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Abstract

Recently, Courbet and Croisille [Math. Model. Numer. Anal., 32, 631–649, 1998] introduced the FV box-scheme for the 2D Poisson problem in the case of triangular meshes. Generalization to higher degree box-schemes has been published by Croisille and Greff [Numer. Methods Partial Differential Equations, 18, 355–373, 2002]. These box-schemes are based on the idea of the finite volume method in that they take the average of the equations on each cell of the mesh. This gives rise to a natural choice of unknowns located at the interface of the mesh. Contrary to the finite volume method, these box-schemes are conservative and use only one mesh. They can be seen as a discrete mixed Petrov-Galerkin formulation of the Poisson problem. In this paper we focus our interest on box-schemes for the Poisson problem in 2D on rectangular grids. We discuss the basic FV box-scheme, and analyse and interpret it as three different box-schemes. The method is demonstrated by numerical examples.

1 Introduction

The aim of this paper is to introduce several box-schemes for elliptic problems on rectangular grids based on the model of [6, 7, 8, 13]. The principle of the box-scheme we intend to discuss here in the case of rectangular grids goes back to H.B. Keller [17], where a box-scheme for the heat equation is introduced. In the case of an elliptic system, the principle of the box-schemes consists of discretizing the mixed form of the equation, by taking the average of the conservation and the constitutive laws on the same mesh. The convenient FE framework is the one of the so-called Petrov-Galerkin methods with two trial function spaces (one for the primitive variable u and one for the flux $p = \nabla u$) and two test function spaces. A comprehensive understanding of this kind of scheme has been introduced in [6, 7, 8] in the case of a triangular mesh. We refer also to [21] for finite volumes methods connected to Petrov-Galerkin formulation.

Here, we consider a rectangular domain $\Omega \subset \mathbb{R}^2$ meshed by a regular grid $\mathcal{T}_h$ made of rectangles (with edges parallel to those of the domain). For the simplicity of the presentation and since we focus on the design principles of different box-schemes, we restrict ourselves to the simple Poisson problem $-\Delta u = f$, for $f \in L^2(\Omega)$ with homogeneous boundary conditions. The mixed form we consider is: Find $(u, p) \in H_0^1(\Omega) \times H_{\text{div}}(\Omega)$ such that

$$\begin{cases} (\operatorname{div} p + f, v)_{0,\Omega} = 0 & \text{for all } v \in L^2(\Omega)\,, \\ (p - \nabla u, q)_{0,\Omega} = 0 & \text{for all } q \in \,(L^2(\Omega))^2. \end{cases} \tag{1}$$

where the space $H_{\rm div}(\Omega)$ is $H_{\rm div}(\Omega) = \{p \in (L^2(\Omega))^2\,;\ {\rm div}\ p \in L^2(\Omega)\}$. As in [6, 8], the discretization of (1) is performed by a mixed Petrov-Galerkin scheme called **box-scheme**. It involves four discrete spaces: $M_{1,h}, X_{1,h}$ as trial spaces, and $M_{2,h},\ X_{2,h}$ as test spaces. The box-scheme reads: Find $(u_h, p_h) \in M_{1,h} \times X_{1,h}$ such that

$$\left\{ \begin{array}{ll} \displaystyle\sum_{K\in\mathcal{T}_h} ({\rm div}\, p_h + f, v_h)_{0,K} = 0 & \text{for all } v_h \in M_{2,h}\,, \\ \displaystyle\sum_{K\in\mathcal{T}_h} (p_h - \nabla u_h, q_h)_{0,K} = 0 & \text{for all } q_h \in X_{2,h}\,. \end{array} \right. \tag{2}$$

The uniqueness of the solution of (2) implies in particular the identity of the dimensions

$$\dim\ M_{1,h} + \dim\ X_{1,h} = \dim\ M_{2,h} + \dim\ X_{2,h}\,. \tag{3}$$

The starting point of this article is the paper [5] by Courbet, where an original box-scheme on quadrangles is introduced for the time dependent diffusive problem. We give the interpretation of that scheme with three different box-schemes, which allows to identify its stability and accuracy properties. As is the case on a triangular mesh, the natural choice for the approximation of the flux p_h is the lowest order Raviart-Thomas space. For the unknown u_h we use the standard Q^1-Lagrange space, or its nonconforming analogue, Q^1_{nc} or the so-called P^1-nonconforming quadrilateral finite element, introduced by Park and Sheen [19]. Due to the properties of the different trial spaces, we can make the link between these three box-schemes explicit. An important characteristic of these box-schemes is their equivalence with a decoupled formulation in the unknowns u_h and p_h separately. This allows the computation of the discrete flux p_h in a cheap way, since it is just given as a function of ∇u_h and the right-hand side f. This local reconstruction of the flux p_h in each cell is in particular of interest for porous media problems, e.g. contaminant transport where the velocity is computed by the Darcy law and introduced in a convection-diffusion equation for the computation of the concentration. This decoupled feature of the box-scheme extends the observation by Marini [18], that the flux in the mixed FEM can be recovered in an inexpensive way. Concerning the *a posteriori* error estimates of the box-scheme, we refer to the recent work by El Alaoui, Ern, [10, 11]. Finally, let us mention that an increasing interest in box-schemes has recently appeared [3, 4]. Note that a different possibility for extending the box-scheme of [6] on rectangles, using the Rannacher-Turek nonconforming FE space, has been studied in [3, 14, 15].

Let us give now some standard notation. We introduce the mesh dependent norms defined respectively on the mesh dependent spaces $H^1_0(\Omega) + M_{1,h}$ and $H_{\rm div}(\Omega) + X_{1,h}$:

$$|u|_{1,h} = \Big(\sum_K |\nabla\, u|^2_{0,K}\Big)^{1/2}, \quad \|u\|_{1,h} = (|u|^2_{0,\Omega} + |u|^2_{1,h})^{1/2} \quad \text{for all } u \in H^1_0(\Omega) + M_{1,h}\,,$$

$$|p|_{{\rm div},h} = \Big(\sum_K |\ {\rm div}\ p|^2_{0,K}\Big)^{1/2}, \quad \|p\|_{{\rm div},h} = (|p|^2_{0,\Omega} + |p|^2_{{\rm div},h})^{1/2} \quad \text{for all } p \in H_{\rm div}(\Omega) + X_{1,h}\,.$$

The geometrical notation is as follows. The rectangles are denoted by K with centre $G_K(x_K, y_K)$, area $|K|$, and diameter h_K. We denote by h the maximum of the diameters of the elements of the mesh. The sizes of the sides of the rectangle K are $|e_{K,x}|$ and $|e_{K,y}|$. We will write ∂K for the set of edges of K. The sets $\mathcal{A}_i$ and $\mathcal{A}_b$ denote the internal and boundary edges respectively. We define $\mathcal{A} = \mathcal{A}_i \cup \mathcal{A}_b$ to be the set of all edges with global numbering. The number of rectangles is NE. The number of edges (respectively internal, boundary edges) is NA (respectively NA_i, NA_b). The number of vertices (respectively internal, boundary vertices) is NV (respectively NV_i, NV_b). The Euler relations are

$$4NE = NA_i + NA \quad \text{and} \quad NE - NA + NV = 1. \tag{4}$$

The outgoing unitary normal vector to an edge e is ν_e. More generally, we write ν. The mid-point of an edge e is x_e and $[u]_e$ denotes the jump of u along e. The gradient of f is $\nabla f = [\partial_x f\,,\,\partial_y f]^T$ and the 2D rotational is $\operatorname{curl} f = [\partial_y f\,,\,-\partial_x f]^T$. The letter C denotes some generic constant independent of the mesh. Let P^0 be the space of piecewise constant functions, P^1 be the space of piecewise affine functions and Q^1 be the space of bilinear functions. We define Π^0 to be the classical projection operator on the piecewise constant functions. Let us recall the definition of RT^0 the lowest order space of Raviart-Thomas [20], useful to discretize the flux $p = \nabla u$:

$$RT^0 = \{q_h \in H_{\rm div}(\Omega)\,;\, q_h \in RT^0(K),\ \ \forall K \in \mathcal{T}_h\}\,,$$

where the local space $RT^0(K)$ is

$$RT^0(K) = P^0(K)^2 + P^0(K)\begin{pmatrix} x \\ 0 \end{pmatrix} + P^0(K)\begin{pmatrix} 0 \\ y \end{pmatrix}.$$

The space RT^0 is of dimension NA, the degrees of freedom being given by the linear forms

$$L_a(q_h) = \frac{1}{|a|}\int_a q_h \cdot \nu_a \, d\sigma \quad \text{for all } a \in \mathcal{A}.$$

Note that the normal component $p_h \cdot \nu_a$ of p along each interior edge is constant.

