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Abstract

We present a construction method for mappings between generalized connections, comprising, e.g., the action of gauge transformations, diffeomorphisms and Weyl transformations. Moreover, criteria for continuity and measure preservation are stated.

1 Introduction

Generalized (or distributional) connections arise naturally when attempting the loop quantization of canonical gravity or other gauge field theories. Often mappings between such connections are used. Examples are gauge transformations, diffeomorphisms and Weyl transformations. Distributional connections are given as homomorphisms from the groupoid of paths in the base manifold of a principal fibre bundle to its structure group. So one typically tries to define transformed connections by modifying the parallel transports of a given one, path by path. This is not always directly possible. Usually one has to break paths down to "simple" pieces, where the mapping can be defined more easily. Afterwards, one patches the simple parts together by homomorphy. However, here one has to take care of the well-definedness. At the end, one is interested in the properties of these mappings, in particular, continuity and measure preservation.

It turns out that most of the transformations considered until now in that framework, follow this pattern. So, in the examples listed above, the parallel transport along a "simple" path is always given by the parallel transports along some possibly other "simple" path plus some conjugation with structure group elements specifying the transformation. Since, therefore, proofs often are very similar for different transformations, we are now going to somewhat unify the treatment in the present paper. We start with some results on the decomposition of paths in Section 2 and then establish the general construction in Section 3. After applying it to connections as in Section 4 and introducing the notion of graphical

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morphisms in Section 5, the main results are presented in Section 6. We close the paper with a bunch of examples in Section 7.

Finally, let us fix some manifold M and some connected Lie group $\mathbf{G}$. If we make any statements on measures, we will assume $\mathbf{G}$ to be compact.

2 Completeness

Recall [4] that a path is a piecewise C^r map from $[0,1]$ to our fixed manifold M. Here, the fixed r is either a positive integer, ∞ or ω. Moreover, we decide whether we restrict ourselves to piecewise embedded paths or not. A path is said to be trivial iff its image is a single point. The inverse path γ^{-1} of a path γ is given by $\gamma^{-1}(t) := \gamma(1-t)$. Two paths γ_1 and γ_2 are composable iff the end point $\gamma_1(1)$ of the first one coincides with the starting point $\gamma_2(0)$ of the second one. If they are composable, their product is given by

$$(\gamma_1\gamma_2)(t) \quad := \quad \begin{cases} \gamma_1(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}.$$

An edge e is a path having no self-intersections, i.e., from $e(t_1) = e(t_2)$ follows that $|t_1 - t_2|$ is either 0 or 1. Two paths γ_1 and γ_2 coincide up to the parametrization iff there is some orientation preserving piecewise C^r diffeomorphism $\phi : [0,1] \longrightarrow [0,1]$, such that $\gamma_1 = \gamma_2 \circ \phi$. A path is called finite iff it equals up to the parametrization a finite product of edges and trivial paths. In what follows, every path will be assumed to be finite. Next, two paths are equivalent iff there is a finite sequence of paths, such that two subsequent paths coincide up to the parametrization or up to insertion or deletion of retracings $\delta\delta^{-1}$. This means, that, e.g., $\gamma_1\gamma_2$ is equivalent to $\gamma_1\delta\delta^{-1}\gamma_2$ for all paths γ_1, γ_2 and δ. Finally, we denote the set of all paths by $\mathcal{P}_{\text{gen}}$, that of all equivalence classes of paths by $\mathcal{P}$. $\mathcal{P}$ is a groupoid.

Definition 2.1 Let γ be some path.
Then a finite sequence $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_n)$ in $\mathcal{P}_{\text{gen}}$ is called **decomposition** of γ iff $\gamma_1 \cdots \gamma_n$ equals γ up to the parametrization.

This definition is well defined, since $\gamma_1(\gamma_2\gamma_3)$ equals $(\gamma_1\gamma_2)\gamma_3$ up to the parametrization. Moreover, observe that every reparametrization of γ gives a decomposition of γ.

If confusion is unlikely, we identify $\gamma_1 \cdots \gamma_n$ and $(\gamma_1, \ldots, \gamma_n)$.

Definition 2.2 Let $\boldsymbol{\gamma} := \gamma_1 \cdots \gamma_I$ and $\boldsymbol{\delta} := \delta_1 \cdots \delta_J$ be decompositions of some path γ.
Then $\boldsymbol{\gamma}$ is a **refinement** of $\boldsymbol{\delta}$ iff there are $0 = I_0 < I_1 < \ldots < I_J = I$, such that $\gamma_{I_{j-1}+1} \cdots \gamma_{I_j}$ is a decomposition of δ_j for all $j = 1, \ldots, J$. We write $\boldsymbol{\gamma} \geq \boldsymbol{\delta}$ iff $\boldsymbol{\gamma}$ is a refinement of $\boldsymbol{\delta}$.

Lemma 2.1 Let γ be some path.
Then the set of all decompositions of γ is directed w.r.t. $\geq$.

Proof Let $\boldsymbol{\gamma} = \gamma_1 \cdots \gamma_I$ be a decomposition of γ, i.e., $(\ldots((\gamma_1\gamma_2)\gamma_3)\cdots)\gamma_I$ equals $\gamma \circ \phi^{-1}$ for some piecewise C^r diffeomorphism ϕ from $[0,1]$ onto itself. Now, the nontrivial end points of the γ_i correspond to the parameter values $\phi(\frac{1}{2^{I-1}})$, …, $\phi(\frac{1}{4})$, and $\phi(\frac{1}{2})$ in γ. In other words, these parameter values decompose γ into the γ_i.[1]

[1]Note that these values need not be uniquely determined. In fact, if there is some interval in $[0,1]$, where γ is constant, then ϕ is not uniquely determined by γ and $\boldsymbol{\gamma}$, since every $\widetilde{\phi}$ coinciding with ϕ outside that interval gives $\gamma \circ \phi^{-1} = \gamma \circ \widetilde{\phi}^{-1}$.

Let now $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ be two decompositions of γ. Then we may find two sets of parameter values that decompose γ according to $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, respectively. Now, decompose γ according to the union of these two sets. This gives a decomposition $\boldsymbol{\gamma}$ of γ. It is easy to check that $\boldsymbol{\gamma}$ is a refinement of both $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$. **qed**

Definition 2.3 A subset $\mathcal{Q}$ of $\mathcal{P}_{\text{gen}}$ is called **hereditary** iff for each $\gamma \in \mathcal{Q}$
1. the inverse of γ is in $\mathcal{Q}$ again, and
2. every decomposition of γ consists of paths in $\mathcal{Q}$.

