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Abstract: The classical Arzela-Ascoli theorem is a compactness result for families of functions depending on bounds on the derivatives of the functions, and is of invaluable use in many fields of mathematics. In this paper, inspired by a result of Corlette, we prove an analogous compactness result for families of immersed submanifolds which depends only on bounds on the derivatives of the second fundamental forms of these submanifolds. We then show how the result of Corlette may be obtained as an immediate corollary.

Key Words: Arzela-Ascoli, Immersed Submanifolds, Compactness, Cheeger/Gromov Convergence

AMS Subject Classification: 57.20, 53.40

1 - Introduction.

In [1], Cheeger proved his famous "Finiteness Theorem" which states that, given positive real numbers $K, \epsilon, R \in \mathbb{R}^+$, there exist only finitely many homeomorphism classes of complete manifolds of a given dimension with sectional curvature bounded above in absolute value by K, injectivity radius bounded below by ϵ and diameter bounded above by R. Gromov later showed how this result may be viewed in terms of (more precisely, as a corollary of) a compactness result for the family of such manifolds (see, for example [3] and [4]). Viewed from this perspective, the conditions imposed by Cheeger become quite satisfying to our common sense. The curvature bound is probably the most fundamental. Indeed, curvature reflects the "derivatives" of the manifold (how fast it turns), and whenever derivatives are bounded, following the philosophy of the classical Arzela-Ascoli theorem, one is entitled to expect to find a compactness result. The other two conditions reflect more geometrical considerations. The lower bound on the injectivity radius excludes "pinching" (one can consider a sequence of cylinders of ever smaller radius: the only intrinsic data that informs us of degeneration is the injectivity radius which tends to zero), and the bound on the diameter excludes the possibility of adding components indefinitely without introducing very high curvatures (to see what happens without this condition, one can consider a sequence of surfaces of ever increasing genus).

In [2], Corlette proved an analogous finiteness theorem for immersed submanifolds of a given Riemannian manifold. He proves that, given positive real numbers $K, R \in \mathbb{R}^+$ and a given compact manifold M, there exists only finitely many C^1 isotopy classes of complete immersed submanifolds of M with second fundamental form bounded above in norm by K and diameter bounded above by R. This result no longer requires the condition on the injectivity radius of the immersed submanifold since such a lower bound is now a product of the bounds on the second fundamental form of the submanifold and the curvature of the ambient manifold (one may consider again the example of a cylinder in Euclidean space: its injectivity radius cannot become small without its second fundamental form becoming large). Following the philosophy of Cheeger's finiteness theorem, one would expect this result to also arise from a compactness result for immersed submanifolds.

The compactness result thus obtained bears a perfect analogy to the classical Arzela-Ascoli theorem, being, in certain aspects, a generalisation of this result, and it is for this reason that we have chosen to name it thus. An example of an application of this result may be found in [5]. The statement of this theorem requires the following definition:

Definition 1.1

Let (M, g) be a Riemannian manifold. Let $X = (Y, i)$ be an immersed submanifold in M. Let ∇^i be the Levi-Civita covariant derivative generated over Y by the immersion i into (M, g). Let $A(X)$ be the second fundamental form of X. For all $k \geqslant 2$, we define $A_k(X)$ using the following recurrence relation:

$$\begin{aligned} A_2(X) &= A(X), \\ A_k(X) &= \nabla^i A_{k-1}(X)\ \forall k \geqslant 3. \end{aligned}$$

We now define $\mathcal{A}_k(X)$ for all $k \geqslant 2$ by:

$$\mathcal{A}_k(X) = \sum_{i=2}^{k} \|A_i(X)\|.$$

The principal result of this paper is now given by the following theorem:

Theorem 1.2

Let $k \geqslant 2, m \leqslant n \in \mathbb{N}$ be positive integers.

Let $(M_n, p_n)_{n\in\mathbb{N}}, (M_\infty, p_\infty)$ be complete pointed Riemannian manifolds of dimension n and of class at least C^k such that $(M_n, p_n)_{n\in\mathbb{N}}$ converges towards (M_∞, p_∞) in the pointed C^k Cheeger/Gromov topology.

For all $n \in \mathbb{N}$, let $\Sigma_n = (S_n, q_n)$ be an m-dimensional pointed immersed submanifold of M_n of type at least C^k such that $i(q_n) = p_n$.

Suppose that for all $R > 0$, there exists B such that, for all n:

$$\left|\mathcal{A}^k_{\Sigma_n}(q)\right| \leqslant B \qquad \forall q \in B_R(q_n).$$

Then, there exists a pointed complete immersed submanifold $\Sigma_\infty = (S_\infty, q_\infty)$ of M_∞ of type $C^{k-1,1}$ such that $i(q_\infty) = p_\infty$ and that, after extraction of a subsequence, $(\Sigma_n, q_n)_{n\in\mathbb{N}}$ converges towards $(\Sigma_\infty, q_\infty)$ in the pointed weak $C^{k-1,1}$ Cheeger/Gromov topology.

The terms used in this theorem are explained in section 2 and appendix A of this paper. What we call the Cheeger/Gromov topology is essentially the canonical topology that one would expect to use for immersed submanifolds. In particular, when $k = 2$, the condition on the submanifolds amounts to a bound on the norms of the second fundamental forms of the immersed submanifolds.

We would like to underline that in [2], Corlette clearly states that he anticipates that his finiteness result should arise from a compactness result in a way analagous to the Cheeger/Gromov case. We consider however that, given the importance of this result to our own work, it was necessary to properly unearth this theorem and to state and prove it in its own right.

In the second section, we introduce various concepts associated to immersed submanifolds, and we describe the Cheeger/Gromov topology for pointed Riemannian manifolds and immersed submanifolds. In the third section, we study the manner in which immersed submanifolds may locally be described in terms of graphs over the tangent space to each point. The objective here being to bound from below the radius of the disk over which the submanifold is a graph, and to bound from above the derivatives of the function of which the submanifold is a graph. In the fourth section, using the technical results of the third section, we prove theorem 1.2 and we then prove Corlette's result [2] as a corollary.

In this paper, we will be working in the $C^{k,\alpha}$ category. Since we geometers are not in general in the habit of using mappings of this type, appendix A reviews the properties required of a class of functions for one to be able to construct a theory of manifolds out of it, and we then show how the class of $C^{k;\alpha}_{\text{loc}}$ mappings satisfies these properties. Finally, in appendix B, we provide a proof of the now classical compactness theorem of Riemannian geometry in a form that is most appropriate for our uses. We also hope (perhaps vainly) that we have provided here a slightly more accessible proof of what is an important, but technically challenging result.

I would like to thank François Labourie for introducing me to the subject and drawing my attention to the utility of this result.

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2 - Convergence of Manifolds.

2.1 Immersed Submanifolds.

We review the basic definitions from the theory of immersed submanifolds and establish the notations that will be used throughout this article.

Let M be a smooth manifold. An **immersed submanifold** is a pair $\Sigma = (S, i)$ where S is a smooth manifold and $i : S \to M$ is a smooth immersion. Let g be a Riemannian metric on M. We give S the unique Riemannian metric i^*g which makes i into an isometry. We say that Σ is **complete** if and only if the Riemannian manifold (S, i^*g) is.

We introduce the following definition:

Definition 1.1

Let (M, g) be a Riemannian manifold. Let $X = (Y, i)$ be an immersed submanifold in M. Let ∇^i be the Levi-Civita covariant derivative generated over Y by the immersion i into (M, g). Let $A(X)$ be the second fundamental form of X. For all $k \geqslant 2$, we define $A_k(X)$ using the following recurrence relation:

$$\begin{aligned} A_2(X) &= A(X), \\ A_k(X) &= \nabla^i A_{k-1}(X)\ \forall k \geqslant 3. \end{aligned}$$

We now define $\mathcal{A}_k(X)$ for all $k \geqslant 2$ by:

$$\mathcal{A}_k(X) = \sum_{i=2}^{k} \|A_i(X)\|.$$

2.2 The Cheeger/Gromov Topology.

A **pointed Riemannian manifold** is a pair (M, p) where M is a Riemannnian manifold and p is a point in M. If (M, p) and (M', p') are pointed manifolds then a **morphism** (or **mapping**) from (M, p) to (M', p') is a (not necessarily even continuous) function from

M to M' which sends p to p' and is C^∞ in a neighbourhood of p. The family of pointed manifolds along with these morphisms forms a category. In this section, we will discuss a notion of convergence for this family. It should be borne in mind that even though this family is not a set, we may consider it as such. Indeed, since every manifold may be plunged into an infinite dimensional real vector space, we may discuss, instead, the equivalent family of pointed finite dimensional submanifolds of this vector space, and this is a set.

Let $(M_n, p_n)_{n\in\mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds. For all n, we denote by g_n the Riemannian metric over M_n. We say that the sequence $(M_n, p_n)_{n\in\mathbb{N}}$ **converges** to the complete pointed manifold (M_0, p_0) in the **Cheeger/Gromov topology** if and only if:

(i) for all n, there exists a mapping $\varphi_n : (M_0, p_0) \to (M_n, p_n)$,

such that, for every compact subset K of M_0, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$:

(i) the restriction of φ_n to K is a C^∞ diffeomorphism onto its image, and

(ii) if we denote by g_0 the Riemannian metric over M_0, then the sequence of metrics $(\varphi_n^* g_n)_{n\geqslant N}$ converges to g_0 in the C^∞ topology over K.

We refer to the sequence $(\varphi_n)_{n\in\mathbb{N}}$ as a sequence of **convergence mappings** of the sequence $(M_n, p_n)_{n\in\mathbb{N}}$ with respect to the limit (M_0, p_0). The convergence mappings are trivially not unique. However, two sequences of convergence mappings $(\varphi_n)_{n\in\mathbb{N}}$ and $(\varphi'_n)_{n\in\mathbb{N}}$ are equivalent in the sense that there exists an isometry ϕ of (M_0, p_0) such that, for every compact subset K of M_0, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$:

(i) the mapping $(\varphi_n^{-1} \circ \varphi'_n)$ is well defined over K, and

(ii) the sequence $(\varphi_n^{-1} \circ \varphi'_n)_{n\geqslant N}$ converges to ϕ in the C^∞ topology over K.

One may verify that this mode of convergence does indeed arise from a topological structure over the space of complete pointed manifolds. Moreover, this topology is Haussdorf (up to isometries).

Most topological properties are unstable under this limiting process. For example, the limit of a sequence of simply connected manifolds is not necessarily simply connected. On the other hand, the limit of a sequence of surfaces of genus k is a surface of genus at most k (but quite possibly with many holes).

Let M be a complete Riemannian manifold. A **pointed immersed submanifold** in M is a pair (Σ, p) where $\Sigma = (S, i)$ is an immersed submanifold in M and p is a point in S.

Let $(\Sigma_n, p_n)_{n\in\mathbb{N}} = (S_n, p_n, i_n)_{n\in\mathbb{N}}$ be a sequence of complete pointed immersed submanifolds in M. We say that $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ **converges** to $(\Sigma_0, p_0) = (S_0, p_0, i_0)$ in the **Cheeger/Gromov topology** if and only if $(S_n, p_n)_{n\in\mathbb{N}}$ converges to (S_0, p_0) in the Cheeger/Gromov topology, and, for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ of convergence mappings of $(S_n, p_n)_{n\in\mathbb{N}}$ with respect to this limit, and for every compact subset K of S_0, the sequence of functions $(i_n \circ \varphi_n)_{n\geqslant N}$ converges to the function $(i_0 \circ \varphi_0)$ in the C^∞ topology over K.

As before, this mode of convergence arises from a topological structure over the space of complete immersed submanifolds. Moreover, this topology is Haussdorf (up to isometries).

2.3 The Class of $C^{k,\alpha}$ Pointed Manifolds.

We define the space of $C^{k,\alpha}_{\text{loc}}$ mappings as in appendix A, and we define the following topological structure over the space $C^{k,\alpha}$:

Definition 2.1

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $(f_n)_{n\in\mathbb{N}}, f$ be functions over Ω of type $C^{k,\alpha}$. We say that $(f_n)_{n\in\mathbb{N}}$ converges to f in the **weak $C^{k,\alpha}$ topology** *if and only if, for all $\beta \in (0,\alpha)$:*

$$(\|f_n - f\|_{C^{k,\beta}(\Omega)})_{n\in\mathbb{N}} \to 0.$$

If $(f_n)_{n\in\mathbb{N}}, f$ are functions of type $C^{k,\alpha}_{\text{loc}}$ over Ω, then we say that $(f_n)_{n\in\mathbb{N}}$ converges to f in the **weak $C^{k,\alpha}$ topology** *if and only if for all $p \in \Omega$ there exists a neighbourhood V of p in Ω such that $(f_n)_{n\in\mathbb{N}}$ converges to f in the weak $C^{k,\alpha}$ topology over V.*

We show in appendix A that for $k \geqslant 1$, addition, multiplication, composition and inversion of $C^{k,\alpha}_{\text{loc}}$ functions are continuous operations with respect to the weak $C^{k,\alpha}_{\text{loc}}$ topology.

We show in appendix A how the class of $C^{k,\alpha}_{\text{loc}}$ functions for $k \geqslant 1$ has sufficient structure for us to construct the class of $C^{k,\alpha}$ manifolds. We thus make the following definition:

Definition 2.2

For $k \in \mathbb{N}$ and $\alpha \in (0,1]$, a $C^{k,\alpha}$ Riemannian manifold is a triplet $(S,\mathcal{A},g)$ where:

(1) S is a connected topological manifold,

(2) $\mathcal{A}$ is a $C^{k,\alpha}$ atlas (i.e. all the transition mappings are of type $C^{k,\alpha}_{\text{loc}}$) and

(3) g is a $C^{k-1,\alpha}_{\text{loc}}$ metric over $(S,\mathcal{A})$ (i.e. g is described locally in every chart by a $C^{k-1,\alpha}_{\text{loc}}$ function).

We define pointed $C^{k,\alpha}$ manifolds and immersed submanifolds as for smooth manifolds. We may also define for such manifolds a topology analogous to the Cheeger/Gromov topology. In the case of convergence of $C^{k,\alpha}$ manifolds, the convergence mappings are all of type $C^{k,\alpha}$ and the metrics converge in the weak $C^{k-1,\alpha}_{\text{loc}}$ topology. We call the resulting topology the **weak $C^{k,\alpha}$ Cheeger/Gromov** topology.

2.4 Uniqueness of The Cheeger/Gromov Limit.

Let $U \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. Let g be a Riemannian metric over U. Let TU be the tangent bundle over U. Let $\pi : TU \to U$ be the canonical projection. Let TTU be the tangent bundle over TU. Let VTU be the vertical subbundle of TTU:

$$VTU = \text{Ker}(\pi).$$

Let HTU be the horizontal subbundle of TTU associated to the Levi-Civita connection of g. We have:

$$HTU \oplus VTU = TTU.$$

We know that:

$$HTU, VTU \cong \pi^* TU.$$

Let us denote by i_H (resp. i_V) the canonical isomorphism which sends HTU (resp. VTU) into π^*TU. We define the metric Tg over TTU such that:

$$\begin{array}{ll} Tg|_{HTU} & = i_H^* \pi^* g, \\ Tg|_{VTU} & = i_V^* \pi^* g, \\ HTU & \perp_{Tg} VTU. \end{array}$$

We call Tg the **Levi-Civita lifting** of g over TU. Since the Levi-Civita connection of g depends on the first derivative of g, a $C^{k,\alpha}$ control on g and g^{-1} yields a $C^{k-1,\alpha}$ control on Tg and Tg^{-1}.

