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by

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Abstract

The equivalence problem under local unitary transformation for n–partite pure states is reduced to the one for $(n-1)$–partite mixed states. In particular, a tripartite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, where $\mathcal{H}_j$ is a finite dimensional complex Hilbert space for $j = A, B, C$, is considered and a set of invariants under local transformations is introduced, which is complete for the set of states whose partial trace with respect to $\mathcal{H}_A$ belongs to the class of generic mixed states.

Keywords: tripartite quantum states, local unitary transformation, entanglement, invariants

Introduction

The importance of a measure to quantify entanglement became evident in the years by the number of applications exploiting nonlocality properties which have been developed: we mention, among others, quantum computation (see, e.g., [1, 2]), quantum teleportation (see, e.g., [3–10]), superdense coding (see, e.g., [11]), quantum cryptography (see, e.g., [12–14]).

Many proposals have been made for a measure of entanglement in the bipartite case, see e.g., [15–22]. Less results are known instead for the tripartite and in general for the n–partite case [20, 23–25], although such systems are important for example in quantum multipartite teleportation or telecloning processings.

One of the properties employed in the bipartite case is the Schmidt-decomposition [26]. However this decomposition is a peculiarity of bipartite systems and does not exist for n–partite ones, a sign of the complexity of the many-partite problem. Generalizations of the Schmidt-decomposition have been proposed [27–30], but the results are not sufficient to provide good measures of entanglement in the n–partite case. In the following, we first

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reduce the n–partite problem to a (n-1)–partite one. To illustrate this, we consider the case of a tripartite system. Then we define invariants under local unitary transformations which form a complete set at least for tripartite states for which a solution of the bipartite problem for entanglement measures is known.

Tripartite states as bipartite ones

Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_C$ be complex Hilbert spaces of finite dimension N_A, N_B, and N_C, respectively, and let $\{|j\rangle_k\}_{j=1}^{N_k}$, $k = A, B, C$, be an orthonormal basis of $\mathcal{H}_k$. A pure state $|\psi\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ can then be written as

$$|\psi\rangle = \sum_{j=1}^{N_A} \sum_{k=1}^{N_B} \sum_{l=1}^{N_C} a_{jkl} |j\rangle_A \otimes |k\rangle_B \otimes |l\rangle_C \,, \qquad \sum_{j=1}^{N_A} \sum_{k=1}^{N_B} \sum_{l=1}^{N_C} a_{jkl} a^*_{jkl} = 1 \,.$$

We denote by $\mathrm{U}(\mathcal{H})$ the group of all unitary operators on the space $\mathcal{H}$.

First of all, we can consider tripartite states as special cases of bipartite ones, by decomposing the system into two subsystems, for example A–BC. The following lemma holds.

Lemma 1 *Let $|\psi\rangle$, $|\psi'\rangle$ be two pure states in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and define $\rho = \mathrm{Tr}_A\,(|\psi\rangle\langle\psi|)$, $\rho' = \mathrm{Tr}_A\,(|\psi'\rangle\langle\psi'|)$, where Tr_A denotes the partial trace with respect to $\mathcal{H}_A$.*

a) The function $I^A_\alpha(|\psi\rangle) = \mathrm{Tr}\,\rho^\alpha$ is invariant under local unitary transformations, for any $\alpha \in \mathbb{N}$;

b) If $I^A_\alpha(|\psi'\rangle) = I^A_\alpha(|\psi\rangle)$ for $\alpha = 1, \ldots, \min\{N_A, N_B \cdot N_C\}$, there exist $U_A \in \mathrm{U}(\mathcal{H}_A)$, $U_{BC} \in \mathrm{U}(\mathcal{H}_B \otimes \mathcal{H}_C)$ such that $|\psi'\rangle = U_A \otimes U_{BC} |\psi\rangle$. In particular, $\rho' = U_{BC}\rho U^\dagger_{BC}$.

Proof. As already shown in [25], *a)* is easily proved as $\mathrm{Tr}_A(|\psi\rangle\langle\psi|) = A^T_A A^*_A$, where A_A is the matrix obtained considering $|\psi\rangle$ as a bipartite state in the A–BC system, with the row (resp. column) indices from the subsystem A (resp. BC). The indices ^{T} resp. * denote transpose resp. complex conjugation. As an example,

$$A_A = \begin{pmatrix} a_{111} & a_{112} & a_{121} & a_{122} \\ a_{211} & a_{212} & a_{221} & a_{222} \end{pmatrix}$$

is the matrix A_A for the case $N_A = N_B = N_C = 2$. Indeed, if $|\psi'\rangle = U_A \otimes U_B \otimes U_C |\psi\rangle$, with $U_i \in \mathrm{U}(\mathcal{H}_i)$, $i = A, B, C$, then A'_A and A_A are related by