Definition 2.4 A subset $\mathcal{Q}$ of $\mathcal{P}_{\text{gen}}$ is called **complete** iff it is hereditary and every path in $\mathcal{P}_{\text{gen}}$ has a decomposition into paths in $\mathcal{Q}$.

A decomposition consisting of paths in $\mathcal{Q}$ only, will be called $\mathcal{Q}$-decomposition.

Lemma 2.2 The set of all edges and trivial paths in $\mathcal{P}_{\text{gen}}$ is complete.

Proof Clear from the definition of $\mathcal{P}_{\text{gen}}$. **qed**

3 Construction

Definition 3.1 Let $\mathcal{Q}$ be some hereditary subset of $\mathcal{P}_{\text{gen}}$.
Then a map $\rho : \mathcal{Q} \longrightarrow \mathbf{G}$ is called **$\mathcal{Q}$-germ** iff for all $\gamma \in \mathcal{Q}$
1. $\rho(\gamma^{-1}) = \rho(\gamma)^{-1}$, and
2. $\rho(\gamma) = \rho(\gamma_1)\rho(\gamma_2)$ for all decompositions $\gamma_1\gamma_2$ of γ.

The set of all $\mathcal{Q}$-germs from $\mathcal{Q}$ to $\mathbf{G}$ is denoted by $\text{Germ}(\mathcal{Q}, \mathbf{G})$.

Observe that $\rho(\gamma)$ and $\rho(\delta)$ coincide if γ and δ coincide up to the parametrization. In fact, since every decomposition $\gamma_1\gamma_2$ of γ is also some for δ, we may apply property 2. above.

Note that we will shortly speak about germs instead of $\mathcal{Q}$-germs, provided the domain $\mathcal{Q}$ is clear from the context.

Proposition 3.1 Let $\mathcal{Q}$ be some complete subset of $\mathcal{P}_{\text{gen}}$, and let $\rho : \mathcal{Q} \longrightarrow \mathbf{G}$ be a germ. Then we have:
- There is a unique germ $\widehat{\rho} : \mathcal{P}_{\text{gen}} \longrightarrow \mathbf{G}$ extending ρ.
- The map $\widehat{\rho}$ is given by
$$\widehat{\rho}(\gamma) \quad = \quad \prod_{i=1}^{I} \rho(\gamma_i)$$
for each $\gamma \in \mathcal{P}_{\text{gen}}$, where $\gamma_1 \cdots \gamma_n$ is any[2] $\mathcal{Q}$-decomposition of γ.
- The map $\widehat{\rho}$ is constant on equivalence classes in $\mathcal{P}_{\text{gen}}$.
- The induced map $[\widehat{\rho}] : \mathcal{P} \longrightarrow \mathbf{G}$ is a homomorphism.

Proof Let us first define the desired map $\widehat{\rho}$ as given in the proposition above and now check its properties.
1. $\widehat{\rho}$ does not depend on the choice of the $\mathcal{Q}$-decomposition.
Let $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ be two $\mathcal{Q}$-decompositions of γ. Since, by assumption, every path in $\mathcal{Q}$ has $\mathcal{Q}$-decompositions only, and since the set of decompositions of a path is directed w.r.t. $\geq$, we may assume $\boldsymbol{\gamma} \geq \boldsymbol{\delta}$. But, in this case the well-definedness follows directly from the definitions and germ property 2. of ρ.

[2] Recall that, by completeness of $\mathcal{Q}$, such a decomposition exists always.

2. $\widehat{\rho}$ is constant on equivalence classes in $\mathcal{P}_{\rm gen}$.
   Let γ and δ in $\mathcal{P}_{\rm gen}$ be equivalent. By definition, it is sufficient to check the following two cases:
   - γ and δ coincide up to the parametrization.
     Since every $\mathcal{Q}$-decomposition of γ is also one of δ, we have $\rho(\gamma) = \rho(\delta)$.
   - There is some ε in $\mathcal{P}_{\rm gen}$ and some decomposition $\gamma_1\gamma_2$ of γ, such that δ equals the product of γ_1, ε, ε^{-1} and γ_2.
     Now, in this case, choose some $\mathcal{Q}$-decompositions $\varepsilon_1 \cdots \varepsilon_K$ of ε and $\gamma_{s1} \cdots \gamma_{sI_s}$ of γ_s with $s = 1, 2$. Then $\gamma_{11} \cdots \gamma_{1I_1}\, \gamma_{21} \cdots \gamma_{2I_2}$ is a $\mathcal{Q}$-decomposition of γ and $\gamma_{11} \cdots \gamma_{1I_1}\, \varepsilon_1 \cdots \varepsilon_K\, \varepsilon_K^{-1} \cdots \varepsilon_1^{-1}\, \gamma_{21} \cdots \gamma_{2I_2}$ one of δ. Hence, we have

$$\begin{aligned}
\widehat{\rho}(\delta) &= \rho(\gamma_{11}) \cdots \rho(\gamma_{1I_1})\, \rho(\varepsilon_1) \cdots \rho(\varepsilon_K) \\
&\qquad \rho(\varepsilon_K^{-1}) \cdots \rho(\varepsilon_1^{-1})\, \rho(\gamma_{21}) \cdots \rho(\gamma_{2I_2}) && \text{(Definition of } \widehat{\rho}) \\
&= \rho(\gamma_{11}) \cdots \rho(\gamma_{1I_1})\, \rho(\gamma_{21}) \cdots \rho(\gamma_{2I_2}) && \text{(Property 1. of } \rho) \\
&= \widehat{\rho}(\gamma). && \text{(Definition of } \widehat{\rho})
\end{aligned}$$

3. $\widehat{\rho}$ is a germ extending ρ, and $[\widehat{\rho}]$ is a homomorphism.
   This is proven as the statements above.
4. $\widehat{\rho}$ is the only germ extending ρ.
   If $\widehat{\rho}'$ is some other germ extending ρ different from $\widehat{\rho}$, then there is some $\gamma \in \mathcal{P}_{\rm gen}$ with $\widehat{\rho}'(\gamma) \neq \widehat{\rho}(\gamma)$. Now, choose a $\mathcal{Q}$-decomposition $\gamma_1 \cdots \gamma_I$ of γ. By the properties of a germ, there is some i with $\widehat{\rho}'(\gamma_i) \neq \widehat{\rho}(\gamma_i)$. However, since both $\widehat{\rho}'$ and $\widehat{\rho}$ extend ρ, both sides are equal to $\rho(\gamma_i)$. Contradiction. **qed**