The outline of the paper is as follows. In Section 2, we recall briefly (in the particular case of the Poisson problem) the design principles of Courbet's scheme. To understand it in a finite element sense, we introduce in Section 3 a finite element box-scheme based on the relation between the Courbet space (a space used by Courbet to approximate the unknown u, which will be defined in the next section) and the standard Q^1-Lagrange finite element space. The approximation of the flux $p = \nabla u$ is done using the Raviart-Thomas space. However, this box-scheme seems to be instable. The idea of Section 4 is to build a new box-scheme generalizing the previous one and based on the inclusion of the space Q^1-Lagrange into Q^1-nonconforming. Both unknowns u and p are discretized in nonconforming spaces with respect to $H^1_0(\Omega)$ and $H^1_{\rm div}(\Omega)$. We perform the numerical analysis of the scheme, and its equivalence to a decoupled problem in u_h and p_h: a nonconforming scheme in u_h and a local reconstruction formula of p_h (in function of u_h and the data f). Consequently, we can make explicit the link with the box-scheme of Section 3. It turns out that the solution u_h of the box-scheme is only affine (and not bilinear) per rectangle. Section 5 is devoted to the development and the analysis of a reduced box-scheme. We conclude this work with numerical results in Section 6. Note that the most part of this paper has been presented in [13]. See also [14].

2 Courbet's box-scheme

2.1 Introduction

In [5], B. Courbet has introduced a box-scheme for the time dependent mixed formulation of the compressible Navier-Stokes equations. The scheme was intended to extend to a rectangular grid the well-known box-scheme of H. B. Keller for the heat equation introduced in [17]. In the case of the Poisson problem, the box-scheme of Courbet is a derivation of the mixed form of the problem taken as mean value on each rectangle K:

$$\left\{\begin{array}{l} \displaystyle\int_K \operatorname{div} p\, dx + \int_K f\, dx = 0\,, \\ \displaystyle\int_K p\, dx - \int_K \nabla u\, dx = 0\,, \\ u = 0 \ \text{ on } \partial\Omega\,. \end{array}\right. \tag{5}$$

The Courbet box-scheme referred later as (BS1) is: Find $u = (u_a)_{a\in\mathcal{A}}$ and $p = (p_a)_{a\in\mathcal{A}}$ such that for all rectangles K of the grid:

$$\text{(BS1)} \quad \left\{\begin{array}{l} \displaystyle\sum_{a\in\partial K} |a|\, p_{a,K} + \int_K f\, dx = 0\,, \\ \displaystyle\frac{(p_{a_{1,K}} - p_{a_{3,K}})}{2} - \frac{(|a_{1,K}| u_{a_{1,K}} - |a_{3,K}| u_{a_{3,K}})}{|K|} = 0\,, \\ \displaystyle\frac{(p_{a_{2,K}} - p_{a_{4,K}})}{2} - \frac{(|a_{2,K}| u_{a_{2,K}} - |a_{4,K}| u_{a_{4,K}})}{|K|} = 0\,, \\ u_a = 0\,, \quad \forall a \in \mathcal{A}_b\,, \end{array}\right. \tag{6}$$

where the subscripts $a_{1,K}$, $a_{2,K}$, $a_{3,K}$ and $a_{4,K}$ are related to the edges $a_{1,K}$, $a_{2,K}$, $a_{3,K}$ and $a_{4,K}$ of each rectangle K (see Figure 1). The unknowns u_a and p_a denote respectively the average of u and the normal component of the flux $p = \nabla u$ along an edge a and are located at the interface of the mesh. This gives $4NE$ unknowns and $3NE$ equations. In contrast with the analogous scheme on triangles introduced in [6], here is a lack of NE equations. Courbet suggests to add the constraint on each rectangle K as a discrete equation:

$$u_{a_{1,K}} + u_{a_{3,K}} = u_{a_{2,K}} + u_{a_{4,K}}\,. \tag{7}$$

In particular the mean value of the solution u in each box coincides with its horizontal and vertical average. Let us denote by C_0 the space introduced by Courbet to discretize the unknown

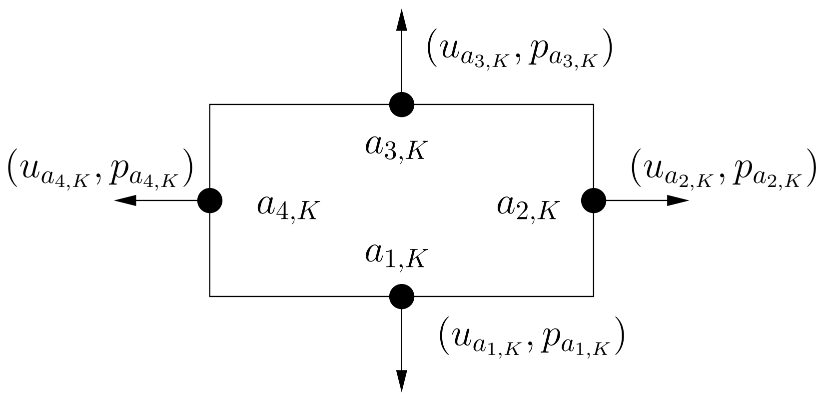

Figure 1: Rectangle K, with edges $a_{i,K}$ and unknowns $(u_{a_{i,K}}, p_{a_{i,K}})$, for $i = 1, \ldots, 4$.

u. It is generated by vectors whose size is the number of edges of the domain Ω, vanishing at the boundary, and satisfying the additional condition (7) on each rectangle of the grid. The space C_0 is defined by:

$$\begin{aligned} C_0 &= \big\{(u_a)_{a\in\mathcal{A}} \in \mathbb{R}^{NA} \text{ such that } u_a = 0 \quad \text{for all } a \in \mathcal{A}_b \text{ and} \\ &\qquad\qquad u_{a_{1,K}} + u_{a_{3,K}} = u_{a_{2,K}} + u_{a_{4,K}} \text{ on each rectangle } K\big\}\,, \\ &= \big\{(u_a)_{a\in\mathcal{A}_i} \in \mathbb{R}^{NA_i}, u_{a_{1,K}} + u_{a_{3,K}} = u_{a_{2,K}} + u_{a_{4,K}} \text{ on each rectangle } K\big\}\,. \end{aligned} \tag{8}$$

However, the dimension of the space C_0 is $\dim C_0 = NV_i = NA_i - NE + 1 > NA_i - NE$. Indeed the boundary degrees of freedom of the space C_0 are not independent. In fact, if $u_a = 0$ for $NA_b - 1$ boundary edges, then $u_a = 0$ holds on the last one. This implies that the box-scheme (BS1) does not define a well-posed problem in the sense that the number of unknowns is larger than the number of equations. Actually, due to the dimension of the space C_0, the number of unknowns is: Number of unknowns $(u_a,\, p_a) \;=\; NV_i + NA = 3NE + 1$ whereas there are only $3NE$ equations. Despite this dimension inconsistency, the numerical results reported in [5] for the time dependent heat equation prove that there is no drawback in practice.
The observation that $\dim C_0 = NV_i$ suggests that the space C_0 is identical to the Q^1-Lagrange space with homogeneous boundary conditions:

Lemma 2.1 *The mapping L defines a bijection between $Q^1_{c,0}$ and the Courbet space*

$$\begin{aligned} L:\, Q^1_{c,0} &\longrightarrow C_0 \\ u &\longmapsto \big(u(x_a)\big)_{a\in\mathcal{A}}\,, \end{aligned}$$

where x_a denotes the mid-edge of a and Q^1_c is the standard Q^1-Lagrange finite element space:

$$Q^1_c = \{u \in \mathcal{C}^0(\Omega)\;;\; u \in Q^1(K)\;\; \forall K \in \mathcal{T}_h\,\}\,, \quad Q^1(K) = \mathrm{Span}\{1,\, x,\, y,\, xy\}\,,$$

and $Q^1_{c,0}$ its restriction to functions vanishing on $\partial\Omega$.

The proof of the lemma follows from the linearity, injectivity (see Proposition 2.1 hereafter) of the mapping L, and the dimension equality of the spaces C_0 and $Q^1_{c,0}$. Before going further with the stabilization of the box-scheme (BS1), we recall some useful properties of the nonconforming Q^1 finite element space and its relation to Q^1_c.