$$A'_A = U_A A_A (U_B \otimes U_C)^T$$

and

$$\begin{aligned} I^A_\alpha(|\psi'\rangle) &= \mathrm{Tr}(A'^T_A A'^*_A)^\alpha = \mathrm{Tr}((U_A A_A (U_B \otimes U_C)^T)^T (U_A A_A (U_B \otimes U_C)^T)^*)^\alpha \\ &= \mathrm{Tr}(U_B \otimes U_C (A^T_A A^*_A)^\alpha (U_B \otimes U_C)^\dagger) = \mathrm{Tr}(A^T_A A^*_A)^\alpha \\ &= I^A_\alpha(|\psi\rangle) \end{aligned}$$

for any power $\alpha \in \mathbb{N}$. The decomposition $|\psi'\rangle = U_A \otimes U_{BC}|\psi\rangle$ follows directly considering $|\psi\rangle$ as a bipartite state of the system A–BC and applying the results of [21]. □

Remark 1 The statement can be generalized to n–partite systems: the equivalence problem for n–partite pure states is reduced in this way to the equivalence problem for $(n-1)$-partite mixed states.

Reduction to bipartite mixed states

Lemma 1 allows us to reduce the tripartite problem on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ to a bipartite problem on $\mathcal{H}_B \otimes \mathcal{H}_C$.

Lemma 2 *Let $|\psi'\rangle = U_A \otimes U_{BC}|\psi\rangle$, with $U_A \in \mathrm{U}(\mathcal{H}_A)$, $U_{BC} \in \mathrm{U}(\mathcal{H}_B \otimes \mathcal{H}_C)$ and define $\rho = \mathrm{Tr}_A\left(|\psi\rangle\langle\psi|\right)$, $\rho' = \mathrm{Tr}_A\left(|\psi'\rangle\langle\psi'|\right)$. If*

$$\rho' = U_B \otimes U_C \rho U_B^\dagger \otimes U_C^\dagger\,,$$

where $U_B \in \mathrm{U}(\mathcal{H}_B)$ and $U_C \in \mathrm{U}(\mathcal{H}_C)$, then there exist matrices $V_A \in \mathrm{U}(\mathcal{H}_A)$, $V_B \in \mathrm{U}(\mathcal{H}_B)$, $V_C \in \mathrm{U}(\mathcal{H}_C)$ such that

$$|\psi'\rangle = V_A \otimes V_B \otimes V_C|\psi\rangle\,,$$

i.e., $|\psi\rangle$ and $|\psi'\rangle$ are equivalent under local unitary transformations.

Proof. On one hand we have

$$\begin{aligned} U_{BC}\,\mathrm{Tr}_A\left(|\psi\rangle\langle\psi|\right)^\alpha U_{BC}^\dagger &= \mathrm{Tr}_A\left(\mathbb{1}\otimes U_{BC}|\psi\rangle\langle\psi|(\mathbb{1}\otimes U_{BC})^\dagger\right)^\alpha \\ &= \mathrm{Tr}_A\left(U_A\otimes U_{BC}|\psi\rangle\langle\psi|(U_A\otimes U_{BC})^\dagger\right)^\alpha\,, \end{aligned}$$

on the other hand

$$U_B\otimes U_C\,\mathrm{Tr}_A\left(|\psi\rangle\langle\psi|\right)^\alpha U_B^\dagger\otimes U_C^\dagger = \mathrm{Tr}_A\left(U_A\otimes U_B\otimes U_C|\psi\rangle\langle\psi|(U_A\otimes U_B\otimes U_C)^\dagger\right)^\alpha\,.$$

Since this holds for any power $\alpha\in\mathbb{N}$, there exist a local unitary transformation W_A on $\mathcal{H}_A$ such that

$$\begin{aligned} &U_A\otimes U_{BC}|\psi\rangle\langle\psi|(U_A\otimes U_{BC})^\dagger \\ &\quad= (W_A\otimes\mathbb{1}\otimes\mathbb{1})U_A\otimes U_B\otimes U_C|\psi\rangle\langle\psi|(U_A\otimes U_B\otimes U_C)^\dagger(W_A\otimes\mathbb{1}\otimes\mathbb{1})^\dagger \\ &\quad= W_AU_A\otimes U_B\otimes U_C|\psi\rangle\langle\psi|(W_AU_A\otimes U_B\otimes U_C)^\dagger\,. \end{aligned}$$

Hence

$$|\psi'\rangle = U_A\otimes U_{BC}|\psi\rangle = \widetilde{U}_A\otimes U_B\otimes U_C|\psi\rangle\,,$$

where $\widetilde{U}_A$ is equal W_AU_A up to a phase factor. □

Lemma 1 and Lemma 2 together give rise to the following proposition.

