownian motion of a rigid fractal particle. Such consideration is useful in colloids as well as biological systems [37]. Usually theoretical studies of Brownian motion are restricted to spherical particles in which case one has, from the Stokes' law, the formula for frictional force in terms of the viscosity and one also assumes that the displacements well separated in time are not correlated. Any deviation from the sphericity makes matters complicated [38]. Berry [39] studied velocity of fractal flakes falling under the gravity. He assumed that the cluster entrains the air inside it and used the Stokes' law. We do not make any such assumption in the following. Also, there exists a formal theoretical treatment of the Brownian motion of particles of arbitrary shapes using hydrodynamical approach [40]. However it is not valid for fractal particles since one needs to solve fluctuating hydrodynamical equations with boundary conditions emanating from the surface of the particle and in the process one needs to integrate the normal components of the fluctuating stress tensors over the surface. This can not be carried out for a fractal particle since the normal can not be defined on a fractal boundary and integrating over a fractal surface would require integrals as in equation (4) necessitating a new approach using the present calculus with the order of the derivative being the local Hölder exponent of the surface. Here we model the irregularity of the suspended particle by the fractality and first, using purely statistical arguments, argue that the Brownian motion of such irregular particles leads to the Lévy flights. Then we arrive at the same conclusion starting from above local fractional Langevin equation. The essential step is to compare the distribution of the resultant force acting along the center of mass on a particle with fractal boundary with the lateral dimension D to that of a spherical particle of diameter D. We treat the problem in two dimensions. First, we consider the ideal situation wherein the boundary is a mathematical fractal without any lower cutoff and the surrounding fluid consists of point particles. We ignore any lower length scales in the problem. In a small time interval Δt, N particles collide with a spherical Brownian particle whereas for a particle with a fractal boundary N^d particles, with d, $1 < d < 2$ being the dimension of the fractal boundary, will impart their momentum to the fractal particle. Given the fact that the case with a spherical particle gives rise to the normal diffusion with finite variance leads us to the conclusion that the fractal particle, under the same assumptions, will undergo diffusion with much larger fluctuations leading it to the basin of attraction of a Lévy process. More precisely, the fractal particle will undergo the same number of collisions in time t as the spherical particle would undergo in time t^d making it a Lévy flight with index $2/d$.

One can use the above formalism of the local fractional Langevin equation to describe this phenomenon. In order to do this we consider the x-component of the displacement of the particle as a function of time and hypothesise that the frictional force is proportional to $x^{(1/d)}(t)$, which we call the "renormalised" velocity instead of the usual velocity which is the case when $d = 1$. Here $1/d$ is again the Hölder exponent of the x-component of the fractal boundary [41]. A way to motivate this is to note that the friction increases with the irregularity of the particle and this diverging situation can be remedied by renormalising the velocity. We use this in the Langevin equation and again assume the high friction limit and neglect the acceleration term. In this way we arrive at the equation (5) and its solution we know is the Lévy process with index $2/d$. It is clear that, since $1 < d < 2$, the resulting process has finite mean.

Clearly we have made many assumptions in order to understand the essence of the problem. Firstly we have assumed a strict mathematical boundary which is irregular down to the finest length scales. In practice, the systems will have a lower cutoff arising from the smoothening of the boundary at the lower length scales or the finite density of surrounding fluid which will give rise to a smaller number of collisions and thus make larger flights less probable. This will lead to truncated Lévy flights [42] instead of the Lévy flights. Also, other constraints, like the size of the system, would also force us to this conclusion. Another assumption we have made is that the fluid consists of point particles. The size and shape of fluid would add to the complications especially if the molecules are large. This is because, as demonstrated in [43] and especially for dimension greater than 1.5, the fractal boundary develops wiggles making some part of it inaccessible to larger molecules. The fractal dimension of this accessible surface, which we here call apparent dimension, may be smaller than actual dimension and may depend on the shape of the approaching molecule. It is only this apparent dimension that will be important. Finally, we have considered a particle with only the boundary that is fractal but this again is not essential and one can have a porous fractal. Once again, it is the apparent dimension which will play the role [44] and this dimension should be greater than the dimension of the spherical surface in order to obtain the present result.