2.2 Some properties of the Q^1 nonconforming space

The nonconforming Q^1 finite element space denoted by Q^1_{nc} is defined by:

$$Q^1_{nc} = \Big\{u_h \in L^2(\Omega)\;;\; u_h \in Q^1(K)\,\forall K \in \mathcal{T}_h\,;\int_a u_{h|K_1}\,d\sigma = \int_a u_{h|K_2}\,d\sigma\,\forall a = \partial K_1 \cap \partial K_2 \in \mathcal{A}_i\Big\}\,.$$

The space $Q^1_{nc,0}$ is the zero boundary space: $Q^1_{nc,0} = \Big\{u_h \in Q^1_{nc}\;;\; \int_a u_h\,d\sigma = 0\;\forall a \in \mathcal{A}_b\Big\}\,.$
Since the edges of the mesh are parallel to the axis of the domain, the mean value of a function in Q^1 along an edge is also the value at the mid-point of the edge. We recall that for all $v_h \in Q^1_{nc}$, the set of values

$$p_a(v_h) = v_h(x_a) \text{ for all } a \in \mathcal{A} \text{ with the associated mid-point } x_a,$$

does not form a unisolvent set of degrees of freedom [1, 2, 12]. Indeed let η be the function defined on $Q^1(K)$ by:

$$\begin{aligned} \eta \,:\, Q^1(K) &\longrightarrow \mathbb{R}^4 \\ p &\longmapsto \big(p(x_a)\big)_{a\in\partial K}\,. \end{aligned}$$

It is well-known that the kernel of η is of dimension 1, generated by the nonconforming bubble b_K:

$$\operatorname{Ker}\eta = \operatorname{Span}\{b_K\}\;, \quad b_K(x,y) = \frac{4}{|K|}(x-x_K)(y-y_K),$$

where (x_K, y_K) is the centre of the rectangle K. It is easy to check that for any $K \in \mathcal{T}_h$ and any $v \in Q^1(K)$, the function b_K has the following properties:

$$\int_{\partial K} b_K \, d\sigma = 0\;, \qquad \int_K \operatorname{curl} b_K \cdot \nabla v \, dx = 0\;, \qquad \int_{\partial K} (\operatorname{curl} b_K \cdot \nu)\, v \, d\sigma = 0. \tag{9}$$

Let Ψ be the vector space generated by the local bubbles:

$$\Psi = \{\psi\,;\psi_{|K} = \alpha_K b_K,\ \alpha_K \in \mathbb{R}\ \ \forall K \in \mathcal{T}_h\}\,.$$

Then, $\dim \Psi = NE$ and $\Psi \subseteq Q^1_{nc,0}$.

Definition 2.1 *Let us define the following element* $\mathcal{B} \in Q^1_c$ *by* $\mathcal{B} = \sum_{K\in\mathcal{T}_h} \operatorname{sgn}(K) b_K$*, where* $\operatorname{sgn}(K)$ *takes alternately the values* -1, $+1$ *as displayed on the Figure 2.*

$-$	$+$	$-$	$+$	$-$
$+$	$-$	K	$-$	$+$
$-$	$+$	$-$	$+$	$-$

Figure 2: Sign of K.

$\mathcal{B}$ is the so called hourglass-mode introduced by Hansbo in [16], which gives rise to some instability. By using this definition and the properties of the previous spaces, we prove the following proposition.

Proposition 2.1 *The spaces* Q^1_c, Q^1_{nc} *and* Ψ *satisfy:*

$$(i)\quad Q^1_c \cap \Psi = \operatorname{Span}\{\mathcal{B}\}, \quad (ii)\quad Q^1_{nc} = Q^1_c + \Psi, \quad (iii)\quad Q^1_{nc,0} = Q^1_{c,0} \oplus \Psi.$$

In particular, $\dim Q^1_{nc} = NA + 1$ *and* $\dim Q^1_{nc,0} = NA_i$*.*

Proof *(i)* Let $\psi = \sum_{K \in \mathcal{T}_h} \alpha_K b_K \in Q^1_c \cap \Psi$. The continuity of ψ through each internal edge $a = \partial K_1 \cap \partial K_2$ implies $\alpha_K = \mathrm{sgn}(K)\,\alpha$, for all $K \in \mathcal{T}_h$, $\alpha \in \mathbb{R}$.
(ii) Let us define the space $M = Q^1_c + \Psi$. Then, $M \subseteq Q^1_{nc}$. To prove $M = Q^1_{nc}$, we will prove that $\dim M = \dim Q^1_{nc}$. Let i be the linear map

$$\begin{array}{rl} i: Q^1_{nc} & \longrightarrow \mathbb{R}^{NA} \\ u & \longmapsto \big(u(x_a)\big)_{a \in \mathcal{A}}\,, \end{array}$$

where x_a is the mid-point of the edge $a \in \mathcal{A}$. Using the definitions of Ψ and b_K, we prove that

$$\begin{aligned} \mathrm{Ker}\ i &= \Big\{u \in Q^1_{nc};\ u(x_a) = 0\ \forall a \in \mathcal{A}\Big\} \subseteq \Psi\,, \qquad \text{and} \\ \mathrm{Im}\ i &= \Big\{(u(x_a))_{a \in \mathcal{A}} \in \mathbb{R}^{NA}\ ;\ u(x_{a_{K_1}}) + u(x_{a_{K_3}}) = u(x_{a_{K_2}}) + u(x_{a_{K_4}})\Big\}\,. \end{aligned}$$

This in turn gives that $\dim Q^1_{nc} = \dim(\mathrm{Ker}\, i) + \dim(\mathrm{Im}\, i) \leq NA$. Moreover, $\dim M = \dim Q^1_c + \dim \Psi - \dim(Q^1_c \cap \Psi) = NV + NE - 1$. The Euler relation gives $\dim M = NA$. $M \subseteq Q^1_{nc}$, so $\dim M \leq \dim Q^1_{nc}$. We deduce that $\dim Q^1_{nc} = NA$, which concludes the proof of *(ii)*.
The statement *(iii)* is directly implied by *(i)* and *(ii)*. ■
Using the property of $\mathcal{B}$ and the continuity of the normal component of the element in RT^0, we deduce the following lemma

Lemma 2.2 *Let Φ be the vector space generated by the curl of the nonconforming bubble*

$$\Phi = \mathrm{curl}\,\Psi = \{\phi = \sum_{K \in \mathcal{T}_h} \beta_K\, \mathrm{curl}\, b_K,\ \ \beta_K \in \mathbb{R}\}\,.$$

Then, $\Phi \cap RT^0 = \mathrm{Span}\{\mathrm{curl}\big(\mathcal{B}\big)\}$.

Note that the box-scheme (BS1) is a derivation of the mixed formulation of the Laplace equation on each rectangle K given by the system (5). Since $\int_K \mathrm{curl}\, b_K\, dx = 0$ and $\mathrm{div}(\mathrm{curl}\, b_K) = 0$ for any $\beta_K \in \mathbb{R}$, we get from the mixed formulation (5) that for any β_K, we can superpose to p_h any function $\sum_K \beta_K\ \mathrm{curl}\, b_K$, which is a parasitic mode. Therefore a stabilization of the scheme has to eliminate that mode.

3 A first stabilization of the Courbet's box-scheme

We will now give a first stabilization of the box-scheme (BS1) using the finite element interpretation of the space C_0 coupled to the Raviart-Thomas space RT^0 in order to discretize the unknowns (u, p). We also need to add one additional test function in order to have the right number of equations. The element $\mathcal{B}$ is the simplest choice according to results of the previous section.

Proposition 3.1 *Let us call (BS2) the box-scheme: Find the solution $(u_h, p_h) \in Q^1_{c,0} \times RT^0$ of*

$$(BS2)\quad \left\{ \begin{array}{l} (\mathrm{div}\, p_h + f, v_h)_{0,\Omega} = 0\ \ \forall v_h \in P^0\,, \\ (p_h - \nabla u_h, q_h)_{0,\Omega} = 0\ \ \forall q_h \in X_{2,h} = (P^0)^2 + \mathrm{Span}\{\mathrm{curl}(\mathcal{B})\}. \end{array} \right. \tag{10}$$

(i) The box-scheme (BS2) has $3NE + 1$ unknowns.
(ii) The box-scheme (BS2) has a unique solution given by:

(a) $u_h \in Q^1_{c,0}$ is the solution of:

$$\sum_{K \in \mathcal{T}_h} (\Pi^0 \nabla u_h, \Pi^0 \nabla v_h)_{0,K} = (\Pi^0 f, v_h)_{0,\Omega} \text{ for all } v_h \in Q^1_{c,0} \,.$$

(b) p_h is given by:

$$p_{h|K} = (\Pi^0 \nabla u_h)_K - \frac{\Pi^0 f_{|K}}{2} \begin{pmatrix} x - x_K \\ y - y_K \end{pmatrix} + \gamma_K \begin{pmatrix} x - x_K \\ -(y - y_K) \end{pmatrix} ,$$

where γ_K is the solution of a certain sparse linear system.