Proposition 1 *For pure states $|\psi\rangle$ and $|\psi'\rangle$, $\rho = \mathrm{Tr}_A\left(|\psi\rangle\langle\psi|\right)$ and $\rho' = \mathrm{Tr}_A\left(|\psi'\rangle\langle\psi'|\right)$, we have that $I_\alpha^A(|\psi'\rangle) = I_\alpha^A(|\psi\rangle)$ for $\alpha = 1,\ldots,\min\{N_A, N_B\cdot N_C\}$ and $\rho' = U_B\otimes U_C\rho U_B^\dagger\otimes U_C^\dagger$ for some $U_B\in\mathrm{U}(\mathcal{H}_B)$, $U_C\in\mathrm{U}(\mathcal{H}_C)$, if and only if $|\psi\rangle$ and $|\psi'\rangle$ are equivalent under local unitary transformations.*

Remark 2 A result corresponding to Lemma 1, Lemma 2, and Proposition 1 holds when tripartite is replaced by n–partite, for any $n\geqslant 3$, by splitting the system $A_1A_2\ldots A_n$ into, e.g., A_1–$A_2\ldots A_n$.

New invariants

The next step is to find further invariants under local unitary transformations which give the same value for two states if and ony if ρ' can be written as $U_B \otimes U_C \rho U_B^\dagger \otimes U_C^\dagger$ for some unitary transformations $U_B \in \mathrm{U}(\mathcal{H}_B)$, $U_C \in \mathrm{U}(\mathcal{H}_C)$, the main obstacle being the fact that in general ρ is a bipartite mixed state and there is no general characterization of entanglement for that case.

The generalization of $I_\alpha^A(|\psi\rangle)$ to bipartite mixed states is $\mathrm{Tr}(\mathrm{Tr}_j(\rho))^\alpha$, where $j = B, C$. For a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ this means to consider the functions

$$\mathrm{Tr}(\mathrm{Tr}_j\,(\mathrm{Tr}_A\,|\psi\rangle\langle\psi|))^\alpha\,.$$

Therefore we introduce the following set of new invariants

$$I_{\alpha,\beta}^{j,k}(|\psi\rangle) = \mathrm{Tr}(\mathrm{Tr}_k\,(\mathrm{Tr}_j\,|\psi\rangle\langle\psi|)^\alpha)^\beta\,, \tag{1}$$

where $j, k \in \{A, B, C\}$, $j \neq k$, and $\alpha, \beta \in \mathbb{N}$.

Lemma 3 *The functions $I_{\alpha,\beta}^{j,k}(|\psi\rangle)$ defined in (1) are invariant under local unitary transformations $U_A \otimes U_B \otimes U_C$.*

Proof. As a model we consider $I_{\alpha,\beta}^{A,B}(|\psi\rangle)$. The other cases can be treated in an analogous manner. We have

$$\mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \sum_{j=1}^{N_A}\sum_{k,p=1}^{N_B}\sum_{l,q=1}^{N_C} a_{jkl}a_{jpq}^*|kl\rangle\langle pq|\,, \tag{2}$$

where $|kl\rangle$ stands for $|k\rangle_B \otimes |l\rangle_C$. Multiplying (2) α times ($\alpha \in \mathbb{N}$) and calculating the partial trace on $\mathcal{H}_B$ of the matrix obtained we get

$$\begin{aligned}&\mathrm{Tr}_B\,(\mathrm{Tr}_A\,|\psi\rangle\langle\psi|)^\alpha \\ &= \sum_{\substack{j_1=1,\\ j_2=1,\\ \ldots,\\ j_\alpha=1}}^{N_A}\sum_{\substack{p_1=1,\\ p_2=1,\\ \ldots,\\ p_\alpha=1}}^{N_B}\sum_{\substack{q_1=1,\\ q_2=1,\\ \ldots,\\ q_\alpha=1,\\ m_1=1}}^{N_C} a_{j_1p_1q_1}^* a_{j_2p_1q_1} a_{j_2p_2q_2}^* a_{j_3p_2q_2}\ldots a_{j_\alpha p_{\alpha-1}q_{\alpha-1}} a_{j_\alpha p_\alpha q_\alpha}^* a_{j_1p_\alpha m_1}|m_1\rangle\langle q_\alpha|\end{aligned}$$

