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The Dirichlet Hopf algebra of arithmetics

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Abstract

Many constructs in mathematical physics entail notational complexities, deriving from the manipulation of various types of index sets which often can be reduced to labelling by various multisets of integers. In this work, we develop systematically the “Dirichlet Hopf algebra of arithmetics” by dualizing the addition and multiplication maps. Then we study the additive and multiplicative antipodal convolutions which **fail** to give rise to Hopf algebra structures, but form only a weaker Hopf geбра obeying a weakened homomorphism axiom. A careful identification of the algebraic structures involved is done featuring subtraction, division and derivations derived from coproducts and chochains using branching operators. The consequences of the weakened structure of a Hopf geбра on cohomology are explored, showing this has major impact on number theory. This features multiplicativity versus complete multiplicativity of number theoretic arithmetic functions. The deficiency of not being a Hopf algebra is then cured by introducing an ‘unrenormalized’ coproduct and an ‘unrenormalized’ pairing. It is then argued that exactly the failure of the homomorphism property (complete multiplicativity) for non-coprime integers is a blueprint for the problems in quantum field theory (QFT) leading to the need for renormalization. Renormalization turns out to be the morphism from the algebraically sound **Hopf algebra** to the physical and number theoretically meaningful **Hopf geбра** (literally: antipodal convolution). This can be modelled alternatively by employing Rota-Baxter operators. We stress the need for a characteristic-free development where possible, to have a sound starting point for generalizations of the algebraic structures. The last section provides three key applications: symmetric function theory, quantum (matrix) mechanics, and the combinatorics of renormalization in QFT which can be discerned as functorially inherited from the development at the number-theoretic level as outlined here. Hence the occurrence of number theoretic functions in QFT becomes natural.

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1 Motivation

Many problems of mathematical physics are tackled by entities which are indexed by (sub)sets of the non-negative integers or sequences of nonnegative integers and Cartesian sets formed out of them. Among them are found such important cases as the partitions of integers, appearing in group representations and symmetric functions, or the occupation numbers of states of a quantum system, which also involves representation theory. All these plentiful examples come with an additional structure when the objects are manipulated, for example using formal power series. This can be translated in many cases into combinatorial properties of the index sets, and sometimes into some arithmetic on them. It is then a natural question to ask what kind of algebraic structure comes with the indices of such objects as generating functions.

It turns out that we are able to develop a coefficient-based approach directly on the index sets. Hence we are dealing directly with the arithmetic on the nonnegative integers \mathbb{Z}_+ . It does not seem to be widely appreciated amongst mathematical physicists or even some number theorists, that it is possible to define additive and multiplicative comonoid structures on \mathbb{Z}_+ , which are dual to addition and multiplication in \mathbb{Z}_+ . In some sense the question here is, in which way a nonnegative integer can be decomposed additively or multiplicatively. Such structures are of course not new – for example the fact, that the non-negative integers admit a partial ordering by divisibility is well-studied via coalgebras on posets [38].

What us interests here is to exhibit clearly the fact that many standard constructions in mathematics and mathematical physics are actually based on additive and multiplicative comonoid constructions without these being made explicit. Our tool will be that of Hopf algebras, and we try to exploit this tool as far as possible in a first exposition. While the additive structure is not so obvious, the multiplicative one is very deeply involved in number theory, combinatorics, representation theory, and last but not least in the renormalization theory of quantum fields.

We call both operations of addition and multiplication (plus, times) $+, \cdot : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ ‘products’, to make direct contact to Hopf algebraic notions. However, both pairs of ‘products’ and ‘coproducts’, i.e. $(+, \Delta^+)$ and (\cdot, Δ^-) , **fail** to come up with a Hopf algebra structure, but form only a weaker **Hopf gebra**, for notation see below and in [26]. That is, in both cases we find an antipodal convolution which allows to introduce *some* crossing by Oziewicz’ theorem [45] such that one comes up with a generalized Hopf algebra fulfilling the homomorphism axiom for product and coproduct. However, the natural crossing, the switch of two adjacent elements in the monoid $\mathbb{Z}_+^{\otimes 2}$, does not allow the Hopf axioms to be fulfilled.

Our aim is to show, that the additive convolution and the multiplicative convolution are intimately connected, as are addition and multiplication in a ring. Furthermore, we want to show that the fact that the multiplicative convolution, which we call **Dirichlet Hopf algebra** in a double abuse of language, fails to be Hopf, having a deep number-theoretic counterpart. Furthermore, we shall try to give arguments, that the structure of renormalization of quantum fields has exactly the same root. This might be one step towards an explanation of why and how number theoretic functions appear inevitably in renormalization, among them multiple zeta values.

There are quite a few loose ends. The further iteration of multiplication to form the noncommutative operation of exponentiation is not treated here. The present structure is more compatible with a 2-category picture, but we refrained here from exploring this in a first exposition. The cohomological considerations are only taken up superficially, and a much deeper study is needed to classify linear forms etc. Topological issues of generating functions, i.e. convergence, have been totally neglected. This leaves us with formal results, which prevent for the moment the use of our methods in analytic number theory where they are of interest. Further comments on these connections are given at the end of the paper.

The paper is plainly structured so as to investigate first the additive convolution, then the multiplicative convolution, the ‘Dirichlet Hopf algebra’. The treatment of the Hopf convolutions is kept at a formal level, in order to exhibit as much as possible the parallels between the additive and multiplicative cases, and the introduction of associated constructions such as generating functions and series is postponed until the Hopf algebraic details have been discussed. Finally the interplay between the various structures is explored and the unrenormalized coproduct is introduced. **Renormalization** is the morphism which maps the Hopf algebra onto the antipodal convolution. The first of these structures is necessary to be able to compute expansion formulae so typical for perturbative QFT (pQFT). The last section provides three key applications of the structure which may exhibit its ubiquitous appearance and importance:

a: Symmetric functions provide the example where Rota and Stein implicitly introduced much of the presently discussed material. There the iterated structure of a plethystic Hopf algebra appears which is based on a 2-vectorspace $\text{Tens Tens}[V]^+$.

b: It is demonstrated that the **normal ordering** of quantum mechanical creation and annihilation operators, which produce the Stirling numbers, can be modelled by the Dirichlet structure *including renormalization*. This can be achieved in a less general setting by the usage of a Rota-Baxter operator. We demonstrate, that the correct identification of algebraic structures is best done in a characteristic free setting.

c: The example which coined the naming ‘renormalized’ for some structures is the application in **renormalization theory of quantum fields**. We show how to make contact to the seminal work of Brouder and Schmitt [13] and thereby to Epstein-Glaser renormalization [24]. We also discuss the connection of this combinatorial approach to the Connes-Kreimer-BPHZ formalism of renormalization [41, 21].

2 The Hopf convolution of addition

2.1 Dualizing addition

We start by considering the monoid of addition on nonnegative integers \mathbb{Z}_+ . The addition map is defined as the common addition of integers

2.1 Definition: The commutative **addition** $+$ of nonnegative integers \mathbb{Z}_+ is defined as usual

$$\begin{aligned} + : \mathbb{Z}_+ \times \mathbb{Z}_+ &\rightarrow \mathbb{Z}_+ \\ n \times m &\mapsto +(n, m) = +(m, n) = n + m \end{aligned} \quad (2-1)$$

Our aim is to dualize this addition, seen as a ‘product’ on the number monoid, and to come up with a ‘coproduct’ related to this addition. This, however, depends on the chosen duality. We will investigate two such choices in the sequel. We distinguish loosely two types of tensor products: \otimes_2 is linear over $\mathbb{Z}_2 = \mathbb{Z}_+/2\mathbb{Z}_+$, and \otimes is linear over \mathbb{Z} . There arises some peculiarity, since we should carefully separate the scalars \mathbb{Z}_2 or \mathbb{Z}_+ from the monoid elements \mathbb{Z}_+ , but that will cause no confusion here.

2.2 Definition: The **renormalized Kronecker duality** K and the **renormalized Kronecker pairing**, also denoted K , are defined as

$$\begin{aligned} K : \mathbb{Z}_+ &\rightarrow \mathbb{Z}_+^* \cong \text{hom}(\mathbb{Z}_+, \mathbb{Z}_2) \\ K(\mathbb{Z}_+)(\mathbb{Z}_+) &= \text{eval}(\mathbb{Z}_+^* \otimes_2 \mathbb{Z}_+) \\ &= K(\mathbb{Z}_+, \mathbb{Z}_+) = \langle \mathbb{Z}_+, \mathbb{Z}_+ \rangle \\ n \times m &\mapsto \langle n | m \rangle = n^*(m) = K(n, m) = \delta_{n,m} \end{aligned} \quad (2-2)$$

where $\delta_{n,m}$ is the usual Kronecker delta. ■

This is our first choice, which might be the most natural to think of at a first glance if coming from numbers. Later it will become clear that the second choice may be natural from a physical modelling point of view. The naming is chosen such that the adjectives ‘renormalized’ and ‘unrenormalized’ fit the the usage in physics. We will have need to introduce an ‘unrenormalized’ pairing later, see theorem 3.22, which give a second duality.

2.3 Definition: The **unrenormalized Kronecker duality** R and the **unrenormalized Kronecker pairing**, also denoted R , is defined as

$$\begin{aligned} R : \mathbb{Z}_+ &\rightarrow \mathbb{Z}_+^\# \cong \text{hom}(\mathbb{Z}_+, \mathbb{Z}_+) \\ R(\mathbb{Z}_+)(\mathbb{Z}_+) &= \text{eval}(\mathbb{Z}_+^\# \otimes_{\mathbb{Z}} \mathbb{Z}_+) \\ &= R(\mathbb{Z}_+, \mathbb{Z}_+) = (\mathbb{Z}_+, \mathbb{Z}_+) \\ n \times m &\mapsto (n | m) = n^\#(m) = R(n, m) \end{aligned} \quad (2-3)$$

where we have introduced the dual $\mathbb{Z}_+^\#$ with a different duality that should not be confused with \mathbb{Z}_+^* . ■

Both *Kronecker pairings* allow us to define duals of addition, the additive coproducts. We still use this terminology despite the fact that we deal with an additive structure. The ‘tensor monoid’ is written using the more natural direct sum symbol \oplus in the additive case. We will have need to carefully distinguish between *units*, *zeros* and *ones*.

2.4 Definition: The **unit of addition** is the zero 0. The injection of zero into the algebra is $\eta^+ : \mathbb{Z}_2 \rightarrow \mathbb{Z}_+$ where $\text{eval}(\eta^+) = 0$. It will be sometimes crucial to distinguish the two notions $\eta^+ \in \mathbb{Z}_+$ and $0 \in \mathbb{Z}_2$ and we will use η^+ to denote 0 injected into \mathbb{Z}_+ . ■

This is evident from $0 + n = n = n + 0$. Using renormalized Kronecker duality, we get

2.5 Theorem: The **renormalized Kronecker addition coproduct** Δ^+ , with respect to K , is given as

$$\begin{aligned} \Delta^+(n) &= \sum_{r=0}^n r \oplus (n - r) \\ &= n_{(1)} \oplus n_{(2)} \end{aligned} \quad (2-4)$$

where the last line gives the notation using Sweedler indices [60]. The addition coproduct Δ^+ is cocommutative and coassociative. ■

Proof: We need to show that the coproduct Δ^+ is Kronecker dual to addition. Therefore we consider

$$\begin{aligned}
\delta_{n,s+t} &= \langle n \mid s+t \rangle \\
&= \langle \Delta^+(n) \mid s \oplus t \rangle \\
&= \langle n_{(1)} \mid s \rangle \langle n_{(2)} \mid t \rangle \\
&= \delta_{n_{(1)},s} \delta_{n_{(2)},t} \quad \forall s, t \text{ such that } s+t=n
\end{aligned} \tag{2-5}$$

From which the definition follows. Cocommutativity is obvious, coassociativity follows either by duality or by a direct short computation. □

The additive case is notationally somewhat peculiar. Strictly speaking we could deal with the monoid structure $(\mathbb{Z}_+, +)$ of the Abelian semigroup of addition. By a slight abuse of notation we do not distinguish Hopf ring and Hopf algebra. Moreover, since we employ scalar valued linear forms¹ we need to define an action of scalars on the monoid and furthermore we want to be able to use the additive monoidal structure \oplus to be able to consider pairs (n, m) and even r -tuples (n_1, \dots, n_r) of nonnegative numbers. The action of the scalars \mathbb{Z}_2 is multiplicative $0 \cdot n = 0$ and $1 \cdot n = n$, turning the monoid \mathbb{Z}_+ into a \mathbb{Z}_2 -module. Therefore we agree to use the following notation

$$\begin{aligned}
\eta^+ \oplus n &\cong n \cong n \oplus \eta^+ \\
\text{eval}(\eta^+ \oplus n) &= \text{eval}(n \oplus \eta^+) = n \\
\text{eval}(0 \oplus n) &= \text{eval}(n \oplus 0) = 0
\end{aligned} \tag{2-6}$$

This defines an isomorphism between tuples having units η^+ with such tuples where the units are omitted. A tuple having a zero 0 will be mapped to 0 under the evaluation. This is the analogy of $1 \otimes V \cong V \cong V \otimes 1$ in the multiplicatively written case. If the η^+ is evaluated in the underlying trivial ring, it will be zero, the additive unit.

2.6 Definition: The renormalized **Kronecker proper cut addition coproduct** $\Delta^{+'}$ is given by the nonzero terms of the coproduct Δ^+ only

$$\begin{aligned}
\Delta^{+'}(n) &= \sum_{r=1}^{n-1} r \oplus (n-r) \\
\Delta^+(n) &= \eta^+ \oplus n + n \oplus \eta^+ + \Delta^{+'}(n)
\end{aligned} \tag{2-7}$$

Sums over proper cuts, hence omitting the terms involving *units*, here the zero, will be denoted \sum' for short.

2.7 Definition: The **counit** ϵ^+ of the additive coproduct Δ^+ is given as

$$\begin{aligned}
\epsilon^+ : \mathbb{Z}_+ &\rightarrow \mathbb{Z}_2 \\
n &\mapsto \epsilon^+(n) = \delta_{0,n}
\end{aligned} \tag{2-8}$$

Proof: We need to check the defining relation of a counit. This reads

$$\begin{aligned}
(\epsilon^+ \oplus \text{Id})\Delta^+(n) &= (\text{Id} \oplus \epsilon^+)\Delta^+(n) \\
&= \epsilon^+(n_{(1)}) \oplus n_{(2)} \\
&= \delta_{0,n_{(1)}} n_{(2)} = \sum \delta_{0,r} (n-r) \\
&= n
\end{aligned} \tag{2-9}$$

¹A further peculiarity of the additive case is, that the only 'scalars' are the 0 and 1 and all linear forms just map all elements to \mathbb{Z}_2 .

showing ϵ^+ to be a left and right counit. \square

Note that the ‘scalars’ \mathbb{Z}_2 act multiplicatively on r -tuples $n_1 \oplus \dots \oplus n_r$. This is necessary to define the notion of a ‘linear map’ and to keep the analogy with the multiplicative case.

A further structural property of interest is that of primitiveness. An element n of a comodule (comonoid) is called (a, b) -primitive, if it fulfils

$$\Delta^+(n) = n \oplus a + b \oplus n. \quad (2-10)$$

Thus from the definition of Δ^+ we have the obvious

2.8 Theorem: There is only one (η^+, η^+) -primitive element, called primitive for short, the one 1. \blacksquare

The deeper meaning of the theorem is that it allows the construction of the whole monoid \mathbb{Z}_+ from this single primitive element 1 and we could address \mathbb{Z}_+ as the module so generated. This is well known from Peano axioms of natural numbers, where the successor map $\sigma : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$, $\sigma \mapsto n \rightarrow n + 1$ allows the construction of all numbers out of zero by iteration: $1 = \sigma(0)$, $2 = \sigma(\sigma(0))$, and so on.

We note further that the counit is a homomorphism of addition as the unit is an homomorphism of the coaddition. In terms of Hopf algebra theory this states that the monoid \mathbb{Z}_+ and comonoid \mathbb{Z}_+ are **connected**. In formulae this reads

$$\begin{aligned} \epsilon^+(n + m) &= \epsilon^+(n)\epsilon^+(m) \in \mathbb{Z}_2 \\ \Delta^+(\eta^+) &= \eta^+ \oplus \eta^+ \quad (\Delta^+(0) = 0 \oplus 0) \end{aligned} \quad (2-11)$$

2.2 Antipodal convolution of addition

From any pair of compatible, i.e. composable, product and coproduct maps one is able to define a convolution for maps from the comonoid B to the monoid C .

2.9 Definition: A **convolution product** \star of maps $f, g : B \rightarrow C$ of a coproduct map $\Delta : A \rightarrow B \otimes B$ and a product map $m : C \otimes C \rightarrow D$ is given as

$$\begin{aligned} (f \star g) &: A \rightarrow D \\ (f \star g)(a) &= m(f \otimes g)\Delta(a) \end{aligned} \quad (2-12)$$

We will consider mainly endomaps $f, g : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ (or $\mathbb{Z}_2 \subset \mathbb{Z}_+$) and complex valued maps $f, g : \mathbb{Z}_+ \rightarrow \mathbb{C}$ where the injection $\iota : \mathbb{Z}_+ \rightarrow \mathbb{C}$ is the canonical embedding $\iota(n) = n + i0 \in \mathbb{C}$. The convolution is a map $*$: $\text{hom}(\mathbb{Z}_+, \mathbb{C}) \times \text{hom}(\mathbb{Z}_+, \mathbb{C}) \rightarrow \text{hom}(\mathbb{Z}_+, \mathbb{C})$ turning this space into an algebra.

2.10 Theorem: The (in general complex) convolution $(+, \Delta^+)$ is unital with unit $u^+ = \eta^+ \circ \epsilon^+$. \blacksquare

Proof: A trivial checking of the convolutional identity

$$f \star u = f = u \star f \quad (2-13)$$

gives the result. The unit is unique due to biassociativity (associativity and coassociativity) \square

2.11 Theorem: The (in general complex) convolution $(+, \Delta^+)$ is antipodal with antipode $\mathbf{S}^+ : \mathbb{Z}_+ \rightarrow \mathbb{C}$, $\mathbf{S}^+(n) = -n$. \blacksquare

Proof: The definition of an antipode is

$$(\mathbf{S}^+ \oplus \mathbf{Id}) = u^+ = (\mathbf{Id} \oplus \mathbf{S}^+) \quad (2-14)$$

Due to bicommutativity (cocommutativity and commutativity) we needed to use only the first equation. Therefrom we get

$$\sum \mathbf{S}^+(r) \oplus (n - r) = \eta^+ \circ \epsilon^+(n) = \delta_{0,n} \cdot \eta^+ \quad (2-15)$$

which can be solved recursively and yields $\mathbf{S}^+(n) = -n$. \square

The antipode does not exist if the codomain is \mathbb{Z}_+ , but needs a codomain containing \mathbb{Z} . If \mathbb{Z} is constructed from pairs $(\mathbb{Z}_+, \mathbb{Z}_+)$ modulo an equivalence relation, then the antipode is realized as the switch of these pairs.

2.12 Theorem: The convolution $(+, \Delta^+)$ together with unit u^+ , counit ϵ^+ and antipode \mathbf{S}^+ does **not** form a Hopf algebra, but only a Hopf gebra [26]. ■

Proof: The compatibility axiom, using the switch $\mathbf{sw} : \mathbb{Z}_+ \otimes_2 \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \otimes_2 \mathbb{Z}_+$, turning a product-coproduct map pair $(+, \Delta^+)$ into a bialgebra maps fails to hold, as can be checked on the element $2 \otimes_2 3$ providing a counterexample.