4 Connections

To be prepared for the main results of this paper, let us briefly recall [1, 4, 2] the basic definitions and properties of generalized connections. Algebraically, the space[3] $\overline{\mathcal{A}}$ of generalized connections equals $\mathrm{Hom}(\mathcal{P}, \mathbf{G})$. To equip $\overline{\mathcal{A}}$ with a topology and measures thereon, we have to go again into the field of paths. Segments of a path are restrictions of that path to connected subintervals, affinely stretched to maps with domain $[0, 1]$. Initial and final segments of paths are defined naturally. We will write $\gamma_1 \uparrow\uparrow \gamma_2$ iff there is some path γ being (possibly up to the parametrization) an initial segment of both γ_1 and γ_2. A hyph υ is now some finite collection $(\gamma_1, \ldots, \gamma_n)$ of edges each having a "free" point. This means, for at least one direction none of the segments of γ_i starting in that point in this direction, is (up to the parametrization) a full segment of some of the γ_j with $j < i$. The decomposition of paths and the inclusion relation generate a directed ordering on the set of hyphs. Now,

$$\overline{\mathcal{A}} \;\equiv\; \mathrm{Hom}(\mathcal{P}, \mathbf{G}) \;=\; \varprojlim_{\upsilon} \overline{\mathcal{A}}_\upsilon,$$

with $\overline{\mathcal{A}}_\upsilon := \mathrm{Hom}(\mathcal{P}_\upsilon, \mathbf{G}) \cong \mathbf{G}^{\#\upsilon}$ given the topology induced by that of $\mathbf{G}$. Here, $\mathcal{P}_\upsilon$ is the subgroupoid of $\mathcal{P}$, generated freely by the (equivalence classes of the) edges in υ. Then $\pi_\upsilon : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^{\#\upsilon}$ with $\pi_\upsilon(\overline{A}) := \overline{A}([\upsilon])$ is always continuous.

Proposition 4.1 Let $\mathcal{Q}$ be some complete subset of $\mathcal{P}_{\rm gen}$. Let X be some topological space, and let $\lambda : X \longrightarrow \mathrm{Germ}(\mathcal{Q}, \mathbf{G})$ be some map. Finally, assume that the map $\big(\lambda(\cdot)\big)(\gamma) : X \longrightarrow \mathbf{G}$ is continuous for all $\gamma \in \mathcal{Q}$.
Then

$$\begin{aligned}
\Theta_\lambda : \quad X &\longrightarrow \overline{\mathcal{A}} \\
x &\longmapsto [\widehat{\lambda(x)}]
\end{aligned}$$

is continuous, where $\widehat{\cdot}$ is given as in Proposition 3.1.

[3] The elements of $\overline{\mathcal{A}}$ are denoted by $\overline{A}$ or, synonymously, by $h_{\overline{A}}$.

Proof It is sufficient [4] to prove that $\pi_\gamma \circ \Theta_\lambda : X \longrightarrow \mathbf{G}$ is continuous for all edges γ. Since the multiplication in $\mathbf{G}$ is continuous and $\mathcal{Q}$ is complete, we even may restrict ourselves to the cases of $\gamma \in \mathcal{Q}$. Here, however, the assertion follows immediately from

$$(\pi_\gamma \circ \Theta_\lambda)(x) \equiv \pi_\gamma([\widehat{\lambda(x)}]) = [\widehat{\lambda(x)}]([\gamma]) = \widehat{\lambda(x)}(\gamma) \equiv \lambda(x)(\gamma),$$

i.e., $\pi_\gamma \circ \Theta_\lambda = \big(\lambda(\cdot)\big)(\gamma)$ for all $\gamma \in \mathcal{Q}$. **qed**

We close with

Lemma 4.2 Two generalized connections coincide iff they coincide for all (equivalence classes of) paths of a complete subset of $\mathcal{P}_{\mathrm{gen}}$.

5 Graphomorphisms

Among the possibilities to modify connections, i.e., mappings from $\mathcal{P}$ to $\mathbf{G}$, those induced by transformations of $\mathcal{P}$ are very important. In particular, they arise in the context of diffeomorphisms that naturally induce an action on paths and graphs. But not only diffeomorphisms give nicely behaving transformations of paths and graphs. Hence, we extend the notion of diffeomorphisms.

Definition 5.1 Let $\varphi : M \longrightarrow M$ be a map.

- φ is called **graphical homomorphism** iff φ induces[4] a groupoid homomorphism on $\mathcal{P}$.
- φ is called **graphical isomorphism** (or shorter: **graphomorphism**) iff φ is bijective and both φ and φ^{-1} are graphical homomorphisms.

The set of all graphomorphisms is denoted by $\mathrm{Grapho}(M)$.

Of course, each diffeomorphism is a graphomorphism.

For technical purposes, it is often convenient to have simpler criteria for φ being a groupoid homomorphism.

Lemma 5.1 Let $\varphi : M \longrightarrow M$ be some map. Consider the following statements:

1. φ maps differentiable edges to paths.
2. φ maps edges to edges.
3. φ maps hyphs to hyphs.
4. φ maps paths to paths.
5. φ induces a groupoid homomorphism on $\mathcal{P}$.
6. φ is injective.

Then we have the following implications:

$$1. \iff 4. \iff 5.$$
$$2. \Longrightarrow 4.$$
$$3. \Longrightarrow 4.$$

If M is connected, then even

$$2. \Longrightarrow 6.$$

If φ is injective (6.), then

$$1. \iff 2. \iff 3. \iff 4. \iff 5.$$

[4]This includes that $\varphi \circ \gamma$ is in $\mathcal{P}_{\mathrm{gen}}$ for all $\gamma \in \mathcal{P}_{\mathrm{gen}}$.

Note that, by definition, 5. $\Longrightarrow$ 4. $\Longrightarrow$ 1.

Proof 1. $\Longrightarrow$ 4. Trivial, since (up to the parametrization) each path is a product of piecewise differentiable edges and trivial paths.

2. $\Longrightarrow$ 4. Trivial as well.

3. $\Longrightarrow$ 4. Given a path γ, choose [2] a hyph υ, such that γ is a path in υ. By assumption, $\varphi \circ \upsilon$ is a hyph again. $\varphi \circ \gamma$ is now a product of elements in $\varphi \circ \upsilon$, their inverses and trivial paths, hence it is a path as well.