and hence

$$\begin{aligned}&\mathrm{Tr}(\mathrm{Tr}_B\,(\mathrm{Tr}_A\,|\psi\rangle\langle\psi|)^\alpha)^\beta \\ &= \prod_{k=1}^{\beta}\Big(\sum_{\substack{j_{k_1}=1,\\ j_{k_2}=1,\\ \ldots,\\ j_{k_\alpha}=1}}^{N_A}\sum_{\substack{p_{k_1}=1,\\ p_{k_2}=1,\\ \ldots,\\ p_{k_\alpha}=1}}^{N_B}\sum_{\substack{q_{k_1}=1,\\ q_{k_2}=1,\\ \ldots,\\ q_{k_\alpha}=1}}^{N_C} a_{j_{k_1}p_{k_1}q_{k_1}}^* a_{j_{k_2}p_{k_1}q_{k_1}}\ldots a_{j_{k_\alpha}p_{k_{\alpha-1}}q_{k_{\alpha-1}}} a_{j_{k_\alpha}p_{k_\alpha}q_{k_\alpha}}^* a_{j_{k_1}p_{k_\alpha}q_{k-1_\alpha}}\Big)\,,\end{aligned}$$

where $q_{0_\alpha} \equiv q_{\beta_\alpha}$. Instead of the one employed in the proof of Lemma 1, an alternative way to consider the factors a_{jkl} is by writing them in matrices $(A^{(j)})_{kl}$: the index j sets the considered matrix and k, l describe the row and column of $A^{(j)}$, respectively. That is, we write $|\psi\rangle = \sum_{j=1}^{N_A} \sum_{k=1}^{N_B} \sum_{l=1}^{N_C} A_{kl}^{(j)} |jkl\rangle$. Using this notation, one obtains

$$\begin{aligned}
&\mathrm{Tr}(\mathrm{Tr}_B\,(\mathrm{Tr}_A\,|\psi\rangle\langle\psi|)^\alpha)^\beta \\
&\quad = \sum_{\substack{j_{1_1},\dots,j_{1_\alpha}=1\\ j_{2_1},\dots,j_{2_\alpha}=1\\ \dots\\ j_{\beta_1},\dots,j_{\beta_\alpha}=1}}^{N_A} \Big(\prod_{k=1}^{\beta} \mathrm{Tr}(A^{(j_{k_1})\dagger} A^{(j_{k_2})})\,\mathrm{Tr}(A^{(j_{k_2})\dagger} A^{(j_{k_3})}) \dots \mathrm{Tr}(A^{(j_{k_{\alpha-1}})\dagger} A^{(j_{k_\alpha})})\Big) \\
&\qquad\qquad \cdot \mathrm{Tr}(A^{(j_{\beta_\alpha})\dagger} A^{(j_{\beta_1})} A^{(j_{\beta-1_\alpha})\dagger} A^{(j_{\beta-1_1})} \dots A^{(j_{1_\alpha})\dagger} A^{(j_{1_1})})\,.
\end{aligned}$$

For a local unitary transformations $U \otimes V \otimes W$ we have

$$|\psi'\rangle \;:= U \otimes V \otimes W |\psi\rangle = \sum_{j=1}^{N_A} \sum_{k=1}^{N_B} \sum_{l=1}^{N_C} A'^{(j)}_{kl} |jkl\rangle$$

$$U \otimes V \otimes W|\psi\rangle \;= \sum_{j,m=1}^{N_A} \sum_{k,p=1}^{N_B} \sum_{l,q=1}^{N_C} A_{kl}^{(j)} U_{mj} V_{pk} W_{ql} |mpq\rangle = \sum_{j,m=1}^{N_A} \sum_{k=1}^{N_B} \sum_{l=1}^{N_C} U_{jm} (V A^{(m)} W^T)_{kl} |jkl\rangle,$$

i.e., $A'^{(j)}_{kl} = \sum_{m=1}^{N_A} U_{jm} (V A^{(m)} W^T)_{kl}$ and

$$\mathrm{Tr}(A'^{(p_{q_r})\dagger} A'^{(p_{q_{r+1}})}) = \sum_{m_1,m_2=1}^{N_A} U^\dagger_{m_1 p_{q_r}} U_{p_{q_{r+1}} m_2}\, \mathrm{Tr}(A^{(m_1)\dagger} A^{(m_2)}).$$