However, due to a theorem of Oziewicz [45, 46, 32, 26], every antipodal convolution can be turned into a **Hopf gebra** with respect to the crossing

$$(+ \otimes_2 +)(\mathbf{Id} \oplus \Delta^+ \oplus \mathbf{Id})(\mathbf{S}^+ \oplus \mathbf{Id} \oplus \mathbf{S}^+)(\mathbf{Id} \oplus + \oplus \mathbf{Id})(\Delta^+ \oplus \Delta^+) \quad (2-16)$$

Note that the $+$ sign stands for the multilinear addition map and not literally for binary addition here. If the crossing is a braid the convolution will form a braided Hopf algebra and if the crossing is the (graded) switch one would be left with a (graded) Hopf algebra. However, we know this fails to hold in our case, so that we remain in the general Hopf gebra case. □

2.13 Theorem: The antipode is an (anti)homomorphism of addition

$$\mathbf{S}^+(n+m) = \mathbf{S}^+(m) + \mathbf{S}^+(n) \quad (2-17)$$

Proof: Trivial, due to the commutativity of addition the (anti) does not make actual sense here. However, one should notice, that the standard Hopf algebraic proof does not apply here, since we are not dealing with a connected Hopf *algebra* but only with a Hopf gebra, see also discussion in [26, 32]. □

2.3 Branching operators and subtraction

As studied in [12] and [29] we introduce **branching operators**. Necessary ingredients for this are cochains and coboundary operators. An n -cochain is a \mathbb{Z}_2 -linear map from the n -tuple $\mathbb{Z}_+^{\oplus n}$ into \mathbb{Z}_2 (or \mathbb{C}). A 1-cochain is hence a map $\mathbb{Z}_+ \rightarrow \mathbb{Z}_2$. A particular 1-cochain is the counit ϵ^+ .

The **convolutive inverse** ϕ^{-1} of a 1-chain ϕ is defined such that $\phi^{-1} \star \phi = \phi \star \phi^{-1} = \epsilon^+$ that is

$$\begin{aligned} +((\phi \oplus \phi^{-1})\Delta^+(n)) &= +((\phi^{-1} \oplus \phi)\Delta^+(n)) = \epsilon^+(n) \\ \sum \phi_r^{-1} + \phi_{n-r} &= \delta_{n,0}\eta^+ \end{aligned} \quad (2-18)$$

This can be recursively solved. Consider an arbitrary series $\phi(i) = \phi_i$ forming a 1-cochain. The inverse reads

$$\begin{aligned} \phi_0^{-1} &= -\phi_0, & \phi_1^{-1} &= -\phi_1, & \phi_2^{-1} &= -\phi_2, \\ \phi_3^{-1} &= -\phi_3, & \phi_4^{-1} &= -\phi_4, & \phi_5^{-1} &= -\phi_5, \dots \end{aligned} \quad (2-19)$$

Hence one gets $\phi_i^{-1} = -\phi_i$. A normalized 1-cochain is defined to have $\phi_0 = 0$.

A coboundary operator may be defined along the lines given in the papers above using Sweedler cohomology [59]. We have

$$\begin{aligned} \partial^i c_n(k_0, \dots, k_n) &= \begin{cases} \epsilon^+(k_0)c_n(k_1, \dots, k_n) & i = 0 \\ c_n(k_0, \dots, k_{i-2}, k_{i-1} \cdot k_i, k_{i+1}, \dots, k_n) & i \neq 0, n+1 \\ c_n(k_0, \dots, k_{n-1})\epsilon^+(k_n) & i = n+1 \end{cases} \\ \partial_n c_n &= \partial_n^0 c_n \star \partial_n^1 c_n^{-1} \star \dots \star \partial_n^n c_n^{\pm 1} = c_{n+1} \end{aligned} \quad (2-20)$$

In the additive setting this specializes as follows: a 1-cochain ϕ is a 1-cocycle if $(\partial_2 \phi)(n, m) = 0$, which is equivalent to

$$0 = (\partial_2 \phi)(n, m) = \sum_{r,s} (\epsilon^+(r) + \phi(n-r)) - \phi(n+m) + (\phi(s) + \epsilon^+(m-s)) \quad (2-21)$$

hence to an additive map

$$\phi(n + m) = \phi(n) + \phi(m) \quad (2-22)$$

A normalized 1-cocycle is then fully defined by the cocycle condition, and its value on 1 .

$$\begin{aligned} \phi(0) &= 0, & \phi(1) &= \phi_1 \\ \phi(n) &= n \cdot \phi_1 & \text{since} \\ \phi(n + m) &= \phi(n) + \phi(m) = n \cdot \phi_1 + m \cdot \phi_1 = (n + m) \cdot \phi_1 \end{aligned} \quad (2-23)$$

An unnormalized 1-cochain cannot be a cocycle, since $\phi(n) = \phi(n + 0) = \phi(n) + \phi(0) \neq \phi(n)$ spoils the cocycle condition.

2.14 Definition: A **branching operator** for the additive convolution is given by a 1-cochain ϕ and the coproduct as

$$\begin{aligned} / \Phi &= \text{eval}((\phi \oplus \text{Id})\Delta^+) = \text{eval}((\text{Id} \oplus \phi)\Delta^+) \\ / \Phi(n) &= \phi(n_{(1)}) \cdot n_{(2)} = \phi(n_{(2)}) \cdot n_{(1)} \end{aligned} \quad (2-24)$$

■

Note the asymmetry in this definition, using the multiplicative action of the ‘scalars’ \mathbb{Z}_2 under evaluation. Special 1-cochains can be derived from the renormalized Kronecker duality $\delta_{n,m}$. Let us introduce $\phi_b(n) = K(b)(n) = \delta_{b,n}$. In this way, via the evaluation map we can **curry** [47] the \mathbb{Z}_2 -linear forms and parameterize them by elements of \mathbb{Z}_+ . Hence we identified the dual \mathbb{Z}_+^* with \mathbb{Z}_+ via the Kronecker delta.

2.15 Theorem: The branching operator $/ \Phi_b$, $b \in \mathbb{Z}_+$ fixed, with respect to the 1-cochain ϕ_b acts as subtraction by b if the argument is greater or equal to b , and as projection to 0 otherwise. ■

Proof: We compute the branching operator as

$$\begin{aligned} / \Phi_b(n) &= \text{eval}((\phi_b \otimes_2 \text{Id})\Delta^+(n)) \\ &= \phi_b(n_{(1)}) \cdot n_{(2)} = \sum_{r=0}^n \phi_b(r) \cdot (n - r) \\ &= \sum \delta_{b,r} \cdot (n - r) = \begin{cases} n - b & \text{if } n \geq b \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (2-25)$$

showing the desired feature. □

Note, that while the antipode \mathbf{S}^+ was a map which necessarily enlarged the codomain, the subtraction established by employing branching operators here can still be established as an endomorphism of \mathbb{Z}_+ .

2.4 Contractions and a derivation

Branchings are related to contractions via an identification of V with V^* using K or $V^\#$ using R . While a linear form ϕ_b acts in a branching the ‘name’ b can be dualized and acts via a contraction $b \lrcorner_K n = \text{eval}(K(b) \oplus n)$. In differential geometry one would write $\phi_b(x) \cong i_b(x)$ since it forms an inner derivation if b is primitive.

Since we have only one primitive element 1, the (η^+, η^+) -primitive element to which we could assign the grade 1, in this convolution the 1, we can expect only one **derivation** acting on the additive convolution. Let us define $1^* = K(1)$ as the Kronecker dual linear form $1^*(n) = \delta_{1,n} = K(1, n)$ and compute

$$\begin{aligned} 1^*(n) &= (\text{eval} \oplus \text{Id})(K \oplus \text{Id})(\text{Id} \oplus \Delta^+)(1 \oplus n) \\ &= \sum_r 1^*(r) \cdot (n - r) = \sum_r \delta_{1,r} \cdot (n - r) = n - 1 \end{aligned} \quad (2-26)$$

Besides the fact that this is not literally a derivation, it cannot fulfil automatically the Leibniz rule, since the homomorphism axiom does not hold in this convolution. Actually a ‘derivation’ of n by 1 would

need to produce as $n(n-1)$ due to the fact that there are n possible ways to extract a 1 out of $n = 1 + \dots + 1$. Since we do not need it here, we postpone the solution to this problem until we have studied the multiplicative case. However, if one thinks in terms of successor maps a derivation would be a much more natural object to think of. It would be the derivative w.r.t. the successor map.

2.5 Ordinary polynomial series generating functions

With the abstract notion developed so far, we want to give a first application, which also sheds light on the potential field of usage of the present ideas.

Let us consider an infinite sequence of nonnegative integers (integers in general) $a_1, a_2, \dots, a_n, \dots$. To cope with such data, one usually resorts to generating functions. We will use ordinary polynomial series generating functions (opsgf, see [64]). Hence we introduce a formal indeterminate, say t . We do *not* specify the domain of this variable (yet), but we demand that the a_n and t commute, and that t is at least power associative and not nilpotent, hence that we can introduce arbitrary powers of t recursively.

2.16 Definition: An **ordinary polynomial series generating function** (opsgf) is an element of the ring of formal power series $\mathbb{Z}_+[[t]]$. Addition in this ring is component wise addition of power series, multiplication is given by pointwise multiplication of power series.

$$\begin{aligned}
 (f + g)(t) &= \sum f_n t^n + g_n t^n = \sum (f_n + g_n) t^n \\
 (f \cdot g)(t) &= \sum_{n \geq 0} \sum_{m \geq 0} f_n t^n g_m t^m \\
 &= \sum_{n \geq 0} \sum_{r=0}^n (f_r \cdot g_{n-r}) t^n = \sum h_n t^n \quad \text{Cauchy product formula} \\
 \text{hence } \Delta^+(n) &= \sum r \oplus (n - r) \quad \text{on the indices.} \tag{2-27}
 \end{aligned}$$

■

A more formal way to look at this is to say (\oplus is our monoid product)

$$\begin{aligned}
 (f + g)(t) &= \sum_{n \geq 0} +(f \oplus g) \Delta^+(n) t^n \\
 (f \cdot g)(t) &= \sum_{n \geq 0} \cdot (f \oplus g) \Delta^+(n) t^n \tag{2-28}
 \end{aligned}$$

This unveils the usage of the coproduct of addition in the Cauchy formula. Note, that the process of forming a formal power series employs a duality. Consider the series $a_0, a_1, a_2, \dots, a_n, \dots$ i.e. an element of \mathbb{Z}_+^∞ and the series $1 = t^0, t = t^1, t^2, \dots, t^n, \dots$ which pair element wise to form an element of $\mathbb{Z}_+[[t]]$. Note further, that multiplying powers of t , due to power associativity, amounts to *adding* the exponents, a basic fact used in combinatorics. Hence one finds here the addition of nonnegative integers. The coefficients have, due to the Cauchy product formula, a quite different law of composition, which we recognize immediately to be the Kronecker coproduct of addition. In a certain sense these two paired series are dual if evaluation is considered.

We may make this more explicit using ‘letter-place’ techniques as often employed by Rota. Let \mathbb{A} be an alphabet, i.e. the formal sum of the a_n , which we take as letter. Further, let \mathbf{T} be a formal alphabet of the t^n , and use it as a ‘place’. The formal power series emerges as a pairing between letters and places, $(\mathbb{A} \mid \mathbf{T}) \in \mathbb{Z}_+[[t]]$. We may now restate the laws governing the structure of the power series rings more

formally as

$$\begin{aligned}
(\mathbb{A} \mid \mathbf{T}) + (\mathbb{B} \mid \mathbf{T}) &= (\mathbb{A} + \mathbb{B} \mid \mathbf{T}) = (\mathbb{A} \mid \mathbf{T}_{\{1\}})(\mathbb{B} \mid \mathbf{T}_{\{2\}}) \\
(\mathbb{A} \mid \mathbf{T}) \cdot (\mathbb{B} \mid \mathbf{T}) &= (\mathbb{A} \cdot \mathbb{B} \mid \mathbf{T}) = (\mathbb{A} \mid \mathbf{T}_{(1)})(\mathbb{B} \mid \mathbf{T}_{(2)}) \\
\Delta_{\emptyset}(\mathbf{T}) = \mathbf{T}_{\{1\}} \otimes \mathbf{T}_{\{2\}} &\Leftrightarrow \Delta_{\emptyset}(t^n) = t^n \otimes 1 + 1 \otimes t^n \\
&(\mathbb{A} \mid 1) = 1 = (\mathbb{B} \mid 1) \\
\Delta(\mathbf{T}) = \mathbf{T}_{(1)} \otimes \mathbf{T}_{(2)} &\Leftrightarrow \Delta(t^n) = \sum_{r \geq 0}^n t^r \otimes t^{n-r}
\end{aligned} \tag{2-29}$$

Hence we see that the antipodal Kronecker Hopf convolution of addition governs the ring structure of the formal power series i.e. that of ordinary polynomial series generating functions. This fact is not a coincidence when the categorical background is taken into account; however a detailed development is beyond the scope of this work.

Noting that the variable t was not fixed for a special domain, we could take t as a placeholder for an irreducible representation space of a group $t \equiv V^\lambda \cong \mathbb{C}^{|\lambda|} = \langle z_1, z_2, \dots \rangle$, and λ an integer partition. In this case, the powers of t would have to be considered as symmetrized powers of such a vector space V^λ . This leads to the well known fact that the algebra of formal power series is equivalent to the symmetric algebra over a vector space over a ring \mathbf{R}

$$\mathbb{Z}_+[[t]] \cong \text{Sym}(V^\lambda) = \mathbf{R} \oplus V^\lambda \oplus \otimes^2 V^\lambda \oplus \dots \tag{2-30}$$

we will employ this in section (5.1).

Finally we take up the opportunity to identify the dual notion of this construction. Introducing ∂_t , the formal derivative with respect to t as a dual of t we find the action

$$\begin{aligned}
\partial_t(t) &= 1 \\
\frac{\partial_t^n}{n!} t^m \Big|_{t=0} &= \delta_{n,m}
\end{aligned} \tag{2-31}$$

Sometimes the procedure to extract the n -th coefficient of a power series $f(t)$ is denoted by $[t^n]f(t) = f_n$, see [64] for notation. Hence one has $[t^n] = 1/n! \partial_t^n \Big|_{t=0}$. However, this explicit construction is possible in characteristic 0 only. If one wants to avoid this complication, one sticks with the abstract notion of a new type of variable. These formal dual elements $[t^n]$ fulfil now a slightly different algebra

$$[t^n] \cdot [t^m] = \binom{n+m}{n} [t^{n+m}] \tag{2-32}$$

This is a **divided powers algebra** $\text{Div}(\mathbf{T})$ and the elements are usually written as $t^{(n)}$ with parentheses around the exponents. We will use in the following $t^{(n)}$ and $t^{\#n}$ synonymously for $[t^n]$. One should note, that a power series algebra based on divided powers has the same properties as an exponential generating function. Indeed in characteristic zero these notions are interchangeable. Once more it is now easy to see the well known fact that the dual of a symmetric algebra in any characteristic is given by a divided powers algebra (and not in general by another symmetric algebra, which is possible in characteristic zero only)

$$\text{Sym}(V)^{\#} \cong \text{Div}(V^{\#}) \tag{2-33}$$

This is in full accord with experience, that derivations, i.e. duals of variables, are exponentiated in group theory or Fourier analysis, which is modelled here employing divided powers. Letter-place techniques and divided powers were exactly employed in these areas for that particular reason, see e.g. [6, 4, 5] and references therein.

We close this discussion by providing the divided powers algebra coproduct of addition, which produces the same terms as (2.5) but with additional binomial weightings. Let f, g be formal power series in the

dual variables $t^\#$ which are divided powers, we find

$$\begin{aligned}
f(t^\#) \cdot g(t^\#) &= \sum_{n,m \geq 0} f_n g_m t^{\#n} t^{\#m} \\
&= \sum_{n \geq 0} \sum_{r=0}^n f_r g_{n-r} \binom{n}{r} t^{\#n} \\
&= \sum_{n \geq 0} (f \oplus g) \underline{\Delta}^+(n) t^n
\end{aligned} \tag{2-34}$$

This gives the ‘unrenormalized coproduct of addition’ as we will see below in theorem (3.20)

$$\underline{\Delta}^+(n) = \sum_{r=0}^n \binom{n}{r} r \oplus (n-r) \tag{2-35}$$

From a combinatorial point of view it is of great importance to know where these coefficients come from to have a proper interpretation of the combinatorial meaning of series in such dual variables. An ingenious usage of these ideas may be found in [53, 52]. For a discussion of polynomials associated with binomial coalgebras see [54] and references therein.

3 The Dirichlet Hopf algebra

3.1 Dualizing multiplication

Since arithmetic comes with addition and multiplication, we proceed to dualize multiplication along the same lines as we did with addition. For the first step we once more use Kronecker duality, definition 2.2, to achieve this.

3.1 Definition: The commutative **multiplication** \cdot of nonnegative integers \mathbb{Z}_+ is defined as

$$\begin{aligned}
\cdot : \mathbb{Z}_+ \times \mathbb{Z}_+ &\rightarrow \mathbb{Z}_+ \\
n \times m &\mapsto \cdot(n, m) = \cdot(m, n) = n \cdot m
\end{aligned} \tag{3-1}$$

The **unit of multiplication** is the one 1. The injection of the unit into the monoid \mathbb{Z}_+ is denoted η . ■

This is evident from $1 \cdot n = n = n \cdot 1$. The product is commutative and associative.

3.2 Theorem: The **renormalized Kronecker multiplication coproduct** Δ' is given as

$$\begin{aligned}
\Delta'(n) &= \sum_{d|n} d \otimes_2 \frac{n}{d} \\
&= n_{[1]} \otimes_2 n_{[2]}
\end{aligned} \tag{3-2}$$

where $d | n$ denotes ‘ d divides n ’, the sum is over all divisors d , and we used the Brouder-Schmitt convention [13] to indicate different coproducts by differently shaped Sweedler brackets. ■

Proof: We need to show that the coproduct Δ' is dual to multiplication. Therefore we compute

$$\begin{aligned}
\delta_{n,s \cdot t} &= \langle n | s \cdot t \rangle \\
&= \langle \Delta'(n) | s \otimes_2 t \rangle \\
&= \langle n_{[1]} | s \rangle \langle n_{[2]} | t \rangle \\
&= \delta_{n_{[1]},s} \delta_{n_{[2]},t} \quad \forall s, t \text{ such that } s \cdot t = n
\end{aligned} \tag{3-3}$$

From which the definition follows. Cocommutativity is obvious from the construction, coassociativity follows by duality or from a short computation. □

Computations involving this coproduct are costly, since the explicit knowledge of all divisors of an integer involves its prime number factorization. However, it will serve us as a formal device very well.

3.3 Definition: The renormalized **Kronecker proper cut multiplication coproduct** Δ' is given by the nontrivial terms of the coproduct only

$$\begin{aligned}\Delta'(n) &= \sum'_{\substack{d|n \\ d, n \neq \{1, n\}}} d \otimes_2 \frac{n}{d} \\ \Delta(n) &= \eta \otimes_2 n + n \otimes_2 \eta + \Delta'(n)\end{aligned}\tag{3-4}$$

We denote sums of proper cuts once more as $\sum'_{d|n}$. ■

It is readily checked that $\frac{n}{d}$ is in \mathbb{Z}_+ since d is a divisor of n and the definition is meaningful.