4. $\Longrightarrow$ 5. First, we have to check whether φ induces a well-defined mapping from $\mathcal{P}$ to $\mathcal{P}$. Let $\gamma, \delta \in \mathcal{P}_{\text{gen}}$. Hence $\varphi \circ \gamma$ and $\varphi \circ \delta$ are in $\mathcal{P}_{\text{gen}}$ again. If γ and δ coincide up to parametrization, then also $\varphi \circ \gamma$ and $\varphi \circ \delta$ do so. If there are $\gamma_1, \gamma_2, \varepsilon \in \mathcal{P}_{\text{gen}}$ with $\gamma = \gamma_1\gamma_2$ and $\delta = \gamma_1\varepsilon\varepsilon^{-1}\gamma_2$, then similarly

$$\varphi \circ \gamma = (\varphi \circ \gamma_1)\,(\varphi \circ \gamma_2)$$

and

$$\begin{aligned} \varphi \circ \delta &= (\varphi \circ \gamma_1)\,(\varphi \circ \varepsilon)\,(\varphi \circ \varepsilon^{-1})\,(\varphi \circ \gamma_2) \\ &= (\varphi \circ \gamma_1)\,(\varphi \circ \varepsilon)\,(\varphi \circ \varepsilon)^{-1}\,(\varphi \circ \gamma_2); \end{aligned}$$

hence, $[\varphi \circ \gamma] = [\varphi \circ \delta]$. This shows that equivalent paths are mapped to equivalent paths. Next, if two paths are composable, their images are composable as well. Now, the homomorphy property is clear.

2. $\Longrightarrow$ 6. Let x and y be two distinct points in M with $\varphi(x) = \varphi(y)$. By connectedness, there is an edge γ in M running through x and y, where at least one of these points is not an endpoint of γ. Now the image of γ w.r.t. φ is not an edge.

1. $\Longrightarrow$ 2. Follows since each path, hence each edge is piecewise differentiable and since φ is one-to-one.

2. $\Longrightarrow$ 3. By assumption, every hyph is mapped to a finite sequence of edges. Observe again that, by injectivity, φ-images of edges (or segments of them) can coincide up to the parametrization only if the origins do so. Hence, free points are mapped to free points, making images of hyphs hyphs again. **qed**

We remark that, in general, 4. neither implies 3. nor 2.

Corollary 5.2 A bijection φ is a graphomorphism iff both φ and φ^{-1} map edges to paths.

If only φ maps edges to paths, then φ may fail to be a graphomorphism. In fact, assume that we are working in the C^1 class on $M = \mathbb{R}^n$ and do not consider embedded paths only. Let φ be the homeomorphism mapping $x \in \mathbb{R}^n$ to $\|x\|\,x$. Of course, φ maps edges to paths. Its inverse $\varphi^{-1}(x) = \|x\|^{-\frac{1}{2}}\,x$, however, does not. For example, the straight line $\gamma(t) = t\,e$ with $e \in \mathbb{R}^n$ having norm 1 is mapped to $(\varphi^{-1} \circ \gamma)(t) = \sqrt{t}\,e$ being not differentiable at $t = 0$.

Finally, we have

Lemma 5.3 Every graphomorphism on M maps (complete) hereditary subsets of $\mathcal{P}_{\text{gen}}$ into (complete) hereditary ones.

6 Main Results

In this section we are going to provide the general scheme, comprising several transformations on $\overline{\mathcal{A}}$. Let us very briefly recall the two most important ones – the gauge transforms and the diffeomorphisms.

The set $\overline{\mathcal{G}}$ of gauge transforms consists of all maps $\overline{g}$ from M to $\mathbf{G}$ acting on $\overline{\mathcal{A}}$ by[5]

$$h_{\overline{A}\circ\overline{g}}(\gamma) \quad := \quad \overline{g}(\gamma(0))^{-1}\, h_{\overline{A}}(\gamma)\, \overline{g}(\gamma(1)) \quad \text{for all } \gamma \in \mathcal{P}_{\text{gen}}$$

and is given the product topology on $\mathrm{Maps}(M, \mathbf{G}) \cong \mathbf{G}^M$. The action of diffeomorphisms on M can be lifted to an action on $\mathcal{P}_{\text{gen}}$ (and $\mathcal{P}$), which again can be lifted to an action on $\overline{\mathcal{A}}$. In fact, each diffeomorphism φ defines a map from $\overline{\mathcal{A}}$ to $\mathrm{Maps}(\mathcal{P}, \mathbf{G})$, again denoted by φ, via

$$h_{\varphi(\overline{A})}(\gamma) \quad := \quad h_{\overline{A}}(\varphi^{-1} \circ \gamma) \quad \text{for all } \gamma \in \mathcal{P}_{\text{gen}}.$$

As we will see also for other examples like the Weyl transformations later, there are two typical features characterizing these transformations Θ on $\overline{\mathcal{A}}$. First, there is given some set $\mathcal{Q}$ of elementary paths (we had $\mathcal{Q} = \mathcal{P}_{\text{gen}}$ in the examples above, but have to reduce this set, e.g., for Weyl transformations). And, second, the modified parallel transport along some path in $\mathcal{Q}$ depends always only on

- the original parallel transport for some, possibly different path in $\mathcal{Q}$, and
- some conjugation-like multiplication by some group elements depending on the path only.

In other words, we have

$$h_{\Theta(\overline{A})}(\gamma) \quad = \quad f_1(\gamma)^{-1}\, h_{\overline{A}}(\varphi(\gamma))\, f_2(\gamma)$$

with some functions $f_1, f_2 : \mathcal{Q} \longrightarrow \mathbf{G}$ and some mapping $\varphi : \mathcal{Q} \longrightarrow \mathcal{Q}$. Let us now check criteria to make such a general transformation well defined on $\overline{\mathcal{A}}$. First of all, φ should map edges to edges and extend to a well-defined mapping from $[\mathcal{Q}]$ to $[\mathcal{Q}]$. For simplicity, let us assume additionally that φ is even induced by a graphical homomorphism and let (just for the next few lines) $\mathbf{G}$ have trivial center. We get

$$\begin{aligned} f_1(\gamma^{-1})^{-1}\, h_{\overline{A}}(\varphi \circ \gamma)^{-1}\, f_2(\gamma^{-1}) \quad &= \quad f_1(\gamma^{-1})^{-1}\, h_{\overline{A}}(\varphi \circ \gamma^{-1})\, f_2(\gamma^{-1}) \\ &= \quad h_{\Theta(\overline{A})}(\gamma^{-1}) \\ &= \quad h_{\Theta(\overline{A})}(\gamma)^{-1} \\ &= \quad f_2(\gamma)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma)^{-1}\, f_1(\gamma) \end{aligned}$$