Therefore

$$\begin{aligned}
&\Big(\prod_{k=1}^{\beta} \mathrm{Tr}(A'^{(j_{k_1})\dagger} A'^{(j_{k_2})})\,\mathrm{Tr}(A'^{(j_{k_2})\dagger} A'^{(j_{k_3})}) \dots \mathrm{Tr}(A'^{(j_{k_{\alpha-1}})\dagger} A'^{(j_{k_\alpha})})\Big) \cdot \\
&\cdot \mathrm{Tr}(A'^{(j_{\beta_\alpha})\dagger} A'^{(j_{\beta_1})} A'^{(j_{\beta-1_\alpha})\dagger} A'^{(j_{\beta-1_1})} \dots A'^{(j_{1_\alpha})\dagger} A'^{(j_{1_1})}) \\
&= \sum_{\substack{m_{1_1},\dots,m_{1_\alpha}=1\\ m_{2_1},\dots,m_{2_\alpha}=1\\ \dots\\ m_{\beta_1},\dots,m_{\beta_\alpha}=1\\ p_{1_\alpha},\dots,p_{\beta_\alpha}=1}}^{N_A} \sum_{\substack{n_{1_1},\dots,n_{1_\alpha}=1\\ n_{2_1},\dots,n_{2_\alpha}=1\\ \dots\\ n_{\beta_1},\dots,n_{\beta_\alpha}=1\\ q_{1_1},\dots,q_{\beta_1}=1}}^{N_A} \Big(\prod_{k=1}^{\beta} U^\dagger_{m_{k_1} j_{k_1}} U_{j_{k_2} n_{k_2}} U^\dagger_{m_{k_2} j_{k_2}} U_{j_{k_3} n_{k_3}} \dots U_{j_{k_\alpha} n_{k_\alpha}} \\
&\qquad\qquad \cdot \mathrm{Tr}(A^{(m_{k_1})\dagger} A^{(n_{k_2})})\,\mathrm{Tr}(A^{(m_{k_2})\dagger} A^{(n_{k_3})}) \dots \mathrm{Tr}(A^{(m_{k_{\alpha-1}})\dagger} A^{(n_{k_\alpha})})\Big) \\
&\qquad\qquad \cdot U^\dagger_{p_{\beta_\alpha} j_{\beta_\alpha}} U_{j_{\beta_1} q_{\beta_1}} U^\dagger_{p_{\beta-1_\alpha} j_{\beta-1_\alpha}} U_{j_{\beta-1_1} q_{\beta-1_1}} \dots U_{j_{1_1} q_{1_1}} \\
&\qquad\qquad \cdot \mathrm{Tr}(A^{(p_{\beta_\alpha})\dagger} A^{(q_{\beta_1})} A^{(p_{\beta-1_\alpha})\dagger} A^{(q_{\beta-1_1})} \dots A^{(p_{1_\alpha})\dagger} A^{(q_{1_1})})\,.
\end{aligned}$$

The result follows, since U is unitary and hence $\sum_k U^\dagger_{jk} U_{kl} = \delta_{jl}$. □

Remark 3 The invariants $I^{j,k}_{\alpha,\beta}(|\psi\rangle)$ can easily be generalized to n–partite systems: the functions

$$I^{j_1,j_2,\dots,j_n}_{\alpha_1,\alpha_2,\dots,\alpha_n}(|\psi\rangle) = \mathrm{Tr}\left(\mathrm{Tr}_{j_1}\left(\mathrm{Tr}_{j_2}\left(\dots(\mathrm{Tr}_{j_n}|\psi\rangle\langle\psi|)^{\alpha_n}\dots\right)^{\alpha_3}\right)^{\alpha_2}\right)^{\alpha_1}\;, \quad \alpha_i \in \mathbb{N}\,, \quad i = 1,\dots,n\,,$$

are invariant under local unitary transformations $U_1 \otimes U_2 \otimes \dots \otimes U_n$.

Unfortunately, the invariants (1) seem to be sufficient only in the case in which the λ_j of the decomposition $\rho = \sum_{j=1}^{n} \lambda_j |\varphi_j\rangle\langle\varphi_j|$, where $n \leqslant N_B \cdot N_C$ and $\varphi_j \in \mathcal{H}_B \otimes \mathcal{H}_C$ for all j, are not degenerated, i.e, $\lambda_j \neq \lambda_k$ for $j \neq k$. Indeed, the following lemma holds.

Lemma 4 *Let $|\psi\rangle$ and $|\psi'\rangle$ be two tripartite pure states such that $I^{j,k}_{\alpha,\beta}(|\psi\rangle) = I^{j,k}_{\alpha,\beta}(|\psi'\rangle)$ for $j,k \in \{A,B,C\}$ and $j \neq k$, $\alpha = 1,\ldots,N_q \cdot N_r$, and $\beta = 1,\ldots,N_r$, where $q,r \in \{A,B,C\}$ and r is different from j, k and q. Then,*

a) there exist $U_p \in \mathrm{U}(\mathcal{H}_p)$ and $U_{q,r} \in \mathrm{U}(\mathcal{H}_q \otimes \mathcal{H}_r)$, with p, q, r different from each other, such that $|\psi'\rangle = U_p \otimes U_{q,r}|\psi\rangle$;

b) for any $|\varphi_m\rangle$ of the decomposition $\mathrm{Tr}_p\left(|\psi\rangle\langle\psi|\right) = \sum_{m=1}^{n} \lambda^{(p)}_m |\varphi^{(p)}_m\rangle\langle\varphi^{(p)}_m|$ for which $\lambda^{(p)}_m$ is not degenerate we have

$$U_{q,r}|\varphi^{(p)}_m\rangle = v^m_q \otimes u_r|\varphi^{(p)}_m\rangle = u_q \otimes v^m_r|\varphi^{(p)}_m\rangle\,,$$

where $v^m_q, u_q \in \mathrm{U}(\mathcal{H}_q)$ and $u_r, v^m_r \in \mathrm{U}(\mathcal{H}_r)$.