3.4 Definition: The **counit** ϵ' of the multiplication coproduct Δ' is given as

$$\begin{aligned}\epsilon' : \mathbb{Z}_+ &\rightarrow \mathbb{Z}_2 \\ n &\mapsto \epsilon'(n) = \delta_{1, n}\end{aligned}\tag{3-5}$$

Proof: For notational convenience, we drop the explicit display of multiplication from now on. We need to check the defining relation of a counit. This reads

$$\begin{aligned}(\epsilon' \otimes_2 \mathbf{Id})\Delta(n) &= (\mathbf{Id} \otimes_2 \epsilon')\Delta(n) \\ &= \epsilon'(n_{[1]}) \otimes_2 n_{[2]} \\ &= \delta_{1, n_{[1]}} n_{[2]} = \sum_{d|n} \delta_{1, d} \frac{n}{d} \\ &= n\end{aligned}\tag{3-6}$$

showing ϵ' to be a left and right counit. This counit is unique due to biassociativity. □

3.5 Theorem: The (η', η') -primitive elements are exactly the prime integers p_i . ■

Proof: The unique prime number decomposition of an integer n is given as $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$. The divisors of n can be formed from similar expressions where the exponents $\{s_1, \dots, s_k\}$ form a sub multiset of the multiset $\{1^{r_1}, \dots, k^{r_k}\}$. The unit is written as $\eta' = p_i^0$ for some i and its coproduct has the form $\Delta(\eta') = \eta' \otimes_2 \eta'$. Hence the unit is not a primitive element (and not a prime). Due to the fact that $\Delta(\prod_i p_i^{r_i}) = \prod_i \Delta(p_i^{r_i})$, only such sets $\{r_1, \dots, r_k\}$ which have *exactly* one 1 and other elements zero give rise to exactly two divisors, n itself and the unit. Hence we find

$$\Delta(p_i) = p_i \otimes_2 \eta' + \eta' \otimes_2 p_i \quad \forall p_i \in \text{primes}\tag{3-7}$$

as the only primitive elements. □

3.6 Theorem: The monoid $\mathbb{Z}_+^{\otimes 2}$ is connected w.r.t. multiplication and comultiplication. ■

Proof: We need to check

$$\begin{aligned}\epsilon'(n \cdot m) &= \epsilon'(n)\epsilon'(m) \\ \Delta(\eta') &= \eta' \otimes_2 \eta'\end{aligned}\tag{3-8}$$

the first is obvious, the second was discussed in the preceding proof. □

3.2 The Dirichlet convolution of arithmetic functions

We will proceed to establish an antipodal convolution as in the case of addition. As in subsection 2.5 we will identify it with the well studied convolution ring of (multiplicative) arithmetic functions. The identification will become clear *a posteriori*. We postpone furthermore the precise definition of the complex functions $f(s) = \sum_{n \geq 1} f_n n^{-s}$ attached to the series of complex numbers (integers) $f_1, f_2, \dots, f_n, \dots$ until we discuss these generating functions in section 3.7. However we use the term ‘function’ for such series from

now on by abuse of language. Note that $f(n) = f_n$ is a series element but $f(s)$, $s \in \mathbb{C}$ will be a complex valued function.

We consider series of complex numbers (integers) $f_1, f_2, \dots, f_n, \dots$ and therewith related functions $f, g, \dots : \mathbb{Z}_+ \rightarrow \mathbb{C}$. These functions are called arithmetic functions in number theory, see e.g. [14]. The addition of these functions is defined term wise

$$(f + g)(n) = f(n) + g(n) = f_n + g_n \quad (3-9)$$

where the multiplication is given by Dirichlet convolution, which we connect with multiplication and the Kronecker multiplication coproduct.

3.7 Definition: The **Hadamard product** $\cdot : f \times g \mapsto f \cdot g$ of arithmetic functions is given as the *term wise* multiplication

$$(f \cdot g)(n) = f(n)g(n) = f_n g_n \quad (3-10)$$

Actually this gives the series a product ring structure, which we will need below. We can define a group like coproduct $\delta : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \otimes_2 \mathbb{Z}_+$ in such a way that all elements just double $\delta(n) = n \otimes_2 n$. The Hadamard product becomes then the convolution product of the pair (\cdot, δ) . These transformations play an important role in the theory of Schur-Weyl duality in the symmetric group [56]. One could make use of them in the present setting too, which we will demonstrate elsewhere.

3.8 Definition: The **Dirichlet convolution** (\star, Δ) of maps $f, g : \mathbb{Z}_+ \rightarrow \mathbb{C}$ or \mathbb{Z}_+ is defined as

$$(f \star g) : \mathbb{Z}_+ \rightarrow \mathbb{C} \\ (f \star g)(\mathbb{Z}_+) \mapsto (f \star g)(n) = \cdot (f \otimes g) \Delta(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right) \quad (3-11)$$

We will see in section 3.7 that this product is intimately related with the the point wise product of Dirichlet generating functions.

3.9 Definition: The **Dirichlet convolution ring** of arithmetic functions is defined via component wise addition of the series and the Dirichlet convolution as product.

3.10 Theorem: The Dirichlet convolution is unital with unit $u \cdot = \eta \cdot \circ \epsilon \cdot$

Proof: Since we know the product to be unital and the coproduct to be counital, it is a matter of checking

$$f \star u = f = u \star f \quad (3-12)$$

to see that the unit $u \cdot = \eta \cdot \circ \epsilon \cdot$ is a unit for the convolution. This unit is unique due to biassociativity. \square

3.11 Definition: The Möbius function $\mu(s)$ is given as Dirichlet series w.r.t. the formal parameter s (usually taken complex valued, definition below Sec. 3.7)

$$\mu(s) = \sum_{n>0} \frac{\mu_n}{n^s} \quad (3-13)$$

defining the series $\{\mu_n\} = \{1, -1, -1, 0, -1, 1, -1, 0, 0, 1, \dots\}$. The coefficients are given by

$$\mu_n = \begin{cases} 1 & n = 1 \\ (-1)^k & n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k} \text{ where } r_i \in \{0, 1\} \forall i \\ 0 & \text{otherwise} \end{cases} \quad (3-14)$$

The Möbius function is hence a (signed) projection onto square-free integers where the sign is negative for an odd number of mutually distinct primes and positive for an even number of mutually distinct primes. By definition one has $\mu_1 = 1$.

3.12 Theorem: The Dirichlet convolution is antipodal with antipode $\mathbf{S}^\cdot = n \cdot \mu(n)$ where $\mu(n)$ is the (series of coefficients of the) Möbius function. Hence the antipode \mathbf{S}^\cdot is given as the Hadamard product of the identity times the Möbius function $(\mathbf{Id} \cdot \mu)(s)$, where $\mathbf{Id}(s) = \sum_{n \geq 1} \mathbf{Id}_n / n^s$ and $\mathbf{Id}_n = n$, hence $\mathbf{Id}(s) = \sum_{n \geq 1} 1/n^{s-1}$, see identification as series in eqn. (3-60). ■

Proof: We use the definition of the antipode to compute

$$\begin{aligned} (\mathbf{S}^\cdot \star \mathbf{Id})(n) &= u^\cdot(n) = (\mathbf{Id} \star \mathbf{S}^\cdot)(n) \\ &= \sum_{[n]} \mathbf{S}^\cdot(n_{[1]})n_{[2]} \\ &= \sum_{d|n} \mathbf{S}^\cdot(d) \frac{n}{d} \\ &= \eta^\cdot \circ \epsilon^\cdot(n) = \delta_{1,n} \eta^\cdot \end{aligned} \quad (3-15)$$

This equation can be recursively solved yielding the result. □

The present result is well known in number theory as Möbius inversion, see e.g. [7].

3.13 Examples:

$$\begin{aligned} \mathbf{S}^\cdot(1) &= 1, \quad \mathbf{S}^\cdot(2) = -2, \quad \mathbf{S}^\cdot(3) = -3, \\ \mathbf{S}^\cdot(4) &= -\mathbf{S}^\cdot(2)2 - \mathbf{S}^\cdot(1)4 + 0 = 0, \quad \mathbf{S}^\cdot(5) = -5, \\ \mathbf{S}^\cdot(6) &= -\mathbf{S}^\cdot(3)2 - \mathbf{S}^\cdot(2)3 - \mathbf{S}^\cdot(1)6 + 0 = 6, \dots \end{aligned} \quad (3-16)$$

Two numbers n, m are called relatively prime, if their greatest common divisor (gcd) is 1, denoted as $\text{gcd}(n, m) = 1$ or $(n \mid m) = 1$ for short.

3.14 Definition: An arithmetic function $f : \mathbb{Z}_+ \rightarrow \mathbb{C}$ is called **multiplicative**, if for any two relative prime numbers n, m

$$f(n \cdot m) = f(n)f(m) \quad (3-17)$$

it is called **complete multiplicative** if this holds true for all pairs of numbers n, m . ■

Another way to say that a function is complete multiplicative is to state that it is a homomorphism of the multiplicative structure. Hence number theoretic functions *fail* in general to be homomorphisms on products of non relative prime integers n, m . Many important number theoretic functions are multiplicative but *not* totally multiplicative. Examples of multiplicative functions are the Euler phi-function, the Legendre symbol, the number of divisors of n function,

3.15 Theorem: The Dirichlet convolution does **not** form a Hopf algebra but only a Hopf gebra. However, on relative prime numbers n, m the compatibility axiom holds true. ■

This means, that the product is a multiplicative map on relative prime elements of the comonoid \mathbb{Z}_+ and the coproduct Δ^\cdot is a multiplicative map on relative prime elements of the monoid \mathbb{Z}_+ .

Proof: We take 2, 2 with $\text{gcd}(2, 2) = 2$ and compute (using percent signs to abbreviate parts of the formula)

$$\begin{aligned} \Delta^\cdot(2 \cdot 2) &= \sum_{d|4} d \otimes_2 \frac{4}{d} = 1 \otimes_2 4 + 2 \otimes_2 2 + 4 \otimes_2 1 \\ (\cdot \otimes_2 \cdot)(\mathbf{Id} \otimes_2 \mathbf{sw} \otimes_2 \mathbf{Id})(\Delta^\cdot \otimes_2 \Delta^\cdot)(2 \otimes_2 2) \\ &= (\%) (\%) ((2, 1, 2, 1) + (1, 2, 2, 1) + (2, 1, 1, 2) + (1, 2, 1, 2)) \\ &= (\%) ((2, 2, 1, 1) + (1, 2, 2, 1) + (2, 1, 1, 2) + (1, 1, 2, 2)) \\ &= 4 \otimes_2 1 + 2 \otimes_2 2 + 2 \otimes_2 2 + 1 \otimes_2 4 \end{aligned} \quad (3-18)$$

showing that there appears a multiplicity and this serves as a counter example for total multiplicativity. However, a similar consideration for general coprime n, m shows the identity to hold. □

We come now to a fact which has major implications on the applicability of the cohomological considerations. This will have also great impact on the potential usage and interpretation in physics.

3.16 Theorem: A 1-cochain ϕ is a 1-cocycle in the Dirichlet convolution ring, if it is a complete multiplicative map, that is

$$\phi(n \cdot m) = \phi(n)\phi(m) \quad (3-19)$$

for all n, m in \mathbb{Z}_+ . ■

Proof: The cocycle identity (2-20) reads

$$(\partial_2\phi)(n, m) = \epsilon'(n)\epsilon'(m) \quad (3-20)$$

From this identity we derive using the convolution product

$$\begin{aligned} \sum_{\substack{d|n \\ s|m}} \phi\left(\frac{n}{d}\right)\phi\left(\frac{m}{s}\right)\phi^{-1}(d \cdot s) &= \epsilon'(n)\epsilon'(m) = \epsilon'(n \cdot m) \\ \sum_{\substack{d|n \\ s|m \\ r|d \cdot s}} \phi\left(\frac{n}{d}\right)\phi\left(\frac{m}{s}\right)\phi^{-1}\left(\frac{d \cdot s}{r}\right)\phi(r) &= \sum_{r|d \cdot s} \epsilon'\left(\frac{n \cdot m}{r}\right)\phi(r) \\ \sum_{\substack{d|n \\ s|m}} \phi\left(\frac{n}{d}\right)\phi\left(\frac{m}{s}\right)\epsilon'(d \cdot s) &= \phi(n \cdot m) \\ \phi(n)\phi(m) &= \phi(n \cdot m) \end{aligned} \quad (3-21)$$

where we used $\epsilon'(n \cdot m) = \delta_{n \cdot m, 1} \Leftrightarrow n = 1, m = 1$. This computation is valid for all n, m since we did not make use of the homomorphism axiom, but the converse is not true for non-complete multiplicative 1-cochains, see example below. □

3.17 Example: To exhibit the peculiarities appearing for non-complete multiplicativity, we give an example of multiplicative functions and their derived 2-cocycles in the case of the Möbius series and the zeta series. We write $\mu(n)$ for the n th element of the series related to the Möbius function $\mu(s)$. Note that the Möbius series is not a homomorphism since $\mu(2)\mu(2) = -1 \cdot -1 = 1$ but $\mu(2 \cdot 2) = \mu(4) = 0$. Let ∂_2 be the coboundary operator as introduced in (2-20), see [59], and employed in [12, 29]. It maps 1-cochains into 2-coboundaries. The related Möbius 2-cocycle is hence given, using $\mu \star \zeta = \eta$, that is $\mu^{-1} = \zeta$, as

$$\begin{aligned} (\partial\mu)(n, m) &= \sum_{\substack{d|n \\ l|m}} \mu(d)\mu(l)\zeta\left(\frac{n}{d} \cdot \frac{m}{l}\right) \\ &= \sum_{\substack{d|n \\ l|m}} \mu(d)\mu(l) \\ &= \sum_{d|n} \mu(d) \sum_{l|m} \mu(l) \\ &= \epsilon'(n)\epsilon'(m) \end{aligned} \quad (3-22)$$

Since $\sum_{d|n} \mu[d] = \delta_{n,1} = \epsilon'(n)$. Thus the Möbius 1-cochain μ , besides not being a homomorphism, nevertheless gives rise to a trivial 2-cocycle, and turns out to be a 1-cocycle! A question which has to be settled is thus if all multiplicative but not complete multiplicative functions give rise to trivial 2-cocycles. However, the calculation for the convolutive inverse 1-cochain ζ amazingly gives

$$\begin{aligned} (\partial_2\zeta)(n, m) &= \sum_{\substack{d|n \\ l|m}} \zeta(d)\zeta(l)\mu\left(\frac{n}{d} \cdot \frac{m}{l}\right) \\ &= \sum_{\substack{d|n \\ l|m}} \mu\left(\frac{n}{d} \cdot \frac{m}{l}\right) \end{aligned} \quad (3-23)$$

which is by no means in an obvious way trivial. If one computes the characteristic polynomials of n by n upper left submatrices, one gets quite nontrivial polynomials. The zeros of these polynomials are in \mathbb{Z} with certain multiplicities. It is hence false that all multiplicative functions form *trivial 2-cocycles*. This sheds some light on how cohomology may help to deal with number theoretic questions. The seeming discrepancy between this example and the theorem is explained by the observation that for *complete multiplicative* ϕ one finds

$$(\partial_2 \phi)^{-1} = (\partial_2 \phi^{-1}) \quad (3-24)$$

The Möbius function hence does not allow to the analogous computation, explaining the astonishing fact that

$$(\partial_2 \mu)^{-1} \neq (\partial_2 \mu^{-1}) = (\partial_2 \zeta) \quad (3-25)$$

despite the fact that $\mu \star \zeta = \epsilon$. This means, that inverses have to be established for any grade by direct computation or other algebraic means. ■

We would like to remark at this point, that it is this failure of complete multiplicativity which was the origin of the present investigation. In quantum field theory one needs to consider renormalization. There, certain integrals need to be regularized by a subtraction process called renormalization which includes compensating singular integrals, the counter terms. There are strong hints, that the failure of total multiplicativity here and its re-establishment using ‘counter terms’ has a deep link to this renormalization process. The work where this might appear to be seen at least with hindsight is the work of Brouder and Schmitt [13], see also section 5.3 and [12]. To remedy the situation we have two options (at least).

Option A: We know the following, using the Oziewicz crossing [45, 46, 32, 26]. Analogous to theorem 2.12 we find:

3.18 Theorem: The antipodal convolution is a **Hopf convolution** and hence a **Hopf algebra** with respect to the following crossing

$$(\cdot \otimes_2 \cdot)(\mathbf{Id} \otimes_2 \Delta' \otimes_2 \mathbf{Id})(\mathbf{S}' \otimes_2 \mathbf{Id} \otimes_2 \mathbf{S}')(\mathbf{Id} \otimes_2 \cdot \otimes_2 \mathbf{Id})(\Delta' \otimes_2 \Delta') \quad (3-26)$$

Proof: The failure for being Hopf has been demonstrated above. The existence of this crossing is guaranteed, due to biassociativity and the existence of the antipode. □

The Dirichlet convolution ring thus forms a Hopf algebra only. It needs further checking if this Hopf algebra could still be braided to form a braided Hopf algebra. It would then be possible to develop this crossing as the ordinary switch plus additional correction terms $\tau = \mathbf{sw} + O(1)$. In [32] we had to learn that this route is rather tedious.

Option B: We can use a slightly altered version of the renormalized multiplication coproduct, which adjusts the multiplicities which are responsible for the failure of being multiplicative. Such an ‘unrenormalized’ coproduct would emerge from a different type of duality, and we are lead to a new duality map beside the Kronecker duality used above. A candidate for such a pairing turns out to be the **unrenormalized Kronecker pairing**²

$$(n | m) = R(n, m) = \delta_{n,m} \prod_i r_i! = \left(\prod_i r_i! \right) \langle n | m \rangle \quad (3-27)$$

where $n = \prod_i p_i^{r_i}$. This pairing dualizes multiplication by employing R in eqn. (2.3) in such a way that the pair $(\cdot, \underline{\Delta})$, see definition 3.20 below, fulfils the **Hopf algebra** axioms. The precise definition will be given in theorem (3.22). Furthermore, our renormalized multiplication coproduct becomes a relative of the renormalization coproduct of [13]. We will discuss this in more detail below.

²This name is chosen using wishful thinking and awaited applications in pQFT, see also [13] and section 5.3. It depends however on the viewpoint which of the coproducts should be addressed as ‘renormalized’. Our naming scheme reflects the usage of the term renormalized in physics. Further more, the Dirichlet convolution is the mathematical interesting structure and not the unrenormalized one, hence our awkward development starting using the unrenormalized coproducts.

3.3 The unrenormalized coproduct

We noticed above, that the coproduct obtained from dualizing multiplication is only an homomorphism if we consider relatively prime integers. Hence this property does not hold for the monoid \mathbb{Z}_+ as a whole. There is a standard method to circumvent this problem, and to define an actual Hopf algebra in such a way that the new coproduct is a homomorphism on \mathbb{Z}_+ . This will turn out to be the unrenormalized coproduct.

3.19 Definition: The monoid \mathbb{Z}_+ can be graded by the number of primes of every $n \in \mathbb{Z}_+$, where multiplicities are counted. Let $n = \prod_i p_i^{r_i}$, $\nu = \sum_i r_i$,

$$\begin{aligned}\mathbb{Z}_+ &= 1 + \mathbb{Z}_+^1 + \mathbb{Z}_+^2 + \dots + \mathbb{Z}_+^\nu + \dots \\ \mathbb{Z}_+^\nu &= \left\langle \prod_j p_j^{r_j} \right\rangle \quad \text{where } \sum r_i = \nu \\ \mathbb{Z}_+^\nu \cdot \mathbb{Z}_+^{\nu'} &= \mathbb{Z}_+^{\nu+\nu'}\end{aligned}\tag{3-28}$$

■

The grade one elements, i.e. the primitive elements, are the infinitely many primes p_i . We can use now the Kronecker duality to dualize the multiplication of a single prime p and obtain as above $\underline{\Delta}(p) = p \otimes_2 1 + 1 \otimes_2 p$. A coproduct which fulfils the homomorphism property can now be defined by *recursion* on the grade forcing the homomorphism property.