We have the following elementary result:

Lemma 2.3

Let U,V be open subsets of $\mathbb{R}^n$. Let $\varphi : U \to V$ be a diffeomorphism. Let g and h be Riemannian metrics over U and V respectively. Let M and N be the matrices representing g and h respectively with respect to Euclidean metric over $\mathbb{R}^n$. There exists a function C_0 such that if $\varphi^* h = g$, then:

$$\|D\varphi\|_{C^0(U)} \leqslant C_0(\|M\|_{C^0(U)}, \|M^{-1}\|_{C^0(U)}, \|N\|_{C^0(U)}, \|N^{-1}\|_{C^0(U)}).$$

Proof: Since $\varphi^* h = g$, we have:

$$D\varphi^t N D\varphi = M.$$

Since the mapping $A \mapsto A^2$ defines a diffeomorphism of the space of symmetric positive definite matrices onto itself, we may write:

$$(N^{1/2} D\varphi)^t (N^{1/2} D\varphi) = M.$$

For any matrix A, we have:

$$\|A^t A\| = \|A\|^2.$$

Thus:

$$\|N^{1/2} D\varphi\|^2 = \|M\|.$$

Consequently:

$$\|D\varphi\| = \|N^{-1/2} N^{1/2} D\varphi\| \leqslant \|N^{-1/2}\| \|M^{1/2}\|.$$

The result now follows. □

Using the Levi-Civita lifting of the metric, we may generalise this result to higher derivatives of φ:

Lemma 2.4

Let U,V be open subsets of $\mathbb{R}^n$. Let $\varphi : U \to V$ be a diffeomorphism. Let g and h be Riemannian metrics over U and V respectively. Let M and N be the metrics representing g and h respectively with respect to the Euclidean metric over $\mathbb{R}^n$. For all $k \in \mathbb{N}$, there exists a function C_k such that if $\varphi^* h = g$, then:

$$\|D^k\varphi\|_{C^0(U)} \leqslant C_k(\|M\|_{C^k(U)}, \|M^{-1}\|_{C^k(U)}, \|N\|_{C^k(U)}, \|N^{-1}\|_{C^k(U)}).$$

Proof: We proof this by induction. By lemma 2.3, the result is true when $k = 0$. Suppose that the result is true for all $k \leqslant m$. Let $T^m\varphi$ be the m'th jet of φ sending T^mU into T^mV. Let T^mg and T^mh be the m-fold Levi-Civita liftings of g and h respectively. We have:

$$(T^m\varphi)^*T^mh = T^mg.$$

Let M_m and N_m be the matrices of T^mg and T^mh respectively with respect to the Euclidean metric. Since T^mU (resp. T^mV) is a bundle over U (resp. V), for $R \in \mathbb{R}^+$, we may consider the subbundle B_R^mU (resp. B_R^mV) of balls of radius R (with respect to the Euclidean metric) in T^mU (resp. T^mV). Since the result is true for all $k \leqslant m$, there exists R which depends only on $\|M\|_{C^m(U)}$,$\|M^{-1}\|_{C^m(U)}$,$\|N\|_{C^m(U)}$, $\|N^{-1}\|_{C^m(U)}$ such that:

$$T^m\varphi(B_1^m(U)) \subseteq B_R^m(V).$$

There exist functions $\hat{C}_{m,R}$ and $\hat{c}_{m,R}$ such that:

$$\begin{aligned} \|N_m^{-1}\|_{C^0(B_R(V))} &\geqslant \hat{c}_{m,R}(\|N\|_{C^m(V)}, \|N^{-1}\|_{C^m(V)}), \\ \|M_m\|_{C^0(B_1(U))} &\leqslant \hat{C}_{m,R}(\|M\|_{C^m(U)}, \|M^{-1}\|_{C^m(U)}). \end{aligned}$$

Finally, there is a function D_{m+1} such that:

$$\|D^{m+1}\varphi\|_{C^0(U)} \leqslant D_{m+1}(\|DT^{m+1}\varphi\|_{C^0(B_1(U))}).$$

The result now follows for $k = m + 1$ by lemma 2.3, and the result follows for all k by induction. □

In particular, we obtain the following corollary:

Corollary 2.5

Let U, V be open subsets of $\mathbb{R}^n$. Let $(g_n)_{n\in\mathbb{N}}$,g be Riemannian metrics over U such that:

$$(\|g_n - g\|_{C^k(U)})_{n\in\mathbb{N}} \to 0.$$

Let $(h_n)_{n\in\mathbb{N}}$,h be Riemannian metrics over V such that:

$$(\|h_n - h\|_{C^k(V)})_{n\in\mathbb{N}} \to 0.$$

Let g_{Euc} be the Euclidean metric over $\mathbb{R}^n$ and suppose that there exists $\Lambda \in \mathbb{R}^+$ such that, for all n:

$$\frac{1}{\Lambda} g_{\mathrm{Euc}} \leqslant g_n, h_n \leqslant \Lambda g_{\mathrm{Euc}}.$$

Let $(\varphi_n)_{n\in\mathbb{N}} : U \to V$ be C^{k+1} mappings such that, for all n:

$$\varphi_n^* h_n = g_n, \qquad \varphi_n(0) = 0.$$

Then, there exists a $C^{k,1}$ mapping $\varphi_0 : U \to V$ such that, after extraction of a subsequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ_0 in the weak $C^{k,1}$ topology. Moreover:

$$\varphi_0^* h = g, \qquad \varphi_0(0) = 0.$$

Proof: This follows from the preceeding lemma and the classical Arzela-Ascoli theorem. □

Remark: This result yields the uniqueness of Cheeger/Gromov limits.

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3 - Immersed Submanifolds Locally as Graphs.

3.1 Immersed Submanifolds as Graphs.

It is trivial that every immersed submanifold may be described everywhere locally as a graph over a ball of a given radius in the tangent space to the submanifold at each point. In this section, we will show how to obtain a bound from below for the radius of such a ball in terms of the norm of the second fundamental form in a neighbourhood of the given point. We then show how bounds on the derivatives of the function of which the submanifold is a graph may be obtained in terms of bounds on the derivatives of the second fundamental form of the submanifold.

In this section we will only consider C^∞ manifolds, although the same reasoning remains valid for $C^{k,\alpha}$ manifolds. Let (S, i) be an immersed submanifold in $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$. Let Ω be an open subset of S and let V be an open subset of $\mathbb{R}^m$. We say that (Ω, i) is a **graph** over V if and only if there exists a diffeomorphism $\alpha : V \to \Omega$ and a function $f : V \to \mathbb{R}^{n-m}$ such that for all $x \in V$:

$$i \circ \alpha(x) = (x, f(x)).$$

We call f the **graph function** of Σ and we call α the **graph reparametrisation** of Ω.

We have the following result:

Lemma 3.1

Let $\Sigma = (S, i)$ be an immersed submanifold of $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$. Let U_1, U_2 be open subsets of $\mathbb{R}^m$. Suppose that there exist open subsets Ω_1, Ω_2 of S such that, for each k

(Ω_k, i) is a graph over U_k. For each k, let $\alpha_k : U_k \to \Omega_i$ be the graph reparametrisation of Ω_k and let $f_k : U_i \to \mathbb{R}^{n-m}$ be the graph function.

Suppose that there exists $p \in U_1 \cap U_2$ such that:

$$\alpha_1(p) = \alpha_2(p),$$

then, for every q in the connected component of $U_1 \cap U_2$ containing p:

$$\alpha_1(q) = \alpha_2(q), \qquad f_1(q) = f_2(q).$$

Remark: In other words, for a given pair of open sets (U, Ω), the graph function and the graph reparametrisation are locally unique.

Proof: Let V be the connected component of $U_1 \cap U_2$ containing p. Let us define X by:

$$X = \{q \in V \text{ s.t. } \alpha_1(q) = \alpha_2(q)\}.$$

The set X is closed. Let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be the canonical projection. For all $q \in V$:

$$(\pi \circ i) \circ \alpha_1(q) = (\pi \circ i) \circ \alpha_2(q) = q.$$

Since $(\pi \circ i)$ is locally invertible, it follows that if $\alpha_1(q) = \alpha_2(q)$ then, for all q' in a neighbourhood Ω of q:

$$\alpha_1(q) = \alpha_2(q).$$

Consequently, X is open. The result now follows. $\square$

We have the following definition:

Definition 3.2

Let (S, i) be an immersed submanifold in $\mathbb{R}^n$. Let K be a closed subset of S, and let Ω be an open subset of $\mathbb{R}^n$. We say that (K, i) is **complete with respect to** Ω *if and only if for every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in S such that $(i(x_n))_{n\in\mathbb{N}}$ converges in Ω, there exists x_0 in S such that $(x_n)_{n\in\mathbb{N}}$ converges to x_0.*

Remark: If Ω is relatively compact, then this definition is independant of the Riemannian metric chosen over a neighbourhood of Ω.

Let $\Sigma = (S, i)$ be an immersed submanifold of $\mathbb{R}^n$. For $\epsilon \in \mathbb{R}^+$, suppose that Σ is a complete with respect to the ball $B_\epsilon(0)$. Let p be a point in S and suppose that:

$$i(p) = 0, \qquad T_p\Sigma = i_* T_p S = \mathbb{R}^m \times \{0\}.$$

We define E to be the set of all $\eta \in (0, \infty)$ such that there exists a neighbourhood Ω of p such that (Ω, i) is a graph over $B_\eta(0)$. We define η_0 by:

$$\eta_0 = \mathrm{Sup}(E).$$

By the inverse function theorem, E is non-empty, and consequently η_0 is well defined. Moreover, by lemma 3.1, $\eta_0 \in E$. We obtain the following result:

Lemma 3.3

Let $r > \epsilon > 0$ be positive real numbers, and suppose that the closed ball of radius r about p in S is complete with respect to $B_\epsilon(0)$. There exists a function $\mu_{\text{graph}}(\epsilon, r)$ such that one of the following must be true:

(1) $\eta_0 \geqslant \epsilon/2$,

(2) $\text{Sup}\,\{\|f(p)\| \text{ s.t. } p \in B_{\eta_0}(0)\} \geqslant \epsilon/2$, *or*

(3) $\text{Sup}\,\{\|Df(p)\| \text{ s.t. } p \in B_{\eta_0}(0)\} \geqslant \mu_{\text{graph}}(\epsilon, r)$.

Remark: There are three ways for the submanifold Σ to stop being a graph:

(1) The submanifold leaves $B_\epsilon(0)$.

(2) The submanifold leaves $B_r(p)$.

(3) The graph becomes vertical.

The first two conditions in the lemma take into account the first form of degenation, wheras the last condition simultaneously takes into account the other two.

Proof: We define $\mu_{\text{graph}}(\epsilon, r)$ by:

$$\mu_{\text{graph}}(\epsilon, r) = \sqrt{\frac{4r^2}{\epsilon^2} - 1}.$$

We will assume the contrary in order to obtain a contradiction. Let Ω be a neighbourhood of p, $\alpha : B_{\eta_0}(0) \to \Omega$ a diffeomorphism and $f : B_{\eta_0}(0) \to \mathbb{R}^{n-m}$ a mapping such that, for all $q \in B_{\eta_0}(0)$:

$$i \circ \alpha(q) = (q, f(q)).$$

For all $q \in B_{\eta_0}(0)$, we have:

$$\|Df(q)\| \leqslant \mu_{\text{graph}}(\epsilon, r).$$

Since its derivative is bounded, f extends to a continuous function on $\overline{B_{\eta_0}(0)}$.

Let us denote by g_m the Euclidean metric over $\mathbb{R}^m$. Using the bound of Df, we find that there exists $\delta_2 > 0$ such that:

$$(i \circ \alpha)^* g_n \leqslant \left(\frac{4r^2}{\epsilon^2} - \delta_2\right) g_m.$$

It follows that there exists $\delta_3 > 0$ such that for all $q \in B_{\eta_0}(0)$:

$$\alpha(q) \in B_{r-\delta_3}(p).$$

Moreover, if $(q_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B_{\eta_0}(0)$ then $(\alpha(q_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Ω. Since the closed ball of radius r about p in Σ is complete with respect to $B_\epsilon(0)$, it follows that α extends to a continuous function over $\overline{B_{\eta_0}(0)}$.

Now, there exists $\delta_1 > 0$ such that, for all $q \in \partial B_{\eta_0}(0)$:

$$\begin{aligned}\|(i\circ\alpha)(q)\| &= \|q, f(q)\|\\ &\leqslant \|q\| + \|f(q)\|\\ &\leqslant \epsilon - \delta_1.\end{aligned}$$

Consequently, for all $q \in \overline{B_{\eta_0}(0)}$:

$$(q, f(q)) = (i\circ\alpha)(q) \in B_\epsilon(0).$$

In otherwords, for all $q \in \overline{B_{\eta_0}(0)}$, $i(\alpha(q))$ is contained in $B_\epsilon(0)$.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ be the canonical projection. Let p be a point in $\partial B_{\eta_0}(0)$. Since, for all $q \in \Omega$:

$$\begin{aligned}\mathrm{Det}(T_q(\pi\circ i)) &= \mathrm{Det}(\delta_{ij} + \langle\partial_i f, \partial_j f\rangle\alpha^{-1}(q))^{-1/2}\\ &\geqslant (1 + \mu_{\mathrm{graph}}(\epsilon, r)^2)^{-\frac{n}{2}}\end{aligned}$$

It follows that:

$$\mathrm{Det}(T_{\alpha(p)}(\pi\circ i)) \neq 0.$$

Thus, by the inverse function theorem, there exists $\eta_p \in \mathbb{R}^+$, a neighbourhood Ω_p of $\alpha(p)$ in S, a diffeomorphism $\alpha : B_{\eta_p}(p) \to \Omega_p$ and a function $f_p : B_{\eta_p}(p) \to \mathbb{R}^{n-m}$ such that for all q in $B_{\eta_p}(p)$:

$$(i\circ\alpha_p)(q) = (q, f_p(q)).$$

Since $\alpha_p(p) \in \overline{\Omega}$, it follows that $\Omega\cap\Omega_p \neq \emptyset$. Let q be a point in $\Omega\cap\Omega_p$. We have:

$$(\alpha\circ(\pi\circ i))(q) = (\alpha_p\circ(\pi\circ i))(q) = q.$$

Since $(\pi\circ i)(q) \in B_{\eta_0}(0)\cap B_{\eta_p}(p)$ it follows by lemma 3.1 that α coincides with α_p over $B_{\eta_0}(0)\cap B_{\eta_p}(p)$.

By compactness, there exists a finite set of points $p_1, ..., p_k \in \partial_{\eta_0}(p)$ and $\delta \in \mathbb{R}^+$ such that:

$$B_{\eta_0+\delta}(0) \subseteq B_{\eta_0}(0) \cup \left(\bigcup_{i=1}^{k} B_{\eta_{p_i}}(p_i)\right).$$

Suppose that $B_{\eta_{p_i}(p_i)} \cap B_{\eta_{p_j}(p_j)} \neq \emptyset$. Since the straight line joining p_i and p_j, which is contained in $\overline{B_{\eta_0}(0)}$ intersects $B_{\eta_{p_i}}(p_i)\cap B_{\eta_{p_j}}(p_j)$ non trivially, we have:

$$B_{\eta_{p_i}}(p_i)\cap B_{\eta_{p_j}}(p_j)\cap\overline{B_{\eta_0}(0)} \neq \emptyset.$$

The mappings α_{p_i} and α_{p_j} coincide in this set. Thus, since $B_{\eta_{p_i}}(p_i)\cap B_{\eta_{p_j}}(p_j)$ is connected, it follows by lemma 3.1 that α_{p_i} and α_{p_j} coincide over this set.

It thus follows that there exists $\delta \in \mathbb{R}^+$ such that we may extend α and f to $C^{k,\alpha}$ mappings defined on $B_{\eta_0+\delta}(0)$. Moreover, since:

$$(\pi\circ i)\circ\alpha = \mathrm{Id},$$

it follows that α is a diffeomorphism onto its image. Consequently:

$$\eta_0 + \delta \in E.$$

We thus obtain a contradiction and the result now follows. □

3.2 A Bound from Below of the Radius of Definition.

Let Ω be an open subset of $\mathbb{R}^n$. Let g be a Riemannian metric over Ω. Let $M : \Omega \to \text{End}(\mathbb{R}^n)$ be a smooth function taking values in the space of positive definite symmetric matrices such that for all $V_p \in T_p\Omega$:

$$g(V_p, V_p) = \langle V_p, M \cdot V_p \rangle.$$

For every $p \in \Omega$, let Exp_p be the exponential mapping of Ω with respect to g defined in a neighbourhood of $0 \in T_p(\Omega)$. We recall the following facts concerning this application:

Lemma 3.4

For all $p \in \Omega$:

$$\begin{aligned} \|D\text{Exp}_p(0)\| &= \|M^{-1}\|^{1/2}, \\ \|D\text{Exp}_p(0)^{-1}\| &= \|M\|^{1/2}. \end{aligned}$$

There exist functions $\mu^1_{\text{exp}}, \mu^{-1}_{\text{exp}}$ such that:

$$\begin{aligned} \|D^2\text{Exp}_p(0)\| &\leqslant \mu^1_{\text{exp}}(\|M\|, \|M^{-1}\|, \|DM\|), \\ \|D^2(\text{Exp}_p)^{-1}(p)\| &\leqslant \mu^{-1}_{\text{exp}}(\|M\|, \|M^{-1}\|, \|DM\|). \end{aligned}$$

Next, we have the following result concerning the transformation of the second fundamental form:

Lemma 3.5

There exists a continuous function $\mu_{II}^{\text{transform}}$ such that if:

(1) U,V are open sets,

(2) $\varphi : U \to V$ is a diffeomorphism,

(3) $\Sigma = (S, i)$ is an immersed hypersurface and,

(4) II (resp. II') is the (Euclidean) second fundamental form of (S, i) (resp. $(S, \varphi \circ i)$),

then, for all $p \in S$:

$$\begin{aligned} \|II(p)\| &\leqslant \mu_{II}^{\text{transform}}(\|II'(p)\|, \|D\varphi(p)\|, \|D\varphi^{-1}(p)\|, \|D^2\varphi(p)\|), \\ \|II'(p)\| &\leqslant \mu_{II}^{\text{transform}}(\|II(p)\|, \|D\varphi(p)\|, \|D\varphi^{-1}(p)\|, \|D^2\varphi(p)\|). \end{aligned}$$

In particular, if $\|D\varphi\|$, $\|D\varphi^{-1}\|$ and $\|D^2\varphi\|$ are bounded, then a bound on $\|II\|$ yields a bound on $\|II'\|$.