Proof *(i)* Using the Euler relation, we get the following identity between the number of unknowns and the number of equations:

$$\dim Q^1_{c,0} + \dim RT^0 = NV_i + NA = 3NE + 1 = \dim P^0 + \dim (X_{2,h}) \,.$$

(ii) Let $(u_h, p_h) \in Q^1_{c,0} \times RT^0$ be a solution of (BS2). We prove that (u_h, p_h) satisfies the system ((a),(b)).
(a) Suppose given $v_h \in Q^1_{c,0}$ then $q_h = \Pi^0(\nabla v_h) \in X_{2,h}$. Introducing this value of q_h in $(10)_2$, afterwards using the decomposition $\nabla v_{h|K} = \Pi^0 \nabla v_{h|K} + \delta_K \nabla b_K$ for any $\delta_K \in \mathbb{R}$ and Green's formula, we get

$$\begin{aligned} \sum_K (\nabla u_h, \Pi^0 \nabla v_h)_{0,K} &= \sum_K (p_h, \Pi^0 \nabla v_h)_{0,K} \\ &= \sum_K (p_h, \nabla v_h - \delta_K \nabla b_K)_{0,K} \\ &= -\sum_K \int_K \operatorname{div} p_h \, v_h \, dx + \sum_K \int_{\partial K} v_h \, p_h \cdot \nu \, d\sigma \\ &\quad + \sum_K \int_K \delta_K \operatorname{div} p_h \, b_K \, dx - \sum_K \int_{\partial K} p_h \cdot \nu \, \delta_K \, b_K \, d\sigma \,. \end{aligned} \tag{11}$$

Since the mean value of the bubble function b_K vanishes and $p_h \in RT^0$, we have that $\int_K \delta_K \operatorname{div} p_h \, b_K \, dx = 0$ and $\int_{\partial K} p_h \cdot \nu \, \delta_K \, b_K \, d\sigma = 0$. On the other hand, the equation $(10)_1$ gives $\operatorname{div} p_{h|K} = -\Pi^0 f_{|K}$ for all $K \in \mathcal{T}_h$. Therefore the equality (11) becomes:

$$\sum_K (\nabla u_h, \Pi^0 \nabla v_h)_{0,K} = \sum_K \int_K \Pi^0 f v_h \, dx - \sum_{a \in \mathcal{A}_i} \int_a p_h \cdot \nu_a \, [v_h]_a \, d\sigma + \sum_{a \in \mathcal{A}_b} \int_a p_h \cdot \nu_a \, v_h \, d\sigma \,. \tag{12}$$

Since $v_h \in Q^1_{c,0}$, $[v_h]_a = 0$ for all $a \in \mathcal{A}_i$ and $v_{h|a} = 0$ for all $a \in \mathcal{A}_b$, the relation (12) becomes

$$\sum_K (\nabla u_h, \Pi^0 \nabla v_h)_{0,K} = \sum_K (\Pi^0 f, v_h)_{0,K} \,,$$

which concludes (a).
(b) Any element p_h in $RT^0(K)$ can be decomposed as

$$p_{h|K} = (\Pi^0 p_h)_{|K} + \frac{\operatorname{div} p_{h|K}}{2} \begin{pmatrix} x - x_K \\ y - y_K \end{pmatrix} + \gamma_K \begin{pmatrix} x - x_K \\ -(y - y_K) \end{pmatrix} , \quad \gamma_K \in \mathbb{R} \,.$$

Using respectively the equations $(10)_1$ and $(10)_2$, we get $\operatorname{div} p_{h|K} = -\Pi^0 f_{|K}$ and $(\Pi^0 p_h)_{|K} = (\Pi^0 \nabla u_h)_{|K}$. Then

$$p_{h|K} = (\Pi^0 \nabla u_h)_{|K} - \frac{\Pi^0 f_{|K}}{2} \begin{pmatrix} x - x_K \\ y - y_K \end{pmatrix} + \gamma_K \begin{pmatrix} x - x_K \\ -(y - y_K) \end{pmatrix} .$$

The computation of the coefficient γ_K is done using (a), the equation $(10)_2$ with $q_h = \sum_K \operatorname{sgn}(K) \operatorname{curl} b_K$, and the continuity of the normal component of p_h.
This implies that any solution of the box-scheme (BS2) is a solution of the system $((a),(b))$, which is unique ($f = 0$ in the system ((a),(b)) implies $u_h = 0$ and $p_h = 0$). The existence of solutions of (BS2) is deduced from the uniqueness of the solution, the linearity of the problem, and the equality between the number of unknowns and equations. ■

Remarks:
(i) As proved by Hansbo, [16], the 1-point integration of the gradient of u_h, (Lemma 3.1 *(ii)*) is not sufficient to obtain stability of the scheme.
(ii) The parasitic perturbation $\sum_K \beta_K \operatorname{curl} b_K \in \Phi$ seems to be controlled globally by the box-scheme but not locally. As a consequence, we do not get a local reconstruction of the flux p_h in each rectangle K.

4 A second stabilization of Courbet's box-scheme

Due to its possible instability, the box-scheme (BS2) is not totally satisfying. So, we want to build a box-scheme using larger spaces for both unknowns u and p. The basic idea is to use the nonconforming space $Q^1_{nc,0}$ containing the Q^1-Lagrange space $Q^1_{c,0}$ (used in (BS2)) for the approximation of u. For the flux, we consider the space RT^0 of Raviart-Thomas, supplemented with the space Φ of the rotational of the bubble. Note that those spaces are both nonconform respectively in $H^1_0(\Omega)$ and $H_{\text{div}}(\Omega)$. Also this choice of spaces gives the advantage to get the number of unknowns proportional to the number of rectangles, i.e. the trial spaces in (2) can be piecewise polynomial spaces.

4.1 Definition of the box-scheme

Proposition 4.1 *Let (BS_{nc}) be the following box-scheme: Find $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ being solution of*

$$(BS_{nc}) \begin{cases} \displaystyle\sum_{K \in \mathcal{T}_h} (\operatorname{div} p_h + f, v_h)_{0,K} = 0 \ \forall v_h \in M_{2,h} = P^0 , \\ \displaystyle\sum_{K \in \mathcal{T}_h} (p_h - \nabla u_h, q_h)_{0,K} = 0 \ \forall q_h \in X_{2,h} = (P^0)^2 + P^0 \begin{pmatrix} y \\ x \end{pmatrix} + P^0 \begin{pmatrix} x \\ -y \end{pmatrix} . \end{cases} \quad (13)$$

(i) The box-scheme (BS_{nc}) has $5NE$ unknowns.
(ii) The box-scheme (BS_{nc}) has a unique solution $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ given by:

(a) u_h is the solution of the variational problem: Find $u_h \in Q^1_{nc,0}$ such that

$$\sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v_h)_{0,K} = (\Pi^0 f, v_h)_{0,\Omega} \quad \textit{for all } v_h \in Q^1_{nc,0} .$$

(b) p_h is locally given by:

$$p_{h|K} = (\nabla u_h)_{|K} - \frac{\Pi^0 f_{|K}}{|e_{x,K}|^2 + |e_{y,K}|^2} \begin{pmatrix} |e_{y,K}|^2 (x - x_K) \\ |e_{x,K}|^2 (y - y_K) \end{pmatrix} .$$

Note that this box-scheme is nonconform for both unknowns u_h and p_h. The test spaces $M_{2,h}$ and $X_{2,h}$ (in the system (2)) are piecewise polynomial functions. Remark that $X_{2,h}$ is also $X_{2,h} = (P^0)^2 + P^0(\nabla b_K) + P^0(\operatorname{curl} b_K)$, in particular, $\nabla(M_{1,h}) \subseteq X_{2,h}$.

Proof *(i)* By the Euler relations, we prove that:
$\dim\, Q^1_{nc,0} + \dim\,(RT^0 + \Phi) = (NA_i + 1) + (NA + NE - 1) = 5\, NE = \dim\, X_{2,h} + \dim\, P^0$.
(ii) Let us prove that any $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ satisfying the equations (BS_{nc}) fulfils the system ((a),(b)).
(a) Let $v_h \in Q^1_{nc,0}$. Let $q_h = \nabla v_h \in X_{2,h}$ in equation $(13)_2$. By integration by parts

$$\sum_K (\nabla u_h \,,\, \nabla v_h)_{0,K} \;=\; -\sum_K \int_K \operatorname{div}\, p_h\, v_h\, dx + \sum_K \int_{\partial K} (p_h \cdot \nu)\; v_h\, d\sigma \,.$$

Moreover, since $\operatorname{div} p_{h|K} \in P^0(K)$, the equation $(13)_1$ gives $\operatorname{div} p_{h|K} = -\Pi^0 f_{|K}$. Hence, $p_h \in RT^0 + \Phi$ can be written as $p_h = \bar{p}_h + \sum_K \beta_K \operatorname{curl} b_K$, with $\bar{p}_h \in RT^0$. This implies that

$$\begin{aligned} \sum_K (\nabla u_h \,,\, \nabla v_h)_{0,K} &= \sum_K (\Pi^0 f, v_h)_{0,K} + \sum_K \int_{\partial K} (\bar{p}_h + \beta_K \operatorname{curl} b_K) \cdot \nu \; v_h\, d\sigma \\ &= (\Pi^0 f, v_h)_{0,\Omega} + \sum_K \int_{\partial K} \bar{p}_h \cdot \nu \; v_h\, d\sigma + \sum_K \int_{\partial K} \beta_K \operatorname{curl} b_K \cdot \nu \; v_h\, d\sigma \,. \end{aligned}$$