3.20 Definition: The **unrenormalized coproduct of multiplication** $\underline{\Delta}$ is defined recursively on the graded monoid \mathbb{Z}_+ as

$$\begin{aligned}\underline{\Delta}(\eta) &= \eta \otimes_2 \eta && \text{on } \mathbb{Z}_+^0 \\ \underline{\Delta}(p) &= p \otimes_2 1 + 1 \otimes_2 p && \text{on } \mathbb{Z}_+^1 \\ \underline{\Delta}(m \cdot n) &= \underline{\Delta}(n)\underline{\Delta}(m) && \text{otherwise,}\end{aligned}\tag{3-29}$$

where $\mathbb{Z}_+^0 = 1$ and $\mathbb{Z}_+^1 = \{p_i\}$ and p_i are prime numbers. ■

3.21 Corollary: The explicit form of the unrenormalized coproduct $\underline{\Delta}$ on $n = \prod_i^k p_i^{r_i}$ is given as

$$\begin{aligned}\underline{\Delta}(n) &= \prod_i \underline{\Delta}(p_i^{r_i}) \\ \underline{\Delta}(p_i^r) &= \sum_{s=0}^r \binom{r}{s} p_i^{r-s} \otimes_2 p_i^s \\ \underline{\Delta}(n) &= \sum_{s_1}^{r_1} \dots \sum_{s_k}^{r_k} \binom{r_1}{s_1} \dots \binom{r_k}{s_k} p_{i_1}^{s_1} \dots p_{i_k}^{s_k} \otimes_2 p_{i_1}^{r_1-s_1} \dots p_{i_k}^{r_k-s_k}\end{aligned}\tag{3-30}$$

The $(r-1)$ -iterated coproduct of a single prime power p^r is given as

$$\begin{aligned}\underline{\Delta}^{(r-1)}(p^r) &= \\ \sum_{s_1=0}^r \sum_{s_2=0}^{s_1} \dots \sum_{s_{r-1}=0}^{s_{r-2}} \binom{r}{s_1} \binom{s_1}{s_2} \dots \binom{s_{r-1}}{0} p^{r-s_1} \otimes_2 p^{s_1-s_2} \otimes_2 \dots \otimes_2 p^{s_{r-1}}\end{aligned}\tag{3-31}$$

■

Proof: We know by definition that the coproduct is a homomorphism, hence we need to consider prime powers only. From

$$\begin{aligned}\underline{\Delta}(p^r) &= (\underline{\Delta}(p))^r = (p \otimes_2 1 + 1 \otimes_2 p)^r \\ &= \sum_{s=0}^r \binom{r}{s} p^s \otimes_2 p^{r-s}\end{aligned}\tag{3-32}$$

we get the first part of the assertion. The second assertion follows by iteration of the coproduct. \square

Note that the appearance of the *binomial coefficients* makes up the difference between the renormalized coproduct of multiplication and the unrenormalized coproduct of multiplication.

$$\begin{aligned}\Delta(p^2) &= \sum_{d|2} p^d \otimes_2 p^{\frac{2}{d}} = p^2 \otimes_2 1 + p \otimes_2 p + 1 \otimes_2 p^2 \quad \text{since } d \in \{1, p, p^2\} \\ \underline{\Delta}(p^2) &= \sum_{s=0}^2 \binom{2}{s} p^s \otimes_2 p^{2-s} = p^2 \otimes_2 1 + 2p \otimes_2 p + 1 \otimes_2 p^2\end{aligned}\tag{3-33}$$

Hence this change actually involves a duality and employs the divided power structure. It should be noted that the iterated coproduct of eqn. (3-31) has an analogous structure to a path-ordered product, see [19] for a discussion.

We are now able to define a new scalar product based on the Laplace pairing, see [53, 12, 26], induced by the renormalized coproduct. The Laplace pairing is obtained by an expansion formula analogy to the expansion of a determinant. One uses the fact that the product can be dualized into the coproduct an vice versa. In our case we demand that

$$\begin{aligned}(\underline{\Delta}(p) \mid r \otimes_2 s) &= (p \mid r \cdot s) \\ (p \otimes_2 q \mid \underline{\Delta}(s)) &= (p \cdot q \mid s)\end{aligned}\tag{3-34}$$

Of course, this cannot be the Kronecker pairing since we know that the pair $(\cdot, \underline{\Delta} \cdot)$ is not Kronecker dual.

3.22 Theorem: The **unrenormalized Kronecker pairing** $(\cdot \mid \cdot) : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ is given by Laplace expansion of $n = \prod_i p_i^{r_i}$ and $m = \prod_j p_j^{s_j}$ as

$$(n \mid m) = \prod_i \delta_{r_i, s_i} r_i! \tag{3-35}$$

■

Proof: We need to show that the pairing can be defined recursively by Laplace expansion of either n or m . To do so we assume $n \geq m$ ($n < m$ follows by symmetry) and denote $q_i = p_i^{s_i}$. Then we compute

$$\begin{aligned}(n \mid m) &= (n \mid q_1 \cdot \dots \cdot q_k) \\ &= (\underline{\Delta}^{\cdot(k-1)}(n) \mid q_1 \otimes_2 \dots \otimes_2 q_k) \\ &= (n_{(1)} \otimes_2 \dots \otimes_2 n_{(k)} \mid q_1 \otimes_2 \dots \otimes_2 q_k) \\ &= (n_{(1)} \mid q_1)^r (n_{(2)} \mid q_2)^r \dots (n_{(k)} \mid q_k)\end{aligned}\tag{3-36}$$

Furthermore we have

$$\begin{aligned}(p_i^{r_i} \mid p_j^{s_j}) &= (\underline{\Delta}^{\cdot(s_j-1)}(p_i^{r_i}) \mid p_j \otimes_2 \dots \otimes_2 p_j) \\ &= \sum_{s_1=0}^{r_i} \sum_{s_2=0}^{s_1} \dots \sum_{s_{r_i-1}=0}^{s_{r_i-2}} \binom{r_i}{s_1} \binom{s_1}{s_2} \dots \binom{s_{r_i-2}}{s_{r_i-1}} \\ &\quad (p_i^{r_i-s_1} \mid p_j)(p_i^{s_1-s_2} \mid p_j) \dots (p_i^{s_{r_i-2}-s_{r_i-1}} \mid p_j) \\ &= \delta_{i,j} \delta_{r_i, s_i} r_i \cdot (r_i - 1) \cdot \dots \cdot 1\end{aligned}\tag{3-37}$$

since only terms of the form $(p_i \mid p_j)$ for $i = j$ survive. \square

The counit ϵ' of Δ' is still the counit of $\underline{\Delta}$, the product remains unchanged and hence also the convolutive unit u^+ is unaltered. We can compute the unrenormalized antipode for the convolution $(\cdot, \underline{\Delta} \cdot)$.

3.23 Theorem: The unrenormalized antipode \mathbf{S}^u of the unrenormalized multiplicative convolution on an element $n = \prod_i p_i^{r_i}$ of grade $\nu = \sum_i r_i$ is given as $\mathbf{S}^u(1) = 1$ and for $n \geq 1$

$$\mathbf{S}^u(n) = (-1)^\nu n \tag{3-38}$$

Proof: We compute recursively the defining equation of the antipode $(\mathbf{S}^u \star \mathbf{Id})(n) = \eta \circ \epsilon^+(n) = \delta_{n1}\eta$ ■

$$\begin{array}{lll}
n = 1) & \mathbf{S}^u(1)1 = 1 & \mathbf{S}^u(1) = +1 \\
n = 2) & \mathbf{S}^u(2) + \mathbf{S}^u(1)2 = 0 & \mathbf{S}^u(2) = -2 \\
n = 3) & \mathbf{S}^u(3) + \mathbf{S}^u(1)3 = 0 & \mathbf{S}^u(3) = -3 \\
n = 4) & \mathbf{S}^u(4) + 2\mathbf{S}^u(2)2 + \mathbf{S}^u(1)4 = 0 & \mathbf{S}^u(4) = +4 \\
n = 5) & \mathbf{S}^u(5) + \mathbf{S}^u(1)5 = 0 & \mathbf{S}^u(5) = -5 \\
n = 6) & \mathbf{S}^u(6) + \mathbf{S}^u(3)2 + \mathbf{S}^u(2)3 + \mathbf{S}^u(1)6 = 0 & \mathbf{S}^u(6) = +6 \\
& \vdots & \\
n = 8) & \mathbf{S}^u(8) + 3\mathbf{S}^u(4)2 + 3\mathbf{S}^u(2)4 + \mathbf{S}^u(1)8 = 0 & \mathbf{S}^u(8) = -8 \\
& \vdots &
\end{array} \tag{3-39}$$

The cases $n = 4$ and $n = 8$ show clearly, that the non square-free numbers now also get a sign grading, and no longer disappear. From this observation and the homomorphism property of the coproduct the conclusion can be drawn. □

3.24 Corollary: The 5-tuple $H = (\mathbb{Z}_+, \cdot, \eta, \epsilon, \mathbf{S}^u)$ is a Hopf algebra. ■

This is true by construction.

3.25 Corollary: The unrenormalized antipode \mathbf{S}^u is involutive $\mathbf{S}^{u2} = \mathbf{Id}$ as a linear operator. ■

This has topological consequences, see [42].

3.4 Branching operators and division

A branching operator for the multiplicative convolution is now defined along the same lines as described above in definition 2.14, consult also [12, 29].

3.26 Definition: A **branching operator** for the multiplicative convolution is given by a 1-cochain ϕ and the coproduct as

$$\begin{aligned}
/\Phi &= \cdot(\phi \otimes_2 \mathbf{Id})\Delta \cdot = \cdot(\mathbf{Id} \otimes_2 \phi)\Delta \cdot \\
/\Phi(n) &= \phi(n_{[1]})n_{[2]} = \phi(n_{[2]})n_{[1]}
\end{aligned} \tag{3-40}$$

We have of course now two duality maps available to define special 1-cochains parameterized by elements b in \mathbb{Z}_+ . These are the renormalized Kronecker duality K and the unrenormalized Kronecker duality R . Hence we have two possibilities $\phi_b(n) = \langle b | n \rangle$ and $\phi_b^r(n) = (b | n)$ to obtain a branching. The first gives us the

3.27 Theorem: The branching operator $/\Phi_b$ w.r.t. the 1-cochain ϕ_b acts as division by b if the argument is divisible by b and as projection to 0 otherwise. ■

Proof: We compute the branching operator as

$$\begin{aligned}
/\Phi_b(n) &= (\phi_b \otimes_2 \mathbf{Id})\Delta(n) \\
&= \phi_b(n_{[1]})n_{[2]} = \sum_{d|n} \phi_b(d) \frac{n}{d} \\
&= \sum \delta_{b,d} \frac{n}{d} = \begin{cases} \frac{n}{b} & \text{if } b | n \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{3-41}$$

showing the desired feature. □

We might wonder, if the branching operators of primitive elements fulfil a Leibniz type rule. Indeed we can derive the following

3.28 Corollary: Let $\epsilon = \phi_1$ be the trivial 1-cochain and $\phi_p(n) = \delta_{p,n}$ a 1-cochain based on a primitive element p (prime number) with branching operator $/\Phi_p$. Let n, m be relatively prime; then one has the Leibniz rule

$$\begin{aligned} /\Phi_p(n \cdot m) &= /\Phi_{p[1]}(n_{[1]})/\Phi_{p[2]}(m_{[1]}) n_{[2]} \cdot m_{[2]} \\ &= n \cdot /\Phi_p(m) + /\Phi_p(n) \cdot m \end{aligned} \quad (3-42)$$

■

This result is however somehow void. Since n, m are relatively prime, a factor containing p occurs either in n or in m or in neither term, and hence one of the two terms or both vanishes, but in general not the other one. If n, m are not relatively prime the result does not hold, since we needed the homomorphism axiom to come up with it. However, division is not expected to be a derivation at all.

3.5 Branching operators and derivation

The second option is employed in a disguised version. Having the unrenormalized coproduct at our disposal, we are now in the position to introduce a different branching. We can compose the unrenormalized coproduct and the renormalized Kronecker evaluation map to define a new type of action of elements on the monoid \mathbb{Z}_+ . This can be seen in two different ways, either by changing the identification between the monoid and its dual, or by keeping this canonical isomorphism and changing the coproduct.

$$\begin{aligned} R(V, V) &\cong \text{eval}(R(V) \otimes V) \cong \text{eval}(V^\# \otimes V) \\ K(V, V) &\cong \text{eval}(K(V) \otimes V) \cong \text{eval}(V^* \otimes V) \end{aligned} \quad (3-43)$$

where $R : V \rightarrow V^\#$ is the duality induced by the unrenormalized pairing and K is the same map induced by the renormalized Kronecker pairing, recall definitions 2.3 and 2.2.

To start, we choose the second possibility keeping evaluation and Kronecker pairing straight. Using the renormalized coproduct we get another interesting map, a contraction map \lrcorner_R . Contractions are related to branching operators as

$$\begin{aligned} \lrcorner : V \otimes V &\rightarrow V \Leftrightarrow /\Phi_V : V \rightarrow V \\ n \lrcorner m &\cong i_n(m) \cong K(n)(m) \rightarrow K(b) \cong /\Phi_b \end{aligned} \quad (3-44)$$

explicitly

$$n \lrcorner_R m = (\text{eval} \otimes_2 \text{Id})(K \otimes_2 \underline{\Delta})(n \otimes_2 m) = /Phi_n^R(m) \quad (3-45)$$

This map has the properties of a *derivation* as we may calculate for p, q different primes

$$\begin{aligned} p \lrcorner_R(p^2 q) &= (\text{eval} \otimes_2 \text{Id})((p^*, p^2 q, 1) + 2(p^*, p, pq) + 2(p^*, pq, p) \\ &\quad + (p^*, q, p^2) + (p^*, p^2, q) + (p^*, 1, p^2 q)) \\ &= 2pq \end{aligned} \quad (3-46)$$

We have used $K(p) = p^*$ as the linear form attached to p . We consider once more \mathbb{Z}_+ as a graded space, the grading induced by the number of primes constituting a number. Note that only grade one elements act as derivations obeying a Leibniz rule while higher grade elements act due to the Hopf algebra structure, see [27]. Since we use the evaluation map and the renormalized coproduct obtained from a different pairing, this Hopf algebra structure is *no longer selfdual*. This may be summarized as

3.29 Theorem: The branching operator $/\Phi_{p_i}^R$ w.r.t. the 1-cochain ϕ_{p_i} , p_i a primitive element (prime number) acts as a derivation by p_i if the argument contains a factor p_i (a non constant function of p_i) and as projection to 0 otherwise. ■

Proof: We compute, using $n = \prod p_i^{r_i}$ and the unrenormalized coproduct, the branching operator as

$$\begin{aligned}
/\Phi_{p_i}^R(n) &= (\phi_{p_i} \otimes_2 \mathbf{Id}) \underline{\Delta}(n) \\
&= \phi_{p_i}(n_{[1]}) n_{[2]} = \sum \binom{r_1}{s_1} \cdots \binom{r_k}{s_k} \phi_{p_i}(\prod p_j^{s_j}) \prod p_j^{r_j - s_j} \\
&= \sum r_i \delta_{p_i, p_j} \frac{n}{p_i} = \begin{cases} r_i \frac{n}{p_i} & \text{if } p_i \mid n \\ 0 & \text{otherwise} \end{cases} \quad (3-47)
\end{aligned}$$

showing the desired feature. \square

3.30 Corollary: A branching by a primitive element fulfils the Leibniz rule. \blacksquare

Proof: Since the homomorphism axiom holds, the proof is standard, see proof given for the division, corollary 3.28. \square

Hence a branching operator can be identified somehow with a derivation $/\Phi_{p_i}^R(n) = \partial_{p_i} n$. This can be seen from

$$\begin{aligned}
\partial_{p_i} n &= \partial_{p_i} \prod p_j^{r_j} \\
&= r_i \prod p_j^{r_j'}, \quad \text{where } r_j' = r_j, \quad i \neq j, \quad \text{and } r_i' = r_i - 1 \text{ or } 0 \text{ if } r_i = 0. \quad (3-48)
\end{aligned}$$

The proof of the preceding theorem shows that the alternative way to define the branching operators using the renormalized pairing and the unrenormalized coproduct property gives an equivalent result

$$\begin{aligned}
/\Phi_p^R(n) &= (\phi_p \otimes_2 \mathbf{Id}) \underline{\Delta}(n) \\
&= (\phi_p^R \otimes_2 \mathbf{Id}) \Delta(n) \quad (3-49)
\end{aligned}$$

since $\phi_{p_i}^R(d) = (p_i \mid d) = r_i \delta_{p_i, d}$ and $\phi_{p_i} = (p_i \mid d) = \delta_{p_i, d}$. Hence we can identify $\phi_b^R = b^\#$ and $\phi_b = b^*$. This shows, that we have two alternatives to incorporate the unrenormalized structure, via the coproduct or via the pairing. The hereby used linear form is the exponential of the grade one linear form.

3.6 Algebraic identification of the duals

We introduced duality maps K and R in definitions 2.2, 2.3. These maps induce an isomorphism between the space V and V^* or $V^\#$. This can be used to dualize the coproduct structures on the mutual dual spaces.

3.31 Theorem: Given the unrenormalized Kronecker coalgebra of multiplication. Under the dualities K and R the induced multiplications on V^* and $V^\#$ the renormalized duality K yields the unrenormalized Kronecker multiplication, which is again ordinary multiplication, and the unrenormalized dualization R implies a divided power multiplication. \blacksquare

Proof: We denote the image of an element a under the map K as $K(a) = a^*$ and the image of a under the map R as $R(a) = a^\#$. We employ the universal action of the dual, the evaluation eval , and compute

$$\text{eval}(n^* \cdot m^* \otimes_2 k) = \langle n \cdot m \mid k \rangle = \delta_{n \cdot m, k} \quad (3-50)$$

and using $k = \prod_i p_i^{r_i}$ we get

$$\text{eval}(n^\# \cdot m^\# \otimes_2 k) = (n \cdot m \mid k) = \prod_i r_i! \delta_{n \cdot m, k} \quad (3-51)$$

Using $n = \prod_i p_i^{r_i}$ and $m = \prod_j p_j^{s_j}$ we might write (formally) $n^\# = (\prod_i r_i!)^{-1} n$ and compute

$$\begin{aligned} n^\# \cdot m^\# &= \left(\prod_i r_i! \right)^{-1} \left(\prod_i s_i! \right)^{-1} n \cdot m \\ &= \frac{\left(\prod_i (r_i + s_i)! \right)}{\prod_i r_i! \prod_j s_j!} \left(\prod_i (r_i + s_i)! \right)^{-1} n \cdot m \\ &= \frac{\left(\prod_i (r_i + s_i)! \right)}{\prod_i r_i! \prod_j s_j!} (n \cdot m)^\# \end{aligned} \quad (3-52)$$

showing that the duals obtained by the renormalized Kronecker pairing forms a divided powers algebra, compare section 5.2 and eqn. (5-19). Examples of such multiplications are $2^\# \cdot 4^\# = 3 \cdot 8^\#$ and $4^\# / 2^\# = 1/2 \cdot 2^\#$, which we will have occasion to need in section 5.1. \square

This is in accord with eqn. (2-33). It should be noted, that the divided powers algebra $\text{Div}(P)$, over the space P spanned by primitive elements p_i cannot be generated multiplicatively in characteristic $\neq 0$. E.g. $1^\# \cdot 1^\# = 2 \cdot 2^\#$ where 2 is not necessarily a unit. To remedy this one can introduce a Rota-Baxter operator R of weight 0 as employed in section 5.2, which establishes $R(n^\#) = (n+1)^\#$.

3.7 Dirichlet series, Dirichlet L -series, Dirichlet generating functions

In several places we have colloquially introduced ‘functions’ $f(s)$ in connection with formal series f_1, f_2, f_3, \dots without further details. For example, the Möbius function $\mu(s)$ is of this type. These notions shall be made more precise in this subsection.

3.32 Definition: A **Dirichlet generating function** is a formal power series constituting an arithmetic function in the following form

$$f(s) = \sum_{n \geq 1} \frac{f_n}{n^s} \quad (3-53)$$

where $f_n, s = \sigma + i\tau \in \mathbb{C}$ are complex numbers. \blacksquare

3.33 Examples:

$f(s)$	f_n	sequence
$\zeta(s)$	1	1, 1, 1, 1, 1, 1, 1, 1, 1, ...
$\zeta(s)^2$	$d(n)$	1, 2, 2, 3, 2, 4, 2, 4, ...
$\frac{1}{2-\zeta(s)}$	$H(n)$	1, 1, 1, 2, 1, 3, 4, 2, 3, 1, 8, ...
$\lambda(s)$	$\frac{1}{2}(1 - (-1)^n)$	1, 0, 1, 0, 1, 0, 1, 0, 1, 0, ...
$\mu(s)$	μ_n	1, -1, -1, 0, -1, 1, -1, 0, 0, 1, ...

Where $\zeta(s)$ is the Riemann zeta-function, $\zeta(s)^2$ is the divisor function, $d_n = d(n)$ is the number of divisors of n , $H(n)$ is the number of ordered factorizations of n , $\lambda(s)$ is the Dirichlet lambda-function and $\mu(s)$ is the Möbius function. Note, that these functions may or may not be (complete) multiplicative. \blacksquare

This definition can be generalized using number theoretic characters $\chi_k(n)$, which are complete multiplicative k -periodic functions, $\chi_k(n+k) = \chi_k(n)$.