Proof: Since translations and rotations are isometries, we may assume that $\Sigma = (S, i)$ (resp. $\Sigma' = (S, \varphi \circ i)$) is the graph of a function f (resp f') such that $f(0) = df(0) = 0$ (resp. $f'(0) = df'(0) = 0$). In this case, II (resp II') coincides with $D^2 f$ (resp $D^2 f'$) and the result now follows from a direct calculation. □

By combining lemmata 3.4 and 3.5 we now obtain the following result:

Lemma 3.6

There exists a continuous function $\mu_{II}^{\text{compare}}$ such that if Σ is an immersed submanifold in Ω and II^g (resp. II) is the second fundamental form of Σ with respect to g (resp. the Euclidean metric over Ω) then:

$$\|II\| \leqslant \mu_{II}^{\text{compare}}(\|II^g\|, \|M\|, \|M^{-1}\|, \|DM\|).$$

In particular, if $\|M\|$,$\|M^{-1}\|$ and $\|DM\|$ are bounded, then a bound on II^g yields a bound on II.

Proof: Let p be a point in S. By applying $(\text{Exp}_{i(p)})^{-1}$, we may work in an exponential chart about $i(p)$. At the origin in such a chart, the second fundamental form of an immersed submanifold with respect to $(\text{Exp}_{i(p)})^*g$ coincides with the Euclidean second fundamental form. Using lemma 3.5, we may thus bound $\|II\|$ in terms of $\|II^g\|$ and the derivatives of $\text{Exp}_{i(p)}$ at p. The result now follows by lemma 3.4. □

We now obtain:

Lemma 3.7

There exists a function $\mu_{II}(K, B, \epsilon, \|M\|, \|M^{-1}\|, \|DM\|) \leqslant \epsilon/2$ such that if:

(1) $\delta \leqslant \mu_{II}$,

(2) $B_\epsilon(0) \subseteq \Omega$,

and if $f : B_\delta(0) \to \mathbb{R}^{n-m}$ is a function such that:

(1) $f(0), df(0) = 0$, and

(2) the norm of the second fundamental form of the graph of f with respect to M is bounded above by K,

then, for all $q \in B_\delta(0)$:

$$\begin{aligned} \|f(q)\| &\leqslant \epsilon/2, \\ \|Df(q)\| &\leqslant B. \end{aligned}$$

Remark: In otherwords, we start by studying the graph of a function f such that $f(0), df(0) = 0$ (these conditions reflect the fact that, in the sequel, we will be studying immersed submanifolds in terms of graphs over the tangent space at each point). Then, given a bound K on the second fundamental form of the graph, and given a desired bound B on the derivative of f, we find a radius δ, depending only on B and K (and various other variables nonetheless independant of f), over which this bound is satisfied (provided, of course, that $M \circ f$ is defined over this radius, hence $\delta < \epsilon/2$). Finally, for no extra cost, we also obtain a bound for f over this radius which will be useful in the sequel.

Proof: By lemma 3.6, the norm of the second fundamental form of the graph of f with respect to the Euclidean metric if bounded above by $K' = \mu_{II}^{\text{compare}}(K, \|M\|, \|M^{-1}\|, \|DM\|)$. For all i, we denote $\hat{\partial}_i$ and $\hat{\mathsf{N}}_i$ by:

$$\hat{\partial}_i = \begin{pmatrix} \partial_i \\ Df \cdot \partial_i \end{pmatrix}, \qquad \hat{\mathsf{N}}_i = \begin{pmatrix} Df^t \cdot \partial_i \\ -\partial_i \end{pmatrix}.$$

$(\hat{\partial}_1, ..., \hat{\partial}_n)$ is a basis of tangent vectors to the graph of f and $(\hat{\mathsf{N}}_1, ..., \hat{\mathsf{N}}_{n-m})$ is a basis of normal vectors to the graph of f. Let II be the second fundamental form of the graph of f with respect to the Euclidean metric. We have:

$$\begin{aligned} \left|\langle II(\hat{\partial}_i, \hat{\partial}_j), \hat{\mathsf{N}}_k\rangle\right| &= \left|\langle D_{\hat{\partial}_i}\hat{\mathsf{N}}_k, \hat{\partial}_j\rangle\right| \\ &= \left|\partial_i\partial_j f^k\right| \end{aligned}$$

Thus:

$$\begin{aligned} & \left|\partial_i\partial_j f^k\right| && \leqslant K'\|\hat{\partial}_i\|\|\hat{\partial}_j\|\|\hat{\mathsf{N}}_k\| \\ & && \leqslant K'(1+\|Df\|^2)^{3/2} \\ \Rightarrow\ & \|D^2 f\| && \leqslant K'(1+\|Df\|^2)^{3/2} \end{aligned}$$

Solving this differential inequality with the intial conditions $f(0), Df(0) = 0$, we obtain the desired result. □

We now obtain the following result as an immediate corollary:

Lemma 3.8

For $r > \epsilon > 0$, there exist functions $\Delta(K, r, \epsilon, \|M\|, \|M^{-1}\|, \|DM\|) \leqslant \epsilon/2$, and $B(r, \epsilon, \|M\|, \|M^{-1}\|)$ such that if $(\Sigma, p) = (S, i, p)$ is a pointed immersed submanifold of $B_\epsilon(0)$ such that:

(1) the closed ball of radius r about p in Σ is complete with respect to $B_\epsilon(0)$,

(2) $i(p) = 0$, and

(3) if II is the second fundamental form of Σ with respect to g, then:

$$\|II\| \leqslant K,$$

then, for all $\delta \leqslant \Delta$, there exists a unique neighbourhood U of p in S such that (U, i) is a graph over a Euclidean ball of radius δ about the origin, and if f is the graph function of Σ over $B_\delta(0)$, then:

$$\|f\| \leqslant \epsilon/2, \qquad \|Df\| \leqslant B.$$

Proof: The ball of radius $\|M\|^{-1/2}r$ about p in S with respect to the Euclidean metric over $\mathbb{R}^n$ is contained within the ball of radius r about p in S with respect to the metric g. We thus choose:

$$B = \mu_{\text{graph}}(\epsilon, \|M^{-1}\|^{1/2}r).$$

We now define:

$$\Delta = \mu_{II}(K, B, \epsilon, \|M\|, \|M^{-1}\|, \|DM\|).$$

The result now follows from lemmata 3.3 and 3.7. □

3.3 Bounds on the Higher Derivatives of the Graph Function.

In this section, we aim to show how bounds on the higher derivatives of a function may be obtained in terms of bounds on the higher derivatives of the second fundamental form of its graph. The results of this section are essentially trivial, but we include them for completeness and clarity.

Let Ω be an open subset of $\mathbb{R}^n$. For every positive integer k, and for every $r \in \mathbb{R}^+$, we define $B_r^k(0)$ to be the ball of radius r about the origin in $\mathbb{R}^k$. Let m be a positive integer not greater than n. Let $\epsilon \in \mathbb{R}^+$ be a small positive real number and let $f : B_\epsilon^m(0) \to \mathbb{R}^{n-m}$ be a smooth function whose graph is contained in Ω. Let us denote the graph of f by Σ.

Let $\langle \cdot, \cdot \rangle$ be the Euclidean metric over $\mathbb{R}^n$. Let g be a metric over Ω and let $M : \Omega \to \mathrm{Symm}(\mathbb{R}^n)$ be the matrix representing g relative to the Euclidean metric. Thus, for all $V_p \in T\Omega$:

$$g(V_p, V_p) = \langle V_p, M(p) \cdot V_p \rangle.$$

Let $\partial_1, ..., \partial_n$ be the canonical basis of $\mathbb{R}^n$. For all $1 \leqslant i \leqslant m$, we define $\hat{\partial}_1, ..., \hat{\partial}_m$ by:

$$\hat{\partial}_i = (0, ..., 1, ..., 0, \partial_i f^{m+1}, ..., \partial_i f^n)^t.$$

For $m+1 \leqslant i \leqslant n$, we define $\hat{\mathcal{E}}_{m+1}, ..., \hat{\mathcal{E}}_n$ by:

$$\hat{\mathcal{E}}_i = (-\partial_1 f^i, ..., -\partial_m f^i, 0, ..., 1, ..., 0)^t.$$

We define $\hat{\mathsf{N}}_{m+1}, ..., \hat{\mathsf{N}}_n$ by:

$$\hat{\mathsf{N}}_i = M^{-1} \hat{\mathcal{E}}_i.$$

We find that $(\hat{\partial}_1, ..., \hat{\partial}_m)$ defines a moving (non-orthonormal) frame for $T\Sigma$. Similarly, $(\hat{\mathsf{N}}_{m+1}, ..., \hat{\mathsf{N}}_n)$ defines a moving (non-orthonormal) frame for the normal bundle to Σ relative to metric g. We define the matrix B by:

$$B = (\hat{\partial}_1, ..., \hat{\partial}_m, \hat{\mathsf{N}}_{m+1}, ..., \hat{\mathsf{N}}_n).$$

Let ∇ and D be the Levi-Civita covariant derivatives of g and the Euclidean metric respectively. Let II be the second fundamental form of Σ, the graph of f.

In the sequel, for all i, j, k, we denote by $\mathcal{F}(J^i f, J^j M, J^k M^{-1})$ any function depending only on the derivatives of f, M and M^{-1} up to orders i,j and k respectively.

We obtain the following result:

Lemma 3.9

For all i, j, k:

$$g(II(\hat{\partial}_i, \hat{\partial}_j), \hat{\mathsf{N}}_k) = \partial_i \partial_j f^k + \mathcal{F}(J^1 f, J^1 M, J^1 M^{-1}).$$

Proof: For all i, j, k, we have:

$$\begin{aligned} g(II(\hat{\partial}_i, \hat{\partial}_j), \hat{\mathsf{N}}_k) &= \langle \nabla_{\hat{\partial}_i} \hat{\partial}_j, M\hat{N}_k \rangle \\ &= \langle D_{\hat{\partial}_i} \hat{\partial}_j, \hat{\mathcal{E}}_k \rangle + \mathcal{F}(J^1 f, J^1 M, J^1 M^{-1}) \\ &= \partial_i \partial_j f + \mathcal{F}(J^1 f, J^1 M, J^1 M^{-1}). \end{aligned}$$

The result now follows. □

Let π be the orthogonal projection onto $T\Sigma$ with respect to g. We have:

Lemma 3.10

For all $V_p \in T\Sigma$:

$$\pi(V_p) = \sum_{i=1}^{n} \langle (B^{-1})^t \partial_i, V_p \rangle \hat{\partial}_i.$$

Proof: Since $(B\partial_1, ..., B\partial_n)$ is a basis for $T_p\Omega$, there exists $a_1, ..., a_n \in \mathbb{R}$ such that:

$$V_p = \sum_{i=1}^{n} a_i B \partial_i.$$

Moreover, since $(\hat{\partial}_1, ..., \hat{\partial}_m) = (B\partial_1, ..., B\partial_m)$ is a basis for $T\Sigma$ and since $(\hat{\mathsf{N}}_{m+1}, ..., \hat{\mathsf{N}}_n) = (B\partial_{m+1}, ..., B\partial_n)$ is a basis for $T\Sigma^\perp$, we have:

$$\begin{aligned} \pi(V_p) &= \textstyle\sum_{i=1}^m a_i B \partial_i \\ &= \textstyle\sum_{i=1}^m a_i \hat{\partial}_i. \end{aligned}$$

However, for $1 \leqslant i \leqslant m$, we have:

$$\begin{aligned} \langle (B^{-1})^t \partial_i, V_p \rangle &= \langle \partial_i, B^{-1} \textstyle\sum_{j=1}^n a_j B \partial_j \rangle \\ &= \langle \partial_i, \textstyle\sum_{j=1}^n a_j \partial_j \rangle \\ &= a_i. \end{aligned}$$

The result now follows. □

In particular, we obtain:

Corollary 3.11

We have:

$$\pi = \sum_{i=1}^{m} \mathcal{F}(J^1 f, J^0 M, J^0 M^{-1}) \hat{\partial}_i.$$

Proof: By definition of B:

$$B = \mathcal{F}(J^1 f, J^0 M, J^0 M^{-1}).$$

The result now follows by the preceeding lemma. □

Let $\hat{\nabla}$ be the Levi-Civita covariant derivative of Σ with respect to g. We obtain the following generalisation of lemma 3.9:

Lemma 3.12

For all $a_1, ..., a_m, i, j, k$:

$$\begin{aligned} g((\hat{\nabla}^m II)(\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}, ..., \hat{\partial}_{a_m}), \hat{\mathsf{N}}_k) &= \partial_{a_1}...\partial_{a_m}\partial_i\partial_j f^k \\ &\quad + \mathcal{F}(J^{m+1} f, J^{m+1} M, J^{m+1} M^{-1}) \end{aligned}$$

.

Moreover, for all $a_1, ..., a_m, i, j, k$:

$$g((\hat{\nabla}^m II)(\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}, ..., \hat{\partial}_{a_m}), \hat{\partial}_k) = \mathcal{F}(J^{m+1} f, J^{m+1} M, J^{m+1} M^{-1}).$$

Proof: We prove this result by induction. By lemma 3.9, the result holds for $m = 0$. For $m \geqslant 1$, we have:

$$\begin{aligned} g((\hat{\nabla}^m II)(\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}, ..., \hat{\partial}_{a_m}), \hat{\mathsf{N}}_k) &= g(\nabla_{\hat{\partial}_{a_m}}(\hat{\nabla}^{m-1} II)(\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}, ..., \hat{\partial}_{a_{m-1}}), \hat{\mathsf{N}}_k) \\ &\quad - g((\hat{\nabla}^{m-1} II)(\hat{\nabla}_{\hat{\partial}_{a_m}}\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}, ..., \hat{\partial}_{a_{m-1}}), \hat{\mathsf{N}}_k) \\ &\quad - ... \\ &= \partial_{a_m} g((\hat{\nabla}^{m-1} II)(\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}, ..., \hat{\partial}_{a_{m-1}}), \hat{\mathsf{N}}_k) \\ &\quad - g((\hat{\nabla}^{m-1} II)(\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}, ..., \hat{\partial}_{a_{m-1}}), \nabla_{\hat{\partial}_{a_m}}\hat{\mathsf{N}}_k) \\ &\quad - g((\hat{\nabla}^{m-1} II)(\hat{\nabla}_{\hat{\partial}_{a_m}}\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}, ..., \hat{\partial}_{a_{m-1}}), \hat{\mathsf{N}}_k) \\ &\quad - ... \end{aligned}$$

However:

$$\nabla_{\hat{\partial}_{a_m}}\hat{\mathsf{N}}_k = \mathcal{F}(J^2 f, J^1 M, J^1 M^{-1}).$$

Similarly:

$$\begin{aligned} \hat{\nabla}_{\hat{\partial}_{a_m}}\hat{\partial}_i &= \pi(\nabla_{\hat{\partial}_{a_m}}\hat{\partial}_i) \\ &= \pi(\mathcal{F}(J^2 f, J^1 M, J^1 M^{-1})). \end{aligned}$$

Thus, by corollary 3.11:

$$\hat{\nabla}_{\hat{\partial}_{a_m}}\hat{\partial}_i = \sum_{j=1}^{m} \mathcal{F}(J^2 f, J^1 M, J^1 M^{-1})\hat{\partial}_i.$$

Finally:

$$\begin{aligned} \partial_{a_m} g((\hat{\nabla}^{m-1} II)(\hat{\partial}_i, \hat{\partial}_j; \hat{\partial}_{a_1}...\hat{\partial}_{a_{m-1}}), \hat{\mathsf{N}}_k) &= \partial_{a_m}\mathcal{F}(J^m f, J^m M, J^m M^{-1}) \\ &= \mathcal{F}(J^{m+1} f, J^{m+1} M, J^{m+1} M^{-1}). \end{aligned}$$

The first result now follows by the induction hypothesis. The second result follows by a similar reasoning. □

We now obtain the following result:

Lemma 3.13

For every positive integer $m \geqslant 2$, there exists a function B_m such that if $K \in \mathbb{R}^+$ satisfies:

$$\|\mathcal{A}_\Sigma^m\| \leqslant K,$$

then:

$$\|D^m f\|_{C^0(B_\epsilon(0))} \leqslant B_m(K, \|f\|_{C^1(B_\epsilon(0))}, \|M\|_{C^{m-1}(\Omega)}, \|M^{-1}\|_{C^{m-1}(\Omega)}).$$

Proof: Let $\|\cdot\|$ denote the Euclidean norm over $\mathbb{R}^n$. For all i, we have:

$$\|\hat{\partial}_i\| \leqslant 1 + \|Df\|.$$

Thus:

$$g(\hat{\partial}_i, \hat{\partial}_i)^{1/2} \leqslant \|M\|^{1/2}(1 + \||Df\|).$$

Similarly, for all i, we have:

$$g(\hat{\mathsf{N}}_i, \hat{\mathsf{N}}_i)^{1/2} \leqslant \|M\|^{1/2}(1 + \|Df\|).$$

By lemma 3.12, we thus obtain, for all $a_1, ..., a_m$ and for all k:

$$\begin{aligned} \left|\partial_{a_1}...\partial_{a_m} f^k\right| &\leqslant \left|g((\hat{\nabla}^{m-2} II)(\hat{\partial}_{a_1}, ..., \hat{\partial}_{a_m}), \hat{\mathsf{N}}_k)\right| \\ &\quad + \left|\mathcal{F}(J^{m-1}f, J^{m-1}M, J^{m-1}M^{-1})\right| \\ &\leqslant \|M\|^{(m+1)/2}(1 + \|Df\|)^{m+1}\mathcal{A}_\Sigma^m + \left|\mathcal{F}(J^{m-1}f, J^{m-1}M, J^{m-1}M^{-1})\right| \end{aligned}$$

The result now follows by induction. □

◇

4 - A Compactness Result for Immersed Submanifolds.