Using the properties (9) of the bubble b_K and the continuity of the normal component of elements $p_h \in RT^0 \subset H_{\mathrm{div}}(\Omega)$, we obtain

$$\sum_K (\nabla u_h \,,\, \nabla v_h)_{0,K} \;=\; (\Pi^0 f, v_h)_{0,\Omega} + \sum_{a \in \mathcal{A}_b} \int_a \bar{p}_h \cdot \nu_a\, u_h\, d\sigma - \sum_{a \in \mathcal{A}_i} \int_a \bar{p}_h \cdot \nu_a\, [u_h]_a\, d\sigma \,.$$

Since $u_h \in Q^1_{nc,0}$ and $p_h \cdot \nu_a \in P^0(a)$,

$$\int_a \bar{p}_h \cdot \nu_a\, u_h\, d\sigma = 0 \quad \text{for all } a \in \mathcal{A}_b \qquad \text{and} \qquad \int_a \bar{p}_h \cdot \nu_a\, [u_h]_a\, d\sigma = 0 \;\; \text{for all } a \in \mathcal{A}_i \,,$$

which concludes (a). In particular, for $v_h = b_K \in Q^1_{nc,0}$,

$$(\nabla u_h \,,\, \nabla b_K)_{0,K} \;=\; (\Pi^0 f, b_K)_{0,K} \,.$$

The mean value of b_K equals 0 on each rectangle, hence $(\Pi^0 f, b_K)_{0,K} = 0$. Also, ∇u_h is locally written as $(\nabla u_h)_{|K} = (\Pi^0 \nabla u_h)_{|K} + d_K \nabla b_K$, where d_K is given by u_h. We deduce that

$$\begin{aligned} 0 \,=\, (\nabla u_h \,,\, \nabla b_K)_{0,K} &= ((\Pi^0 \nabla u_h)_{|K} + d_K \nabla b_K ,\, \nabla b_K)_{0,K} \\ &= \underbrace{((\Pi^0 \nabla u_h)_{|K} ,\, \nabla b_K)_{0,K}}_{=0} + d_K |\nabla b_K|^2_{0,K} \,. \end{aligned}$$

It means that $d_K = 0$ or equivalently that the bubble component of the solution u_h vanishes. In particular, $(\nabla u_h)_{|K} = (\Pi^0 \nabla u_h)_{|K}$.
(b) Since $p_h \in RT^0 + \Phi$, we have $p_h = \bar{p}_h + \sum_K \beta_K \operatorname{curl} b_K$ with $\bar{p}_h \in H_{\mathrm{div}}(\Omega)$. Again for each rectangle K, $\operatorname{div} p_{h|K} = -\Pi^0 f_{|K}$ and $\operatorname{div}(\operatorname{curl} b_K) = 0$, so that

$$\operatorname{div} p_h = \operatorname{div} \bar{p}_h = -\Pi^0 f\,.$$

The equation $(13)_2$ implies $(\Pi^0 p_h)_{|K} = (\Pi^0 \nabla u_h)_{|K} = (\nabla u_h)_{|K}$. On the other hand,

$$p_{h|K} = (\Pi^0 p_h)_{|K} + \frac{\operatorname{div} p_{h|K}}{2}\begin{pmatrix} x - x_K \\ y - y_K \end{pmatrix} + \tilde{\beta}_K \begin{pmatrix} x - x_K \\ -(y - y_K) \end{pmatrix}, \quad \tilde{\beta}_K \in \mathbb{R}\,.$$

This is equivalent to

$$p_{h|K} = \underbrace{(\nabla u_h)_{|K} - \frac{\Pi^0 f_{|K}}{2}\begin{pmatrix} x - x_K \\ y - y_K \end{pmatrix}}_{p_{h,1}} + \underbrace{\tilde{\beta}_K \begin{pmatrix} x - x_K \\ -(y - y_K) \end{pmatrix}}_{p_{h,2}}.$$

For evaluating the coefficient $\tilde{\beta}_K$, we use the equation $(13)_2$ with $q_h = \operatorname{curl} b_K$:

$$\int_K (p_{h,1} + p_{h,2} - \nabla u_h) \cdot \operatorname{curl} b_K \, dx = 0\,. \tag{14}$$

We know by (9) that $\int_K \nabla u_h \cdot \operatorname{curl} b_K \, dx = 0$. Inserting the value of $p_{h,1}$, the equation (14) becomes

$$\int_K p_{h,2} \cdot \operatorname{curl} b_K \, dx = \frac{2\Pi^0 f_{|K}}{|K|} \int_K (x - x_K)^2 - (y - y_K)^2 \, dx\,.$$

Using the following identities

$$\textstyle\int_K (x - x_K)^2 = \frac{|K|}{12}|e_{x,K}|^2\,, \qquad \int_K (y - y_K)^2 = \frac{|K|}{12}|e_{y,K}|^2\,, \tag{15}$$

and the definition of $p_{h,2}$, give

$$\tilde{\beta}_K = \frac{\Pi^0 f_{|K}}{2}\frac{|e_{x,K}|^2 - |e_{y,K}|^2}{|e_{x,K}|^2 + |e_{y,K}|^2}.$$

We have proved that a solution (u_h, p_h) of the box-scheme (BS_{nc}) is also solution of the problem ((a),(b)), which is unique. This proves the uniqueness of the solution of the box-scheme (BS_{nc}). The linearity and the equality between the number of unknowns and the number of equations permit to conclude existence and uniqueness of the solution of the box-scheme (BS_{nc}) and its equivalence with the formulation ((a),(b)). This concludes *(ii)*. ■

The previous result states that the box-scheme (BS_{nc}) is well-posed and equivalent to a single scheme in u_h alone and an explicit reconstruction formula for p_h. More precisely, u_h is the solution of the nonconforming variational formulation for the problem $-\Delta u = \Pi^0 f$. It also generalizes the previous box-scheme (BS2) and addresses the above instability problem. This box-scheme seems to be a generalization on rectangles of the box-scheme ($(u_h, p_h) \in P^1_{nc,0} \times RT^0$)

of Courbet and Croisille, [6]. Contrary to the triangles case, here the unknowns are not located at the interface of the mesh. Nevertheless in the particular case of a uniform grid consisting of squares, $\tilde{\beta}_K = 0$ on each K, p_h can be written in the square K as

$$p_{h|K} = (\nabla u_h)_{|K} - \frac{\Pi^0 f_{|K}}{2} \begin{pmatrix} x - x_K \\ y - y_K \end{pmatrix},$$

which is the formulation of p_h in the box-scheme of Courbet-Croisille on triangles.

4.2 Numerical analysis

In this section, we provide the stability and the optimal *a priori* error estimates for the box-scheme (BS_{nc}).

Lemma 4.1 (Discrete Poincaré lemma) *There exists a constant $C > 0$ independent of Ω such that for all $u \in Q^1_{nc,0} + H^1_0(\Omega)$,*

$$|u|_{0,\Omega} \leq C|u|_{1,h}\,.$$

Proof ([13]) Let $u \in Q^1_{nc,0} + H^1_0(\Omega)$. Then

$$|u|_{0,\Omega} = \sup_{g \in L^2(\Omega)} \frac{|(u,g)_{0,\Omega}|}{|g|_{0,\Omega}}\,. \tag{16}$$

For $g \in L^2(\Omega)$, there exists $p \in H^1(\Omega)^2$ such that $\operatorname{div} p = g$ and $\|p\|_{1,\Omega} \leq C|g|_{\Omega}$. By replacing g by this value in (16) and using Green's formula, we get

$$(u,g)_{0,\Omega} = (u, \operatorname{div} p)_{0,\Omega} = \underbrace{-\sum_K \int_K \nabla u \cdot p\,dx}_{(A)} + \underbrace{\sum_K \int_{\partial K} p \cdot \nu\, u\, d\sigma}_{(B)}\,. \tag{17}$$

First, we obtain $|(A)| = |\sum_K \int_K \nabla u \cdot p\,dx| \leq |u|_{1,h}\,|p|_{1,\Omega}$. Let us estimate $|(B)|$. Since $p \in (H^1(\Omega))^2 \cap H_{\text{div}}(\Omega)$,

$$(B) = \sum_K \int_{\partial K} p \cdot \nu\, u\, d\sigma = \sum_{a \in \mathcal{A}_b} \int_a p \cdot \nu_a\, u\, d\sigma - \sum_{a \in \mathcal{A}_i} \int_a p \cdot \nu_a\, [u]_a\, d\sigma\,. \tag{18}$$

Let $\overline{p \cdot \nu_a} = \frac{1}{|a|} \int_a p \cdot \nu_a\, d\sigma$ be the mean value of $p \cdot \nu_a$ along the edge a. Since $u \in H^1_0(\Omega) + Q^1_{nc,0}$, by the property of $Q^1_{nc,0}$ to satisfy the *patch-test*, we have

$$\int_a \overline{p \cdot \nu_a}\, u\, d\sigma = 0 \quad \text{for all } a \in \mathcal{A}_b \quad \text{and} \quad \int_a \overline{p \cdot \nu_a}\, [u]_a\, d\sigma = 0 \quad \text{for all } a \in \mathcal{A}_i\,.$$