3.34 Definition: A **Dirichlet L -series** is a Dirichlet series of the form

$$L_k(s, k) = \sum_{n \geq 1} \chi_k(n) n^{-s} \quad (3-55)$$

where the number theoretic character χ_k is an integer function with period k . \blacksquare

There is a close connection here to modular forms etc. on which we will not dwell, as we in general refrain here from making number theoretic statements. We are now able to use some basic facts to obtain a more concise form of the formula for the renormalized antipode.

3.35 Theorem: The Möbius function and the Riemann zeta-function are mutually convolutive inverse Dirichlet series.

$$(\mu \star \zeta)(s) = \sum_{n \geq 1} \sum_{d|n} \mu_d \zeta_{n/d} n^{-s} \quad (3-56)$$

$$= \sum_{n \geq 1} \delta_{n1} n^{-s} = 1 \quad (3-57)$$

■

Proof: Its well known that $\sum_{d|n} \mu_d = \delta_{n1}$. Another interesting proof uses the **Euler product** of the Dirichlet series. This is a product over all primes p_i

$$\begin{aligned} \frac{1}{\zeta(s)} &= \prod_{i \geq 1} \left(1 - \frac{1}{p_i^s}\right) \\ &= \left(1 - \frac{1}{p_1^s}\right) \left(1 - \frac{1}{p_2^s}\right) \left(1 - \frac{1}{p_3^s}\right) \dots \\ &= 1 - \left(\frac{1}{p_1^s} + \frac{1}{p_2^s} + \frac{1}{p_3^s} + \dots\right) \\ &\quad + \left(\frac{1}{p_1^s p_2^s} + \frac{1}{p_1^s p_3^s} + \dots + \frac{1}{p_2^s p_3^s} + \frac{1}{p_2^s p_4^s} + \dots\right) - \dots \\ &= 1 - \sum_{i \geq 1} \frac{1}{p_i^s} + \sum_{j > i \geq 1} \frac{1}{p_i^s p_j^s} - \sum_{k > j > i \geq 1} \frac{1}{p_i^s p_j^s p_k^s} + \dots \\ &= \sum_{n \geq 1} \frac{\mu_n}{n^s} \end{aligned} \quad (3-58)$$

Since we do only formal algebra here, we need not to bother about convergence and the claim is proved. □

The reader should note the quite close relationship between series of symmetric functions and the Euler products. In fact, one has $E = \prod_{i \geq 1} (1 + x_i)$ as the generating function of the elementary symmetric functions. The product form yields the series

$$\prod_{i \geq 1} (1 + x_i) = \sum_{n \geq 0} e_n(x_1, x_2, \dots) \quad (3-59)$$

and Dirichlet series emerge this way as a specialization of the elementary symmetric functions on the values $x_i = -p_i^{-s}$ of complex prime powers. Since one knows, that such specializations assign the energy eigenvalues of a Hamiltonian system $x_i = e^{E_i t}$, this is in accord with recent attempts to find (quantum) Hamiltonian systems having a ‘Riemann operator’ which generates the zeros of the Riemann zeta function. However, our approach is somehow inverse to that.

3.36 Theorem: The antipode of the Kronecker multiplication coproduct convolution is given by the shifted Möbius Dirichlet series $\mu(s-1)$ ■

Proof: We derived in theorem 3.12 that the antipode had as coefficients the Hadamard product of the identity series $\mathbf{Id}_n = n$ and that of the Möbius series μ_n . This amounts to saying

$$\begin{aligned} \mathbf{S}(s) &= (\mathbf{Id} \cdot \mu)(s) = \sum_{n \geq 1} \frac{n \cdot \mu_n}{n^s} = \sum_{n \geq 1} \frac{\mu_n}{n^{s-1}} \\ &= \mu(s-1) \end{aligned} \quad (3-60)$$

showing the claim. □

It should be noted, that Dirichlet series converge for sufficiently large real part of the complex parameter s , hence the antipode using $s-1$ is more singular than the Möbius function itself, converging only for $\Re(s) > 2$. Also note that the identification of series as functions is not unique. Before closing this

subsection we should mention that the Möbius series, for example, can be expanded as a Lambert series too.

$$\begin{aligned}\text{Lambert}(a_n, x) &= \sum_{n \geq 1} a_n \frac{x^n}{1 - x^n} \\ \text{Lambert}(\mu_n, x) &= \sum_{n \geq 1} \mu_n \frac{x^n}{1 - x^n} = x\end{aligned}\tag{3-61}$$

This gives an additional interesting functional relation attached to the Möbius series and implies other coalgebraic structures, which we cannot investigate here. Another potential generalization of Dirichlet series is the Hurwitz-Riemann zeta-function, given as

$$\zeta_k(s) = \sum_{n \geq 1} \frac{1}{(n+k)^s}\tag{3-62}$$

Note, that the Dirichlet lambda-function can be written as

$$\lambda(s) = \sum_{n \geq 1} \frac{1}{(2n+1)^s} = (1 - 2^{-s})\zeta(s).\tag{3-63}$$

3.8 Polylogarithms

As a short but interesting aside, we consider another Dirichlet-like series of interest in physics, the polylogarithms.

$$\text{Li}_s(z) = \sum_{n \geq 1} \frac{z^n}{n^s}\tag{3-64}$$

For $z = 1$ one stays with the Riemann zeta-function. In the sense of combinatorics, this function should be looked at as a twofold generating series, where one part is of ordinary polynomials series type and the other is of Dirichlet type. In this sense we should study series of the form of an ordinary series-Dirichlet generating function (odg)

$$\text{odg}(a_{n,m}, z, s) = \sum_{n \geq 1, m \geq 0} a_{n,m} \frac{z^m}{n^s}\tag{3-65}$$

which is the product of two such series. The polylogarithms are then those series for which the $a_{n,m} = \delta_{n,m}$. The scheme presented here encodes mappings from the additive into the multiplicative realm and vice versa, hence exponential and logarithm maps respectively.

Polylogarithms occur as complete integrals over Fermi-Dirac and Bose-Einstein distributions while evaluating Feynman diagrams. The generalization to many variables, i.e. to Euler-Zagier sums will not be considered, but emerges as a natural generalization of the present structure to (co)monoids of the form $\mathbb{Z}_+^{\otimes 2}$.

4 Relation between addition and multiplication

4.1 The renormalized case

From arithmetic it is clear that addition and multiplication have an intimate relation. This can be made even clearer using categorial methods [43]. First we recognize, that multiplication is the repeated application of addition. Define an operator $\text{add}_n : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$, $\text{add}_n(m) = n + m$ then we find

$$\cdot(n, m) = \text{add}_n^m(0)\tag{4-1}$$

This suggests, that the coproduct of addition Δ^+ and the coproduct of multiplication Δ^\cdot have to be related by similar arguments in a dual fashion. Indeed we find very easily the following relation

4.1 Theorem: The renormalized coproduct of multiplication Δ^\cdot is related to the renormalized coproduct of addition Δ^+ by exponentiation

$$\begin{aligned} n &= \prod_i p_i^{r_i} \\ \Delta^\cdot(n) &= \delta^{\Delta^\cdot}(n) = \sum_{r'_i+r''_i=r_i, \forall i} \prod_i p_i^{r'_i} \otimes_2 \prod_i p_i^{r''_i} \end{aligned} \quad (4-2)$$

where δ is the group like coproduct $\delta(n) = n \otimes_2 n$ on primes and the r'_i, r''_i run in \mathbb{Z}_+ (including zero).

Proof: The theorem is clear for $1 = p_i^0$ and for a single prime $\Delta^\cdot(p) = p \otimes_2 \eta + \eta \otimes_2 p = \sum_{r+s=1} p^r \otimes_2 p^s$. Since the coproduct is not in general multiplicative, we need to check the relation for all p^r separately.

$$\begin{aligned} \Delta^\cdot(p^r) &= \sum_{d|p^r} d \otimes_2 \frac{n}{d} \\ &= \sum_{d=p^s, s=0}^r p^s \otimes_2 p^{r-s} \\ &= \delta(p)^{\Delta^\cdot(r)} = (p \otimes_2 p)^{\Delta^\cdot(r)} \\ &= \sum_{s+t=r} p^s \otimes_2 p^t \end{aligned} \quad (4-3)$$

Now, all integers have a unique prime number factorization, so we have $n = \prod_i p_i^{r_i}$. Furthermore we know that the coproduct of multiplication is a homomorphism of multiplication for *relatively prime* integers n, m . Hence we find the important formula

$$\begin{aligned} \Delta^\cdot(n) &= \Delta^\cdot\left(\prod_i p_i^{r_i}\right) = \prod_i \Delta^\cdot(p_i^{r_i}) \\ &= \prod_i \delta(p_i)^{\Delta^+(r_i)} \\ &= \sum_{r'_i+r''_i=r_i, \forall i} \prod_i p_i^{r'_i} \otimes_2 \prod_i p_i^{r''_i} \end{aligned} \quad (4-4)$$

which proves the theorem. \square

As a mnemonic we may state that $\Delta^\cdot = \delta^{\Delta^+}$, the assumed exponential relationship we were after. There would be much need to say more about distributivity and its failure, but we postpone this for further investigations.

4.2 The unrenormalized case

The unrenormalized case is best treated reversing the argumentation. We already have a formulation of the unrenormalized coproduct in terms of an additive expression in the exponents of the two factors, see eqn. (3-30). We make the

4.2 Definition: The unrenormalized coproduct of addition $\underline{\Delta}^+$ is given as

$$\underline{\Delta}^+(n) = \sum_{0 \leq r \leq n} \binom{n}{r} r \oplus (n-r) \quad (4-5)$$

which allows us to state the

4.3 Theorem: The unrenormalized coproduct of multiplication $\underline{\Delta}$ is related to the unrenormalized coproduct of addition DPR by exponentiation

$$n = \prod_i p_i^{r_i}$$

$$\underline{\Delta}(n) = \delta^{\underline{\Delta}^+}(n) = \sum_{r'_i + r''_i = r_i, \forall i} \binom{r'_i + r''_i}{r'_i} \prod_i p_i^{r'_i} \otimes_2 \prod_i p_i^{r''_i} \quad (4-6)$$

where δ is the group like coproduct $\delta(n) = n \otimes_2 n$ on primes and the r'_i, r''_i run in \mathbb{Z}_+ (including zero).

■

In fact this is at the same time a definition of the unrenormalized coproduct of addition. The difference between the unrenormalized and the renormalized coproducts is given by the binomial prefactors. The way in which way linearity is to be handled is a somewhat delicate point. We could as prefactor adopt the exponent of the rhs. in

$$\binom{r'_i + r''_i}{r'_i} = p_i^{\ln_{p_i} \left(\frac{r'_i + r''_i}{r'_i} \right)} \quad (4-7)$$

where the \ln is taken with respect to the base p_i . From the point of view of a characteristic free development, this looks unnatural. The proper definition of the unrenormalized coproduct of addition therefore requires further work, with a relation to the von Mangoldt function $\Lambda(s)$ expected.

4.3 Lambda ring structure, Witt vectors and the Witt functor

This section is somewhat more abstract and not strictly needed for the applications in section 5. In a characteristic free development one needs these considerations.

Thus far, we have implicitly used more advanced constructions than just addition and multiplication. In this subsection we make some of this mathematics explicit. Our aim is to present development in other areas of mathematics and physics, and induced there is an underlying categorical background which guarantees the universality of our results. For example, it is well known that the binomial coefficients are related to a lambda ring structure on the ring \mathbb{Z} of all integers. Here we elaborate on this in order to set the scene for generalizations of applications presented in the next section to finite characteristics. We begin by recalling a few facts from Knutson's book about lambda rings [40]. Given a ring \mathbf{R} , we associate to it in a functorial way a new ring $1 + \mathbf{R}[[t]]^+$, where $\mathbf{R}[[t]]^+$ is the augmentation ideal, i.e. the kernel of the counit, that is to say power series in t with a zero constant term. This ring is defined by the pair (\mathbf{R}, λ_t) called a lambda ring. The lambda operations fulfil for all $x, y \in \mathbf{R}$

$$\begin{aligned} \lambda^0(x) &= 1 \\ \lambda^1(x) &= x \\ \lambda^n(x + y) &= \sum_{r=0}^n \lambda^r(x) \lambda^{n-r}(y) \\ \lambda^n(xy) &= \mathbf{P}_n(\lambda^1(x), \lambda^2(x), \dots, \lambda^n(x), \lambda^1(y), \lambda^2(y), \dots, \lambda^n(y)) \\ \lambda^n(\lambda^m(x)) &= \mathbf{P}_{n \cdot m}(\lambda^1(x), \lambda^2(x), \dots, \lambda^{n \cdot m}(x)) \\ \lambda_t(1) &= 1 + t, \quad \lambda_t(x) = \sum \lambda^n(x) t^n \end{aligned} \quad (4-8)$$

where \mathbf{P}_n and $\mathbf{P}_{n \cdot m}$ are universal polynomials specifying the lambda ring structure. From the series $\lambda_t(1)^m = (1 + t)^m = \sum \binom{m}{r} t^r$, we see that the binomial coefficients are actually lambda operations on the ring of integers. The binomial coefficients can be obtained as specializations of the elementary symmetric functions of m variables at $x_i = 1$. More explicitly, this reads for a formal alphabet $X =$

$x_1 + x_2 + x_3 + \dots$

$$\begin{aligned}
\lambda_t(X) &= \sum_{i \geq 0} \lambda_t(1 + x_i t) = 1 + \lambda^1(x)t + \lambda^2(x)t^2 + \dots \\
&= 1 + (x_1 + x_2 + x_3 + \dots)t + (x_1x_2 + x_1x_3 + \dots x_2x_3 + x_2x_4 + \dots)t^3 \dots \\
&= \sum_{n=0}^{\infty} e_n(x)t^n
\end{aligned} \tag{4-9}$$

The polynomial \mathbf{P}_n is derived from the inverse Cauchy kernel $\lambda_t(xy) = \prod_{i,j} (1 - x_i y_j t)$ and the second polynomial $\mathbf{P}_{n,m}$ is obtained by expanding $\lambda_t(\lambda^q(x)) = \prod_i (1 - (x_{i_1} \dots x_{i_q})t)$ where the indices run in $1 \leq i_1 < \dots < i_q \leq n$ and the inductive limit $n \rightarrow \infty$ is understood.

4.4 Example: Consider the following two binomial identities which emerge from the lambda structure

$$\begin{aligned}
\lambda^2(xy) &= \binom{xy}{2} = x^2 \binom{y}{2} + y^2 \binom{x}{2} - 2 \binom{x}{2} \binom{y}{2} \\
&= x^2 \lambda^2(y) + y^2 \lambda^2(x) - 2\lambda(x)\lambda(y) \\
\lambda^2(\lambda^2(x)) &= \binom{\binom{x}{2}}{2} = \binom{x}{3} \binom{x}{1} - \binom{x}{4} \\
&= \lambda^3(x)\lambda^1(x) - \lambda^4(x)
\end{aligned} \tag{4-10}$$

■

While in the Hopf algebraic treatment of series one acts on coefficients (ring elements), the lambda structure lifts this to the ring $1 + R[[t]]^+$ of formal power series. This motivates our simultaneous treatment of series and generating functions in the additive and multiplicative case. Lambda rings contain information about the representation theory of the group structure at hand, in particular the ring of symmetric functions is a special lambda ring and contains information about the representation theory of the general linear group and the symmetric group. From our point of view, we should keep in mind that primitive elements are related with the Adams operations employed for example in K -theory. The Adams operations Ψ actually also define a lambda ring structure, but in general not vice versa. The Ψ -ring is then isomorphic to a lambda-ring. Definition and compatibility with the lambda operations is given for $x, y \in \mathbf{R}$ in the lambda ring $1 + R[[t]]^+$ as

$$\begin{aligned}
\Psi^1(x) &= x \\
\Psi^n(1) &= 1 \\
\Psi^n(x + y) &= \Psi^n(x) + \Psi^n(y) \\
\Psi^n(xy) &= \Psi^n(x)\Psi^n(y) \\
\Psi^n(\lambda^n(x)) &= \lambda^n(\Psi^n(x)) \\
\Psi^n(\Psi^m(x)) &= \Psi^{n \cdot m}(x) = \Psi^m(\Psi^n(x))
\end{aligned} \tag{4-11}$$

The last equation defines the operation of plethysm or composition of representations. The relation between lambda and Adams operations is given by

$$\frac{d}{dt} \log(\lambda_t(x)) = \sum_{n \geq 0} (-1)^n \Psi^{n+1}(x) t^n \tag{4-12}$$

A lambda ring element is called binomial if it fulfils $\lambda_t(x) = (1 + x)^a$. It turns out, that an element is binomial if $\Psi^n(x) = x$ for all $n \in \mathbb{Z}$, hence for our integers we find $\Psi^n(m) = m$ and all integers are binomial. However, the Adams operations will act nontrivially on the ring of formal power series over \mathbb{Z} .

Recalling a standard construction, we define an \mathbf{R}^ω ring as the set of countable sequences $[r_1, r_2, r_3, \dots]$ with addition *and* multiplication defined component wise. Define for every integer the Adams operations

on \mathbf{R}^ω as $\Psi^n([r_1, r_2, r_3, \dots]) = [r_n, r_{2n}, r_{3n}, \dots]$. It can be shown that \mathbf{R}^ω is a Ψ -ring. Now, one can define a morphism L relating the lambda ring $1 + \mathbf{R}[[t]]^+$ with \mathbf{R}^ω such that

$$L \circ \lambda_t = \Psi \quad (4-13)$$

L can be defined as $\frac{d}{dt} \log(1 + e_1 t + e_2 t^2 + \dots) = \sum (-1)^n r_{n+1} t^n$. In characteristic zero an inverse exists $L^{-1} : \mathbf{R}^\omega \rightarrow 1 + \mathbf{R}[[t]]^+$ and can be given by³

$$\begin{aligned} L^{-1}([b_1, b_2, b_3, \dots]) &= \exp(-g(t)) \\ g(t) &= \sum_{n+1} (-1)^n b_n t^n, \end{aligned} \quad (4-14)$$

see [40, page51]. Besides the fact that the map L plays an important role in the description of the K -theory of central functions on G with values in K , we are interested in the case where L is related to universal Witt rings.

Let $W_{\mathbf{R}}$ be the set of all elements $[w_1, w_2, w_3, \dots]$, $w_i \in \mathbf{R}$. Hence as sets $W_{\mathbf{R}} = \mathbf{R}^\omega$. We put a new ring structure on $W_{\mathbf{R}}$ defined by a map $M : W_{\mathbf{R}} \rightarrow \mathbf{R}^\omega$ which is given as

$$\begin{aligned} M([w_1, w_2, w_3, \dots]) &= [r_1, r_2, r_3, \dots] \\ r_n &= \sum_{d|n} d w_d^{\frac{n}{d}} \end{aligned} \quad (4-15)$$

4.5 Example:

$$\begin{aligned} r_1 &= w_1, & r_2 &= w_1^2 + 2w_2, & r_3 &= w_1^3 + 3w_3 \\ r_4 &= w_1^4 + 2w_2^2 + 4w_4, & r_5 &= w_1^5 + 5w_5, & \dots \end{aligned} \quad (4-16)$$

■

The set $W_{\mathbf{R}}$ is closed under sum and product in \mathbf{R}^ω which establishes the ring structure. $W_{\mathbf{R}}$ is called Witt ring. In fact there are universal polynomials F_i, G_j with integer coefficients such that

$$\begin{aligned} M([w_1, w_2, w_3, \dots] + [v_1, v_2, v_3, \dots]) &= M([F_1(w_1, v_1), F_2(w_1, w_2, v_1, v_2), \dots]) \\ M([w_1, w_2, w_3, \dots] \cdot [v_1, v_2, v_3, \dots]) &= M([G_1(w_1, v_1), G_2(w_1, w_2, v_1, v_2), \dots]) \end{aligned} \quad (4-17)$$

and the variables in the F_i, G_i run in the set of variables w_d, v_l where d, l are divisors of i . The functor which relates these rings is called the Witt functor.

