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Abstract

We discuss the existence of generalized solutions of the flow of two immiscible, incompressible, viscous Newtonian and Non-Newtonian fluids with and without surface tension in a domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$. In the case without surface tension, the existence of weak solutions is shown, but little is known about the interface between both fluids. If surface tension is present, the energy estimates gives an a priori bound on the $(d-1)$-dimensional Hausdorff measure of the interface, but the existence of weak solutions is open. This might be due to possible oscillation and concentration effects of the interface related to instabilities of the interface as for example fingering, emulsification or just cancellation of area, when two parts of the interface meet. Nevertheless we will show the existence of so-called measure-valued varifold solutions, where the interface is modeled by an oriented general varifold $V(t)$ which is a non-negative measure on $\Omega \times \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. Moreover, it is shown that measure-valued varifold solutions are weak solution if an energy equality is satisfied.

Key words: Two-phase flow, free boundary value problems, varifold solutions, measure-valued solutions, surface tension
AMS-Classification: 35Q30, 35Q35, 76D27, 76D45, 76T99

1 Introduction and Main Results

We study the flow of two incompressible, viscous and immiscible fluids like oil and water inside a bounded domain Ω or in $\Omega = \mathbb{R}^d$, $d = 2, 3$. The fluids fill domains $\Omega_+(t)$ and $\Omega_-(t)$, $t > 0$, and the interface between both fluids is denoted by $\Gamma(t)$. The flow is described using the velocity $v\colon \Omega \times (0, \infty) \to \mathbb{R}^d$ and the pressure $p\colon \Omega \times (0, \infty) \to \mathbb{R}$ in both fluids in Eulerian coordinates. We assume the fluids to be of a generalized

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Newtonian type, i.e., the stress tensors are of the form $T^{\pm}(v,p) = 2\nu^{\pm}(|Dv|)Dv - pI$ with viscosities $\nu^{\pm}$ depending on the shear rate $|Dv|$ of the fluid, $2Dv = \nabla v + \nabla v^T$. Moreover, we consider the cases with and without surface tension at the interface. Precise assumptions are made below. Under suitable smoothness assumptions, the flow is obtained as solution of the system

$$\partial_t v + v \cdot \nabla v - \operatorname{div} T^{\pm}(v,p) = 0 \qquad \text{in } \Omega^{\pm}(t), t > 0, \tag{1.1}$$
$$\operatorname{div} v = 0 \qquad \text{in } \Omega^{\pm}(t), t > 0, \tag{1.2}$$
$$n \cdot T^{+}(v,p) - n \cdot T^{-}(v,p) \ = \kappa H n \quad \text{on } \Gamma(t), t > 0, \tag{1.3}$$
$$V = n \cdot v \quad \text{on } \Gamma(t), t > 0, \tag{1.4}$$
$$v = 0 \qquad \text{on } \partial\Omega, t > 0, \tag{1.5}$$
$$v|_{t=0} = v_0 \qquad \text{in } \Omega, \tag{1.6}$$

together with $\Omega^+(0) = \Omega_0^+$. Here V and H denote the normal velocity and mean curvature, resp., of $\Gamma(t)$ taken with respect to the exterior normal n of $\partial\Omega^+(t)$, and $\kappa \geq 0$ is the surface tension constant ($\kappa = 0$ means no surface tension present). Equations (1.1)-(1.2) describe the conservation of linear momentum and mass in both fluids, (1.3) is the balance of forces at the boundary, (1.4) is the kinematic condition that the interface is transported with the flow of the mass particles, and (1.5) is the non-slip condition at the boundary of Ω. Moreover, it is assumed that the velocity field v is continuous along the interface.

Most publications on the mathematical analysis of free boundary value problems for viscous incompressible fluids study quite regular solutions and often deal with well-posedness locally in time or global existence close to equilibrium states, cf. e.g. Solonnikov [29, 30], Beale [3, 4], Tani and Tanaka [33], Shibata and Shimizu [24] or Abels [1]. These approaches are a priori limited to flows, in which the interface does not develop singularities and the domain filled by the fluid does not change its topology. In the present contribution we consider certain classes of generalized solutions, which allow singularities of the interface and which exist globally in time for general initial data. For this purpose, we need a suitable weak formulation of the system above. Testing (1.1) with a divergence free vector field φ and using in particular the jump relation (1.4), we obtain