Proof. Part a) was already proved in Lemma 1, since

$$I^p_\alpha(|\psi\rangle) = I^{p,k}_{\alpha,1}(|\psi\rangle) = I^{p,k}_{\alpha,1}(|\psi'\rangle) = I^p_\alpha(|\psi'\rangle)\,.$$

Further we know that $\mathrm{Tr}_p\left(|\psi'\rangle\langle\psi'|\right) = U_{q,r}\,\mathrm{Tr}_p\left(|\psi\rangle\langle\psi|\right) U^\dagger_{q,r}$. Since $I^{i,k}_{\alpha,\beta}(|\psi'\rangle) = I^{i,k}_{\alpha,\beta}(|\psi\rangle)$ for $\beta = 1,\ldots,N_r$, with r different from i and k, there exists a $u_r \in \mathrm{U}(\mathcal{H}_r)$ such that

$$\mathrm{Tr}_k\left(\mathrm{Tr}_i\,|\psi'\rangle\langle\psi'|\right)^\alpha = u_r\,\mathrm{Tr}_k\left(\mathrm{Tr}_i\,|\psi\rangle\langle\psi|\right)^\alpha u^\dagger_r\,.$$

Therefore, since this result holds for all $\alpha = 1,\ldots,N_q \cdot N_r$, where i, q, r are different from each other, and the $\lambda^{(p)}_m$ are not degenerated, for any m there exists a $u^{(m)}_q \in \mathrm{U}(\mathcal{H}_q)$ such that

$$\begin{aligned} U_{q,r}|\varphi^{(p)}_m\rangle\langle\varphi^{(p)}_m|U^\dagger_{q,r} &= (u^{(m)}_q \otimes \mathbb{1})(\mathbb{1} \otimes u_r)|\varphi^{(p)}_m\rangle\langle\varphi^{(p)}_m|(\mathbb{1} \otimes u_r)^\dagger(u^{(m)}_q \otimes \mathbb{1})^\dagger \\ &= (u^{(m)}_q \otimes u_r)|\varphi^{(p)}_m\rangle\langle\varphi^{(p)}_m|(u^{(m)}_q \otimes u_r)^\dagger\,. \end{aligned}$$

The statement follows, as $|\langle\varphi^{(p)}_m|(u^{(m)}_q \otimes u_r)^\dagger U_{q,r}|\varphi^{(p)}_m\rangle| = 1$. □

Remark 4 1. Lemma 4 b) is only sufficient if ρ is a pure state.

2. For n-partite pure states the condition $I^{j,k}_{\alpha,\beta}(|\psi\rangle) = I^{j,k}_{\alpha,\beta}(|\psi'\rangle)$ for $j,k \in \{A_1,A_2,\ldots,A_n\}$ implies $|\psi'\rangle = U_{p_1} \otimes U_{p_2,p_3,\ldots,p_n}|\psi\rangle$ for some $U_{p_1} \in \mathrm{U}(\mathcal{H}_{p_1})$, $U_{p_2,p_3,\ldots,p_n} \in \mathrm{U}(\mathcal{H}_{p_2} \otimes \cdots \otimes \mathcal{H}_{p_n})$ and
$$U_{p_2,\ldots,p_n}|\varphi^{(p_1)}_j\rangle = v^j_{p_2} \otimes u_{p_3,\ldots,p_n}|\varphi^{(p_1)}_j\rangle$$
for any $|\varphi^{(p_1)}_j\rangle$ of the decomposition $\mathrm{Tr}_{p_1}\left(|\psi\rangle\langle\psi|\right) = \sum_{j=1}^{n_1} \lambda^{(p_1)}_j |\varphi^{(p_1)}_j\rangle\langle\varphi^{(p_1)}_j|$ such that $\lambda^{(p_1)}_j$ is not degenerated. Further
$$u^j_{p_3,\ldots,p_n}|\varphi^{(p_2)}_{j,k}\rangle = v^{j,k}_{p_3} \otimes u^j_{p_4,\ldots,p_n}|\varphi^{(p_2)}_{j,k}\rangle$$
for $\mathrm{Tr}_{p_2}\left(|\varphi^{(p_1)}_j\rangle\langle\varphi^{(p_1)}_j|\right) = \sum_{k=1}^{n_2} \lambda^{(p_2)}_{j,k} |\varphi^{(p_2)}_{j,k}\rangle\langle\varphi^{(p_2)}_{j,k}|$, if $\lambda^{(p_2)}_{j,k}$ and $\lambda^{(p_1)}_j$ are not degenerated, and so on. Note that only the invariants $I^{j,k}_{\alpha,\beta}$ were considered, and not $I^{j_1,j_2,\ldots,j_n}_{\alpha_1,\alpha_2,\ldots,\alpha_n}$.