for all $\gamma \in \mathcal{Q}$. This implies[6] $f_1(\gamma) = f_2(\gamma^{-1}) =: f(\gamma)$, since $\pi_\gamma : \overline{\mathcal{A}} \longrightarrow \mathbf{G}$ is surjective and $\mathbf{G}$ has trivial center. Next, if a nontrivial path in $\mathcal{Q}$ can be decomposed into two nontrivial paths γ_1 and γ_2 in $\mathcal{Q}$, we get

$$\begin{aligned} & f(\gamma_1\gamma_2)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma_1)\, h_{\overline{A}}(\varphi \circ \gamma_2)\, f((\gamma_1\gamma_2)^{-1}) \\ = \quad & f(\gamma_1\gamma_2)^{-1}\, h_{\overline{A}}(\varphi \circ (\gamma_1\gamma_2))\, f((\gamma_1\gamma_2)^{-1}) \\ = \quad & h_{\Theta(\overline{A})}(\gamma_1\gamma_2) \\ = \quad & h_{\Theta(\overline{A})}(\gamma_1)\, h_{\Theta(\overline{A})}(\gamma_2) \\ = \quad & f(\gamma_1)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma_1)\, f(\gamma_1^{-1})\;\; f(\gamma_2)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma_2)\, f(\gamma_2^{-1}). \end{aligned}$$

Typically, γ_1 and γ_2 form a hyph. Hence, the corresponding parallel transports can be assigned independently. By the triviality of the center of $\mathbf{G}$, we get $f(\gamma_1\gamma_2) = f(\gamma_1)$ and $f(\gamma_1^{-1}) = f(\gamma_2)$.

This motivates (now back to the case of an arbitrary connected Lie group $\mathbf{G}$)

Definition 6.1 Let $\mathcal{Q}$ be some hereditary subset of $\mathcal{P}_{\text{gen}}$.
Then a map $\kappa : \mathcal{Q} \longrightarrow \mathbf{G}$ is called **admissible** iff

- $\kappa(\delta_1) = \kappa(\delta_2)$ for all $\delta_1, \delta_2 \in \mathcal{Q}$ with $\delta_1 \uparrow\uparrow \delta_2$, and
- $\kappa(\gamma_1^{-1}) = \kappa(\gamma_2)$ for all $\gamma \in \mathcal{Q}$ and all decompositions $\gamma_1\gamma_2$ of γ.

[5]Until Theorem 6.1, we simply drop the square brackets in expressions like $h_{\overline{A}}([\gamma])$, since confusions are not to be expected. That all that is well defined will be proven below.

[6]Let G be a group and H a subgroup of G. Let, moreover, a_1, a_2, b_1, b_2 be in G, such that $a_1 h a_2 = b_1 h b_2$ for all $h \in H$. Then $a_1 \in b_1 Z_G(H)$ and $a_2 \in Z_G(H) b_2$. In fact, we have $h^{-1} b_1^{-1} a_1 h = b_2 a_2^{-1} = b_1^{-1} a_1$ for all $h \in H$, where the second equality follows for $h = e_G$. Hence, $b_1^{-1} a_1$ and H commute.

Now, we may state the main

Theorem 6.1 Let $\mathcal{Q}$ be some complete subset of $\mathcal{P}_{\rm gen}$, and let φ be some graphical homomorphism of M. Moreover, let $\kappa : \mathcal{Q} \longrightarrow \mathbf{G}$ be some admissible map.
Then we have:

1. There is a unique continuous map $\Theta : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$, such that, for all $\gamma \in \mathcal{Q}$,
$$h_{\Theta(\overline{A})}([\gamma]) \;=\; \kappa(\gamma)^{-1}\, h_{\overline{A}}([\varphi \circ \gamma])\, \kappa(\gamma^{-1}).$$
2. If φ is injective, then Θ even preserves the Ashtekar-Lewandowski measure μ_0. Hence, the induced operator on $\mathcal{B}(L_2(\overline{\mathcal{A}}, \mu_0))$ is well defined and unitary. Moreover, the pull-back $\Theta^* : C(\overline{\mathcal{A}}) \longrightarrow C(\overline{\mathcal{A}})$ is an isometry.
3. If φ is a graphomorphism, then Θ is even a homeomorphism.

Recall that, for compact $\mathbf{G}$, the Ashtekar-Lewandowski measure μ_0 is the unique regular Borel measure on $\overline{\mathcal{A}}$ whose push-forward $(\pi_\upsilon)_*\mu_0$ to $\overline{\mathcal{A}}_\upsilon \cong \mathbf{G}^{\#\upsilon}$ coincides with the Haar measure there for every hyph υ.

Proof 1. Θ exists uniquely and is continuous.