4.1 Normalised Charts..

We use the results of the preceeding section in order to obtain an Arzela-Ascoli type theorem which is the principal result of this paper. We begin with the following definition:

Definition 4.1

Let $k \in \mathbb{N}$ be a positive integer and let $\rho, \Gamma \in (0, \infty)$ be positive real numbers. Let (M, g) be a Riemannian manifold of type at least C^k. For $p \in M$, a (Γ, k)**-normalised chart of radius** ρ *about p is a C^k coordinate chart (φ, U, V) of M such that:*

(1) $\varphi(p) = 0$,

(2) $B_\rho(0)$, *the closed Euclidean ball of radius* ρ *about* 0 *in* $\mathbb{R}^n$, *is contained in* V, *and*

(3) if A *is the matrix of the metric* $\varphi_* g$ *with respect to the Euclidean metric over* V, *then:*

$$\|A\|_{C^{k-1}(V)}, \|A^{-1}\|_{C^{k-1}(V)} \leqslant \Gamma.$$

For $p \in M$, *we say that* M *is* (Γ, k)**-normalisable over a radius** ρ **about** p *if and only if there exists a* (Γ, k)*-normalised chart of* M *of radius* ρ *about* p. *For* K *a subset of* M, *we say that* M *is* (Γ, k)**-normalisable over a radius** ρ **about** K *if and only if* M *is* (Γ, k)*-normalisable over a radius* ρ *about every point of* K.

We have the following elementary result:

Lemma 4.2

For all $k \geqslant 1$, *there exists a function* C_k *such that if* $(\varphi_i, U_i, V_i)_{i\in\{1,2\}}$ *are* (Γ, k)*-normalised charts, then, for every Euclidean ball* B *contained in* $\varphi_1(U_1 \cap U_2)$, *we have:*

$$\|\varphi_2 \circ \varphi_1^{-1}\|_{C^k(B)} \leqslant C_k(\Gamma).$$

Proof: This follows directly from lemma 2.4. □

For all k, for all r, and for all $x \in \mathbb{R}^k$, we denote by $B^k_r(x)$ the Euclidean ball of radius r about x in $\mathbb{R}^k$. Using the results of the preceeding sections, we show that, for any Riemannian manifold, M, given an immersed submanifold $\Sigma = (S, i)$ of M, we may show that Σ may be described everywhere locally as a graph of a function over a disc in a normalised chart where the radius of the disc may be bounded below and where the C^k norm of the function may be bounded from above. Formally, we have the following technical lemma:

Lemma 4.3

Let $k \geqslant 2, m \leqslant n \in \mathbb{N}$ *be positive integers. Let* $\Gamma, B, \rho > 0$ *be positive real numbers. There exist positive real numbers* $C, r > 0$ *such that if:*

(1) (M, g) *is a complete Riemannian manifold of dimension* n *and of type at least* C^k,

(2) K *is a subset of* M *about which* M *is* (Γ, k)*-normalisable over a radius* ρ, *and*

(3) $\Sigma = (S, i)$ *is a complete immersed submanifold of* M *of dimension* m *and of type* C^k,

and if p *is a point in* S *such that:*

(1) $i(p) \in K$, *and*

(2) $\left|\mathcal{A}^k_\Sigma(q)\right| \leqslant B$ *for all* $q \in B_\rho(p)$,

then, for every (Γ, k)*-normalised chart* (φ, U, V) *of* M *of radius* ρ *about* $i(p)$, *there exists:*

(1) a function $f : B^n_r(0) \to \mathbb{R}^{m-n}$ *of type* C^k *such that* $f(0), df(0) = 0$,

(2) an open set Ω *of* S *about* p, *and*

(3) a diffeomorphism $\alpha : (B^n_r(0), 0) \to (\Omega, p)$,

such that:

(1) (Ω, i) *is the graph of* f *over* $B^m_r(0)$ *in the chart* (φ, U, V) *with graph diffeomorphism* α*. In otherwords, there exists a rotation* A *of* $\mathbb{R}^n$ *such that, for all* $x \in B^n_r(0)$*:*

$$\varphi \circ \alpha(x) = A(x, f(x)),$$

and,

(2) $\|f\|_{C^k(B^m_r(0))} \leqslant C$.

Remark: This result amounts to a "globalisation" of the previous section. To be precise, in the previous section, we obtained a local description of submanifolds of bounded second fundamental form, whereas, in this section, we transform this data into an atlas of controlled charts.

Remark: It is very important to observe that the graph diffeomorphism α also allows us to construct a normalised chart for Σ about p.

Proof: Let (φ, U, V) be a (Γ, k)-normalised chart of M of radius ρ about p. By composing φ with an isometry if necessary, we may suppose that:

$$(T\varphi \circ Ti) \cdot T_pS = \mathbb{R}^m \oplus \{0\}.$$

We choose r such that:

$$r < \Delta(B, \rho', \rho, \Gamma, \Gamma, \Gamma),$$

as in lemma 3.8. Since Σ is complete, the existence of f now follows. Moreover:

$$\|Df\| \leqslant B(\rho', \rho, \Gamma, \Gamma).$$

We now choose C such that:

$$C > B_m(K, B(\rho', \rho, \Gamma, \Gamma), \Gamma, \Gamma).$$

The bound on the derivatives of f now follows from lemma 3.13, and the result now follows. □

We now have the following result:

Lemma 4.4

Let $k \geqslant 1$ *be a positive integer. Let* $(M_n, p_n)_{n\in\mathbb{N}}, (M_\infty, p_\infty)$ *be complete pointed Riemannian manifolds of type at least* C^k*, and suppose that* $(M_n, p_n)_{n\in\mathbb{N}}$ *coverges towards* (M_∞, p_∞) *in the pointed* C^k *Cheeger/Gromov topology.*

For all $R > 0$*, there exists* $D, r > 0$ *such that for all* $n \in \mathbb{N} \cup \{\infty\}$*, the manifold* M_n *is* (D, k)*-normalisable about a radius* r *over* $B_R(p_n)$*.*

Proof: By compactness of the closure of $B_{R+1}(p_\infty)$, there exists $D_1, r_1 > 0$ such that M_∞ is (Δ_1, k)-normalisable about a radius r_1 over $B_{R+1}(p_\infty)$.

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of C^k convergence mappings of $(M_n, p_n)_{n\in\mathbb{N}}$ with respect to (M_∞, p_∞). There exists $N \in \mathbb{N}$ such that for $n \geqslant N$:

(1) the restriction of φ_n to $B_{R+1}(p_\infty)$ is a diffeomorphism onto its image, and

(2) $B_R(p_n) \subseteq \varphi_n(B_{R+1}(p_\infty))$.

Since $(\varphi_n^* g_n)_{n\in\mathbb{N}} \to g_\infty$ in the C^k_{loc} topology, there exists $D_2 \geqslant D_1$ and $r_2 \leqslant r_1$ such that for all $n \geqslant N$, the manifold M_n is (D_2, k)-normalisable about a radius r_2 over $B_R(p_n)$. The result for $n < N$ follows trivially by compactness and the result now follows. □

4.2 The Compactness Result.

We now obtain the following result:

Theorem 1.2

Let $k \geqslant 2, m \leqslant n \in \mathbb{N}$ be positive integers.

Let $(M_n, p_n)_{n\in\mathbb{N}}, (M_\infty, p_\infty)$ be complete pointed Riemannian manifolds of dimension n and of class at least C^k such that $(M_n, p_n)_{n\in\mathbb{N}}$ converges towards (M_∞, p_∞) in the pointed C^k Cheeger/Gromov topology.

For all $n \in \mathbb{N}$, let $\Sigma_n = (S_n, q_n)$ be an m-dimensional pointed immersed submanifold of M of class at least C^k such that $i(q_n) = p_n$.

Suppose that for all $R > 0$, there exists B such that, for all n:

$$\left|\mathcal{A}^k_{\Sigma_n}(q)\right| \leqslant B \qquad \forall q \in B_R(q_n).$$

Then, there exists a pointed complete immersed submanifold $\Sigma_\infty = (S_\infty, q_\infty)$ of M_∞ of type $C^{k-1,1}$ such that $i(q_\infty) = p_\infty$ and that, after extraction of a subsequence, $(\Sigma_n, q_n)_{n\in\mathbb{N}}$ converges towards $(\Sigma_\infty, q_\infty)$ in the pointed weak $C^{k-1,1}$ Cheeger/Gromov topology.

Proof: For all n, let g_n be the metric over M_n and let g_∞ be the metric over M_∞. By lemmata 4.3 and 4.4, for all $R \in \mathbb{R}$, there exists $K, D, D', r, r' \in \mathbb{R}^+$ and, for all n, an atlas $\mathcal{A}_{n,R} = (x_q, U_q, B_r(0))_{q\in B_R(q_n)}$ of (D, k)-normalised charts over a radius r of S_n and an atlas $\mathcal{B}_{n,R} = (y_q, V_q, B_{r'}(0))_{q\in B_R(q_n)}$ of (D', k)-normalised charts over a radius r' of M_n such that:

(1) for every n and for every q in $B_R(q_n)$:

$$x_q(q) = 0,$$

(2) for every n and for every q in $B_R(q_n)$:

$$y_q(i(q)) = 0,$$

(3) for every n and for every q in $B_R(q_n)$, $(y_q \circ i_n \circ x_q^{-1})$ is defined over $B_r(0)$ and:

$$\|y_q \circ i_n \circ x_q^{-1}\|_{C^k(B_r(q))} \leqslant K.$$

Using lemma 4.2, we find that there exists $B \in \mathbb{R}^+$ such that, for every $n \in \mathbb{N}$, $\mathcal{A}_{n,R}$ is a (B, r)-optimal $C^{k-1,1}$-atlas of (S_n, q_n) over a radius R (see definition B.1). Thus, by the compactness theorem of Riemannian geometry (theorem B.3), there exists a complete pointed $C^{k-1,1}$-manifold (S_∞, q_∞) to which $(S_n, q_n)_{n\in\mathbb{N}}$ converges, after extraction of a subsequence, in the weak $C^{k-1,1}$ Cheeger/Gromov topology.

Let ξ be a point in S_∞. Let $(\xi_n)_{n\in\mathbb{N}} \in (S_n)_{n\in\mathbb{N}}$ be a sequence of points converging to ξ. By compactness, we may assume that $i_n(\xi_n)$ converges to a point in M_∞ that we will refer to, slightly abusively, as $i_\infty(\xi_\infty)$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of optimal $C^{k-1,1}$ convergence mappings of $(S_n, q_n)_{n\in\mathbb{N}}$ with respect to (S_∞, q_∞). Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of convergence mappings of $(M_n, p_n)_{n\in\mathbb{N}}$ with respect to (M_∞, p_∞). By definition of $(\varphi_n)_{n\in\mathbb{N}}$, we may suppose that $(x_{\xi_n} \circ \varphi_n \circ x_\xi^{-1})_{n\in\mathbb{N}}$ converges to the identity in the weak $C^{k-1,1}_{\text{loc}}$ topology. Similarly, we may suppose that $(y_{\xi_n} \circ \psi_n \circ y_\xi^{-1})_{n\in\mathbb{N}}$ also converges to the identity in the weak $C^{k-1,1}_{\text{loc}}$ topology. By the classical Arzela-Ascoli theorem, we may suppose that there exists a function $i_\infty : B_r(0) \to B_{r'}(0)$ of type $C^{k-1,1}_{\text{loc}}$ to which $(y_{\xi_n} \circ i_n \circ x_{\xi_n}^{-1})_{n\in\mathbb{N}}$ converges in the weak $C^{k-1,1}_{\text{loc}}$ topology. By composition and inversion, we thus find that $(y_\xi \circ \psi_n^{-1} \circ i_n \circ \varphi_n \circ x_\xi^{-1})_{n\in\mathbb{N}}$ converges to i_∞ in the weak $C^{k-1,1}_{\text{loc}}$ topology. By repeating this operation with ξ taking values in a countable dense subset of S_0, we find that there exists $i_\infty : S_\infty \to M_\infty$ to which $(i_n)_{n\in\mathbb{N}}$ converges, after extraction of a subsequence, in the weak $C^{k-1,1}_{\text{loc}}$ topology.

Let $\hat{g}_\infty$ and g_∞ be the Riemannian metrics over S_∞ and M_∞ respectively. Since $(i_n^* g_n)_{n\in\mathbb{N}}$ converges to $\hat{g}_\infty$ and to $i_\infty^* g_\infty$, we find that i_∞ is an isometric immersion, and the result now follows. □

We obtain as a corollary to this result the following theorem of Corlette:

Theorem 4.5 [Corlette, 1990]

Let M be a compact manifold of dimension n. Let $B, K \in \mathbb{R}$ be two positive real numbers. Let $m \leqslant n \in \mathbb{N}$ be positive integers. For all $\alpha \in (0, 1]$, there exist only finitely many $C^{1,\alpha}$ isotopy classes of compact immersed submanifolds $\Sigma = (S, i)$ of M such that:

$$\begin{array}{ll} \|II_\Sigma\| & \leqslant B, \\ \text{Diam}(\Sigma) & \leqslant K. \end{array}$$

Remark: We observe that if $\|II_\Sigma\| \leqslant B$, then a bound on the volume of Σ is equivalent to a bound on the diameter of Σ. We thus obtain an analogous result for $\text{Vol}(\Sigma) \leqslant K$.

Proof: We suppose the contrary and reason by absurdity. Let $(\Sigma_n, q_n)_{n\in\mathbb{N}} = (S_n, i_n, q_n)$ be a sequence of pointed immersed submanifolds of M no two of which are $C^{1,\alpha}$ isotopy equivalent such that:

$$\begin{array}{ll} \|II_\Sigma\| & \leqslant B, \\ \text{Diam}(\Sigma) & \leqslant K. \end{array}$$

By the compactness of M, after extraction of a subsequence, there exists p_∞ such that $(i_n(q_n))_{n\in\mathbb{N}}$ converges to p_∞. By theorem 1.2, there exists a complete pointed immersed submanifold $(\Sigma_\infty, p_\infty)$ of M of type $C^{1,1}$ such that, after extraction of a subsequence, $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ converges to $(\Sigma_\infty, p_\infty)$ in the weak $C^{1,1}_{\text{loc}}$ Cheeger/Gromov topology. To begin with, we have:

$$\text{Diam}(\Sigma_\infty) \leqslant K.$$

Consequently, Σ_∞ is compact. It follows that there exists $N \in \mathbb{N}$ such that, for all $n \geqslant N$ there exists:

(1) a $C^{1,\alpha}$ diffeomorphism $\varphi_n : S_\infty \to S_n$, and

(2) an immersion $i'_n : S_\infty \to M$,

such that:

(1) for all $n \geqslant N$, $i'_n = i_n \circ \varphi_n$ and

(2) $(i'_n)_{n\in\mathbb{N}} \to i_\infty$ in the weak $C^{1,\alpha}$ topology.