Therefore, the equality (18) becomes:

$$\begin{aligned} \sum_K \int_{\partial K} p \cdot \nu\, u\, d\sigma &= \sum_{a \in \mathcal{A}_b} \int_a (p \cdot \nu_a - \overline{p \cdot \nu_a})\, u\, d\sigma - \sum_{a \in \mathcal{A}_i} \int_a (p \cdot \nu_a - \overline{p \cdot \nu_a})\, [u]_a\, d\sigma \\ &= \sum_K \sum_{e \in \partial K} \int_e (p \cdot \nu_e - \overline{p \cdot \nu_e})\, u\, d\sigma\,. \end{aligned}$$

The Lemma of Crouzeix-Raviart [9], gives

$$|\int_e (p\cdot\nu_e - \overline{p\cdot\nu_e})\,u\,d\sigma| \le C\,h_K |u|_{1,K}\,|p|_{1,K}\,.$$

Then,

$$|(II)| = |\sum_K \int_{\partial K} p\cdot\nu\,u\,d\sigma| \le 4\,Ch|u|_{1,h}\,|p|_{1,\Omega}\,.$$

Finally,

$$|(u,g)_{0,\Omega}| \le (4Ch+1)|u|_{1,h}\,|p|_{1,\Omega} \le (4\,Ch+1)|u|_{1,h}\,\underbrace{\|p\|_{1,\Omega}\,.}_{\le C(\Omega)|g|_{0,\Omega}}$$

■

Proposition 4.2 (Stability) *The solution* $(u_h,p_h)\in Q^1_{nc,0}\times(RT^0+\Phi)$ *of the problem* (BS_{nc}) *satisfies the stability estimate:*

$$\|u_h\|_{1,h} + \|p_h\|_{\mathrm{div},h} \le C|f|_{0,\Omega}.$$

Proof Using the formulation of Proposition 4.1 with $v_h = u_h$, applying Cauchy-Schwarz inequality and Poincaré inequality give

$$\|u_h\|_{1,h} \le C(\Omega)\,|f|_{0,\Omega}\,.$$

On the other hand, the local formula (b) from Proposition 4.1 for p_h and the identity $\mathrm{div}\,p_h = -\Pi^0 f$ imply $\|p_h\|_{\mathrm{div},h} \le C|f|_{0,\Omega}$. This concludes the proof. ■

Proposition 4.3 (*A priori* error estimates) *Let* $(u,p)\in H^1_0(\Omega)\times H_{\mathrm{div}}(\Omega)$ *be the solution of the continuous problem (1) and* $(u_h,p_h)\in Q^1_{nc,0}\times(RT^0+\Phi)$ *be the solution of the box-scheme* (BS_{nc})*. If* $f\in H^1(\Omega)$*, we have:*

$$\begin{array}{llll} (i) & |u-u_h|_{1,h} \le Ch|f|_{0,\Omega} & (ii) & |u-u_h|_{0,\Omega} \le Ch^2(|f|_{0,\Omega}+|f|_{1,\Omega}) \\ (iii) & |p-p_h|_{0,\Omega} \le Ch|f|_{0,\Omega} & (iv) & |p-p_h|_{\mathrm{div},h} \le Ch|f|_{1,\Omega}\,. \end{array} \qquad (19)$$

Proof *(i)* Let us introduce the bilinear form a_h defined for all u, $v\in H^1_0(\Omega)+Q^1_{nc,0}$ by

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} (\nabla u,\nabla v)_{0,K}\,.$$

Then we obtain the classical inequality

$$|u-u_h|_{1,h} \le 2\inf_{w_h\in Q^1_{nc,0}} |u-w_h|_{1,h} + \sup_{w_h\in Q^1_{nc,0}} \frac{|a_h(u_h-u,w_h)|}{|w_h|_{1,h}}\,.$$

The estimation of the consistency error is deduced from the variational formulation from Proposition 4.1:

$$\sup_{w_h\in Q^1_{nc,0}} \frac{|a_h(u_h-u,w_h)|}{|w_h|_{1,h}} \le C\,h\,|f|_{0,\Omega}\,. \qquad (20)$$

By using the Q^1-Lagrange interpolation, we get

$$\inf_{w_h \in Q^1_{nc,0}} |u - w_h|_{1,h} \leq Ch|u|_{2,\Omega}\,.$$

This concludes *(i)*.
(ii) is proved by using the Aubin-Nitsche argument and the result *(i)*.
(iii) is a deduction of the local formula p_h given by Proposition 4.1, *(ii)*.
(iv) results from $\operatorname{div} p = -f$ and $\operatorname{div} p_h = -\Pi^0 f$. ■

4.3 Link to the box-scheme (BS2)

We already mentioned that the 1-point integration of the gradient of u_h is not sufficient to obtain the stability of the scheme (see Section 2.1). Nevertheless, the addition of the local bubble in both trial and test spaces permits to overcome the previous difficulty, as we have just observed. In this sense, the nonconforming bubble is a stabilization parameter. Moreover, from the decomposition of the space $Q^1_{nc,0}$ given in Proposition 2.1, we deduce the following result:

Lemma 4.2 (Link to the box-scheme (BS2)) *The solution $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ of the box-scheme (BS_{nc}) is given as a function of the solution $(\bar{u}_h, \bar{p}_h) \in Q^1_{c,0} \times RT^0$ of the box-scheme (BS2) by*

$$u_h = \bar{u}_h + \sum_{K \in \mathcal{T}_h} \alpha_K\, b_K \quad \text{and} \quad p_h = \bar{p}_h + \sum_{K \in \mathcal{T}_h} \beta_K \operatorname{curl} b_K\,,$$

where

$$\begin{aligned}
\alpha_K &= \frac{3|K|}{4} \frac{1}{|e_{x,K}|^2 + |e_{y,K}|^2} (\bar{p}_h - \nabla \bar{u}_h, \nabla b_K)_{0,K}\,,\\
\beta_K &= -\frac{3|K|}{4} \frac{1}{|e_{x,K}|^2 + |e_{y,K}|^2} (\bar{p}_h - \nabla \bar{u}_h, \operatorname{curl} b_K)_{0,K}\,.
\end{aligned}$$

Proof Let $(\bar{u}_h, \bar{p}_h) \in Q^1_{c,0} \times RT^0$ be the solution of the box-scheme (BS2). We are looking for $(\alpha_K, \beta_K)_{K \in \mathcal{T}_h}$ such that

$$u_h = \bar{u}_h + \sum_K \alpha_K\, b_K \quad , \quad p_h = \bar{p}_h + \sum_K \beta_K \operatorname{curl} b_K$$

define the solution $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ of the box-scheme (BS_{nc}). Due to $\operatorname{div} p_h = \operatorname{div} \bar{p}_h$, the equation $(BS_{nc})_1$ is valid for p_h. Let us suppose that (u_h, p_h) satisfies the equation $(BS_{nc})_2$. By the definition of (u_h, p_h) and since $(\bar{u}_h, \bar{p}_h)$ satisfies the equation $(BS2)_2$,

$$\sum_{K \in \mathcal{T}_h} (p_h - \nabla u_h, q_h)_{0,K} = 0 \quad \text{for all } q_h \in P^0(\nabla b_K) + P^0(\operatorname{curl} b_K). \tag{21}$$

By taking $q_h = \nabla b_K$ in (21), we get

$$0 = (p_h - \nabla u_h, q_h)_{0,K} = (\bar{p}_h - \nabla \bar{u}_h, \nabla b_K)_{0,K} + (\beta_K \operatorname{curl} b_K, \nabla b_K)_{0,K} - (\alpha_K \nabla b_K, \nabla b_K)_{0,K}\,.$$

Since $(\operatorname{curl} b_K, \nabla b_K)_{0,K} = 0$, we deduce the formula of α_K on each rectangle K. Then for each K, α_K is uniquely determined by the unique solution $(\bar{u}_h, \bar{p}_h)$ of the box-scheme (BS2). In the same way, by taking $q_h = \operatorname{curl} b_K$ in the equation (21), we get

$$0 = (p_h - \nabla u_h, \operatorname{curl} b_K)_{0,K} = (\bar{p}_h - \nabla \bar{u}_h, \operatorname{curl} b_K)_{0,K} + (\beta_K \operatorname{curl} b_K, \operatorname{curl} b_K)_{0,K}$$

and deduce the formula for β_K. Then with this definition of the coefficients α_K, β_K, we prove that $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ is the unique solution of the box-scheme (BS_{nc}). ■