We can model the basis change from the w into the r basis by using our coproduct of multiplication. We define an action of M on w by means of multiples of ‘Adams operations’ as

$$M(w_n) = \sum_{d|n} d \cdot [d | w]^{\frac{n}{d}} \quad (4-18)$$

Note that after the universal polynomials have been computed they can be used to define a ring structure on arbitrary rings, torsion free or not (hence in any characteristic). The explicit construction is as follows. First construct a lambda map $f : W_{\mathbf{R}} \rightarrow 1 + \mathbf{R}[[t]]^+$, $f([w]) = \prod (1 + w_d (-t)^d)$ and then apply the map

³We do just mention that this is the place to think of moment-cumulant relations and Spitzer’s identity [48, 49]. A few more glimpses will come up in section 5.2.

L

$$\begin{aligned}
L \circ f([w]) &= \frac{d}{dt} \log \prod_d (1 - w_d(-t)^d) \\
&= \sum_d \frac{dw_d(-t)^{d-1}}{1 - w_d(-t)^d} \\
&= \sum_d \frac{-dw_d(-t)^{d-1}}{t} (1 + w_d(-t)^d + w_d(-t)^{2d} + \dots) \\
&= \sum_d (-1)^{n+1} \left(\sum_{d|n} dw_d^{\frac{n}{d}} \right) t^{n-1} \\
&= \sum_d (-1)^{n+1} r_n t^{n-1}
\end{aligned} \tag{4-19}$$

From the expansion of $f(w)$ one obtains an explicit relation between the elementary symmetric functions e_n and the w_n , compare with eqn. (4-16), as well as with eqn. (4-9)

$$\begin{aligned}
\sum_n e_n t^n &= f(w) = \prod_d (1 - w_d(-t)^d) \\
&= (1 + w_1 t)(1 - w_2 t^2)(1 + w_3 t^3)(1 - w_4 t^4) \dots \\
&= 1 + (w_1)t + (-w_2)t^2 + (w_3 - w_1 w_2)t^3 + (-w_4 + w_1 w_3)t^4 + \dots,
\end{aligned} \tag{4-20}$$

which can be solved recursively for the w_n . One obtains

$$\begin{aligned}
w_1 &= e_1, & w_2 &= -e_2, & w_3 &= e_3 + e_1 e_2, & w_4 &= -e_4 + e_3 e_1 + e_2 e_1^2 \\
w_5 &= e_5 - e_4 e_1 - e_3 e_2 - e_1 e_2^2 + e_1^2 e_3 + e_1^3 e_2, & \dots
\end{aligned} \tag{4-21}$$

These polynomials play a major role in combinatorics of necklaces, and in the study of the Burnside ring [22, 57]. Our interest is in employing these techniques in the theory of plethysms, as will be shown elsewhere.

5 Applications:

Before we turn to the applications, we should remark that the presented constructions are universal in a categorical sense. Hence one can have the hope to employ these methods in seemingly remote mathematical fields as long as the same functorial relations hold. It is hence more that a hope that such elementary structures will arise again in operator algebras of quantum fields, or in representation theoretic issues.

5.1 Symmetric function theory

Consider the ring of symmetric functions Λ , as described for example in [44]. This ring is graded by degree $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$. The ring structure is given by addition of polynomials and multiplication of polynomials in the usual fashion. A basis for this ring is indexed by partitions of integers. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ with parts λ_i such that $\lambda_1 \geq \dots \geq \lambda_k$ and $\sum_{r=0}^k \lambda_r = n$. One can give a multiset description by counting the multiplicity of the parts as $\lambda = [1^{r_1} 2^{r_2} 3^{r_3} \dots n^{r_n} \dots]$. The r_i give the number of parts $\lambda_j = i$. Standard bases of the ring of symmetric functions are the elementary symmetric functions e_λ , the complete symmetric functions h_λ the power sum basis p_λ , the Schur function basis s_λ and the monomial symmetric functions m_λ among others. For our interest, we want to concentrate on the monomial symmetric functions. These functions can be defined for any partition λ as (we use the variable or later alphabet a from now on to comply with [53, 52])

$$m_\lambda = \sum_{i_1 < \dots < i_{r_1}} \sum_{j_1 < \dots < j_{r_2}} \dots \sum_{l_1 < \dots < l_{r_k}} a_{i_1} \dots a_{i_{r_1}} a_{j_1}^2 \dots a_{j_{r_2}}^2 \dots a_{l_1}^k \dots a_{l_{r_k}}^k \tag{5-1}$$

where all indices i_s, j_t, \dots are distinct. With Rota-Stein [53, 52] we abbreviate this as

$$m_\lambda = (a^1)^{(r_1)}(a^2)^{(r_2)} \dots (a^k)^{(r_k)} \quad (5-2)$$

Where we made the following identification

$$(a^i)^{(r_i)} = \sum a_{j_1}^i \dots a_{j_{r_i}}^i, \quad j_{i_1} < \dots < j_{r_i} \quad (5-3)$$

and the a is now from a formal alphabet A , also called letter. Such a monomial is an element of the space $\text{Pleth}[A]$ which is defined in the following way: Let A be an alphabet (formal variable). Construct the symmetric tensor algebra over A as $\text{Tens}[A]$ spanned by monomials (a^n) . Again $\text{Tens}[A]^+$ is the augmentation ideal with zero constant part. This is the module underlying the algebra of formal power series (in the commuting variables a^1, a^2, a^3, \dots). Then form the divided powers algebra over this *module*. This module is generated by $(a^n)^{(k)}$ for all $n \geq 1, k \geq 0$. Hence the module underlying $\text{Pleth}[A]$ is $\text{Div Tens}[A]^+$. The algebra structure is given by the divided power algebra rules for the indices in parentheses. We want to induce, following Rota and Stein [53, 52], a new multiplicative structure on this module in such a way that the monomials m_λ multiply in the same fashion as the monomial symmetric functions. This is done using a cliffordization as follows: Recall that the $[r_1, r_2, r_3, \dots]$ are the multiplicities of the parts λ_i in λ . The coproduct of an element m_λ in $\text{Pleth}[A]$ is given as

$$\begin{aligned} \Delta(m_\lambda) &= m_{\lambda[1]} \otimes m_{\lambda[2]} \\ &= \sum_{r'_i + r''_i = r_i, \forall i} (a^1)^{(r'_1)}(a^2)^{(r'_2)} \dots (a^k)^{(r'_k)} \otimes (a^1)^{(r''_1)}(a^2)^{(r''_2)} \dots (a^k)^{(r''_k)} \\ &= \delta^{\Delta^+}((a^1)^{(r_1)}(a^2)^{(r_2)} \dots (a^k)^{(r_k)}) \end{aligned} \quad (5-4)$$

as in the case of the unrenormalized coproduct. The product is defined for the divided powers component wise on the monomials as

$$\begin{aligned} (a^i)^{(r_i)} \cdot (a^j)^{(r_j)} &= (a^i)^{(r_i)}(a^j)^{(r_j)} \quad i \neq j \\ (a^i)^{(r_i)} \cdot (a^i)^{(s_i)} &= \binom{r_i + s_i}{r_i} (a^i)^{(r_i + s_i)} \end{aligned} \quad (5-5)$$

The next step is to define a pairing, sic a bilinear form. This pairing is assumed to be a Laplace pairing, which literally provides an expansion formula in terms of coproducts, see [53, 26, 12]. The particular pairing will be a 2-cocycle in terms of the cohomology used above eqn. (2-20), which guarantees that the deformed product is associative. One sets:

$$\langle (a^i)^{(r)} \mid (a^j)^{(s)} \rangle = \delta_{r,s} (a^{i+j})^{(s)} \quad (5-6)$$

extended by bilinearity. Actually this pairing is used to reintroduce the algebra structure of formal power series in the ‘inner space’ $\text{Tens}[A]^+$. Using this pairing we introduce the Drinfeld twisted or star product which we call with Rota-Stein a **Clifford product** or circle product in the usual way

$$\begin{aligned} m_\lambda \circ m_\mu &= \langle m_{\lambda(1)} \mid m_{\mu(1)} \rangle m_{\lambda(2)} m_{\mu(2)} \\ &= \mathbf{R}_{\lambda,\mu}^\pi m_\pi \end{aligned} \quad (5-7)$$

where the $\mathbf{R}_{\lambda,\mu}^\pi$ are the structure constants of this algebra in the monomial symmetric function basis. In fact Rota and Stein showed that this process comes up with the same algebraic structure as the monomial symmetric functions have under the point wise product. It is noteworthy to state, that the Clifford product gives a direct route to the **Littlewood-Richardson coefficients** $\mathbf{C}_{\lambda,\mu}^\pi$ of Schur function multiplication. A basis change from monomial symmetric functions to Schur functions is given by the Kostka matrix $K_{\lambda,\mu}$, which can be combinatorially obtained, see [44]. Hence one finds

$$\begin{aligned} s_\lambda &= K_\lambda^\mu m_\mu \\ s_\lambda \circ s_\mu &= K_\lambda^{\lambda'} K_\mu^{\mu'} \mathbf{R}_{\lambda',\mu'}^{\pi'} (K^{-1})_{\pi',s_\pi}^\pi = \mathbf{C}_{\lambda,\mu}^\pi s_\pi \end{aligned} \quad (5-8)$$

This exhibits that the circle product is the ordinary product of polynomials and symmetric functions as claimed. We can summarize this as

5.1 Rota-Stein Theorem: The module $\text{Div Tens}[A]^+$ underlying the plethystic Hopf algebra together with the circle product (5-7) forms the symmetric function algebra (in monomial basis). ■

This result demonstrates in a rather intriguing fashion, via Hopf algebraic mechanisms, the combinatorial origins of symmetric function theory.

5.2 Example: Let $m_1 = (a^1)^{(1)}$ and compute

$$\begin{aligned}
m_1 \circ m_1 &= (a^1)^{(1)} \circ (a^1)^{(1)} \\
&= \sum_{r=0}^1 \sum_{s=0}^1 \langle (a^1)^{(r)} \mid (a^1)^{(s)} \rangle (a^1)^{(1-r)} (a^1)^{(1-s)} \\
&= \sum_{r=0}^1 \sum_{s=0}^1 \delta_{r,s} (a^2)^{(s)} (a^1)^{(1-r)} (a^1)^{(1-s)} \\
&= \sum_{r=0}^1 (a^2)^{(r)} (a^1)^{(1-r)} (a^1)^{(1-r)} \\
&= (a^2)^{(1)} (a^1)^{(0)} (a^1)^{(0)} + (a^2)^{(0)} (a^1)^{(1)} (a^1)^{(1)} \\
&= 2(a^1)^{(2)} + (a^2)^{(1)} \\
&= 2m_{11} + m_2
\end{aligned} \tag{5-9}$$

A more complicated case is

$$\begin{aligned}
m_5 \circ m_{22} &= (a^5)^{(1)} \circ (a^2)^{(2)} \\
&= \langle (a^5)^{(0)} \mid (a^2)^{(0)} \rangle (a^5)^{(1)} (a^2)^{(2)} + \langle (a^5)^{(1)} \mid (a^2)^{(1)} \rangle (a^5)^{(0)} (a^2)^{(1)} \\
&= (a^7)^{(0)} (a^5)^{(1)} (a^2)^{(2)} + (a^7)^{(1)} (a^5)^{(0)} (a^2)^{(1)} \\
&= m_{522} + m_{72}
\end{aligned} \tag{5-10}$$

And checking a result of Rota and Stein we get

$$\begin{aligned}
m_{111} \circ m_{11} &= (a^1)^{(3)} \circ (a^1)^{(2)} \\
&= \langle (a^1)^{(2)} \mid (a^1)^{(2)} \rangle (a^1)^{(1)} (a^1)^{(0)} \\
&\quad + \langle (a^1)^{(1)} \mid (a^1)^{(1)} \rangle (a^1)^{(2)} (a^1)^{(1)} \\
&\quad + \langle (a^1)^{(0)} \mid (a^1)^{(0)} \rangle (a^1)^{(3)} (a^1)^{(2)} \\
&= (a^2)^{(2)} (a^1)^{(1)} (a^1)^{(0)} + (a^2)^{(1)} (a^1)^{(1)} (a^1)^{(1)} + (a^2)^{(0)} (a^1)^{(3)} (a^1)^{(2)} \\
&= (a^2)^{(2)} (a^1)^{(1)} + 2(a^2)^{(1)} (a^1)^{(2)} + \binom{3+2}{2} (a^1)^{(5)} \\
&= m_{221} + 2m_{211} + 10m_{11111}
\end{aligned} \tag{5-11}$$

Since we know that the deformation yields an associative product it follows that the pairing introduced on $\text{Pleth}[A] = \text{Div Tens}^+$ is a 2-cocycle. There is hence a chance that this 2-cocycle $\langle \cdot \mid \cdot \rangle$ is actually derived from a 1-cochain η , giving $\langle \cdot \mid \cdot \rangle = (\partial_2 \eta^{-1})(\cdot, \cdot)$, since we know that the actual product of monomial symmetric functions is isomorphic to the original products of polynomials in the variable(s) A . Indeed, Rota and Stein loc. cit. provide the map $\eta : \text{Tens}[A]^+ \rightarrow \text{Pleth}[A] \cong \text{Div Tens}[A]^+$ such that

$$\eta((a)^{(n)}) = \sum_{\lambda \vdash n} (a^1)^{(r_1)} (a^2)^{(r_2)} (a^3)^{(r_3)} \dots (a^k)^{(r_k)} \tag{5-12}$$

where the sum is over all partitions λ of n . This resembles a classical identity of symmetric functions

$$h_n = \sum_{\lambda \vdash n} m_\lambda \quad (5-13)$$

The complete symmetric functions and monomial symmetric functions form mutually dual bases $\langle h_\lambda | m_\mu \rangle = \delta_{\lambda,\mu}$. The Clifford product and the original product on $\text{Pleth}[A]$ are hence related by

$$\eta((a^i)^{(r_i)}) \circ \eta((a^j)^{(r_j)}) = \eta((a^i)^{(r_i)}(a^j)^{(r_j)}) \quad (5-14)$$

compare with eqn. (5-7). Inspection shows that the scalarproduct introduced on $\text{Pleth}[A]$ is related to the convolutive inverse η^{-1} of η as

$$\langle (a^i)^{(r_i)} | (a^j)^{(r_j)} \rangle = (\partial\eta^{-1})((a^i)^{(r_i)}, (a^j)^{(r_j)}) \quad (5-15)$$

We close this discussion by saying that the $(a)^{(n)}$ have been refereed to by Rota and Stein as complete symmetric functions in the inverse alphabet $A^\#$ under the map η , explicitly $h_n = \eta((a^\#)^{(n)})$. A detailed exposition of these facts is beyond the scope of the present paper, and is postponed to another paper. Also we will not go into the details of plethysms, which is a major point of interest in the present development, but see [31, 33].

5.2 Occupation number representations

Before we try to match our results to quantum field theory, we give a short exposition of occupation number states in quantum mechanics. For the sake of simplicity we will use one label type, quanta of type a , and a bosonic scenario. The discussion will also be insensitive to the basis change between p, q and a, a^\dagger . Physically some of the algebras need to be interpreted in the p, q -basis however.⁴ The precise setting will be outlined elsewhere.

Algebraic setup: Let \mathcal{H} be a L^2 Hilbert space. We construct a countable basis, the occupation number basis, for this space out of a cyclic vector $|0\rangle$ called the ‘vacuum’. Hence we demand that there is a non nilpotent map $a^\dagger : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\mathcal{H} = \oplus_n (a^\dagger)^n |0\rangle = \oplus_n |n\rangle \quad (5-16)$$

Each state $|n\rangle$ is interpreted as having n quanta of type a , while $|0\rangle$ is the state having no such quanta. A general state $|\psi\rangle$ of the Hilbert space \mathcal{H} can be given as a linear combination of the occupation number basis

$$|\psi\rangle = \sum_{n \geq 0} \psi_n |n\rangle = \sum_{n \geq 0} \psi_n (a^\dagger)^n |0\rangle \quad (5-17)$$

Forgetting about convergence issues, one might reinterpret this as an element in the formal power series algebra generated by a^\dagger with coefficients in \mathbb{C} , equivalently $\mathbb{C}[[a^\dagger]]$. The action of a^\dagger on an element $|n\rangle$ is given by

$$\begin{aligned} a^\dagger |n\rangle &= a^\dagger (a^\dagger)^n |0\rangle = (a^\dagger)^{n+1} |0\rangle = |n+1\rangle, \\ (a^\dagger)^n (a^\dagger)^m &= (a^\dagger)^{n+m} \end{aligned} \quad (5-18)$$

where the last line makes the algebra product explicit. Of course, the product of operators a^\dagger amounts to an addition in the exponent. Finally it should be noted that in eqn. (5-18) no explicit normalization factors are included.

Quantum physics needs not only the creation of modes but also annihilation of modes. This is done by the utilization of annihilation operators a . It is commonly accepted that these operators fulfil the same

⁴One has to study the algebras $\mathbb{k}[[q]] \otimes \text{Div}[p] \cong \mathbb{k}[[a]] \otimes \text{Div}[a^\dagger] \cong \mathbb{k}[[t]][t^{-1}]$, where the last has to be interpreted as the localization of $\mathbb{k}[[t]]$ at $t = 0$, and constitutes the quotient field of the power series ring.

type of algebra. However we know from our discussion above (eqn. (2-33)), and as is well known in mathematical literature, see e.g. [63], that the dual of a formal power series algebra (polynomial algebra) is a divided powers algebra. Hence we deal with two types of structures here which are related as in the following diagram⁵

$$\begin{array}{ccc}
 \mathbb{k}[[a]] & \xleftrightarrow{\text{herm. conj.}} & \mathbb{k}[[a^\dagger]] \\
 \uparrow \text{duality} & \begin{array}{c} \text{QM} \\ \text{QM}^* \end{array} & \uparrow \text{duality} \\
 \text{Div}[a] & \xleftrightarrow{\text{herm. conj.}} & \text{Div}[a^\dagger]
 \end{array} \tag{5-19}$$

That quantum mechanics (QM) resides on the diagonals will become clearer below. QM* is the version of QM where p and q or a and a^\dagger are interchanged in their meaning. In characteristic zero, the algebras $\mathbb{k}[[x]]$ and $\text{Div}[x]$ are related by an isomorphism. Of course, the complex number ring \mathbb{C} is of characteristic zero and such an isomorphism is at our disposal, but, our point is that one loses insight into the combinatorial setting of QM by adopting it implicitly, as is usually done. Moreover, a possible generalization to finite characteristic is prevented.

Dropping habitual reflexes, it is at a first glance not totally obvious how to relate annihilation and creation operators. To implement this relation we introduce a duality. Hence we define linear forms as

$$\text{eval}(f_{a^\dagger n} \otimes a^{\dagger m}) = \delta_{n,m} \tag{5-20}$$

We know by now that the algebra of the duals is a divided powers algebra. For notational convenience we adopt $f = f_1 = f_{a^\dagger}$, and $f_n = f_{a^\dagger n}$, and demand that f_1 generates⁶ the divided powers algebra $\text{Div}[f]$

$$\begin{aligned}
 n!f_n &= (n-1)!f_1f_{n-1} = f_1^n \\
 f_n f_m &= \binom{n+m}{n} f_{n+m}
 \end{aligned} \tag{5-21}$$

We can now introduce dual states $\mathcal{H}^* = \text{hom}(\mathcal{H}, \mathbb{k})$ as

$$\begin{aligned}
 \langle 0|f_n &= \langle n| \\
 \langle \psi| &= \sum_n \psi_n \langle n| = \sum_n \psi_n \langle 0|f_n
 \end{aligned} \tag{5-22}$$

Usually operators a are introduced which fulfil the **same** type of power series algebra $\mathbb{k}[[a]]$ as the creation operators assuming the isomorphism $f_i \cong a^i/i!$. The multiplication law of the a^i becomes then $a^i a^j = a^{i+j}$. The evaluation map $\text{eval} : \mathbb{k}[a] \otimes \mathbb{k}[[a^\dagger]] \rightarrow \mathbb{k}$ allows then to introduce a coalgebra structure on the ordinary power series algebra $\mathbb{k}[[a^\dagger]]$ relating to standard notation of QM

$$\begin{aligned}
 \Delta^+((a^\dagger)^0) &= \Delta^+(1) = 1 \otimes 1 \\
 \Delta^+(a^\dagger) &= a^\dagger \otimes 1 + 1 \otimes a^\dagger \\
 \Delta^+((a^\dagger)^n) &= \sum_{r=0}^n (a^\dagger)^r \otimes (a^\dagger)^{n-r}
 \end{aligned} \tag{5-23}$$

This is the **unrenormalized coproduct of addition**, indexing the a -mode creation operator a^\dagger . However, doing so destroys the duality between these two algebras, as we have seen studying the additive Hopf convolution. This forces one to introduce **normalization factors** which are artificial. The linear forms

⁵In fact we are dealing here with a pair of dual Hopf algebras, hence with a Drinfeld quantum double.