Let ϵ be the radius of convergence of M. We may assume that for all $m, n \geqslant N$:

$$\begin{array}{rlll} & d(i'_n(p), i'_m(p)) & \leqslant \epsilon & \forall p \in S_\infty \\ \Rightarrow & d(i_n \circ (\varphi_n \circ \varphi_m^{-1})(p), i_m(p)) & \leqslant \epsilon & \forall p \in S_m \end{array}$$

Consequently, if we denote by $N\Sigma_m$ the normal bundle over Σ_m in M and by Exp the exponential mapping of M_∞, we find that there exists a section X of type $C^{1,\alpha}$ of $N\Sigma_m$ over S_m such that:

$$i_n \circ (\varphi_n \circ \varphi_m^{-1})(p) = \text{Exp}(X(p)) \qquad \forall p \in S_m.$$

We define $(i_t)_{t\in[0,1]}$ by:

$$i_t(p) = \text{Exp}(tX(p)) \qquad \forall p \in S_m.$$

This defines a $C^{1,\alpha}$ isotopy between i_m and $i_n \circ (\varphi_n \circ \varphi_m^{-1})$ and we thus obtain a contradiction. □

◇

A - Hölder Spaces.

A.1 Building Manifolds out of Classes of Functions.

The family of C^∞ Riemannian manifolds is not sufficiently closed for our purposes. We are thus required to introduce the notion of $C^{k,\alpha}$ Riemannian manifolds, where $k \geqslant 1$ and α is a real number in $(0, 1]$. We begin by reviewing the properties required of a class of functions for one to be able to construct a theory of manifolds out of it. Let $\mathcal{C}$ be a functor which associates to every pair of open sets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ a topological space of functions $\mathcal{C}(U, V)$ sending U into V such that:

(1) $\mathcal{C}$ is **contravariant in** U and **covariant in** V. Thus if $U' \subseteq U$ and if $V \subseteq V'$, then restriction defines a continuous mapping $\mathrm{Rest} : \mathcal{C}(U,V) \to \mathcal{C}(U',V')$.

(2) $\mathcal{C}(U,V)$ is contained in $C^1(U,V)$, the space of continuously differentiable mappings from U to V. Moreover, this inclusion is continous.

(3) $\mathcal{C}$ is **locally defined**. In otherwords, if $\mathcal{F}(U,V)$ denotes the family of functions sending U into V, and if, for any open subset U' of U we denote:

$$\mathcal{F}(U,V)\cap\mathcal{C}(U',V) = \{f \in \mathcal{F}(U,V) \text{ s.t. } f|_{U'} \in \mathcal{C}(U',V)\},$$

then, for any family $(U_a)_{a\in A}$ of open subsets of U, we obtain:

$$U = \underset{a\in A}{\cup} U_a \Rightarrow \mathcal{C}(U,V) = \underset{a\in A}{\cap} \mathcal{F}(U,V)\cap\mathcal{C}(U_a,V).$$

(4) $\mathcal{C}$ is **reflexive**. In otherwords, if $f \in \mathcal{C}(U,V)$ is a diffeomorphism, then $f^{-1} \in \mathcal{C}(V,U)$. Moreover, this operation is **continuous** in the sense that if $(f_n)_{n\in\mathbb{N}}, f_0 \in \mathcal{C}(U,V)$, are diffeomorphisms such that $(f_n)_{n\in\mathbb{N}}$ converges to f_0, then, for every relatively compact subset W of $\mathrm{Im}(f_0)$, there exists $N \in \mathbb{N}$ such that:

$$n \geqslant N \Rightarrow W \subseteq \mathrm{Im}(f_n),$$

and $(f_n^{-1}|_W)_{n\geqslant N}$ converges to $(f_0^{-1})|_W$.

(5) $\mathcal{C}$ is **transitive**. In otherwords, if $f \in \mathcal{C}(U,V)$ and if $g \in \mathcal{C}(V,W)$, then $g\circ f \in \mathcal{C}(U,W)$. Moreover, this operation is continuous. Thus, if $(f_n)_{n\in\mathbb{N}}$ converges to f_0 and if $(g_n)_{n\in\mathbb{N}}$ converges to g_0, then $(g_n \circ f_n)_{n\in\mathbb{N}}$ converges to $g_0 \circ f_0$.

Given such a functor $\mathcal{C}$, we construct a class of manifolds, $\mathcal{M}(\mathcal{C})$, whose transition maps are always in $\mathcal{C}$.

(A) Condition (2) concerning differentiability could be replaced by continuity (one would then replace “diffeomorphism” by “homeomorphism” in condition (4)). A meaningful manifold theory can be constructed with simple continuity, but differentiability is essential if one wishes to introduce such tools as Riemannian metrics or the implicit function theorem.

(B) Since maps in $\mathcal{C}$ are locally defined (condition (3)), we do not need to worry about the topology of intersections of coordinate charts when we aim to show that a transition map is in $\mathcal{C}$.

(C) By reflexivity (condition (4)), it suffices to show that a transition map φ is in $\mathcal{C}(U,V)$ to know that its inverse φ^{-1} is also in $\mathcal{C}(U,V)$. This simplifies the construction of atlases.

(D) By transitivity (condition (5)), in order to show that a given chart is compatible with a given atlas, it suffices to show that it is compatible with a minimal family of charts in that atlas covering that chart. This also simplifies the construction of atlases.

(E) Let M and N be two manifolds in $\mathcal{M}(\mathcal{C})$. Let $\varphi : M \to N$ be a continuous mapping. Let $(U_a,x_a)_{a\in A}$ and $(V_b,y_b)_{b\in B}$ be atlases of M and N respectively such that:

$$\forall a \in A\ \exists b(a) \in B \text{ s.t. } \varphi(U_a) \subseteq V_{b(a)}.$$

By transitivity, if $(y_{b(a)} \circ \varphi \circ x_a^{-1})$ is in $\mathcal{C}$ for all $a \in A$, then, for every pair of charts (U,x) and (V,y) in M and N respectively such that $\varphi(U) \subseteq V$, the mapping $y \circ \varphi \circ x^{-1}$ is also in $\mathcal{C}$. This simplifies the construction of mappings of type $\mathcal{C}$ between manifolds in $\mathcal{M}(\mathcal{C})$. Moreover, transitivity allows us to show that the composition of two such mappings is also in $\mathcal{C}$.

Continuity of transitivity permits us to consider convergence of mappings between manifolds in $\mathcal{M}(\mathcal{C})$. Moreover, composition of mappings between manifolds in $\mathcal{M}(\mathcal{C})$ is continuous. By continuity of reflexivity, if $M, N \in \mathcal{M}(\mathcal{C})$ are of the same dimension, and if $(\varphi_n)_{n\in\mathbb{N}}, \varphi_0 : M \to N$ are homeomorphisms of type $\mathcal{C}$ such that $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ_0, then $(\varphi_n^{-1})_{n\in\mathbb{N}}$ converges to φ_0^{-1}. In otherwords, if U is a relatively compact open subset of $\mathrm{Im}(\varphi_0)$, then there exists $N \in \mathbb{N}$ such that:

$$n \geqslant N \Rightarrow U \in \mathrm{Im}(\varphi_n),$$

and $(\varphi_n^{-1}|_U)_{n\in\mathbb{N}}$ converges to $(\varphi_0^{-1}|_U)$.

This allows us to contruct a basic theory of manifolds. We now impose the following technical conditions on $\mathcal{C}$:

(6) If $\mathcal{A}(U,V)$ denotes the space of all affine mappings from U to V then:

$$\mathcal{A}(U,V) \subseteq \mathcal{C}(U,V).$$

Moreover, this inclusion is continuous.

(7) $\mathcal{C}$ is **closed under Cartesian products**. In otherwords, if $f \in \mathcal{C}(U,V)$ and $g \in \mathcal{C}(U',V')$, then:

$$f \times g \in \mathcal{C}(U \times U', V \times V').$$

Moreover, the inclusion $\mathcal{C}(U,V) \times \mathcal{C}(U',V') \to \mathcal{C}(U \times U', V \times V')$ is continuous.

These conditions allow us to use the implicit function theorem and Cartesian products to construct manifolds.

The following conditions are also useful for technical local constructions:

(8) $\mathcal{C}(U,\mathbb{R})$ is an algebra over $\mathbb{R}$. Moreover, addition and multiplication are continuous operations. This algebra contains automatically the multiplicative identity since $\mathcal{C}(U,\mathbb{R})$ contains all affine mappings.

(9) For all n, $\mathcal{C}(U,\mathbb{R}^n)$ is a vector space over $\mathbb{R}$ and $\mathcal{C}(U,\mathbb{R})$. Moreover the action of $\mathcal{C}(U,\mathbb{R})$ on $\mathcal{C}(U,\mathbb{R}^n)$ is continuous.

We also wish to study families of functions and tensors over manifolds in $\mathcal{M}(\mathcal{C})$. Let $\mathcal{F}$ be a functor which associates to every open subset U of $\mathbb{R}^n$ a **topological algebra** of functions from U to $\mathbb{R}$ such that:

(1) $\mathcal{F}$ is **contravariant**. In otherwords, if $U' \subseteq U$, then $\mathcal{F}(U)$ is contained in $\mathcal{F}(U')$.

(2) $\mathcal{F}$ contains the constant mappings (and thus the multiplicative identity).

(3) $\mathcal{C}$ acts on $\mathcal{F}$ by **pull back**. In otherwords, if $\varphi \in \mathcal{C}(U,V)$ and if $f \in \mathcal{F}(V)$, then $f \circ \varphi \in \mathcal{F}(U)$.

(4) The action of pull back is **continuous**. In otherwords, if $(\varphi_n)_{n\in\mathbb{N}}, \varphi_0 \in \mathcal{C}(U,V)$ are such that $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ_0, and if $(f_n)_{n\in\mathbb{N}}, f_0 \in \mathcal{F}(V)$ are such that $(f_n)_{n\in\mathbb{N}}$ converges to f_0, then $(f_n \circ \varphi_n)_{n\in\mathbb{N}}$ converges to $f_0 \circ \varphi_0$.

These conditions permit us to define, for a given $M \in \mathcal{M}(\mathcal{C})$, the topological algebra $\mathcal{F}(M)$ of real functions of class $\mathcal{F}$ over M. Moreover, if $\varphi : M \to N$ is a mapping of type $\mathcal{C}$, then pull back (composition) defines a continuous mapping from $\mathcal{F}(N)$ to $\mathcal{F}(M)$.

Tensors over manifolds in $\mathcal{M}(\mathcal{C})$ are defined in a similar way.

A.2 Lipschitz and Locally Lipschitz Mappings.

Let Ω be an open subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}^n$ be a function defined over Ω. For $\alpha \in (0,1]$, we define $\|f\|_{\mathrm{Lip}^\alpha(\Omega)}$ by:

$$\|f\|_{\mathrm{Lip}^\alpha(\Omega)} = \mathop{\mathrm{Sup}}_{x\neq y\in\Omega} \frac{\|f(x)-f(y)\|}{\|x-y\|^\alpha}.$$

We say that f is an **α-Lipschitz** mapping over Ω if and only if:

$$\|f\|_{\mathrm{Lip}^\alpha(\Omega)} < \infty.$$

We say that f is a **locally α-Lipschitz** mapping if and only if for every $p \in \Omega$, there exists a neighbourhood V of p in Ω such that f is an α-Lipschitz mapping over V. We define $\mathrm{Lip}^\alpha(\Omega)$ (resp. $\mathrm{Lip}^\alpha_{\mathrm{loc}}(\Omega)$) to be the space of α-Lipschitz (resp. locally α-Lipschitz) mappings over Ω.

Remark: We recall the following properties concerning Lipschitz mappings:

(1) Every α-Lipschitz mapping is continuous. Similarly, every locally α-Lipschitz mapping is continuous.

(2) $\|f\|_{\mathrm{Lip}^\alpha(\Omega)} = 0$ if and only if f is constant.

(3) Let us define $\mathrm{Diam}(\Omega)$ to be the diameter of Ω:

$$\mathrm{Diam}(\Omega) = \mathop{\mathrm{Sup}}_{x,y\in\Omega} \|x-y\|.$$

For all $\beta \in (0,\alpha]$, we have:

$$\|f\|_{\mathrm{Lip}^\beta(\Omega)} \leqslant \|f\|_{\mathrm{Lip}^\alpha(\Omega)}\mathrm{Diam}(\Omega)^{\alpha-\beta}.$$

Consequently, if $\mathrm{Diam}(\Omega) < \infty$, then:

$$\beta \leqslant \alpha \Rightarrow \mathrm{Lip}^\alpha(\Omega) \subseteq \mathrm{Lip}^\beta(\Omega).$$

However, in general:

$$\beta \leqslant \alpha \Rightarrow \mathrm{Lip}^{\alpha}_{\mathrm{loc}}(\Omega) \subseteq \mathrm{Lip}^{\beta}_{\mathrm{loc}}(\Omega).$$

(4) We consider Ω with the Euclidean metric as a Riemannian manifold. Let d be the distance function on Ω generated by this Riemannian metric. We define $\mathrm{Dil}(\Omega)$ by:

$$\mathrm{Dil}(\Omega) = \mathop{\mathrm{Sup}}_{x\neq y\in\Omega} \frac{d(x,y)}{\|x-y\|}.$$

If $f \in C^1(\Omega)$ then:

$$\|f\|_{\mathrm{Lip}^1(\Omega)} \leqslant \|f\|_{C^1(\Omega)}\mathrm{Dil}(\Omega).$$

Thus, if $\mathrm{Dil}(\Omega) < \infty$, then:

$$C^1(\Omega) \subseteq \mathrm{Lip}^1(\Omega).$$

However, in general:

$$C^1(\Omega) \subseteq \mathrm{Lip}^1_{\mathrm{loc}}(\Omega).$$

(5) The two preceeding remarks show that the structure of $\mathrm{Lip}^{\alpha}(\Omega)$ depends on the geometry of Ω wheras the structure of $\mathrm{Lip}^{\alpha}_{\mathrm{loc}}(\Omega)$ does not. In the sequel, we define:

$$\Delta(\Omega) = \mathrm{Max}(\mathrm{Dil}(\Omega), \mathrm{Diam}(\Omega)).$$

We following trivial lemma summarises the operations under which the space of Lipschitz mappings is closed:

Lemma A.1

We have the following composition rules:

(1) If $f, g \in Lip^{\alpha}(\Omega)$ then $f+g \in Lip^{\alpha}(\Omega)$ and:

$$\|f+g\|_{\mathrm{Lip}^{\alpha}(\Omega)} \leqslant \|f\|_{\mathrm{Lip}^{\alpha}(\Omega)} + \|g\|_{\mathrm{Lip}^{\alpha}(\Omega)}.$$

(2) If $f, g \in Lip^{\alpha}(\Omega) \cap C^0(\Omega)$ then $f\cdot g \in Lip^{\alpha}(\Omega)$ and:

$$\|f\cdot g\|_{\mathrm{Lip}^{\alpha}(\Omega)} = \|f\|_{C^0(\Omega)}\|g\|_{\mathrm{Lip}^{\alpha}(\Omega)} + \|f\|_{\mathrm{Lip}^{\alpha}(\Omega)}\|g\|_{C^0(\Omega)}.$$

(3) Let Ω' be another open set. If $f \in Lip^{\alpha}(\Omega')$, if $g \in Lip^{\beta}(\Omega)$ and if $g(\Omega) \subseteq \Omega'$ then $f\circ g \in Lip^{\alpha\beta}(\Omega)$ and:

$$\|f\circ g\|_{\mathrm{Lip}^{\alpha\beta}(\Omega)} \leqslant \|f\|_{\mathrm{Lip}^{\alpha}(\Omega')}\|g\|^{\alpha}_{\mathrm{Lip}^{\beta}(\Omega)}.$$

A.3 Hölder Spaces.

In this section, we introduce $C^{k,\alpha}$ and $C^{k,\alpha}_{\text{loc}}$ mappings for $k \geqslant 1$, and provide a brief review of their properties. In particular, we show how $C^{k,\alpha}_{\text{loc}}$ mappings satisfy all the conditions specified in section A.1.