5 A simplified stabilized box-scheme

In this section, we investigate a new way to stabilize the box-scheme (BS1). In fact it seems that the solution of the previous box-scheme is locally in $P^1(K) \times (RT^0(K) + \Phi)$ (see the proof of Proposition 4.1). We are looking for a space locally in $P^1(K)$ instead of $Q^1(K)$ with the same continuity properties as $Q^1_{nc,0}$. The space $\widetilde{M}_{1,h}$ recently introduced by Park and Sheen [19] fulfils those conditions:

$$\widetilde{M}_{1,h} = \Big\{ v \in L^2(\Omega); v_{|K} \in P^1(K) \forall K \in \mathcal{T}_h \; ; \int_a v_{|K_1}\, dx = \int_a v_{|K_2}\, dx \; \forall a = \partial K_1 \cap \partial K_2 \in \mathcal{A}_i \Big\}.$$

Its dimension is $\dim \widetilde{M}_{1,h} = 3NE - NA_i = NV - 1$, since there are three unknowns for each rectangle subject to NA_i independent continuity relations. The corresponding space with homogeneous boundary is

$$\widetilde{M}_{1,h,0} = \{ v \in \widetilde{M}_{1,h}; \int_a v\, dx = 0 \, \forall a \in \mathcal{A}_b \}\,.$$

Its dimension is also $\dim \widetilde{M}_{1,h,0} = NA - NE - (NA_b - 1) = NV_i$. Note that this space satisfies the additional condition (7) of Courbet. However in contrast to the space Q^1_c, it does not contain the nonconforming bubble. The space $\widetilde{M}_{1,h,0}$ is by definition included into $Q^1_{nc,0}$. Similarly to Lemma 2.1, we deduce from the linearity and the injectivity of L and the equality $\dim \widetilde{M}_{1,h,0} = \dim C_0$ the following lemma.

Lemma 5.1 *The mapping L defines a bijection between $\widetilde{M}_{1,h,0}$ and the Courbet space C_0:*

$$\begin{array}{rcl} L : \widetilde{M}_{1,h,0} & \longrightarrow & C_0 \\ u & \longmapsto & \big(u(x_a)\big)_{a \in \mathcal{A}} \,. \end{array}$$

Definition 5.1 *Let (BS3) be the box-scheme: Find $(u_h, p_h) \in \widetilde{M}_{1,h,0} \times (RT^0 + \Phi)$ such that:*

$$(BS3) \quad \left\{ \begin{array}{rcl} \displaystyle\sum_{K \in \mathcal{T}_h} (\operatorname{div} p_h + f, v_h)_{0,K} & = & 0 \;\; \forall v_h \in P^0 \,, \\ \displaystyle\sum_{K \in \mathcal{T}_h} (p_h - \nabla u_h, q_h)_{0,K} & = & 0 \;\; \forall q_h \in X_{2,h} = (P^0)^2 + \Phi \,. \end{array} \right. \tag{22}$$

The box-scheme has $4NE$ unknowns.

Indeed $\dim \widetilde{M}_{1,h} + \dim(RT^0 + \Phi) = NV_i + NA + NE - 1 = 4NE = \dim P^0 + \dim X_{2,h}$.

Lemma 5.2 Link to the box-scheme (BS_{nc}) *The solution* $(\tilde{u}_h, \tilde{p}_h) \in \widetilde{M}_{1,h,0} \times (RT^0 + \Phi)$ *of the box-scheme* $(BS3)$ *is unique and given as a function of the solution* $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ *of the box-scheme* (BS_{nc}) *by*

$$\tilde{u}_h = u_h \quad \text{and} \quad \tilde{p}_h = p_h \, .$$

Proof Any solution $(\tilde{u}_h, \tilde{p}_h) \in \widetilde{M}_{1,h,0} \times (RT^0 + \Phi)$ of the box-scheme (BS3) is included in $Q^1_{nc,0} \times (RT^0 + \Phi)$ and satisfies the equations (13). By uniqueness of the solution of the box-scheme (BS_{nc}) and the linearity of the scheme $(BS3)$, we deduce the result. ■

Note that we rediscover that $u_h \in Q^1_{nc,0}$ in the scheme (BS_{nc}) is locally in $P^1(K)$ (see the proof of Proposition 4.1). In particular, this means that the bilinear term "$x\,y$" is not needed. In fact the solution of the box-scheme (BS_{nc}) is already the solution of the box-scheme (BS3).

Corollary 5.1 *The box-scheme (BS3) has a unique solution* $(u_h, p_h) \in \widetilde{M}_{1,h,0} \times (RT^0 + \Phi)$ *such that*

(a) $u_h \in \widetilde{M}_{1,h,0}$ *is the solution of:*

$$\sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v_h)_{0,K} = (\Pi^0 f, v_h)_{0,\Omega} \quad \text{for all } v_h \in \widetilde{M}_{1,h,0} \, .$$

(b) p_h *is locally given by:*

$$p_{h|K} = (\nabla u_h)_{|K} - \frac{\Pi^0 f_{|K}}{|e_{x,K}|^2 + |e_{y,K}|^2} \begin{pmatrix} |e_{y,K}|^2 (x - x_K) \\ |e_{x,K}|^2 (y - y_K) \end{pmatrix} .$$

Proof This result is deduced from the previous lemma and Proposition 4.1, since $\widetilde{M}_{1,h,0} \subset Q^1_{nc,0}$.

Corollary 5.2 (*A priori* error estimates) *Let* $(u, p) \in H^1_0(\Omega) \times H_{\mathrm{div}}(\Omega)$ *be the solution of the continuous problem (1) and* $(u_h, p_h) \in \widetilde{M}_{1,h,0} \times (RT^0 + \Phi)$ *be the solution of the box-scheme* $(BS3)$. *If* $f \in H^1(\Omega)$, *we have:*

$$\begin{array}{ll} (i) \quad |u - u_h|_{1,h} \leq Ch|f|_{0,\Omega} & (ii) \quad |u - u_h|_{0,\Omega} \leq Ch^2(|f|_{0,\Omega} + |f|_{1,\Omega}) \\ (iii) \quad |p - p_h|_{0,\Omega} \leq Ch|f|_{0,\Omega} & (iv) \quad |p - p_h|_{\mathrm{div},h} \leq Ch|f|_{1,\Omega} \, . \end{array} \tag{23}$$

6 Numerical results

In this section we present several numerical results which demonstrate the theoretical convergence rates obtained for the box-scheme of Section 4. We compute the error estimates for the unknown u and the flux p of the box-scheme (BS_{nc}) on two different domains Ω meshed by rectangles. The solution of the box-scheme (BS_{nc}) is computed according to the decoupled formulation given in Proposition 4.1. The unknown u is the solution of the variational formulation, whereas p is deduced from the local reconstruction on each rectangle. From the computed error, we deduce numerical convergence rate for each solution of each test. The results for each test case are reported in the Tables 1 to 5.

The test cases 1 and 2 of Section 6.1 are given on the unit square domain $\Omega = [0, 1]^2$ meshed by squares. Whereas Section 6.2 is devoted to the computation of the error estimates of the box-scheme (BS_{nc}) on $\Omega = [0,1]^2$ meshed by rectangles. Finally in Paragraph 6.3, we present two test cases on the L-shaped domain $\Omega = [0,2] \times [0,1] \cup [1,2] \times [1,2]$, meshed by squares. All the computed convergence rates are in agreement with the theoretical ones given in Proposition 4.3.

6.1 Square domain meshed by squares

The domain Ω is meshed by four different regular grids made of 100, 225, 400 and 900 squares.

1. *Test case* 1: In this first example, the source term f and the Dirichlet data g are chosen such that $u(x,y) = x\,(1-x)\,\sin(\pi\,y)$ is the exact solution of the Poisson problem

$$\begin{cases} -\Delta u = f \quad on \quad \Omega\,, \\ \quad u = g \quad on \quad \partial\Omega\,. \end{cases} \tag{24}$$

   The results for the box-scheme $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ defined by (BS_{nc}), are given in Table 1. The error for the unknown u is of order 1 in the semi-norm $|\cdot|_{1,h}$ and of order 2 for the L^2-norm. For p we get also order 1 in the L^2-norm. The numerical results are of order of those computed theoretically in Proposition 4.3.

nb rect.	$\lvert u - u_h\rvert_{0,\Omega}$	$\lvert u - u_h\rvert_{1,h}$	$\lvert p - p_h\rvert_{0,\Omega}$	space step h
100	2.261×10^{-3}	7.567×10^{-2}	7.976×10^{-2}	0.1414
225	1.008×10^{-3}	5.053×10^{-2}	5.326×10^{-2}	0.09428
400	5.677×10^{-4}	3.792×10^{-2}	3.997×10^{-2}	0.07071
900	2.525×10^{-4}	2.529×10^{-2}	2.665×10^{-2}	0.04714
conv. rate	1.996	0.9977	0.9979	

Table 1: Box-scheme (BS_{nc}): $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ for Test 1.