A special case for tripartite states

Complete sets of invariants for the case of bipartite mixed states are known only for some special cases. For example, in [21] a complete set was presented for the case in which the state $\rho = \sum_{m=1}^{n} \lambda_m |\varphi_m\rangle\langle\varphi_m|$ is a generic mixed state. To define this set, we need further invariants:

$$\Theta(\rho)_{jk} = \mathrm{Tr}\left(\mathrm{Tr}_B(|\varphi_j\rangle\langle\varphi_j|)^* \,\mathrm{Tr}_B(|\varphi_k\rangle\langle\varphi_k|)^*\right) , \quad \Omega(\rho)_{jk} = \mathrm{Tr}\left(\mathrm{Tr}_C(|\varphi_j\rangle\langle\varphi_j|)\,\mathrm{Tr}_C(|\varphi_k\rangle\langle\varphi_k|)\right) .$$

Assume without loss of generality that $N_B \leqslant N_C$ and complete $\Theta(\rho)$ and $\Omega(\rho)$ to $(N_B^2 \times N_B^2)$-matrices by defining $\Theta(\rho)_{jk} = \Omega(\rho)_{jk} = 0$ for $n < j, k \leqslant N_B^2$. A bipartite mixed state is called generic if the $(N_B^2 \times N_B^2)$-matrices $\Theta(\rho)$ and $\Omega(\rho)$ are non-degenerate.

If ρ is a generic mixed state and $U\rho U^\dagger$, with U unitary, gives the same values as ρ for the invariants $J^j_\alpha(\rho) = \mathrm{Tr}(\mathrm{Tr}_j(\rho^\alpha))$, where $j \in \{B, C\}$, $\Theta(\rho)$, $\Omega(\rho)$, and

$$\begin{aligned} Y(\rho)_{jkl} &= \mathrm{Tr}\left(\mathrm{Tr}_B(|\varphi_j\rangle\langle\varphi_j|)^* \,\mathrm{Tr}_B(|\varphi_k\rangle\langle\varphi_k|)^* (\mathrm{Tr}_B(|\varphi_l\rangle\langle\varphi_l|)^*\right) , \\ X(\rho)_{jkl} &= \mathrm{Tr}\left(\mathrm{Tr}_C(|\varphi_j\rangle\langle\varphi_j|)\,\mathrm{Tr}_C(|\varphi_k\rangle\langle\varphi_k|)(\mathrm{Tr}_C(|\varphi_l\rangle\langle\varphi_l|)\right) , \end{aligned}$$

where $j, k, l = 1, \ldots, n$, then ρ and $U\rho U^\dagger$ are equivalent under local unitary transformations [21]. That is, if $\mathrm{Tr}_A(|\psi\rangle\langle\psi|)$ is a generic mixed state and the above invariants give the same results for $\mathrm{Tr}_A(|\psi\rangle\langle\psi|)$ and $\mathrm{Tr}_A(|\psi'\rangle\langle\psi'|)$, as well as $I^A_\alpha(|\psi\rangle) = I^A_\alpha(|\psi'\rangle)$ for $\alpha = 1, \ldots, \min\{N_A, N_B^2\}$, $|\psi\rangle$ and $|\psi'\rangle$ are equivalent under local unitary transformations. The number of invariants one needs to calculate can be diminished if one considers (1) and takes into account Lemma 4.

Proposition 2 *Let $|\psi\rangle$ and $|\psi'\rangle$ be two pure states of $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and assume that $\rho = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$ is a generic mixed state. $|\psi\rangle$ is equivalent to $|\psi'\rangle$ under local unitary transformations if and only if*

$$I^{A,s}_{\alpha,\beta}(|\psi\rangle) = I^{A,s}_{\alpha,\beta}(|\psi'\rangle) \tag{3}$$

for $s \in \{B, C\}$, $\alpha = 1, \ldots, \min\{N_B^2, N_C^2\}$, $\beta = 1, \ldots, N_r$, where $r \in \{B, C\}$ but is different from s, and for $\rho' = \mathrm{Tr}_A(|\psi'\rangle\langle\psi'|)$

$$\Theta(\rho)_{jk} = \Theta(\rho')_{jk} , \quad \Omega(\rho)_{jk} = \Omega(\rho')_{jk} , \quad Y(\rho)_{jkl} = Y(\rho')_{jkl} , \quad X(\rho)_{jkl} = X(\rho')_{jkl} \tag{4}$$

for the j, k such that $\lambda_j = \lambda_k$.