We recall that $\mathbb{N}$ denotes the set of positive integers, and, in particular, that $0 \notin \mathbb{N}$. For $k \in \mathbb{N}$, for $\alpha \in (0,1]$ and for $f : \Omega \to \mathbb{R}^m$ a C^k function, we define $\|f\|_{C^{k,\alpha}(\Omega)}$, the $\mathrm{C}^{k,\alpha}$ norm of f by:

$$\|f\|_{C^{k,\alpha}(\Omega)} = \sum_{0 \leqslant i \leqslant k} \|D^i f\|_{C^0(\Omega)} + \|D^k f\|_{\mathrm{Lip}^\alpha(\Omega)}.$$

We observe that $\|\cdot\|_{C^{k,\alpha}(\Omega)}$ does indeed define a norm over the space of C^k functions over Ω. We say that a function **is of type $\mathbf{C}^{k,\alpha}$** if and only if it is C^k and its $\mathrm{C}^{k,\alpha}$ norm is finite. Similarly, we say that a function **is of type $\mathbf{C}^{k,\alpha}_{\text{loc}}$** if and only if it is C^k and, for every $p \in \Omega$ there exists a neighbourhood V of p in Ω over which f is of type $\mathrm{C}^{k,\alpha}$. We denote by $\mathrm{C}^{k,\alpha}(\Omega)$ (resp. $\mathrm{C}^{k,\alpha}_{\text{loc}}(\Omega)$) the space of $\mathrm{C}^{k,\alpha}$ (resp. $\mathrm{C}^{k,\alpha}_{\text{loc}}$) functions over Ω. $C^{k,\alpha}_{\text{loc}}$ trivially satisfies condition (1). All functions in $C^{k,\alpha}_{\text{loc}}$ are differentiable, and so $C^{k,\alpha}_{\text{loc}}$ satisfies condition (2). It is locally defined, and thus satisfies condition (3). It contains all affine mappings and is closed under Cartesian products, and thus satisfies conditions (6) and (7). It thus remains to show reflexivity (condition (4)), transitivity (condition (5)), the fact that $C^{k,\alpha}_{\text{loc}}(\Omega,\mathbb{R})$ is a topological algebra (condition (8)) and the fact that $C^{k,\alpha}_{\text{loc}}(\Omega,\mathbb{R}^n)$ is a topological vector space upon which $C^{k,\alpha}_{\text{loc}}(\Omega,\mathbb{R})$ acts continuously (condition (9)). Using lemma A.1, we now obtain the following trivial result:

Lemma A.2

For all $k \in \mathbb{N}$ there exist continuous functions $\nu^{\text{bnd}}_{\text{prod}}$, $\nu^{\text{bnd}}_{\text{comp}}$ and $\nu^{\text{bnd}}_{\text{inv}}$ such that, for all Ω, Ω' and for all $\alpha \in (0,1]$:

(1) If $f, g \in C^{k,\alpha}(\Omega)$ then $f + g \in C^{k,\alpha}(\Omega)$ and:

$$\|f+g\|_{C^{k,\alpha}(\Omega)} \leqslant \|f\|_{C^{k,\alpha}(\Omega)} + \|g\|_{C^{k,\alpha}(\Omega)}.$$

(2) If $f, g \in C^{k,\alpha}(\Omega)$ then $f \cdot g \in C^{k,\alpha}(\Omega)$ and:

$$\|f\cdot g\|_{C^{k,\alpha}(\Omega)} \leqslant \nu^{\text{bnd}}_{\text{prod}}(\|f\|_{C^{k,\alpha}(\Omega)}, \|g\|_{C^{k,\alpha}(\Omega)}, \alpha, \Delta(\Omega)).$$

(3) If $f \in C^{k,\alpha}(\Omega')$, if $g \in C^{k,\alpha}(\Omega)$ and if $g(\Omega) \subseteq \Omega'$, then $f \circ g \in C^{k,\alpha}(\Omega)$ and:

$$\|f\circ g\|_{C^{k,\alpha}(\Omega)} \leqslant \nu^{\text{bnd}}_{\text{comp}}(\|f\|_{C^{k,\alpha}(\Omega)}, \|g\|_{C^{k,\alpha}(\Omega)}, \alpha, \Delta(\Omega)).$$

(4) If $f : \Omega \to \Omega'$ is a $C^{k,\alpha}$ diffeomorphism, then:

$$\|f^{-1}\|_{C^{k,\alpha}(\Omega)} \leqslant \nu^{\text{bnd}}_{\text{inv}}(\|f\|_{C^{k,\alpha}(\Omega)}, \|f^{-1}\|_{C^1(\Omega)}, \alpha, \Delta(\Omega), \Delta(\Omega')).$$

This now yields the following trivial corollary:

Corollary A.3

(1) If $\Delta(\Omega) < \infty$, then $C^{k,\alpha}(\Omega)$ is an algebra over $\mathbb{R}$.

(1') $C^{k,\alpha}_{\text{loc}}(\Omega)$ is an algebra over $\mathbb{R}$.

(2) If $\Delta(\Omega), \Delta(\Omega') < \infty$ and if $\varphi : \Omega \to \Omega'$ is a $C^{k,\alpha}$ mapping, then:

$$\varphi^* C^{k,\alpha}(\Omega') \subseteq C^{k,\alpha}(\Omega).$$

(2')If $\varphi : \Omega \to \Omega'$ is a $C^{k,\alpha}_{\text{loc}}$ mapping, then:

$$\varphi^* C^{k,\alpha}_{\text{loc}}(\Omega') \subseteq C^{k,\alpha}_{\text{loc}}(\Omega).$$

(3) If $\Delta(\Omega), \Delta(\Omega'), \Delta(\Omega'') < \infty$ and if $\varphi : \Omega \to \Omega'$ and $\psi : \Omega' \to \Omega''$ are $C^{k,\alpha}$ mappings, then $\psi \circ \varphi$ is also a $C^{k,\alpha}$ mapping.

(3') If $\varphi : \Omega \to \Omega'$ and $\psi : \Omega' \to \Omega''$ are $C^{k,\alpha}_{\text{loc}}$ mappings, then $\psi \circ \varphi$ is a also a $C^{k,\alpha}_{\text{loc}}$ mapping.

(4) If $\Delta(\Omega), \Delta(\Omega') < \infty$ and if $\varphi : \Omega \to \Omega'$ is a $C^{k,\alpha}$ diffeomorphism, and if $\|\varphi^{-1}\|_{C^1(\Omega)} < \infty$, then φ^{-1} is a $C^{k,\alpha}$ diffeomorphism.

(4') If $\varphi : \Omega \to \Omega'$ is a $C^{k,\alpha}_{\text{loc}}$ diffeomorphism, then φ^{-1} is a $C^{k,\alpha}_{\text{loc}}$ diffeomorphism.

Remark: In particular, we see that $C^{k,\alpha}_{\text{loc}}(\Omega, \mathbb{R})$ is an algebra and that $C^{k,\alpha}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is a vector space upon which $C^{k,\alpha}_{\text{loc}}(\Omega, \mathbb{R})$ acts by multiplication. Moreover, we see that $C^{k,\alpha}_{\text{loc}}$ is reflexive and transitive. It remains to show, however, that $C^{k,\alpha}_{\text{loc}}(\Omega, \mathbb{R})$ is a topological algebra, that $C^{k,\alpha}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is a topological vector space, and that $C^{k,\alpha}_{\text{loc}}(\Omega, \mathbb{R})$ acts continuously on this vector space. We must also show that reflexivity and transitivity are continuous. This will be done in the next section.

Remark: Although we are more interested in studying $C^{k,\alpha}_{\text{loc}}$ functions, when we are dealing with lemmata comparing norms, it is preferable to study the family of $\mathrm{C}^{k,\alpha}$ functions, since, in this case, the results are slightly simpler to state.

A.4 Continuity of Addition, Multiplication, Composition and Inversion.

We now study the topologies of $\mathrm{C}^{k,\alpha}(\Omega)$ and $\mathrm{C}^{k,\alpha}_{\text{loc}}(\Omega)$. Using lemma A.1, we obtain the following result:

Lemma A.4

(1) If $f, f', g, g' \in Lip^\alpha(\Omega)$, then:

$$\|(f+g) - (f'+g')\|_{\mathrm{Lip}^\alpha(\Omega)} \leqslant \|f - f'\|_{\mathrm{Lip}^\alpha(\Omega)} + \|g - g'\|_{\mathrm{Lip}^\alpha(\Omega)}.$$

(2) If $f, f', g, g' \in Lip^\alpha(\Omega) \cap C^0(\Omega)$, then:

$$\begin{aligned}\|f \cdot g - f' \cdot g'\|_{\mathrm{Lip}^\alpha(\Omega)} \quad &\leqslant \|f\|_{C^0(\Omega)}\|g - g'\|_{\mathrm{Lip}^\alpha(\Omega)} + \|f\|_{\mathrm{Lip}^\alpha(\Omega)}\|g - g'\|_{C^0(\Omega)} \\ &\quad + \|f - f'\|_{C^0(\Omega)}\|g'\|_{\mathrm{Lip}^\alpha(\Omega)} + \|f - f'\|_{\mathrm{Lip}^\alpha(\Omega)}\|g'\|_{C^0(\Omega)}.\end{aligned}$$

This yields the following result which shows us that addition and multiplication are continuous in the $\mathrm{C}^{k,\alpha}$ (resp. $\mathrm{C}^{k,\alpha}_{\mathrm{loc}}$) topology over $\mathrm{C}^{k,\alpha}(\Omega)$ (resp. $\mathrm{C}^{k,\alpha}_{\mathrm{loc}}(\Omega)$).

Lemma A.5

(1) If $f, f', g, g' \in C^{k,\alpha}(\Omega)$, then:

$$\|(f+g)-(f'+g')\|_{C^{k,\alpha}(\Omega)} \leqslant \|f-f'\|_{C^{k,\alpha}(\Omega)} + \|g-g'\|_{C^{k,\alpha}(\Omega)}.$$

(2) There exists a continuous function $\nu^{\mathrm{cty}}_{\mathrm{prod}}$ such that:

$$\begin{aligned}\|f\cdot g - f'\cdot g'\|_{C^{k,\alpha}(\Omega)} \quad &\leqslant \nu^{\mathrm{cty}}_{\mathrm{prod}}(\|f-f'\|_{C^{k,\alpha}(\Omega)}, \|g-g'\|_{C^{k,\alpha}(\Omega)},\\ &\qquad \|f\|_{C^{k,\alpha}(\Omega)}, \|f'\|_{C^{k,\alpha}(\Omega)}, \|g\|_{C^{k,\alpha}(\Omega)},\\ &\qquad \|g'\|_{C^{k,\alpha}(\Omega)}, \alpha, \Delta(\Omega)).\end{aligned}$$

Moreover, $\nu^{\mathrm{cty}}_{\mathrm{prod}}(0,0,\cdot,\cdot,\cdot,\cdot,\cdot) = 0$.

In order to define a topology with respect to which composition of functions is continuous, we require the following definition:

Definition A.6

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $(f_n)_{n\in\mathbb{N}}, f$ be functions over Ω of type $C^{k,\alpha}$. We say that $(f_n)_{n\in\mathbb{N}}$ converges to f in the **weak $C^{k,\alpha}$ topology** *if and only if, for all $\beta \in (0,\alpha)$:*

$$(\|f_n - f\|_{C^{k,\beta}(\Omega)})_{n\in\mathbb{N}} \to 0.$$

If $(f_n)_{n\in\mathbb{N}}, f$ are functions of type $C^{k,\alpha}_{\mathrm{loc}}$ over Ω, then we say that $(f_n)_{n\in\mathbb{N}}$ converges to f in the **weak $C^{k,\alpha}_{\mathrm{loc}}$ topology** *if and only if for all $p \in \Omega$ there exists a neighbourhood V of p in Ω such that $(f_n)_{n\in\mathbb{N}}$ converges to f in the weak $C^{k,\alpha}$ topology over V.*

We immediately see that $C^{k,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R})$ is a topological algebra with respect to the weak $C^{k,\alpha}_{\mathrm{loc}}$ topology. Moreover, $C^{k,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ is a topological vector space, and $C^{k,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R})$ acts continuously on this space by multiplication. We now have the following result:

Lemma A.7

(1) If $f \in Lip^{\alpha}(\Omega')$, if $g, g' \in \mathrm{Lip}^1(\Omega)\cap C^0(\Omega)$ and if $g(\Omega), g'(\Omega) \subseteq \Omega'$, then, for all $\beta \in (0,\alpha)$:

$$\|f\circ g - f\circ g'\|_{\mathrm{Lip}^{\beta}(\Omega)} \leqslant 2^{1-\frac{\beta}{\alpha}}\|f\|_{\mathrm{Lip}^{\alpha}(\Omega')}\|g-g'\|^{\alpha-\beta}_{C^0(\Omega)}(\|g\|^{\alpha}_{\mathrm{Lip}^1(\Omega)} + \|g'\|^{\alpha}_{\mathrm{Lip}^1(\Omega)})^{\frac{\beta}{\alpha}}.$$

(2) If $f, f' \in \mathrm{Lip}^{\alpha}(\Omega')\cap C^0(\Omega')$, if $g \in \mathrm{Lip}^1(\Omega)$ and if $g(\Omega) \subseteq \Omega'$, then, for all $\beta \in (0,\alpha)$:

$$\|f\circ g - f'\circ g\|_{\mathrm{Lip}^{\beta}(\Omega)} \leqslant 2^{1-\frac{\beta}{\alpha}}\|f-f'\|^{1-\frac{\beta}{\alpha}}_{C^0(\Omega')}(\|f\|_{\mathrm{Lip}^{\alpha}(\Omega')} + \|f'\|_{\mathrm{Lip}^{\alpha}(\Omega')})^{\frac{\beta}{\alpha}}\|g\|^{\beta}_{\mathrm{Lip}^1(\Omega)}.$$

Remark: The preceeding lemma is a special case of the following more general result:

Lemma A.8

(1) If $f \in \mathrm{Lip}^{\alpha}(\Omega)$, if $g, g' \in \mathrm{Lip}^{\beta}(\Omega') \cap C^0(\Omega)$ and if $g(\Omega), g'(\Omega) \subseteq \Omega'$, then, for all $\lambda \in [0,1]$:

$$\|f \circ g - f \circ g'\|_{\mathrm{Lip}^{\alpha\beta\lambda}(\Omega)} \leqslant 2^{1-\lambda}\|f\|_{\mathrm{Lip}^{\alpha}(\Omega')}\|g - g'\|_{C^0(\Omega)}^{\alpha(1-\lambda)}(\|g\|_{\mathrm{Lip}^{\beta}(\Omega)}^{\alpha} + \|g'\|_{\mathrm{Lip}^{\beta}(\Omega)}^{\alpha})^{\lambda}.$$

(2) If $f \in \mathrm{Lip}^{\alpha}(\Omega') \cap C^0(\Omega')$, if $g \in \mathrm{Lip}^{\beta}(\Omega)$ and if $g(\Omega) \subseteq \Omega'$ then for all $\lambda \in [0,1]$:

$$\|f \circ g - f' \circ g\|_{\mathrm{Lip}^{\alpha\beta\lambda}} = 2^{1-\lambda}\|f - f'\|_{C^0(\Omega')}^{1-\lambda}(\|f\|_{\mathrm{Lip}^{\alpha}(\Omega')} + \|f'\|_{\mathrm{Lip}^{\alpha}(\Omega')})^{\lambda}\|g\|_{\mathrm{Lip}^{\beta}(\Omega)}^{\lambda\alpha}.$$

We now have:

Lemma A.9

(1) There exists a continuous function $\nu_{\mathrm{comp},1}^{\mathrm{cty}}$ such that if $f \in C^{k,\alpha}(\Omega')$, if $g, g' \in C^{k,\alpha}(\Omega)$, if $g(\Omega), g'(\Omega) \subseteq \Omega'$, and if $\beta < \alpha$, then:

$$\begin{aligned}\|f \circ g - f \circ g'\|_{C^{k,\beta}(\Omega)} \quad &\leqslant \nu_{\mathrm{comp},1}^{\mathrm{cty}}(\|g - g'\|_{C^{k,\alpha}(\Omega)}, \|g\|_{C^{k,\alpha}(\Omega)}, \|g'\|_{C^{k,\alpha}(\Omega)},\\ &\qquad \|f\|_{C^{k,\alpha}(\Omega')}, \alpha, \beta, \Delta(\Omega), \Delta(\Omega')).\end{aligned}$$

Moreover $\nu_{\mathrm{comp},1}^{\mathrm{cty}}(0, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) = 0$.

Consequently, if $(g_n)_{n\in\mathbb{N}}$ converges to g_0 in the $C^{k,\alpha}$ (resp. $C_{\mathrm{loc}}^{k,\alpha}$) topology, and if $\Delta(\Omega), \Delta(\Omega') < \infty$, then $(f \circ g_n)_{n\in\mathbb{N}}$ converges to $(f \circ g_0)$ in the $C^{k,\beta}$ (resp. $C_{\mathrm{loc}}^{k,\beta}$) topology.