- Define $\lambda : \overline{\mathcal{A}} \longrightarrow \mathrm{Maps}(\mathcal{Q}, \mathbf{G})$ by[7]
$$\big(\lambda(\overline{A})\big)(\gamma) \;=\; \kappa(\gamma)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma)\, \kappa(\gamma^{-1}).$$
- First we show that $\lambda(\overline{A})$ is indeed in $\mathrm{Germ}(\mathcal{Q}, \mathbf{G})$ for all $\overline{A} \in \overline{\mathcal{A}}$.
In fact, for all $\gamma \in \mathcal{Q}$ and all decompositions $\gamma_1\gamma_2$ of γ, we have
$$\begin{aligned}\big(\lambda(\overline{A})\big)(\gamma^{-1}) &= \kappa(\gamma^{-1})^{-1}\, h_{\overline{A}}(\varphi \circ \gamma^{-1})\, \kappa(\gamma) \\ &= \big(\kappa(\gamma)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma)\, \kappa(\gamma^{-1})\big)^{-1} \;=\; \big(\lambda(\overline{A})(\gamma)\big)^{-1}\end{aligned}$$
and
$$\begin{aligned}\big(\lambda(\overline{A})\big)(\gamma) &= \kappa(\gamma)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma)\, \kappa(\gamma^{-1}) \\ &= \kappa(\gamma_1\gamma_2)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma_1)\, h_{\overline{A}}(\varphi \circ \gamma_2)\, \kappa(\gamma_2^{-1}\gamma_1^{-1}) \\ &= \kappa(\gamma_1)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma_1)\, \kappa(\gamma_1^{-1})\, \kappa(\gamma_2)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma_2)\, \kappa(\gamma_2^{-1}) \\ &= \big(\lambda(\overline{A})\big)(\gamma_1)\, \big(\lambda(\overline{A})\big)(\gamma_2).\end{aligned}$$
Here, we used that $\gamma_1 \uparrow\uparrow \gamma_1\gamma_2$ and $\gamma_2^{-1}\gamma_1^{-1} \uparrow\uparrow \gamma_2^{-1}$.
- Next, observe that for every fixed $\gamma \in \mathcal{Q}$,
$$\big(\lambda(\overline{A})\big)(\gamma) \;=\; \kappa(\gamma)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma)\, \kappa(\gamma^{-1}) \;\equiv\; \kappa(\gamma)^{-1}\, \pi_{\varphi\circ\gamma}(\overline{A})\, \kappa(\gamma^{-1})$$
depends continuously on $\overline{A}$, by definition of the projective-limit topology on $\overline{\mathcal{A}}$.
- Now, by Proposition 4.1, $\Theta := \widehat{[\lambda(\cdot)]} : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$ is continuous, whereas for $\gamma \in \mathcal{Q}$
$$h_{\Theta(\overline{A})}(\gamma) \;\equiv\; \big(\Theta(\overline{A})\big)([\gamma]) \;=\; \widehat{[\lambda(\overline{A})]}([\gamma]) \;=\; \kappa(\gamma)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma)\, \kappa(\gamma^{-1}).$$
The uniqueness of Θ follows from the completeness of $\mathcal{Q}$ and Lemma 4.2.

2. Θ preserves the Ashtekar-Lewandowski measure and has isometric pull-back for injective φ.

- In fact, let υ be an arbitrary, but fixed hyph. By completeness, there is some hyph $\upsilon' \geq \upsilon$ with Y' edges, such that every $\gamma_i \in \upsilon'$ is in $\mathcal{Q}$: Indeed, first decompose each path $\gamma \in \upsilon$ into a product of paths in $\mathcal{Q}$. Collect the paths used there, in some set $\boldsymbol{\gamma} \geq \upsilon$. Since possibly $\boldsymbol{\gamma}$ is not a hyph again, decompose, if necessary, the paths in $\boldsymbol{\gamma}$ further to get a hyph $\upsilon' \geq \boldsymbol{\gamma} \geq \upsilon$ [2]. By construction, $\boldsymbol{\gamma}$ is contained in $\mathcal{Q}$. Now, by heredity of $\mathcal{Q}$, so does υ'.

[7] From now on, we will drop the square brackets in all $h_{\overline{A}}([\ldots])$.

By construction, we have

$$\pi_{\upsilon'} \circ \Theta \quad = \quad (\Theta_{\gamma_1} \times \cdots \times \Theta_{\gamma_{Y'}}) \circ \pi_{\varphi \circ \upsilon'}$$

with $\Theta_\gamma(g) := \kappa(\gamma)^{-1}\, g\, \kappa(\gamma^{-1})$ for $\gamma \in \mathcal{Q}$. In other words, each Θ_γ consists of a left and a right translation, whence the Haar measure on $\mathbf{G}$ is Θ_γ-invariant.

- Since $\pi^{\upsilon'}_{\upsilon} \circ \pi_{\upsilon'} = \pi_\upsilon$ with continuous $\pi^{\upsilon'}_{\upsilon} : \overline{\mathcal{A}}_{\upsilon'} \longrightarrow \overline{\mathcal{A}}_{\upsilon}$, since $(\pi_{\upsilon'})_* \mu_0$ is the Y'-fold product of the Haar measure on $\mathbf{G}$ and since $\varphi \circ \upsilon'$ is again a hyph of Y' edges by Lemma 5.1, we get

$$\begin{aligned}
(\pi_\upsilon)_*(\Theta_* \mu_0) &= (\pi^{\upsilon'}_{\upsilon})_*(\pi_{\upsilon'} \circ \Theta)_* \mu_0 \\
&= (\pi^{\upsilon'}_{\upsilon})_*(\Theta_{\gamma_1} \times \cdots \times \Theta_{\gamma_{Y'}})_*(\pi_{\varphi \circ \upsilon'})_* \mu_0 \\
&= (\pi^{\upsilon'}_{\upsilon})_*(\Theta_{\gamma_1} \times \cdots \times \Theta_{\gamma_{Y'}})_* \mu^{Y'}_{\mathrm{Haar}} \\
&= (\pi^{\upsilon'}_{\upsilon})_* \mu^{Y'}_{\mathrm{Haar}} \\
&= (\pi^{\upsilon'}_{\upsilon})_*(\pi_{\upsilon'})_* \mu_0 \\
&= (\pi_\upsilon)_* \mu_0.
\end{aligned}$$

Since finite regular Borel measures on $\overline{\mathcal{A}}$ coincide iff their push-forwards w.r.t. all π_υ coincide, we get the assertion.

- Let f be some cylindrical function on $\overline{\mathcal{A}}$ w.r.t. υ, i.e., we have $f = f_\upsilon \circ \pi_\upsilon$ for some continuous f_υ on $\mathbf{G}^Y$. Now,

$$\Theta^* f \quad \equiv \quad f \circ \Theta \quad = \quad f_\upsilon \circ \pi^{\upsilon'}_{\upsilon} \circ (\Theta_{\gamma_1} \times \cdots \times \Theta_{\gamma_{Y'}}) \circ \pi_{\varphi \circ \upsilon'}.$$

Since $\pi_{\widetilde{\upsilon}}$ is surjective for all hyphs $\widetilde{\upsilon}$, since $\Theta_{\gamma_1} \times \cdots \times \Theta_{\gamma_{Y'}}$ is surjective and since $\pi^{\upsilon'}_{\upsilon}$ is surjective, we have $\|\Theta^* f\|_\infty = \|f_\upsilon\|_\infty = \|f\|_\infty$. Since cylindrical functions are dense in $C(\overline{\mathcal{A}})$ and since Θ^* is continuous, we get the assertion.