⁶Strictly speaking, the f_i cannot be multiplicatively generated in finite characteristic due to the numerical factors appearing. This can be overcome by the usage of a Baxter operator $R(f_i) = f_{i+1}$. A divided powers algebra together with this operator form a Rota-Baxter algebra of weight 0, see also page 35.

f_n can be described by 1-cochains on this algebra. Especially we can assign (the action of) f_1 to the annihilation operator a as the 'name'⁷ of the linear form with action given by evaluation. For an alternative development of generalized (boson) algebras viewed as Hopf algebras, see [62].

Branchings with respect to polynomial algebra and divided powers: Let us study with more care contraction maps, as developed in section 2.3, 2.4, 3.4 and, 3.5, related to branching operators, starting with division $\underline{\lrcorner}/\dot{\div}$

$$\begin{aligned}
\underline{\lrcorner}/\dot{\div}(a \otimes (a^\dagger)^n) &= (\text{eval} \otimes \mathbf{Id})(\mathbf{Id} \otimes \Delta^+)(K \otimes \mathbf{Id})(a \otimes (a^\dagger)^n) \\
&= (\text{eval} \otimes \mathbf{Id})(\mathbf{Id} \otimes \Delta^+)(f_1 \otimes (a^\dagger)^n) \\
&= \sum_{r=0}^n f_1((a^\dagger)^r) \otimes (a^\dagger)^{n-r} \\
&= (a^\dagger)^{n-1}
\end{aligned} \tag{5-24}$$

Using the unrenormalized coproduct we note that $a \underline{\lrcorner}/\dot{\div}$ **fails** to be a derivation, and as an operator would therefore **commute** with a^\dagger . Dualizing now correctly the divided powers algebra, we have to use the renormalized coproduct of addition here.

$$\begin{aligned}
a((a^\dagger)^n) &= \underline{\lrcorner}/\dot{\div}(a \otimes (a^\dagger)^n) \\
&= (\text{eval} \otimes \mathbf{Id})(\mathbf{Id} \otimes \underline{\Delta}^+)(K \otimes \mathbf{Id})(a \otimes (a^\dagger)^n) \\
&= (\text{eval} \otimes \mathbf{Id})(\mathbf{Id} \otimes \underline{\Delta}^+)(f_1 \otimes (a^\dagger)^n) \\
&= \sum_{r=0}^n \binom{n}{r} f_1((a^\dagger)^r) \otimes (a^\dagger)^{n-r} \\
&= n(a^\dagger)^{n-1}
\end{aligned} \tag{5-25}$$

Hence the dualized elements give rise to 'names' a which **act as derivations**. In fact we proved already earlier that these branchings fulfil the Leibniz rule.

Canonical commutation relations: Using the branching operators more systematically we will now show how the two coproducts, renormalized and unrenormalized intertwine to build the core features of QM. We employ Kronecker duality, a power series algebra structure for the a^\dagger and a divided powers algebra for the duals f_n . This allows us to compute immediately the **canonical commutation relations**

$$\begin{aligned}
a a^\dagger (a^\dagger)^n - a^\dagger a (a^\dagger)^n &= \mathbf{Id}(a^\dagger)^n \\
a(a^\dagger)^{n+1} - a^\dagger a (a^\dagger)^n &= (n+1)(a^\dagger)^n - n(a^\dagger)^n = 1 (a^\dagger)^n \\
\text{so that } [a, a^\dagger] &= \mathbf{Id}
\end{aligned} \tag{5-26}$$

Since this is a major point of our development, we will give a Hopf version of this calculation too. We use the following Sweedler index notation $\Delta^+(n) = n_{(1)} \otimes n_{(2)}$ and $\underline{\Delta}^+(n) = n_{[1]} \otimes n_{[2]}$ and compute the action of a^n on the product of $a^\dagger p a^\dagger q$ for arbitrary $m = p + q$. The general formula, see [26] for a graded version, is given as

$$\begin{aligned}
\underline{\lrcorner}/\dot{\div}(a^n \otimes (a^\dagger p a^\dagger q)) &= f_n(a^\dagger p_{[1]} a^\dagger q_{[1]}) a^\dagger p_{[2]} a^\dagger q_{[2]} \\
&= f_{n(1)}(a^\dagger p_{[1]}) f_{n(2)}(a^\dagger q_{[1]}) a^\dagger p_{[2]} a^\dagger q_{[2]}
\end{aligned} \tag{5-27}$$

⁷The concept of a 'name' of a map stems from category theory [43]. One might however stick to a more mechanical picture. Let f_1, f_2, \dots be the buttons on a calculator, the enter key the evaluation map, then strictly the button is (tagged by) the name of the function you want to use. Hence a is the name of the function f_1 acting by evaluation (Kronecker pairing).

which specializes for $m = n + 1$, $p = 1$, $q = n$ to

$$\begin{aligned}
\triangleleft_{/\div} (a \otimes (a^{\dagger 1} a^{\dagger n})) &= \sum_{\substack{p'+p''=1 \\ n'+n''=n}} \binom{p'+p''}{1} \binom{n'+n''}{n} f_1(a^{\dagger p'}) f_0(a^{\dagger n'}) a^{\dagger p''} a^{\dagger n''} \\
&\quad + \binom{p'+p''}{1} \binom{n'+n''}{n} f_0(a^{\dagger p'}) f_1(a^{\dagger n'}) a^{\dagger p''} a^{\dagger n''} \\
&= 1 \cdot 1 \cdot a^{\dagger n} + 1 \cdot (n-1) \cdot a^{\dagger} a^{\dagger(n-1)}
\end{aligned} \tag{5-28}$$

as in eqn.(5-25). This process does **only** work due to the usage of the **unrenormalized coproduct of addition** dualized from the divided powers product of the duals, since only this pair of algebraic structures fulfil the homomorphism axiom which allows Laplace expansions.

We are now able to compute the scalar product between two states. Note the usage of **two different coproducts**:

$$\begin{aligned}
\langle n | m \rangle &= f_n(a^{\dagger})^m = \langle 0 | \frac{1}{n!} a_n(a^{\dagger})^m | 0 \rangle \\
&= \frac{1}{n!} \sum_{s_1=0}^m \sum_{s_2=0}^{s_1} \dots \sum_{s_{n-1}=0}^{s_{n-2}} \binom{m}{s_1} \binom{s_1}{s_2} \dots \binom{s_{n-2}}{s_{n-1}} \\
&\quad \langle 0 | a((a^{\dagger})^{m-s_1}) a((a^{\dagger})^{s_1-s_2}) \dots a((a^{\dagger})^{s_{n-2}-s_{n-1}}) | 0 \rangle \\
&= \delta_{n,m} \frac{n(n-1)(n-2) \dots 1}{n!} \langle 0 | 0 \rangle \\
&= \delta_{n,m}
\end{aligned} \tag{5-29}$$

We thus come up with properly normalized states, even without using the common normalization using square roots of n . Evaluating the coproducts directly in $\text{Div}[f] \otimes \mathbb{k}[[a^{\dagger}]]$ does not need the fractions at all, and could be generalized to finite characteristic. The computation also sheds some light on the diagram (5-19), explaining why QM is sitting on the diagonal line. We want to emphasize the similarity between this calculation and S -matrix calculations in quantum field theory (of course here $S = \text{Id}$, look at line 3 of eqn. (5-29)). From the above calculation it is clear that we can deduce expectation values of all operators which are described by creation and annihilation operators.

Normal ordering and Rota-Baxter operators: The divided powers algebra is a Baxter algebra of weight $\lambda = 0$, that is a commutative algebra Div together with an Rota-Baxter operator $R : \text{Div} \rightarrow \text{Div}$ such that the following identity holds with $\lambda = 0$

$$R(R(x)y + xR(y) + \lambda R(xy)) = R(x)R(y) \tag{5-30}$$

This identity plays a prominent role in many areas of mathematics,⁸ notably in statistics and the path integral formalism of quantum field theory [61]. For divided powers one finds that

$$R(f_i) = f_{i+1} \tag{5-31}$$

The Rota-Baxter operator is required to generate the f -basis of the dual states, which cannot be multiplicatively generated in characteristic zero. The Baxter relation for $\lambda = 0$ is the functional equation for integration and embodies the integration by parts formula. The Rota-Baxter operator can be used to introduce a deformed convolution and is deeply linked to Lie theory and logarithmic functions [16].

We will point out here only the relevance of Rota-Baxter operators to the problem of normal ordering and Stirling numbers of the second kind. Let t be a formal variable of a polynomial series generating

⁸It is instructive to read the original papers by Baxter [9] Cartier [18] and Rota with collaborators [48, 49, 51, 50]. Especially in the last two references, one will find useful remarks about Spitzer's identity and Warring's formula and their derivation from the 'main theorem' of a free Rota-Baxter algebra.

function, i.e. $t^n t^m = t^{n+m}$. Furthermore, let $t_{(n)}$ be the formal variables related to t by $t_{(n)} = t(t-1)\dots(t-n+1)$, the falling factorial. It seems to be known since the seventies that the normal ordering process of creation and annihilation operators provides another way to obtain these numbers, for example see [58], the results of Bender et al [10], and also [35]. Identifying $t =: a^\dagger a :$ one has

$$\begin{aligned} (: a^\dagger a :)^n &= t^n = \sum_{k=0}^n S(n, k) t_{(n)} = \sum_{k=0}^n S(n, k) : a^{\dagger k} a^k : \\ S(n, k) &= \sum_{j=1}^k \frac{(-)^{k-j} j^n}{j!(k-j)!} \end{aligned} \quad (5-32)$$

where the $S(n, k)$ are the Stirling number of the second kind. Let \mathcal{D} be the forward difference operator $\mathcal{D}(t^n) = t^{n+1} - t^n$. Then one finds $k!S(n, k) = \mathcal{D}^k(t^n)|_{t=0}$. One can identify $R(1)^n = t^n = (: a^\dagger a :)^n$ and $n!R^n(1) = n!R(\dots R(1)\dots) = t_{(n)} =: a^{\dagger n} a^n :$. Iterating the Rota-Baxter identity, eqn. (5-30), one can prove the ‘main theorem’:

5.3 Theorem: Let (A, R) be a commutative algebra and R be a Rota-Baxter operator of weight 1. Let $R^k(1) = R(\dots R(1)\dots)$, k -iterations and $R(1)^k = R(1)\dots R(1)$ k -factors. For $n \geq 1$ we have

$$R(1)^n = \sum_{k=1}^n k!S(n, k)R^k(1) \quad (5-33)$$

and

$$e^{R(1)u} = \sum_{k=0}^n R(1)^k \frac{u^k}{k!} = \sum_{n \geq 0} \sum_{k=0}^n k!S(n, k)R^k(1) \frac{u^n}{n!} \quad (5-34)$$

is the exponential (divided powers) generating function for the Stirling number of the second kind. ■

We will finish this subsection by rederiving such results showing that the mechanism developed for quantum field theory in [25, 12] to describe the transition from time ordering to normal ordering and vice versa, can be used in QM too if the combinatorially correct coproducts are employed. Especially lookup the formulae (12) and (13) loc. cit. where all signs are plus signs since we are dealing with bosons. Let $W = \langle a^\dagger, a \rangle$ and consider the symmetric algebra $\text{Sym}(W) = \bigoplus_n W^n = \text{Sym}(a, a^\dagger)$ spanned by normal ordered monomials $: a^{\dagger n} a^m :$. $\text{Sym}(W)$ is graded. We combine the two coproducts in a single notation

$$\hat{\Delta} = \begin{cases} \delta^{\Delta^+} & \text{if acting on } a^{\dagger n} \Rightarrow \hat{\Delta}(a^{\dagger n}) = \sum a^{\dagger r} \otimes a^{\dagger n-r} \\ \delta^{\Delta^-} & \text{if acting on } a^n \Rightarrow \hat{\Delta}(a^n) = \sum a^r \otimes a^{n-r} \end{cases} \quad (5-35)$$

This is to say, that the Laplace pairing defined below is expanded with respect to the renormalized coproducts. We define the Laplace pairing \mathcal{F} as follows:

$$\begin{aligned} \mathcal{F}(W^n, W^m) &= 0 \quad \text{if } n \neq m, \\ \mathcal{F}(1, 1) &= 1, \\ \mathcal{F}(a, a) &= 0, \quad \mathcal{F}(a, a^\dagger) = 1, \quad \mathcal{F}(a^\dagger, a) = 0, \quad \mathcal{F}(a^\dagger, a^\dagger) = 0, \\ \mathcal{F}(u' \cdot u'', v) &= \mathcal{F}(u' \otimes u'', \hat{\Delta}(v)), \\ \mathcal{F}(u, v' \cdot v'') &= \mathcal{F}(\hat{\Delta}(u), v' \otimes v'') \end{aligned} \quad (5-36)$$

We can now expand the lhs. of the preceding theorem 5.3 using a cliffordization with respect to the Laplace pairing eqn. (5-36), where in the cliffordization we use the unrenormalized coproduct⁹.

$$(: a^{\dagger r} a^s :) \circ (: a^{\dagger m} a^n :) = \mathcal{F} \left(: a^{\dagger r[1]} a^{s[1]} :, : a^{\dagger m[1]} a^{n[1]} : \right) : a^{\dagger r[2]+m[2]} a^{s[2]+n[2]} : \quad (5-37)$$

⁹To achieve normal ordering, it would be sufficient to use the unrenormalized coproduct on the \hat{d} algebra only.

It should be noted, that this process is not only applicable to balanced terms having an equal number of creation and annihilation operators, but applies generally. It can be shown, that the bilinear form is zero unless one considers terms $\mathcal{F}(a^n, a^{\dagger m}) = \delta_{n,m}$, so two coproducts readily drop out.

$$\begin{aligned} (: a^{\dagger r} a^s :) \circ (: a^{\dagger m} a^n :) &= \mathcal{F}\left(: a^{s[1]} :, : a^{\dagger m[1]} : \right) : a^{\dagger r+m[2]} a^{s[2]+n} : \\ &=: a^{\dagger r+m[2]} a^{s-m[1]+n} : \end{aligned} \quad (5-38)$$

5.4 Example: We compute in detail a couple of products to show how the normal ordering appears automatically as in [25], but this time with two different coproducts employed!

$$\begin{aligned} a \circ a &= \mathcal{F}(a, a) 1 + \mathcal{F}(1, 1) : a^2 :=: a^2 : \\ a^{\dagger} \circ a^{\dagger} &= \mathcal{F}(a^{\dagger}, a^{\dagger}) 1 + \mathcal{F}(1, 1) : a^{\dagger 2} :=: a^{\dagger 2} : \\ a^{\dagger} \circ a &= \mathcal{F}(a^{\dagger}, a) 1 + \mathcal{F}(1, 1) : a^{\dagger} a :=: a^{\dagger} a : \\ a \circ a^{\dagger} &= \mathcal{F}(a, a^{\dagger}) 1 + \mathcal{F}(1, 1) : a^{\dagger} a :=: a^{\dagger} a : + 1 \end{aligned} \quad (5-39)$$

We obtain for $R(1)R(1) =: a^{\dagger} a : \circ : a^{\dagger} a :$

$$\begin{aligned} : a^{\dagger} a : \circ : a^{\dagger} a : &= (\mathcal{F} \otimes \cdot)(\mathbf{Id} \otimes \mathbf{sw} \otimes \mathbf{Id}) \\ &\quad (a^{\dagger} a \otimes 1 + a^{\dagger} \otimes a + a \otimes a^{\dagger} + 1 \otimes a^{\dagger} a) \\ &\quad \otimes (a^{\dagger} a \otimes 1 + a^{\dagger} \otimes a + a \otimes a^{\dagger} + 1 \otimes a^{\dagger} a) \\ &\quad \mathcal{F}(a, a^{\dagger}) : a^{\dagger} a : + \mathcal{F}(1, 1) : a^{\dagger 2} a^2 : \\ &\quad : a^{\dagger 2} a^2 : + : a^{\dagger} a : \end{aligned} \quad (5-40)$$

A more delicate computation is the following

$$\begin{aligned} : a^{\dagger} a : \circ : a^{\dagger 2} a^2 : &= (\mathcal{F} \otimes \cdot)(\mathbf{Id} \otimes \mathbf{sw} \otimes \mathbf{Id}) \\ &\quad (a^{\dagger} a \otimes 1 + a^{\dagger} \otimes a + a \otimes a^{\dagger} + 1 \otimes a^{\dagger} a) \\ &\quad \otimes (a^{\dagger 2} a^2 \otimes 1 + 2a^{\dagger} a^2 \otimes a^{\dagger} + a^2 \otimes a^{\dagger 2} \\ &\quad \quad 2a^{\dagger 2} a \otimes a + 4a^{\dagger} a \otimes a^{\dagger} a + 2a \otimes a^{\dagger 2} a \\ &\quad \quad a^{\dagger 2} \otimes a^2 + 2a^{\dagger} \otimes a^{\dagger} a^2 + 1 \otimes a^{\dagger 2} a^2) \\ &= \mathcal{F}(1, 1) : a^{\dagger 3} a^3 : + 2\mathcal{F}(a, a^{\dagger}) : a^{\dagger 2} a^2 : \\ &=: a^{\dagger 3} a^3 : + 2 : a^{\dagger 2} a^2 : \end{aligned} \quad (5-41)$$

and similarly combining the above results one obtains

$$: a^{\dagger} a : \circ : a^{\dagger} a : \circ : a^{\dagger} a : =: a^{\dagger 3} a^3 : + 3 : a^{\dagger 2} a^2 : + : a^{\dagger} a : \quad (5-42)$$

In terms of Rota-Baxter operators using eqn. (5-30) this reads as follows

$$R(1)^3 = 6R^3(1) + 6R^2(1) + R(1) = [3!R^3(1)] + 2 \cdot [2!R^2(1)] + [1!R^1(1)] \quad (5-43)$$

showing clearly that we can model the Rota-Baxter action using twisted products induced by a Laplace pairing. Along similar lines we obtain

$$\begin{aligned} : a : \circ : a^{\dagger n} : &= \sum_{r=0}^n \mathcal{F}(a, a^{\dagger r}) \binom{n}{r} : a^{\dagger n-r} : + \mathcal{F}(1, 1) : a^{\dagger n} a : \\ &= n : a^{\dagger n-1} : + : a^{\dagger n} a : \\ : a^2 : \circ : a^{\dagger n} : &= \sum_{r=0}^n \mathcal{F}(a^2, a^{\dagger r}) \binom{n}{r} : a^{\dagger n-r-2} : + \mathcal{F}(a, a^{\dagger r}) \binom{n}{r} \binom{2}{1} : a^{\dagger n-r-1} a : + \mathcal{F}(1, 1) : a^{\dagger n-r} a^2 : \\ &= 2 \binom{n}{2} : a^{\dagger n-2} : + 2n : a^{\dagger n-1} a : + : a^{\dagger n} a^2 : \end{aligned} \quad (5-44)$$

Commutation methods may seem to be computationally more efficient, however, we know that software implementations [1, 3, 2] of the Hopf algebraic procedure can be faster in many cases. ■

As a disclaimer, the reader should remember that we implicitly operated in both, the p, q and a, a^\dagger basis, which are related in a non trivial way if divided powers algebras are concerned. A more careful exposition will be presented elsewhere.