(2) There exists a continuous function $\nu_{\mathrm{comp},2}^{\mathrm{cty}}$ such that, if $f, f' \in C^{k,\alpha}(\Omega')$, if $g \in C^{k,\alpha}(\Omega)$, if $g(\Omega) \subseteq \Omega'$, and if $\beta < \alpha$, then:

$$\begin{aligned}\|f \circ g - f' \circ g\|_{C^{k,\beta}(\Omega)} \quad &\leqslant \nu_{\mathrm{comp},2}^{\mathrm{cty}}(\|f - f'\|_{C^{k,\alpha}(\Omega')}, \|f\|_{C^{k,\alpha}(\Omega')}, \|f'\|_{C^{k,\alpha}(\Omega')},\\ &\qquad \|g\|_{C^{k,\alpha}(\Omega)}, \alpha, \beta, \Delta(\Omega), \Delta(\Omega')).\end{aligned}$$

Moreover, $\nu_{\mathrm{comp},2}^{\mathrm{cty}}(0, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) = 0$.

Consequently, if $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the $C^{k,\alpha}$ (resp. $C_{\mathrm{loc}}^{k,\alpha}$) topology, and if $\Delta(\Omega), \Delta(\Omega') < \infty$, then $(f_n \circ g)_{n\in\mathbb{N}}$ converges to $(f_0 \circ g)$ in the $C^{k,\beta}$ (resp. $C_{\mathrm{loc}}^{k,\beta}$) topology.

This result shows us that the transitivity of $C_{\mathrm{loc}}^{k,\alpha}$ is continuous. We observe that in order for transitivity to be continuous, we must have $k \geqslant 1$.

Finally, using the chain rule, and the result for composition, we may also obtain the following result concerning inverses:

Lemma A.10

Let $\Omega, \Omega' \subseteq \mathbb{R}^n$ be open sets. Let $(\varphi_n)_{n\in\mathbb{N}}, \varphi_0 : \Omega \to \Omega'$ be $C^{k,\alpha}$ mappings which are diffeomorphisms onto their images. Suppose that $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ_0 in the weak $C_{\mathrm{loc}}^{k,\alpha}$ topology. Then, for every compact subset, K, of $\mathrm{Im}(\varphi_0)$, there exists $N \in \mathbb{N}$ such that:

$$n \geqslant N \Rightarrow K \subseteq \mathrm{Im}(\varphi_n),$$

and $(\varphi_n^{-1})_{n\geqslant N}$ converges to φ_0^{-1} in the weak $C_{\mathrm{loc}}^{k,\alpha}$ topology over K.

It thus follows that reflexivity of $C^{k,\alpha}_{\text{loc}}$ is continuous. Again, in order for this to work, we require that $k \geqslant 1$.

We have thus shown that for $k \geqslant 1$, $C^{k,\alpha}_{\text{loc}}$ functions satisfy all the conditions required for us to be able to construct a theory of $C^{k,\alpha}_{\text{loc}}$ manifolds. Similarly, the same results permit us to construct spaces of $C^{l,\beta}$ functions over such manifolds for $l+\beta \leqslant k+\alpha$ and spaces of $C^{l,\beta}_{\text{loc}}$ tensors over such manifolds for $l+\beta \leqslant (k-1)+\alpha$.

◇

B - Cheeger/Gromov Convergence of Abstract Manifolds.

B.1 Cheeger/Gromov Convergence.

In this appendix, we prove the classical compactness theorem of Riemannian geometry in a form which is most appropriate for our uses.

We begin by making the following definition:

Definition B.1

Let (M,p) be a pointed Riemannian manifold, and let g be the Riemannian metric over M.

*Let $k \in \mathbb{N}$ be a positive integer, and let α be a real number in $(0,1]$. Let $K,\rho,R > 0$ be positive real numbers. A (K,ρ)***-optimal** $C^{k,\alpha}$ **atlas of** (M,p) **over a radius** R *is a family $\mathcal{A} = (x_q,\Omega_q,B_\rho(0))_{q\in B_R(p)}$ of $C^{k,\alpha}$ coordinate charts of M such that:*

(1) for all $q \in B_R(p)$, if A is the matrix representation of $(x_q)_*g$ with respect to the Euclidean metric over $B_\rho(0)$, then:

$$\begin{aligned}\|A\|_{C^{k-1,\alpha}(B_\rho(0))} &\leqslant K,\\ \|A^{-1}\|_{C^{k-1,\alpha}(B_\rho(0))} &\leqslant K,\end{aligned}$$

and,

(2) for all $q,q' \in B_R(p)$, and for every ball B contained in $x_q(\Omega_q\cap\Omega_{q'})$:

$$\|x_{q'}\circ x_q^{-1}\|_{C^{k,\alpha}(B)} \leqslant K.$$

*For all $q \in B_R(p)$, we will refer to $(x_q,\Omega_q,B_\rho(0))$ abusively as a (K,ρ)***-optimal** $C^{k,\alpha}$ **chart of** M **about** q.

*We say that (M,p) is (K,ρ)***-optimisable over a radius** R *if such an atlas exists.*

Let $\mathcal{A}$ be such an atlas of M. Let d be the metric (distance structure) generated over M by g. For all $q \in B_R(p)$, let D_q be the distance function generated over $B_\rho(0)$ by $(x_q)_*g$. For every $x \in B_\rho(0)$, there exists $\rho' < \rho$ which only depends on ρ, $|x|$ and K such that, if we denote $\Omega'_q = x_q^{-1}(B_{\rho'}(x))$, then x_q defines an isometry between (Ω'_q, d) and $(B_{\rho'}(x), D_q)$. For example, if we choose $x = 0$, and if we choose ρ' such that:

$$\rho' \leqslant \frac{\rho}{2K+1},$$

then, we find that, for all $y \in B_{\rho'}(0)$:

$$2D_q(y, 0) < D_q(y, \partial B_\rho(0)).$$

In this case, if $y, y' \in B_\rho(0)$, then the shortest curve in M joining $x_q^{-1}(y)$ to $x_q^{-1}(y')$ remains within Ω_q, and thus:

$$D_q(y, y') = d(x_q^{-1}(y), x_q^{-1}(y')).$$

Definition B.2

We refer to ρ' as the **isometric radius of ρ about $|x|$ with resect to K**.

Let $(M_n, p_n)_{n\in\mathbb{N}}$ be a sequence of complete Riemannian manifolds. For all n, let g_n be the Riemannian metric on M_n and let d_n be the metric (distance structure) generated over M_n by g_n.

Let $k \in \mathbb{N}$ be a positive whole number and let α be a real number in $(0, 1)$. We assume that there exists:

(1) a sequence $(R_n)_{n\in\mathbb{N}}$ of positive real numbers such that $(R_n)_{n\in\mathbb{N}} \uparrow \infty$,

(2) a sequence $(N_n)_{n\in\mathbb{N}}$ of positive integers such that $(N_n)_{n\in\mathbb{N}} \uparrow \infty$,

(3) a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers, and

(4) a sequence $(K_n)_{n\in\mathbb{N}}$ of positive real numbers,

such that, for all n, for all $m \geqslant N_n$, there exists a (K_n, ϵ_n)-optimal $C^{k,\alpha}$ atlas $\mathcal{A}_{m,n}$ of (M_m, p_m) over a radius $R_n + 1$.

We obtain the following result:

Theorem B.3

There exists a complete pointed Riemannian manifold (M_0, p_0) of type $C^{k,\alpha}$ such that $(M_n, p_n)_{n\in\mathbb{N}}$ converges to (M_0, p_0) in the pointed weak $C^{k,\alpha}$ Cheeger/Gromov topology.

Moreover, for all n, there exists a (K_n, ϵ_n)-optimal $C^{k,\alpha}$ atlas $\mathcal{A}_{0,n}$ of (M_0, p_0) over a radius $R_n + 1$.

Finally, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of weak $C^{k,\alpha}$ convergence mappings of $(M_n, p_n)_{n\in\mathbb{N}}$ with respect to (M_0, p_0) such that, for all $m \in \mathbb{N}$, if:

(1) $(q_n)_{n\in\mathbb{N}} \in (M_n)_{n\in\mathbb{N}}$ and $q_0 \in M$ are such that $(q_n)_{n\in\mathbb{N}}$ converges to q_0,

(2) for all $n \in \mathbb{N} \cup \{0\}$, $(x_{q_n}, \Omega_{q_n}, B_{\epsilon_n}(0))$ is a (K_m, ϵ_m)-optimal $C^{k,\alpha}$ chart about q_n, and

(3) ϵ'_m is the isometric radius of ϵ_m with respect to K_m,

then, there exists a (distance preserving) $C^{k,\alpha}$ *mapping* $\alpha : B_{\epsilon'_m}(0) \to B_{\epsilon'_m}(0)$ *such that, after extraction of a subsequence,* $(x_{q_n} \circ \varphi_n \circ x_{q_0}^{-1})_{n\in\mathbb{N}}$ *converges to* α *in the weak* $C^{k,\alpha}$*-topology. In otherwords, for every compact subset* K *of* $B_{\epsilon'_m}(0)$*, there exists* $M \in \mathbb{N}$ *such that:*

(1) for all $n \geqslant M$*,* $(x_{q_n} \circ \varphi_n \circ x_{q_0}^{-1})$ *is defined over* K *and its restriction to* K *is a diffeomorphism onto its image, and*

(2) for all $\beta < \alpha$*:*

$$(\|x_{q_n} \circ \varphi_n \circ x_{q_0}^{-1} - \alpha\|_{C^{k,\beta}(K)})_{n\geqslant M} \to 0.$$

Remark: We will refer to such a sequence of convergence mappings as a sequence of **optimal convergence mappings of** (M_0, p_0) **with respect to** $(M_n, p_n)_{n\in\mathbb{N}}$.

We obtain this result in many steps. We begin by constructing the limit as a metric space:

Proposition B.4

There exists a sequence $(\delta_n)_{n\in\mathbb{N}} \in \mathbb{R}^+$ *and a sequence of functions* $(\mathrm{Cov}_n)_{n\in\mathbb{N}} : (0, \delta_n) \to \mathbb{N}$ *such that, for all* n*, for all* $\delta < \delta_n$*, and for all* $m \geqslant N_n$*, the ball* $\overline{B_{R_n+1}(p_m)}$ *may be covered by* $\mathrm{Cov}_n(\delta)$ *balls of radius* δ*.*

Remark: This result is proved by bounding the volumes of small balls from above and below.

Proof: A proof of this proposition may be found in [4]. □

This yields:

Proposition B.5

There exists a family $(M_{0,n}, d_{0,n}, p_{0,n})_{n\in\mathbb{N}}$ *of pointed compact metric spaces such that, for all* n*:*

$$(\overline{B_{R_n+1}(p_m)})_{m\geqslant N_n} \to (M_{0,n}, d_{0,n}, p_{0,n}),$$

in the Gromov/Haussdorf topology.

Proof: A proof of this proposition may be found in [4]. □

By uniqueness of convergence in the Gromov/Haussdorf topology, for all $n \geqslant n'$, there exists a distance preserving map, sending $(M_{0,n'}, p_{0,n'})$ into the ball of radius $R_{n'+1}$ about $p_{0,n'}$ in $(M_{0,n}, p_{0,n})$. Moreover, this map is unique up to isometries of the domain. Consequently, we may take the union of these limiting spaces to obtain:

Proposition B.6

There exists a pointed locally compact metric space (M_0, d_0, p_0) *such that* $(M_n, d_n, p_n)_{n\in\mathbb{N}}$ *converges to* (M_0, d_0, p_0) *in the pointed Gromov/Haussdorf topology.*

Remark: This space is seperable since it is a union of countably many compact sets.

Remark: We define the set X by:

$$X = M_0 \cup (\bigcup_{n\in\mathbb{N}} M_n).$$

By definition of the Gromov/Haussdorf topology, we may suppose that there exists a complete metric (distance structure) d over X which coincides over M_n with d_n for every $n \in \mathbb{N} \cup \{0\}$. We thus adopt the convention that if $(q_n)_{n\in\mathbb{N}} \in (M_n)_{n\in\mathbb{N}}$ is a sequence and if $q_0 \in M_0$, then the sequence $(q_n)_{n\in\mathbb{N}}$ converges to q_0 if and only if it converges to q_0 in X with respect to the metric d.

Since M_0 is seperable, it contains a countable dense subset which we will denote by Q. For every $q \in Q$, we define k_q by:

$$k_q = \text{Inf}\left\{k \in \mathbb{N} \text{ s.t. } q \in B_{R_k+1}(p_0)\right\}.$$

For all q, let ϵ'_{k_q} be the isometric radius of ϵ_{k_q} with respect to K_{k_q}. Since Q is dense, we have:

$$M_0 = \bigcup_{q\in Q} B_{\epsilon'_{k_q}}(q).$$

For every $q \in Q$, we define $(q_n)_{n\in\mathbb{N}} \in (M_n)_{n\in\mathbb{N}}$ to be a sequence which converges towards q. For all sufficiently large n, we have:

$$q_n \in B_{R_{k_q}+1}(p_n).$$

These definitions will be of use to us in the sequel.

We now furnish the metric space underlying the limiting manifold with a $C^{k,\alpha}$ differential structure and a $C^{k-1,\alpha}$ Riemannian metric:

Proposition B.7

There exists a canonical maximal $C^{k,\alpha}$ atlas $\mathcal{A}$ and a canonical $C^{k-1,\alpha}$ Riemannian metric g_0 over M_0 such that the metric (distance structure) generated by g_0 over M_0 coincides with d_0.

Moreover, for all n, the pointed manifold (M_0, p_0) is (K_n, ϵ_n)-normalisable over a radius R_n.

Proof: Without loss of generality, we may suppose that, for all n:

$$q_n \in B_{R_{k_q}+1}(p_n).$$

For all $n \geqslant N_{k_q}$, let $(x_{q,n}, \Omega_{q,n}, B_{\epsilon_{k_q}}(0))$ be a (K_q, ϵ_{k_q})-optimal $C^{k,\alpha}$ chart of M_n about q_n.

For all n, we define the metric $g_{q,n}$ over $B_{\epsilon_{k_q}}(0)$ by:

$$g_{q,n} = (x_{q,n})_* g_n.$$

By the classical Arzela-Ascoli theorem, after extraction of a subsequence, there exists a $C^{k-1,\alpha}$ metric g_q over $B_{\epsilon_{k_q}}(0)$ such that $(g_{q,n})_{n\in\mathbb{N}}$ converges to g_q in the weak $C^{k-1,\alpha}$ topology. Since the metrics $(g_{q,n})_{n\in\mathbb{N}}$ are uniformly bounded below, it follows that g_q is positive definite.

For all n, let $d_{q,n}$ be the distance structure generated over $B_{\epsilon_{k_q}}(0)$ by $g_{q,n}$. Let d_q be the distance structure generated by g_q.

Since, for all n, $(x_{q,n})^{-1}$ is locally distance preserving, by the classical Arzela-Ascoli theorem, there exists a locally distance preserving mapping $\xi_q : B_{\epsilon'_{k_q}}(0) \rightarrow M_0$ such that $(x_{q,n}^{-1})_{n\in\mathbb{N}}$ converges locally uniformly to ξ_q.

Let x be a point in $B_\epsilon(0)$. Let ρ be the isometric radius of $\epsilon_{k,q}$ about $|x|$ with respect to K. Since the restriction of ξ_q to $B_\rho(x)$ preserves distance, it is a homeomorphism onto its image. Moreover, the image of ξ_q is an open set. Indeed, let y be a point in $B_\rho(x)$. Let δ be such that the closed ball of radius δ about y with respect to d_q is a compact subset of $B_\rho(x)$. We find that the image of the interior of this ball under the action of ξ_q is precisely the open ball of radius δ about $\xi_q(y)$ in M_0, and the openness of ξ_q now follows. Since, for every n, $(x_{q,n})^{-1}$ is a homeomorphism, it follows that ξ_q, being the locally uniform limit of a sequence of homeomorphisms, is also a homeomorphism (see, for example, lemma 2.2.2 of [6]). We thus define $x_q = \xi_q^{-1}$ and $\Omega_q = \xi_q(B_{\epsilon'_{k_q}}(0))$ and we obtain a chart $(x_q, \Omega_q, B_{\epsilon'_{k_q}}(0))$ of M_0 about q.