3. Θ is a homeomorphism if φ is a graphomorphism.
   - Observe that $\varphi(\mathcal{Q})$ is complete by Lemma 5.3. Now, define $\kappa' : \varphi(\mathcal{Q}) \longrightarrow \mathbf{G}$ by $\kappa'(\gamma) := \kappa(\varphi^{-1} \circ \gamma)^{-1}$. It is easy to check that κ' is admissible w.r.t. $\varphi(\mathcal{Q})$. As already proven above, there is a unique continuous map $\Theta' : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$ with

$$h_{\Theta'(\overline{A})}(\gamma) \quad = \quad \kappa'(\gamma)^{-1}\, h_{\overline{A}}(\varphi^{-1} \circ \gamma)\, \kappa'(\gamma^{-1})$$

for all $\gamma \in \varphi(\mathcal{Q})$. Altogether, this gives

$$\begin{aligned}
& h_{\Theta'(\Theta(\overline{A}))}(\gamma) \\
= \;& \kappa'(\gamma)^{-1}\, h_{\Theta(\overline{A})}(\varphi^{-1} \circ \gamma)\, \kappa'(\gamma^{-1}) \\
= \;& \kappa'(\gamma)^{-1}\, \kappa(\varphi^{-1} \circ \gamma)^{-1}\, h_{\overline{A}}(\varphi \circ \varphi^{-1} \circ \gamma)\, \kappa((\varphi^{-1} \circ \gamma)^{-1})\, \kappa'(\varphi \circ \gamma^{-1}) \\
= \;& h_{\overline{A}}(\gamma)
\end{aligned}$$

for all $\gamma \in \varphi(\mathcal{Q})$. The completeness of $\varphi(\mathcal{Q})$ and Lemma 4.2 prove $\Theta' \circ \Theta = \mathrm{id}_{\overline{\mathcal{A}}}$. Analogously, one shows $\Theta \circ \Theta' = \mathrm{id}_{\overline{\mathcal{A}}}$. **qed**

We get immediately

Corollary 6.2 Let $\mathcal{Q}$ be some complete subset of $\mathcal{P}_{\mathrm{gen}}$, let φ be some graphical homomorphism of M, let Y be some topological space and let $\kappa : \mathcal{Q} \times Y \longrightarrow \mathbf{G}$ be some map, such that

- $\kappa(\cdot, y) : \mathcal{Q} \longrightarrow \mathbf{G}$ is admissible for all $y \in Y$, and
- $\kappa(\gamma, \cdot) : Y \longrightarrow \mathbf{G}$ is continuous for all $\gamma \in \mathcal{Q}$.

Then there is a unique map $\Theta : \overline{\mathcal{A}} \times Y \longrightarrow \overline{\mathcal{A}}$ with

$$h_{\Theta(\overline{A}, y)}(\gamma) \quad = \quad \kappa(\gamma, y)^{-1}\, h_{\overline{A}}(\varphi \circ \gamma)\, \kappa(\gamma^{-1}, y)$$

for all $\gamma \in \mathcal{Q}$. Moreover, Θ is continuous.

7 Applications

Let us consider three basic examples.

Corollary 7.1 Gauge Transforms
There is a unique map $\Theta : \overline{\mathcal{A}} \times \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{A}}$, such that

$$h_{\Theta(\overline{A},\overline{g})}(\gamma) \quad = \quad \overline{g}(\gamma(0))^{-1}\, h_{\overline{A}}(\gamma)\, \overline{g}(\gamma(1)) \quad \text{for all } \gamma \in \mathcal{P}_{\text{gen}}.$$

Θ is continuous. Moreover, $\Theta_{\overline{g}} := \Theta(\cdot, \overline{g}) : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$ is a homeomorphism and preserves the Ashtekar-Lewandowski measure for each $\overline{g} \in \overline{\mathcal{G}}$. The inverse of $\Theta_{\overline{g}}$ is given by $\Theta_{\overline{g}^{-1}}$.

Usually, one writes $\overline{A} \circ \overline{g}$ instead of $\Theta(\overline{A}, \overline{g})$.

Proof Set $\mathcal{Q} := \mathcal{P}_{\text{gen}}$ and $\varphi := \text{id}_M$. Moreover, set $Y := \overline{\mathcal{G}}$, and define $\kappa(\gamma, \overline{g}) := \overline{g}(\gamma(0))$. Of course, κ fulfills the requirements of Corollary 6.2. The assertion now follows from $\overline{g}(\gamma^{-1}(0)) = \overline{g}(\gamma(1))$, Theorem 6.1 and Corollary 6.2. **qed**

Corollary 7.2 Diffeomorphisms
There is a unique map $\Theta : \overline{\mathcal{A}} \times \text{Grapho}(M) \longrightarrow \overline{\mathcal{A}}$, such that

$$h_{\Theta(\overline{A},\varphi)}(\gamma) \quad = \quad h_{\overline{A}}(\varphi^{-1} \circ \gamma) \quad \text{for all } \gamma \in \mathcal{P}_{\text{gen}}.$$

Moreover, $\Theta_\varphi := \Theta(\cdot, \varphi) : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$ is a homeomorphism and preserves the Ashtekar-Lewandowski measure for each graphomorphism $\varphi \in \text{Grapho}(M)$. The inverse of Θ_φ is given by $\Theta_{\varphi^{-1}}$.

Usually, one writes $\varphi(\overline{A})$ instead of $\Theta(\overline{A}, \varphi)$.

Proof Define $\mathcal{Q} := \mathcal{P}_{\text{gen}}$ and $\kappa(\gamma) := e_{\mathbf{G}}$ for all $\gamma \in \mathbf{G}$. Theorem 6.1 gives the proof with inverted φ. **qed**

Corollary 7.3 Weyl transformations
Let S be a quasi-surface, and let $\mathcal{Q}$ consist of all edges and trivial paths γ being S-external (i.e., $\text{int}\,\gamma \cap S = \varnothing$) or S-internal (i.e., $\text{int}\,\gamma \subseteq S$). Moreover, let σ_S be some intersection function for S.
Then there is a unique map $\Theta^{S,\sigma_S} : \overline{\mathcal{A}} \times \text{Maps}(M, \mathbf{G}) \longrightarrow \overline{\mathcal{A}}$, such that

$$h_{\Theta^{S,\sigma_S}(\overline{A},d)}(\gamma) \; = \; \begin{cases} d(\gamma(0))^{\sigma_S^-(\gamma)}\, h_{\overline{A}}(\gamma)\, d(\gamma(1))^{\sigma_S^+(\gamma)} & \text{if } \gamma \text{ is } S\text{-external} \\ h_{\overline{A}}(\gamma) & \text{if } \gamma \text{ is } S\text{-internal} \end{cases}.$$

Moreover, the map $\Theta_d^{S,\sigma_S} : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$, given by $\Theta_d^{S,\sigma_S}(\overline{A}) := \Theta^{S,\sigma_S}(\overline{A}, d)$, is a homeomorphism and preserves the Ashtekar-Lewandowski measure for each $d \in \text{Maps}(M, \mathbf{G})$. The inverse of Θ_d^{S,σ_S} is given by $\Theta_{d^{-1}}^{S,\sigma_S}$. Finally, if $\text{Maps}(M, \mathbf{G}) \cong \mathbf{G}^M$ is given the product topology, then Θ^{S,σ_S} is continuous.