Now we consider the local fractional Langevin equation with additional noise term

$$x^{(\alpha)}(t) = \zeta(t) + \eta(t) \tag{8}$$

where $\zeta(t)$ is as before the white noise and $\eta(t)$ are pulses of finite height and zero width distributed on a random

Cantor-like fractal set with dimension equal to α. This could be a result of some self-organised critical process. Its solution is given by

$$x(t) = L(t) + P_\eta(t) \tag{9}$$

where $L(t)$ is the Lévy process and $P_\eta(t)$ is as defined after eqn. (4). The second part of the solution has log-periodic oscillations embedded at the fractal set [45]. Such log-periodic oscillations have been observed in stockmarket data [46] and in other fields [47, 48].

Finally we consider two more local stochastic differential equations in order to demonstrate the generality of the formalism. The first equation we consider is

$$x^\alpha(t) + \eta(t)x(t) = \zeta(t) \tag{10}$$

and its solution is given by

$$x(t) = x_0 e^{-P_\eta(t)} + \int_0^t e^{-(P_\eta(t)-P_\eta(t'))}\zeta(t')d^\alpha t' \tag{11}$$

with $x(0) = x_0$. And the last equation we consider generalises the simple Kubo oscillator

$$x^\alpha(t) = a\zeta(t)x(t) \tag{12}$$

with the solution

$$x(t) = x_0 \exp\left(a \int_0^t \zeta(t')d^\alpha t'\right) \tag{13}$$

This demonstrates that many different complex stochastic signals can be generated using various local stochastic differential equations.

To conclude, we have nontrivially and significantly expanded the concept of stochastic differential equation formalism as instituted by Langevin by replacing the first order derivative with a local fractional derivative of order α between zero and one. It should be emphasized that this procedure does not add any extra correlations in the system as a nonlocal fractional derivative would do. This naturally gives rise to Lévy processes as a solution when the input noise is usual white noise putting Lévy processes and fBm on symmetrical dynamical footing. Essentially this result is a consequence of the integration of the white noise over a fractal boundary and should have wider implications. Here, as an application, we have considered the Brownian motion of a fractal particle and argued that the fractality of the Brownian particle would give rise to the Lévy flights. More precisely, a particle with fractal boundary of dimension d performs a Lévy flight of index $2/d$ when immersed in fluid with point particles. In the case of fluid with larger molecules results may be different and will have to be worked out for that case. Furthermore we can describe this statistical effect using the above generalised Langevin equation by hypothesising that the frictional force is proportional to the "renormalised" velocity. This has given a new way of obtaining superdiffusive behaviour wherein the environment or the heat bath has normal dynamics but it is the fractality of the system that makes it anomalous. Careful experiments should be carried out in order to confirm this prediction. A way to do this would be to study the Brownian motion of an aggregate, whose dimension is known, by tagging it with a fluorescent probe and measure the self-diffusion coefficient [49]. We have restricted our attention only to the translational motion. The behaviour of the angular velocity should also be interesting in itself. Clearly, there are many other factors affecting the Brownian motion of a real biological molecule in living cell environment but the implications of this study should be taken into account.

This work also illustrates the use of local fractional calculus to describe phenomena involving fractals. Such a tool is badly needed for better understanding of structures and processes involving fractals (see [50] for another notable approach with the same aim). This makes it necessary to develop this formalism further to its full potential, especially the present generalization of the stochastic differential equations. The corresponding extension of the delta function might lead to a new way of characterising the correlations.

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[51] As of now, we do not know any example of a continuous everywhere but nowhere differentiable function for which the LFD exist at the critical order. But here we take a point of view that such functions nevertheless exist.
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