Proof. As remarked above, the invariants (4) are sufficient to establish whether two states for which the partial trace on $\mathcal{H}_A$ is a generic mixed state are equivalent or not. It remains to prove that (4) is fulfilled when λ_j, λ_k, and λ_l are non-degenerate, if (3) holds. This follows from Lemma 4 Indeed, for example

$$\begin{aligned} \mathrm{Tr}_C(|\varphi'_j\rangle\langle\varphi'_j|) &= \mathrm{Tr}_C(U_{BC}|\varphi_j\rangle\langle\varphi_j|U^\dagger_{BC}) = \mathrm{Tr}_C(u_B \otimes v^j_C|\varphi_j\rangle\langle\varphi_j|(u_B \otimes v^j_C)^\dagger) \\ &= u_B\,\mathrm{Tr}_C(\mathbb{1} \otimes v^j_C|\varphi_j\rangle\langle\varphi_j|(\mathbb{1} \otimes v^j_C)^\dagger)u^\dagger_B = u_B\,\mathrm{Tr}_C(|\varphi_j\rangle\langle\varphi_j|)u^\dagger_B , \end{aligned}$$

hence

$$\begin{aligned} \Omega(\rho')_{jk} &= \mathrm{Tr}\left(\mathrm{Tr}_C(|\varphi'_j\rangle\langle\varphi'_j|)\,\mathrm{Tr}_C(|\varphi'_k\rangle\langle\varphi'_k|)\right) = \mathrm{Tr}(u_B\,\mathrm{Tr}_C(|\varphi_j\rangle\langle\varphi_j|)u^\dagger_B u_B\,\mathrm{Tr}_C(|\varphi_k\rangle\langle\varphi_k|)u^\dagger_B) \\ &= \mathrm{Tr}\left(\mathrm{Tr}_C(|\varphi_j\rangle\langle\varphi_j|)\,\mathrm{Tr}_C(|\varphi_k\rangle\langle\varphi_k|)\right) = \Omega(\rho)_{jk} . \end{aligned}$$

The same holds for $\Theta(\rho)$, $Y(\rho)$, and $X(\rho)$. $\square$

Remark 5 We know that the rank of ρ is smaller than $\min\{N_A, N_B \cdot N_C\}$ (see, e.g., [31]). On the other hand, the assumption that ρ is a generic mixed state implies that ρ has full rank, i.e., $N_B \cdot N_C$. Therefore, in order to fulfill the conditions of Proposition 2, we need $N_A \geqslant N_B \cdot N_C$.

In this last section we have seen that a criterion for equivalence of a class of bipartite mixed states gives rise to a criterion of equivalence for a class of pure tripartite states. In [32], the complete invariants for another two classes of bipartite mixed states are given. For bipartite mixed states on $\mathbb{C}^m \times \mathbb{C}^n$,

$$\rho = \sum_{l=0}^{N} \mu_l |\xi_l\rangle\langle\xi_l|,$$

where the rank of ρ is $N+1$ $(N \geq 1)$, μ_l are eigenvalues with corresponding eigenvectors $|\xi_l\rangle = \sum_{ij} \xi_{ij}^{(l)} |ij\rangle$. Let $A_l := (\xi_{ij}^{(l)})$, $\rho_l := A_l A_l^*$, and $\theta_l := A_l^* A_l$, for $l = 0, 1, ..., N$. If each eigenvalue of ρ_0 and θ_0 has multiplicity one (i.e., is "multiplicity free"), then ρ belongs to the class of density matrices to which a complete set of invariants can be explicitly given. For rank two mixed states on $\mathbb{C}^m \times \mathbb{C}^n$ such that each of the matrices ρ_0, ρ_1, θ_0, and θ_1 has at most two different eigenvalues, an operational criterion can be also found. From these criteria for bipartite mixed states, by using Lemma 4 we can similarly obtain criteria for some classes of pure tripartite states.

Conclusion

We have reduced the equivalence problem for n–partite pure states to the one for (n-1)–partite mixed states and in the special case $n = 3$ we have constructed a set of invariants under local unitary transformations which is complete for the states with partial trace on $\mathcal{H}_A$ which is a generic mixed state.

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