For the definition of quasi-surfaces and intersection functions, see [3]. Moreover, note that the Weyl operators are the pull-backs of the corresponding Weyl transformations Θ_d^{S,σ_S} to $C(\overline{\mathcal{A}})$ (or their induced action on $L_2(\overline{\mathcal{A}}, \mu_0)$).

Proof
- $\mathcal{Q}$ is complete.
  This follows since S is a quasi-surface, i.e., every edge (hence any finite path) can be decomposed (up to the parametrization) into a product of edges and trivial paths being S-external or S-internal. [3]

- Θ^{S,σ_S} exists uniquely and is continuous for the product topology on $\mathrm{Maps}(M, \mathbf{G})$.
  Let φ be the identity on M, and let $Y := \mathrm{Maps}(M, \mathbf{G})$. Moreover, let

$$\kappa(\gamma, d) \;:=\; \begin{cases} d(\gamma(0))^{-\sigma_S^-(\gamma)} & \text{if } \gamma \text{ is } S\text{-external} \\ e_{\mathbf{G}} & \text{if } \gamma \text{ is } S\text{-internal} \end{cases}.$$

  The only nontrivial property of κ in Corollary 6.2 to be checked is $\kappa(\gamma_1^{-1}, d) = \kappa(\gamma_2, d)$ for decompositions $\gamma_1\gamma_2$ of S-external γ. Observe, however, that here $\gamma_1^{-1}(0) \equiv \gamma_1(1) \equiv \gamma_2(0)$ is not contained in S, hence $\kappa(\gamma_1^{-1}, d) = e_{\mathbf{G}} = \kappa(\gamma_2, d)$. The claim now follows from $\sigma_S^-(\gamma) + \sigma_S^+(\gamma^{-1}) = 0$ and Corollary 6.2.
- Θ_d^{S,σ_S} is a homeomorphism and leaves μ_0 invariant.
  This now follows from Theorem 6.1. **qed**

Finally, we may use the theorem above to rederive and extend results known from [2].

Corollary 7.4 Let N be some set of points in M having no accumulation point, and let $\mathcal{Q}$ be the set of all paths that are N-external or trivial. For every $x \in N$, let E_x be some set of edges starting at x, such that
1. $N \cap \mathrm{im}\, E_x = \{x\}$ and
2. $\gamma_1 \uparrow\uparrow \gamma_2$ for $\gamma_1, \gamma_2 \in E_x$ implies $\gamma_1 = \gamma_2$.

Define E to be the union of all E_x. Moreover, let $f : E \longrightarrow \mathbf{G}$ be some function and define for $\gamma \in \mathcal{Q}$

$$\kappa(\gamma, \overline{A}) \;:=\; \begin{cases} h_{\overline{A}}(e)\, f(e)^{-1} & \text{if } \gamma \uparrow\uparrow e \text{ for some } e \in E \\ e_{\mathbf{G}} & \text{otherwise} \end{cases}.$$

Then there is a unique map $\Theta : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}$, such that

$$h_{\Theta(\overline{A})}(\gamma) \;=\; \kappa(\gamma, \overline{A})^{-1}\, h_{\overline{A}}(\gamma)\, \kappa(\gamma^{-1}, \overline{A})$$

for all $\gamma \in \mathcal{Q}$. Moreover, Θ is continuous.

Proof
- $\mathcal{Q}$ is complete.
  Since N does not contain accumulation points, every path can be decomposed into finitely many subpaths either not containing any point of N in their interior or being trivial. The heredity is clear.
- κ is well defined.
  If there exist $e_1, e_2 \in E$ with $\gamma \uparrow\uparrow e_1$ and $\gamma \uparrow\uparrow e_2$, then $e_1 \uparrow\uparrow e_2$ and $e_1, e_2 \in E_{\gamma(0)}$, hence $e_1 = e_2$ by assumption.
- $\kappa(\cdot, \overline{A})$ is admissible for all $\overline{A} \in \overline{\mathcal{A}}$.
  Let $\delta_1, \delta_2 \in \mathcal{Q}$ with $\delta_1 \uparrow\uparrow \delta_2$. If they are trivial, then even $\delta_1 = \delta_2$. Otherwise, they are N-external. Then one of them starts as some $e \in E$ iff the other does. Hence, in both cases, $\kappa(\delta_1, \overline{A}) = \kappa(\delta_2, \overline{A})$ for all $\overline{A} \in \overline{\mathcal{A}}$. Let now γ be in $\mathcal{Q}$ and $\gamma_1\gamma_2$ be a decomposition of γ. If γ is N-external, then, in particular, neither γ_1^{-1} nor γ_2 starts as any $e \in E$, since $\gamma_1^{-1}(0) \equiv \gamma_2(0) = \gamma(\frac{1}{2}) \notin N$. Hence, we have $\kappa(\gamma_1^{-1}, \overline{A}) = e_{\mathbf{G}} = \kappa(\gamma_2, \overline{A})$. The case of trivial γ is clear.
- Θ is continuous.
  By Corollary 6.2, it is sufficient to show that $\kappa(\gamma, \overline{A})$ is continuous in $\overline{A}$. This however follows, because, by construction, there is at most one $e \in E$ with $\gamma \uparrow\uparrow e$. **qed**

Note, that, in general, Θ does not leave the Ashtekar-Lewandowski measure invariant. In fact, we have $h_{\Theta(\overline{\mathcal{A}})}(e) = f(e)$ for every $e \in E$. Therefore, $\Theta(\overline{\mathcal{A}}) \subseteq \pi_e^{-1}(\{f(e)\})$, hence

$$0 \;\leq\; \mu_0(\Theta(\overline{\mathcal{A}})) \;\leq\; \mu_0\bigl(\pi_e^{-1}(\{f(e)\})\bigr) \;=\; \mu_{\mathrm{Haar}}(\{f(e)\}) \;=\; 0,$$

unless **G** is trivial.

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