Before moving to quantum field theory in general, it should be borne in mind that the first two examples, symmetric function theory, and quantum oscillators, are closely related via two-dimensional quantum field theory and the fermion-boson correspondence [39]. From the line of development we have followed, the symbolic role played by vertex operators (fields), as exponentials of oscillator modes [36, 37, 55], could be expected to be intimately related to that of plethysms [17, 8] as functors at a categorical level in symmetric function theory [44].

5.3 Combinatorics of renormalization in quantum field theory

In this paragraph we exhibit the implications of the Dirichlet Hopf algebra in the theory of renormalization of quantum fields. We concentrate on the combinatorial side, but want to emphasize that due to the categorical structure of our considerations we expect the scheme to be generally valid. A functorial description supporting this view will be given elsewhere.

Our mathematical approach makes contact with renormalization through the work of Brouder and Schmitt [13]. A Hopf algebraic formulation of pQFT was given in [12]. The arena of renormalization is the bialgebra B of normal ordered (scalar) fields $\phi^n(x)$. Here $x \in \mathbb{R}^{1,3}$ is a space-time point¹⁰ and $n \in \mathbb{Z}_+$. The algebra and coalgebra structures as used in physics are given as

$$\begin{aligned}\phi^n \phi^m &= \phi^{n+m} \\ \delta_B \phi^n &= \sum_{k=0}^n \binom{n}{k} \phi^k \otimes \phi^{n-k} \\ \epsilon_B(\phi^n) &= \delta_{n,0}\end{aligned}\tag{5-45}$$

This pair of operations thus comprises addition and the unrenormalized coproduct of addition. The choice reflects the fact that the dual $\phi^\#(x)$ is assumed to obey a divided powers algebra structure and the coproduct is the dualized product of the divided powers algebra of the dual fields. Then one defines the symmetric algebra $\text{Sym}[B]$ over the *module* underlying B . A monom in $\text{Sym}[B]$ reads

$$a = : \phi^{n_1}(x_1) \dots \phi^{n_k}(x_k) := (\phi^1)^{(n_1)} (\phi^2)^{(n_2)} \dots (\phi^k)^{(n_k)}\tag{5-46}$$

where the right hand side is rewritten in the same type of notation as we employed for symmetric functions, compare eqn. (5-2), only the ‘letter’ of our alphabet, generating the algebra, is now the quantum field ϕ . The coproduct induced from B is given as

$$\begin{aligned}\delta_B \left((\phi^1)^{(n_1)} \dots (\phi^k)^{(n_k)} \right) &= \sum_{i_1}^{n_1} \dots \sum_{i_k}^{n_k} \binom{n_1}{i_1} \dots \binom{n_k}{i_k} \\ & (\phi^1)^{(i_1)} \dots (\phi^k)^{(i_k)} \otimes (\phi^1)^{(n_1-i_1)} \dots (\phi^k)^{(n_k-i_k)}\end{aligned}\tag{5-47}$$

these are **normal ordered fields**. The time ordered product of such a monomial in the field ϕ is given [12] in analogy with the treatment of Epstein-Glaser, as

$$T(a) = \sum t(a_{(1)})a_{(2)}\tag{5-48}$$

where $t : \text{Sym}[B^+] \rightarrow \mathbb{k}$ is an appropriate 1-cochain. Note that this is a branching operator as defined above in sections 2.3, 2.4, 3.4 and 3.5, explicitly reading $T = \cdot(t \otimes) \delta_B$. Such operators play a fundamental

¹⁰Strictly speaking we need to consider test functions sufficiently smooth and localized, but that does not matter for the present algebraic discussion. So we use the loose notation.

role in the theory of group branchings [29, 30] from where we borrow their name. The **Wick theorem** [25] takes the form

$$T(: (\phi^1)^{(n_1)} \dots (\phi^k)^{(n_k)} :) = \sum_{i_1}^{n_1} \dots \sum_{i_k}^{n_k} \binom{n_1}{i_1} \dots \binom{n_k}{i_k} t(: (\phi^1)^{(i_1)} \dots (\phi^k)^{(i_k)} :) : (\phi^1)^{(n_1-i_1)} \dots (\phi^k)^{(n_k-i_k)} : \quad (5-49)$$

in accordance with [25, 15, 24] and the development in [13]. In physics the renormalization is then introduced via a second time ordered product \tilde{T} as follows. For a as in eqn. (5-46) one defined the b^i are the nonempty parts of a . Define further a map O as

$$O : \text{Sym}[B] \rightarrow B$$

$$\tilde{T}(a) = \sum_{\lambda} T(: O(b^1) \dots O(b^l) :) \quad (5-50)$$

where λ is a setpartition of $\sum_i n_i$ into l nonempty parts. The operator O , encoding an renormalization scheme e.g. extracting pole parts, was introduced in Bogoliubov-Shirkov [11] (under a different name). It is convenient to set $\tilde{T}(1) = O(1) = 1$ and for $a \in B$ set $\tilde{T}(a) = a$, hence one gets $O(a) = a$. Using the standard recursion, employing the proper cut coproduct, one comes up with

$$\tilde{T}(a) = T(a) + O(a) + \sum_{\lambda} T(: O(b^1) \dots O(b^l) :) \quad (5-51)$$

Brouder and Schmitt show that O is also a branching operator, and can be defined using a 1-cochain $c(a) = \epsilon_B(O(a))$ as¹¹

$$O(a) = \sum c(a_{(1)})a_{(2)}$$

$$c(a) = \tilde{t}(a) - t(a) - \sum_{\lambda} \sum c(b_{(1)}^1) \dots c(b_{(1)}^l) t(: c(b_{(2)}^1) \dots c(b_{(2)}^l) :) \quad (5-52)$$

Formally one obtains a new 1-cochain $c = \tilde{t} - t$ allowing to write the abstract form of the branching operators O as

$$O(a) = \sum c(a_{(1)}) \prod a_{(2)} \quad (5-53)$$

The relation to renormalization is now given by combining eqn. (5-50) and (5-53) to obtain

$$\begin{aligned} \tilde{T}(a) &= \sum T(: O(b^1) \dots O(b^l) :) \\ &= \sum_{\lambda} \sum c(b_{(1)}^1) \dots c(b_{(1)}^l) T(: \prod b_{(1)}^1, \dots, \prod b_{(2)}^l :) \\ &= \sum C(a_{[1]})T(a_{[2]}) \end{aligned} \quad (5-54)$$

where this is related to the coproduct in $\text{Sym Sym}[B]^+$. It is remarkable that the renormalization map is given by $C(a) = c(a)$ for $a \in \text{Sym}[B]^+$ and $C(uv) = C(u)C(v)$ for $u, v \in \text{Sym Sym}[B]^+$. This is exactly the form of maps obtained by the branchings induced by plethysms as obtained in [31]. This formula, via some steps of identification, carried out in [13], finally makes contact to the Epstein-Glaser framework of renormalization.

While our treatment remained formal and followed closely the approach of Brouder and Schmitt, we hope to have shown that the lift from the coproduct of addition to the coproduct of multiplication (causing the group like coproduct action on a indicated by the Sweedler indices $a_{[i]}$) is reminiscent of the same

¹¹The first equality can be proved for logarithmic divergences, while in the general case it may look different [private communication Ch. Brouder].

process as lifting addition to multiplication. We denoted this mnemonically as $\Delta' = \delta^{\Delta^+}$. A similar process can be found in group theory and describes the branchings of characters of subgroups of $GL(n)$ which fix a tensor of arbitrary Young symmetry type π [31]. The insight which can be drawn from this work is, that branchings in group theory and the reorderings induced by branching operators, as demonstrated in [25], are in general based on 1-cochains which are **not product homomorphisms**. In the case of addition and multiplication it was seen that many number theoretic functions are not complete multiplicative. The cure was to introduce an unrenormalized (binomial) coproduct and to invent a subtraction scheme to remove the superfluous terms which were introduced to establish the homomorphism property at the end of a calculation. Our discussion sheds some light on our naming scheme of the arithmetic coproducts and pairings. Actually the non Hopf convolutions $(+, \Delta^+)$ and (\cdot, Δ') are of mathematical and physical interest. However they have bad algebraic behaviour forming a Hopf algebra only, so that it is more natural to adopt another coproduct and/or pairing map imposing a nice algebraic behaviour (Hopf algebra and homomorphism property). To get rid of the artificially introduced terms one invents then a subtraction scheme, called ‘renormalization’, which extracts the searched for results. We named our coproducts and pairings so that they comply with their potential usage in renormalization theory in pQFT.

6 Concluding remarks

It is our hope, to have convinced the reader, that the intimately linked structures of addition and multiplication also have a perfectly justified dual life. The coproducts of addition and multiplication play important roles in various branches of mathematics. It was the impetus of this work to make this obvious. The relation between addition and multiplication found its mnemonic counterpart in the formula $\Delta' = \delta^{\Delta^+}$.

The subtle distinction between multiplicative and complete multiplicative functions in arithmetic number theory has an algebraic counterpart in the fact that the additive and multiplicative convolutions are antipodal but **fail to be Hopf algebras**. This deficiency may be seen as the major source of hard problems in (analytic) arithmetic number theory. Since numbers can be addressed in various ways it is difficult to disentangle the role they perform in various settings. Numbers may just count something (states in physics). Numbers can operate on other numbers (operators in physics). Numbers may act as linear forms (dual states) etc. Our investigations have shown that, if a careful identification is done, one obtains new insight into the machinery of symmetric functions, quantum mechanics and quantum field theory. This has some implications for number theory also, since it implies that such methods as ‘renormalization’, after being turned into proper mathematics, may help to solve some long standing problems.

In this work ‘renormalization’ was formalized as the following procedure:

a: Observe that the positive selfadjoint convolutions of addition and multiplication, where products and coproducts are related by Kronecker duality, are antipodal convolutions but fail to be Hopf and form a **Hopf algebra** only [26, 32]. This stage is missing in physics, where the modelling is done using the mathematical structures appearing in stage **b:**, e.g. in number theory we need an ‘unrenormalization map’ to proceed to step **b:**.

b: Introduce a second duality by imposing a new coproduct which is identical on primitive elements to the dualized one but extended as an algebra homomorphisms. This new coproduct forms with the related product by construction a Hopf algebra with all the nice algebraic properties.

c: Compute via Laplace expansion [34, 53, 26, 12], which is available due to construction of the unrenormalized coproducts, a new unrenormalized pairing. It is this pairing which reflects the algebraic relation between variables and derivations, which is typical for the description of quantum systems.

d: Invent a **subtraction scheme**, called ‘renormalization’, which extracts those terms in the Hopf algebraic convolution which have been introduced by the unrenormalized coproduct to obtain the nice algebra. Prove that the Hopf algebra together with the subtraction scheme gives the same results as the original antipodal convolution (Hopf algebra). This subtraction scheme can be employed via a deformation or a Rota-Baxter operator, as we demonstrated in our quantum mechanical example in section 5.2.

To solidify our point of view we have applied our framework to number theory, symmetric function theory etc. An amazing fact is further, that our treatment of renormalization benefits from a number theoretic approach, and a characteristic free approach. In this sense it is not a miracle that number theoretic functions such as multiple zeta-values and Clausen functions appear in the evaluation of a renormalized

quantum field theory. The idea, proposed by Cartier [20], of a motivic Galois group seems hence to be related to the theory of Witt vectors (sec. 4.3) and the Witt functor, which actually provided us in the theory of symmetric functions with the correct basis.

This paper also provided some starting points for a characteristic free approach to quantum mechanics. The appearance of complex numbers, an algebraically closed field, is often argued to be a key feature, e.g. for producing interference effects, but we doubt this. A ‘phase’ may be modelled by a finite cyclic group also. The expectation values can be obtained in a topos theoretic setting using more general truth objects and therewith related subobject classifiers. This will be explored elsewhere. We think that the present work shows at least, that for the identification of the algebraic structures involved in quantum mechanics and quantum field theory, a characteristic free approach to quantum mechanics would be of great help. Especially the interpretation of combinatorial factors, normalizations etc. would benefit from such a view, even if the complex number field is finally adopted. The appearance of number theoretic functions in renormalization supports this point of view.,

While the present paper provided a detailed exposition of the Dirichlet Hopf algebra, it is necessary to point out, what should be done to unveil the underlying categorial structure. A careful study of branching operators is necessary. These operators not only appear in the theory of group branchings [29], but emerge also in the treatment of random walks [23] and in the study of entanglement of quantum states. There is also a close relation to categorial logic. We noticed, that the Hopf algebra cohomology becomes delicate if the homomorphism axiom no longer holds, even if it is not explicitly needed in the proof of a particular statement. Hence this cohomology has to be revisited, to show where the homomorphism axiom enters, and what can be still obtained having only the weakened multiplicativity axiom. The failure of the homomorphism axiom (complete multiplicativity) provides the key feature for deducing new branching rules and character formulae for non semisimple subgroups of $gl(n)$, as is shown in [31, 30]. The related problem of computing plethysms is a long standing one and algorithmically not solved in a satisfactory way. An analysis along the same lines as of the present paper is in progress, and a Hopf algebraic version of the plethysm may help to develop better algorithms if the coalgebra structure is at hand also. This is ongoing work [33], but recall section 4.3 on Witt vectors.

Of course loose ends are left with renormalization. Most of them need an identification of the proper algebraic structures involved, and combinatorics is a beautiful help in doing so. Indeed, in our opinion one needs to step back and to describe the categorial backbone of the structure, for more efficient identification of the separate algebras involved. In the modelling of quantum field theory algebraic structures are mostly used without semantic or syntactic distinctions. Such a categorial picture will naturally include 2-categories: we have already used $\text{Pleth}[A]$ and $\text{Sym Sym}[B]^+$ whose underlying modules form 2-vector space structures. In terms of operations, multiplication is the iterated addition, addition is the iterated successor map of Peano. In this sense, ‘renormalization’ seems to be necessary to interlock those iterated structures properly. To understand this process in a mathematically rigorous way is therefore of great importance.

It would hence be an interesting task to figure out the proper identification of the Bogoliubov map C back in the theory of symmetric functions. Actually this has to be a basis transformation between the monomial symmetric basis, which we used for the construction of $\text{Sym Sym}[B]^+$ and a basis of the form

$$\begin{aligned} \tilde{T}(a) &= C(a_{[1]})T(a_{[2]}) \\ &= \sum_{\lambda}' \sum c(b_{(1)}^1) \dots c(b_{(1)}^l) T(: \prod b_{(1)}^1, \dots, \prod b_{(2)}^l) \\ &= \sum_{\lambda}' \sum (b)^{(1)} \dots (b)^{(1)} T(: \prod b_{(1)}^1, \dots, \prod b_{(2)}^l) \end{aligned} \tag{6-1}$$

where the prefactors are evaluated under the Gessel map at monomials of type

$$h_{\lambda} = (a^{\#})^{(\lambda_1)} \circ \dots \circ (a^{\#})^{(\lambda_n)} \tag{6-2}$$

as can be guessed from [52] page 13063, in-line formula before proposition 1 (notions also taken from those sources). Rota and Stein identify the h_{λ} in a dual setting, therefore our usage of $a^{\#}$. Actually one

expects these functions to be related to the r_λ basis and its dual q_λ of the Witt ring. It is however very difficult to match these objects since common treatments do not take care of the proper algebraic distinction between a , $a^\#$ and the various algebraic structures which arise from the $\text{Pleth}[A]$ Hopf algebra and its dual $\text{Brace}\{A\}$, see loc. cit.

Our list of further opportunities and open problems could be prolonged considerably, and the attentive reader will have noticed these during the course of reading.

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A Matrix analogy for addition

In [28] we investigated some facts about positive selfadjoint Hopf algebras using singular value decomposition (SVD) methods. We will use this device here too, to establish a few relations about the product and coproduct of addition. First we introduce the rectangular ‘multiplication table’ m of addition

$$\begin{array}{c|cccccccccccc}
 + & (0,0) & (1,0) & (0,1) & (2,0) & (1,1) & (0,2) & (3,0) & (2,1) & (1,2) & (0,3) & \dots \\
 \hline
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\
 \vdots & \vdots & & & & & & & & & \vdots & \ddots
 \end{array} \tag{1-1}$$

This table is infinite and reads like this: The addition of the pair in the first row gives the sum (one term in fact) of the first column weighted by the column below the pair. E.g. $+(2,1) = 0 \cdot 0 + 0 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 + \dots$. The Kronecker addition coproduct has as its comultiplication table m^T , the **section coefficients**, the transposed infinite matrix.

Using ideas from SVD we may form the matrices $A = m m^T$ and $B = m^T m$ both rectangular and symmetric by construction. A and B have up to a kernel the same eigenvalues, which are squares of the singular values attached to m and m^T . We compute

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 4 & \dots \\ \vdots & & & \vdots & \ddots \end{bmatrix} \\
 B &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ \vdots & & & & & & & \vdots & \ddots \end{bmatrix}
 \end{aligned} \tag{1-2}$$

It is easily checked that B has also eigenvalues $1, 2, 3, 4, \dots$ and an infinite kernel. Hence one sees, that the addition has singular values $\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots$ which do **not** belong to \mathbb{Z}_+ .

From the above cited paper *Fausser, loc. cit.* we know, that the following two operators have then up to the kernel the same spectrum

$$\begin{aligned}
(+ \circ \Delta^+)(n) &= n_{(1)} + n_{(2)} = (n+1)n \\
\prod_{k \geq 0} ((+ \circ \Delta^+) - (k+1)) &= 0 \\
(\Delta^+ \circ +)^\infty \prod_{k \geq 0} ((\Delta^+ \circ +) - (k+1)) &= 0
\end{aligned} \tag{1-3}$$

The operator in the last line is, however, not diagonal. Eigenvectors are provided by the Kronecker coproduct of addition due to construction, see [28].

B Matrix analogy for multiplication

In the same way as we did with addition, we proceed with multiplication and compute the first elements of the multiplication table m

\cdot	(1, 1)	(2, 1)	(1, 2)	(3, 1)	(1, 3)	(4, 1)	(2, 2)	(1, 4)	(5, 1)	(1, 5)	\dots
1	1	0	0	0	0	0	0	0	0	0	\dots
2	0	1	1	0	0	0	0	0	0	0	\dots
3	0	0	0	1	1	0	0	0	0	0	\dots
4	0	0	0	0	0	1	1	1	0	0	\dots
5	0	0	0	0	0	0	0	0	1	1	\dots
\vdots	\vdots								\vdots	\ddots	

(2-1)

Note the different indexing of the first row by pairs of divisors of n . Once more, the section coefficients of the coproduct obtained by Kronecker dualization $\Delta \cdot = m^T$ is given by the transposition. In the SVD style, we can once more form the symmetric matrices $A = m m^T$ and $B = m^T m$, which read

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 2 & \dots \\ \vdots & & & \vdots & \ddots & \end{bmatrix} \\
B &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & & & & & & & & \vdots & \ddots & \end{bmatrix}
\end{aligned} \tag{2-2}$$

From this description one sees that the linearly ordered basis is a particular poor choice for ordering the involved basis elements. The monoid \mathbb{Z}_+ with adjoined multiplication map should be considered as a poset

under the partial order of mutual divisibility. Such a poset is graded by the numbers of prime factors of a number. The above matrix A would then come up with an infinite list of eigenvalues $1, 2^\infty, 3^\infty, 4^\infty, 5^\infty, \dots$ where ∞ notifies the countable infinity of primes. The same would be true for the matrix B , since it has the same eigenvalue structure as A up to a kernel due to SVD theory. The characteristic polynomial has hence the primes as roots and all of them have an infinite degeneracy

$$\begin{aligned} A : \quad & \prod_i (\cdot \circ \Delta' - p_i)^\infty = 0 \\ B : \quad & (\cdot \circ \Delta')^\infty \prod_i (\cdot \circ \Delta' - p_i)^\infty = 0, \end{aligned} \tag{2-3}$$

where the products run over all primes. Of course, these formulae make only formal sense due to the infinite degeneracy.

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