Let q' be another point in Q. We construct $(x_{q'}, \Omega_{q'}, B_{\epsilon_{k_{q'}}}(0))$ as for q. We suppose that $\Omega_q \cap \Omega_{q'} \neq \emptyset$. Let K be a compact subset of $\Omega_q \cap \Omega_{q'}$. We have $x_q(K) \subseteq B_{\epsilon_{k_q}}(0)$. For all sufficiently large n, we have:

$$x_{q_n}^{-1}(x_q(K)) \subseteq \Omega_{q'_n}.$$

Indeed, let y be a point in $x_q(K)$. Let us define $z \in B_{\epsilon_{q'}}(0)$ by:

$$z = (x_{q'} \circ x_q^{-1})(y).$$

Let ϵ_1 be such that $B_{\epsilon_1}(z) \subseteq B_{\epsilon'_{k_{q'}}}(0)$. Since the mappings $(x_{q'_n})_{n\geqslant N_{q'}}$ are uniformly bilipschitzian, there exists ϵ_2 such that, for all $n \geqslant N_{q'}$:

$$\begin{aligned} B_{\epsilon_2}(x_{q'_n}^{-1}(z)) &\subseteq x_{q'_n}^{-1}(B_{\epsilon_1}(z)) \\ &\subseteq \Omega_{q'_n}. \end{aligned}$$

Since $(x_{q_n}^{-1}(y))_{n\in\mathbb{N}}$ and $(x_{q'_n}^{-1}(z))_{n\in\mathbb{N}}$ both converge towards $x_q^{-1}(y) = x_{q'}^{-1}(z)$, it follows that, for all sufficiently large n:

$$(x_{q_n}^{-1}(y)) \subseteq B_{\epsilon_2/2}(x_{q'_n}^{-1}(z)).$$

Consequently, there exists ϵ_3 such that, for all sufficiently large n:

$$\begin{aligned} x_{q_n}^{-1}(B_{\epsilon_3}(y)) &\subseteq B_{\epsilon_2/2}(x_{q_n}^{-1}(y)) \\ &\subseteq B_{\epsilon_2}(x_{q'_n}^{-1}(z)) \\ &\subseteq \Omega_{q'_n}. \end{aligned}$$

Since y is arbitrary, and since K is compact, it follows that for sufficiently large n:

$$x_{q_n}^{-1}(x_q(K)) \subseteq \Omega_{q'_n},$$

and we obtain the desired result.

Let B be an open ball whose closure is contained in $x_q(\Omega_q \cap \Omega_{q'}) \subseteq B_{\epsilon_q}(0)$. In particular, $x_q^{-1}(\overline{B})$ is compact, and thus, for sufficiently large n, $x_{q'_n} \circ x_{q_n}^{-1}$ is defined over B. By our hypotheses, for sufficiently large n:

$$\|x_{q'_n} \circ x_{q_n}^{-1}\|_{C^{k,\alpha}(B)} \leqslant K.$$

Consequently, by the classical Arzela-Ascoli theorem, there exists $\varphi \in C^{k,\alpha}(B)$ such that, after extraction of a subsequence, $(x_{q'_n} \circ x_{q_n}^{-1})_{n\in\mathbb{N}}$ converges to φ in the weak $C^{k,\alpha}$ topology over B. However, for all sufficiently large n:

$$x_{q'_n}^{-1} \circ (x_{q'_n} \circ x_{q_n}^{-1}) = x_{q_n}^{-1}.$$

Consequently, by taking limits, we obtain:

$$\begin{array}{ll} & \xi_{q'} \circ \alpha \quad = \xi_q \\ \Rightarrow & \alpha \qquad\quad = x_{q'} \circ x_q^{-1}. \end{array}$$

In particular, it follows that $(x_{q'} \circ x_q^{-1})$ is of type $C^{k,\alpha}_{\text{loc}}$.

It thus follows that $(x_q, \Omega_q, B_{\epsilon_{k_q}}(0))_{q\in Q}$ is a $C^{k,\alpha}$ atlas over M_0.

For all $q, q' \in Q$, and for all sufficiently large n:

$$((x'_{q_n})_* g_n)|_{x_{q'_n}(\Omega_{q'_n} \cap \Omega_{q_n})} = (x_{q'_n} \circ x_{q_n}^{-1})_*((x_{q_n})_* g_n)|_{x_{q_n}(\Omega_{q'_n} \cap \Omega_{q_n})}.$$

Thus, taking limits, we obtain:

$$g_{q'}|_{x'_q(\Omega_{q'} \cap \Omega_q)} = (x_{q'} \circ x_q^{-1})_* g_q|_{(x_q(\Omega_{q'} \cap \Omega_q)}.$$

It follows that the family $(g_q)_{q\in Q}$ defines a $C^{k-1,\alpha}$ Riemannian metric g_0 over M_0. Moreover, with these differential and Riemannian structures, for all $n \in \mathbb{N}$, (M_0, p_0) is (K_n, ϵ_n)-normalisable over a radius of R_n.

Let p_0,p_1 be two points in M_0. Let $\epsilon \in \mathbb{R}^+$ be a positive real number. There exist sequences $(p_{0,n})_{n\in\mathbb{N}}, (p_{1,n})_{n\in\mathbb{N}} \in (M_n)_{n\in\mathbb{N}}$ such that $(p_{0,n})_{n\in\mathbb{N}}$ and $(p_{1,n})_{n\in\mathbb{N}}$ converge respectively to p_0 and to p_1. For sufficiently large n, we may suppose that:

$$d_n(p_{0,n}, p_{1,n}) \leqslant d_0(p_0, p_1) + \epsilon/2.$$

For all n, there exists a continuous path $\gamma_n : I \to M_n$ joining $p_{0,n}$ to $p_{1,n}$ such that:

$$\text{Length}(\gamma_n) \leqslant d_n(p_{0,n}, p_{q,n}) + \epsilon/2.$$

Thus, for sufficiently large n:

$$\mathrm{Length}(\gamma_n) \leqslant d_n(p_0, p_1) + \epsilon.$$

We may suppose that every γ_n is parametrised by a constant factor of arc length. The classical Arzela-Ascoli theorem now tells us that there exists $\gamma_0 : I \to M_0$ such that $(\gamma_n)_{n\in\mathbb{N}}$ converges uniformly to γ_0 over I. In particular:

$$\gamma_0(0) = p_0, \qquad \gamma_0(1) = p_1,$$

and:

$$\mathrm{Length}(\gamma_0) \leqslant d_0(p_0, p_1) + \epsilon.$$

Since $\epsilon \in \mathbb{R}^+$ is arbitrary, it follows that d_0 is a length metric.

However, for all q, if ϵ'_{k_q} is the isometric radius of ϵ_{k_q} about 0 with respect to K, then $x_q : (\Omega_q, d_0) \to (B_{\epsilon'_{k_q}}(0), d_q)$ is an isometry of metric spaces. Thus d_0 coincides everywhere locally with the distance structure generated over M_0 by g_0. Since d_0 is a length metric, the result now follows. □

For all $q \in Q$, we construct $(x_q, \Omega_q, B_{\epsilon_{k_q}(0)})$ as in the proof of proposition B.7. We now make the following definition:

Definition B.8

Let U be an open subset of M_0 containing p_0. Let $k \in \mathbb{N}$ be a positive integer and let α be a real number in $(0, 1]$. A sequence of $C^{k,\alpha}$ **strong convergence mappings of** *$(M_n, p_n)_{n\in\mathbb{N}}$* **over** *$U$* **with respect to** *$Q$ is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ such that:*

(1) for all n, $\varphi_n : (U, p_0) \to (M_n, p_0)$ is a $C^{k-1,\alpha}_{\mathrm{loc}}$ diffeomorphism onto its image, and

(2) for all $q, q' \in Q$, the sequence $(x_{q'_n} \circ \varphi_n \circ x_q^{-1})_{n\in\mathbb{N}}$ converges to $(x_{q'} \circ x_q^{-1})$ in the weak $C^{k,\alpha}_{\mathrm{loc}}$ topology. In other words, for every compact set $K \subseteq x_q(\Omega_q \cap \Omega_{q'} \cap U) \subseteq B_{\epsilon'_{k_q}}(0)$:

(a) there exists N such that, for all $n \geqslant N$:

$$\begin{array}{ll} x_q(K) & \subseteq \Omega_q, \\ \varphi_n(x_q(K)) & \subseteq \Omega_{q'_n}. \end{array}$$

and,

(b) $(x_{q'_n} \circ \varphi_n \circ x_q^{-1})_{n\geqslant N}$ converges to $x_{q'} \circ x_q^{-1}$ in the weak $C^{k,\alpha}$ topology over K.

Remark: By taking subsequences, we may suppose that, for all $q \in Q$, $(x_{q_n}^{-1} \circ x_q)$ is a sequence of $C^{k,\alpha}$ strong convergence mappings of $(M_n, p_n)_{n\in\mathbb{N}}$ over U with respect to Q.

We now prove theorem B.3 by induction. The induction step is guaranteed by the following result:

Proposition B.9

Let U be an open subset of M containing p_0. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of strong convergence mappings of (M_n, p_n) over U with respect to Q_0. For every $q \in Q$ and for every relatively compact open subset $V \subseteq U \cup \Omega_q$, after extraction of a subsequence of $(M_n, p_n)_{n\in\mathbb{N}}$, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of strong convergence mappings of $(M_n, p_n)_{n\in\mathbb{N}}$ over V with respect to Q.

Proof: Let (ϕ_U, ϕ_q) be a $C^{k,\alpha}$-partition of unity of $U \cup \Omega_q$ subordinate to the cover (U, Ω_q). Define $K \subseteq U \cap \Omega_q$ by:

$$K = \overline{V} \cap \mathrm{Supp}(\phi_q) \cap \mathrm{Supp}(\phi_U).$$

Since $(\varphi_n)_{n \geqslant N}$ is a sequence of strong convergence mappings, there exists M_1 such that for $n \geqslant M_1$:

$$\varphi_n(K) \subseteq \Omega_{q_n}.$$

For $n \geqslant M_1$, we define $\psi_n : V \to M_n$ by:

$$\psi_n(p) = \begin{cases} \varphi_n(p) & \text{if } p \in \mathrm{Supp}(\phi_q)^C \cap V, \\ (x_{q_n}^{-1} \circ x_q)(p) & \text{if } p \in \mathrm{Supp}(\phi_U)^C \cap V, \\ x_{q_n}^{-1}(\phi_U(p)(x_{q_n} \circ \varphi_n)(p) + \phi_q(p) x_q(p)) & \text{otherwise (i.e., if } p \in K). \end{cases}$$

For all n, since φ_n is of type $C^{k,\alpha}_{\mathrm{loc}}$, so is ψ_n.

Let p be a point in $\mathrm{Supp}(\phi_q)^C \cup \mathrm{Supp}(\phi_U)^C$. Let Ω be a neighbourhood of p in $\mathrm{Supp}(\phi_q)^C \cup \mathrm{Supp}(\phi_U)^C$. Since, for all n, ψ_n coincides over Ω either with φ_n or with $(x_{q_n}^{-1} \circ x_q)$, it follows trivially that $(\psi_n)_{n \in \mathbb{N}}$ is a sequence of strong convergence mappings of $(M_n, p_n)_{n \in \mathbb{N}}$ over Ω with respect to Q. It thus remains to study what happens near points in K.

Let p be a point in K. Let Ω be a relatively compact neighbourhood of p in Ω_q. There exists $N \geqslant N_{k_q}$ such that, for all $n \geqslant N$:

$$\Omega \subseteq \Omega_q \cap \varphi_n^{-1}(\Omega_{q_n}).$$

For $n \geqslant N$, and for $z \in x_q(\Omega)$, we have:

$$(x_{q_n} \circ \psi_n \circ x_q^{-1})(z) = (\phi_U \circ x_q^{-1})(z)(x_{q_n} \circ \varphi_n \circ x_q^{-1})(z) + (\phi_q \circ x_q^{-1})(z) z.$$

Since $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence of $C^{k,\alpha}$ strong convergence mappings over U with respect to Q, it follows that, since $(x_{q_n} \circ \varphi_n \circ x_q^{-1})(z)_{n \in \mathbb{N}}$ converges to $x_q \circ x_q^{-1} = \mathrm{Id}$ in the weak $C^{k,\alpha}_{\mathrm{loc}}$ topology over $x_q(\Omega)$. Consequently $(x_{q_n} \circ \psi_n \circ x_q^{-1})(z)_{n \in \mathbb{N}}$ also converges to the identity in the weak $C^{k,\alpha}_{\mathrm{loc}}$ topology over $x_q(\Omega)$.

Let q',q'' be two other points in Q. Let us suppose that $p \in \Omega_q \cap \Omega_{q'} \cap \Omega_{q''}$ and let $\hat{\Omega}$ be a relatively compact neighbourhood of p in $\Omega_q \cap \Omega_{q'} \cap \Omega_{q''}$.

For sufficiently large n, $x_{q_n''} \circ \psi_n \circ x_{q'}^{-1}$ is defined over $\hat{\Omega}$, and, for all $z \in \hat{\Omega}$:

$$(x_{q_n''} \circ \psi_n \circ x_{q'}^{-1})(z) = (x_{q_n''} \circ x_{q_n}^{-1}) \circ (x_{q_n} \circ \psi_n \circ x_q^{-1}) \circ (x_q \circ x_{q'}^{-1}).$$

It thus follows that $(x_{q_n''} \circ \psi_n \circ x_{q'}^{-1})_{n \in \mathbb{N}}$ converges to $(x_{q''} \circ x_{q'}^{-1})$ over $\hat{\Omega}$ in the weak $C^{k,\alpha}_{\mathrm{loc}}$ topology.

It thus follows that for all $q', q'' \in Q$, $(x_{q_n''} \circ \psi_n \circ x_{q'}^{-1})_{n \in \mathbb{N}}$ converges to $(x_{q''} \circ x_{q'}^{-1})$ in the weak $C^{k,\alpha}$ topology over $x_{q'}(\Omega_{q'} \cap \Omega_{q''} \cap V)$. It thus follows that, if we can show that the, for sufficiently large n, the restriction of ψ_n to V is a diffeomorphism onto its image, then $(\psi_n)_{n\in\mathbb{N}}$ will be a sequence of strong convergence mappings of (M_n, p_n) over V with respect to Q.

For all $q', q'' \in Q$, for sufficiently large n, $x''_{q_n} \circ \psi_n \circ x_{q'}^{-1}$ is everywhere a local diffeomorphism. Consequently, for all sufficiently large n, ψ_n is everywhere a local diffeomorphism. It now remains to show that, for all sufficiently large n, ψ_n is injective.

Suppose that there exists $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in V$ such that, for all $n \in \mathbb{N}$:

$$x_n \neq y_n, \qquad \psi_n(x_n) = \psi_n(y_n).$$

By compactness, we may assume that there exists $x_0, y_0 \in \overline{V}$ such that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge respectively to x_0 and y_0. Since, for all $q', q'' \in Q$, the sequence $(x_{q''_n} \circ \psi_n \circ x_{q'}^{-1})_{n\in\mathbb{N}}$ converges to $x_{q''} \circ x_{q'}^{-1}$, we find that $x_0 = y_0$. Since this sequence converges in the weak $C^{k,\alpha}_{\text{loc}}$ topology and since $k \geqslant 1$, it follows that $x_n = y_n$ for all sufficiently large n (see, for example, lemma 2.2.3 of [6]). This is absurd, and it follows that there exists N such that for $n \geqslant N$, the application $\psi_n : V \to M_n$ is a diffeomorphism onto its image. The result now follows. □

We may now prove theorem B.3:

Proof of theorem B.3: By proposition B.9, for every finite family $Q' \subseteq Q$, and for every relatively compact subset V of $\cup_{q\in Q'} \Omega_q$, we may find a positive integer N and a sequence of $C^{k,\alpha}$ strong convergence mappings $(\varphi_n)_{n\geqslant N}$ over V. Using a diagonal argument (à la Cantor), we may thus find:

(1) a sequence of real numbers $(\rho_n)_{n\in\mathbb{N}}$ such that $(\rho_n)_{n\in\mathbb{N}} \uparrow \infty$, and

(2) for all $n \in \mathbb{N}$, a local diffeomorphism $\varphi_n : (B_{\rho_n}(p_0), p_0) \to (M_n, p_0)$,

such that, for all $N \in \mathbb{N}$, the sequence $(\varphi_n)_{n\geqslant N}$ is a sequence of strong convergence mappings over $B_{\rho_N}(p_0)$.

Consequently, for all $N \in \mathbb{N}$, and for all $n \geqslant N$, the restriction of φ_n to $B_{\rho_n}(p_0)$ is a diffeomorphism onto its image. Moreover, since $(\varphi_n)_{n\geqslant N}$ is a sequence of strong convergence mappings over $B_{\rho_N}(p_0)$, we find that $(\varphi_n^* g_n)_{n\geqslant N}$ converges to g_0 in the weak $C^{k,\alpha}_{\text{loc}}$ topology over $B_{\rho_N}(p_0)$ and the first result now follows.

The second result follows directly from proposition B.7 and the third result is an immediate consequence of the fact that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of strong convergence mappings. □

◇

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