2. *Test case* 2: Our second example is a test case proposed by Douglas et al., [1]. The source term and the boundary conditions are chosen such that $u(x,y) = \exp\bigl(-100\,((x-1/4)^2 + (y-1/3)^2)\bigr)$ is the exact solution of the problem (24). It concerns a Gaussian pulse centred at the point $(x_0, y_0) = (\frac{1}{4}, \frac{1}{3})$. The error estimates for both unknowns u and $p = \nabla u$ are given in Table 2 for the box-scheme $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$. The convergence rates are a little bit lower than expected (1.8 instead of 2 for u in the L^2-norm and 0.97 instead of 1 for p in the L^2-norm), but still close to the *a priori* error estimates of Proposition 4.3. This is due to the high gradient of the exact solution at the point (x_0, y_0).

6.2 Square domain meshed by rectangles

In this example, we consider the domain $\Omega = [0,1]^2$ meshed by rectangles and the solution $u(x,y) = x\,(1-x)\,y\,(1-y)\,\exp(5\,x)$ of the problem (24), where the right-hand side and the Dirichlet conditions are computed using the exact solution u. The grid is made of $n_x \times n_y$ rectangles where n_x and n_y are the number of subdivisions of the segment $[0,1]$ in each direction (O_x) and (O_y). We compute the solution (u_h, p_h) for (n_x, n_y) taking the values (20,5), (40,10),

nb rect.	$\|u - u_h\|_{0,\Omega}$	$\|u - u_h\|_{1,h}$	$\|p - p_h\|_{0,\Omega}$	space step h
100	3.927×10^{-2}	0.9945	1.035	0.1414
225	1.990×10^{-2}	0.6885	0.7112	0.09428
400	1.174×10^{-2}	0.5148	0.5333	0.07071
900	5.401×10^{-3}	0.3422	0.3553	0.04714
conv. rate	1.808	0.9737	0.9751	

Table 2: Box-scheme (BS_{nc}): $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ for Test 2.

(80,20) and (100,25), i. e. 100, 400, 1600 and 2500 rectangles. The exact solution presents a boundary layer at $x = 1$. Nevertheless the computed solution u_h and the discrete flux p_h of (BS_{nc}) seem to take it into account. The convergence rate between the exact and the discrete solution for both unknowns u and $p = \nabla u$ are assembled in Table 3. The numerical results are really satisfying the theoretical estimates of Proposition 4.3.

nb rect.	$\|u - u_h\|_{0,\Omega}$	$\|u - u_h\|_{1,h}$	$\|p - p_h\|_{0,\Omega}$	space step h
20×5	3.386×10^{-2}	1.979	1.586	0.2061
40×10	8.488×10^{-3}	1.001	0.7991	0.1031
80×20	2.124×10^{-3}	0.5020	0.4003	0.05154
100×25	1.359×10^{-3}	0.4017	0.3203	0.04123
conv. rate	1.998	0.9911	0.9942	

Table 3: Box-scheme (BS_{nc}): $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ for Test 3.

6.3 Tests cases on an L-shaped domain

In this case we consider a different domain Ω_L, given by the square $[0, 2] \times [0, 2]$ without the part $[0, 1] \times [1, 2]$. We obtain an L-shaped domain. We compute the solution (u_h, p_h) of the box-scheme (BS_{nc}) associated with the Poisson problem (24). The data f and g are chosen such that u is the exact solution of (24). The computed results conform to the theoretical ones.

1. *Test case* 4: The exact solution is $u(x, y) = x\,(2 - x)\,y\,(2 - y)$.
   Convergence rates of the error for both unknowns u and $p = \nabla u$ are given in Table 4.

nb rect.	$\|u - u_h\|_{0,\Omega}$	$\|u - u_h\|_{1,h}$	$\|p - p_h\|_{0,\Omega}$	space step h
75	1.351×10^{-2}	0.2799	0.2804	0.2828
300	3.402×10^{-3}	0.1399	0.1400	0.1414
675	1.506×10^{-3}	9.325×10^{-2}	9.326×10^{-2}	0.09428
conv. rate	1.996	1.000	1.002	

Table 4: Box-scheme (BS_{nc}): $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ for Test 4.

2. *Test case* 5: The exact solution is $u(x,y) = x\,(2-x)\,y^2\,(2-y)\,\sin(x+2y)$.
   Convergence rate and error for both unknowns u and $p = \nabla u$ are given in Table 5.

nb rect.	$\lvert u - u_h\rvert_{0,\Omega}$	$\lvert u - u_h\rvert_{1,h}$	$\lvert p - p_h\rvert_{0,\Omega}$	space step h
75	3.368×10^{-2}	0.5089	0.5215	0.2828
300	8.613×10^{-3}	0.2543	0.2616	0.1414
675	3.813×10^{-3}	0.1688	0.1739	0.09428
conv. rate	1.981	1.004	0.9993	

Table 5: Box-scheme (BS_{nc}): $(u_h, p_h) \in Q^1_{nc,0} \times (RT^0 + \Phi)$ for Test 5.

6.4 Conclusion

The numerical results for the box-scheme (BS_{nc}) are really consistent with the *a priori* error estimates of Proposition 4.3. The results obtained for the box-scheme (BS_{nc}) could be computed analogously by the box-scheme (BS3).
The local formulation of the flux p_h of each box-scheme suggests consideration of a more relevant finite element space for the approximation of the flux. It might be interesting to consider a flux space locally in $(P^0(K))^2 + P^0(K)\begin{pmatrix} |e_{K,y}|(x - x_K) \\ |e_{K,x}|(y - y_K) \end{pmatrix}$ for each rectangle K, submitted to some continuity constraints.

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References

[1] D. N. Arnold, D. Boffi, and R. S. Falk. Approximation by quadrilateral finite elements. *Math. Comp.*, 71(239):909–922 (electronic), 2002.

[2] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2002.

[3] S-H. Chou, D. Y. Kwak, and K. Y. Kim. Mixed finite volume methods on nonstaggered quadrilateral grids for elliptic problems. *Math. Comp.*, 72(242):525–539 (electronic), 2003.

[4] S-H Chou and S. Tang. Comparing two approaches of analyzing mixed finite volume methods. *KSIAM J.*, 5(1):55–78, 2001.

[5] B. Courbet. Two-point schemes for computational fluid dynamics. *La Recherche Aérospatiale*, 5:21–46, 1990.

[6] B. Courbet and J-P. Croisille. Finite volume box schemes on triangular meshes. *M2AN Math. Model. Numer. Anal.*, 32(5):631–649, 1998.

[7] J-P. Croisille. Finite volume box-schemes and mixed methods. *Math. Model. and Numer.*, 34(5):1087–1106, 2000.

[8] J-P. Croisille and I. Greff. Some nonconforming mixed box schemes for elliptic problems. *Numer. Methods Partial Differential Equations*, 18(3):355–373, 2002.

[9] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, 7(R-3):33–75, 1973.

[10] L. El Alaoui. *Analyse d'erreur a priori et a posteriori pour des méthodes d'éléments finis mixtes non-conforme.* PhD thesis, Ecole Nationale des Ponts et Chaussées, France, 2005.

[11] L. El Alaoui and A. Ern. Residual and hierarchical a posteriori error estimates for nonconforming mixed finite element methods. *M2AN Math. Model. Numer. Anal.*, 38(6):903–929, 2004.

[12] V. Girault. Problèmes variationnels et méthodes d'élements finis. Notes de cours de DEA, Université Paris 6.

[13] I. Greff. *Schémas boîte: Etude théorique et numérique.* PhD thesis, Université de Metz, France, 2003.

[14] I. Greff. Some box-schemes for elliptic problems on rectangular meshes. In *Proceeding LUXFEM03*, Luxembourg, November, 13–14, 2003.

[15] I. Greff. An interpretation of the Raviart-Thomas flux on rectangular grids via box-scheme. 2006.

[16] P. Hansbo. A new approach to quadrature for finite elements incorporating hourglass control as a special case. *Comput. Methods Appl. Mech. Engrg.*, 158(3-4):301–309, 1998.

[17] H. B. Keller. A new difference scheme for parabolic problems. In *Numerical Solution of Partial Differential Equations, II (SYNSPADE 1970) (Proc. Sympos., Univ. of Maryland, College Park, Md., 1970)*, pages 327–350. Academic Press, New York, 1971.

[18] L. D. Marini. An inexpensive method for the evaluation of the solution of the lowest order Raviart-Thomas mixed method. *SIAM J. Numer. Anal.*, 22(3):493–496, 1985.

[19] C. Park and D. Sheen. P_1-nonconforming quadrilateral finite element methods for second-order elliptic problems. *SIAM J. Numer. Anal.*, 41(2):624–640 (electronic), 2003.

[20] P.-A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical Aspects of Finite Element Methods. Lecture Notes in Math.*, volume 606, pages 292–315. Springer, Berlin, 1977.

[21] J.-M. Thomas and D. Trujillo. Mixed finite volume methods. *Internat. J. Numer. Methods Engrg.*, 46(9):1351–1366, 1999. Fourth World Congress on Computational Mechanics (Buenos Aires, 1998).