$$\begin{aligned} -(v, \partial_t\varphi)_Q - (v_0, \varphi|_{t=0})_\Omega - (v \otimes v, \nabla\varphi)_Q \\ +(S(\chi, Dv), D\varphi)_Q = \kappa \int_0^\infty \left\langle H_{\Gamma(t)}, \varphi(t) \right\rangle dt \end{aligned} \tag{1.7}$$

for all $\varphi \in C^\infty_{(0)}(\Omega \times [0,\infty))^d$, $\operatorname{div}\varphi = 0$, where $Q = \Omega \times (0,\infty)$, $\chi = \chi_{\Omega^+}$, $S(1, Dv) = 2\nu^+(|Dv|)Dv$, $S(0, Dv) = 2\nu^-(|Dv|)Dv$, and

$$\left\langle H_{\Gamma(t)}, \varphi(t) \right\rangle = \int_{\Gamma(t)} H n \cdot \varphi(x,t) \, d\mathcal{H}^{d-1}(x) \tag{1.8}$$

Now the aim is to construct generalized solutions in a class of functions determined by the energy estimate: If v and $\Gamma(t)$ are sufficiently smooth, then choosing $\varphi = v\chi_{[0,T]}$ in (1.7) one obtains the *energy equality*

$$\frac{1}{2}\|v(T)\|^2_{L^2(\Omega)} + \kappa\mathcal{H}^{d-1}(\Gamma(T)) + \int_0^T \int_\Omega S(\chi, Dv)Dv\,dx\,dt = \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \kappa\mathcal{H}^{d-1}(\Gamma_0) \tag{1.9}$$

for all $T > 0$, where $\Gamma_0 = \partial\Omega_0^+$. – Note that $\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma(t)) = -\int_{\Gamma(t)} HV\,d\mathcal{H}^{d-1} = -\langle H_{\Gamma(t)}, v(t)\rangle$ due to (1.4), cf. Lemma 2.3 below. – Now assuming that

$$\nu^\pm(|Dv|) \geq c|Dv|^{q-2}$$

for $q > 1$ the equality above gives a uniform bound of

$$v \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^q(0,\infty; \dot{W}^1_q(\Omega)),$$

where we refer to Section 2.2 below for the precise definitions the function spaces in this section. Moreover, we note that

$$-\langle\nabla\chi(t), \varphi\rangle = \int_{\Omega(t)} \operatorname{div}\varphi(x)\,dx = \int_{\Gamma(t)} n\cdot\varphi(x)\,d\mathcal{H}^{d-1}(x)$$

for all $\varphi \in C_0^\infty(\Omega)^d$. Hence the distributional gradient $\nabla\chi(t)$ is a finite Radon measure and $\|\nabla\chi(t)\|_{\mathcal{M}(\Omega)} = \mathcal{H}^{d-1}(\Gamma(t))$. Thus, if $\kappa > 0$, $\chi(t) \in BV(\Omega)$ for all $t > 0$ and the energy equality above gives an a priori estimate of

$$\chi \in L^\infty(0,\infty; BV(\Omega)).$$

In the case without surface tension, $\kappa = 0$, we only obtain that $\chi \in L^\infty(Q)$ is a priori bounded by one. This motivates to look for weak solutions (v,χ) lying in the function spaces above, satisfying (1.9) with a suitable substitute of (1.8), such that (v,χ) solve (1.7) as well as the transport equation

$$\partial_t\chi + v\cdot\nabla\chi = 0 \qquad \text{in } Q, \tag{1.10}$$

$$\chi|_{t=0} = \chi_0 \qquad \text{in } \Omega \tag{1.11}$$

for $\chi_0 = \chi_{\Omega_0^+}$ in a suitable weak sense, where (1.10) is a weak formulation of (1.4), cf. [17, Lemma 1.2].

In the case without surface tension and for Newtonian fluids, i.e., $\nu^\pm(|Dv|) \equiv \nu^\pm > 0$, the existence of weak solutions (even for N-fluids with different densities) was proven by Nouri and Poupaud [17]. Moreover, Giga and Takahashi [10] consider the case of a two-phase Stokes flow with ν^+ close to ν^-. The main difference in their approach is that (1.10)-(1.11) is replaced by a transport equation for a level set function, which is solved in the sense of viscosity solutions. Due to a lack of regularity in the velocity v only sub- and super-solutions exists, which may differ. This causes the possibility of "boundary fattening", cf. [10] for details. – In [17] and the present contribution the transport equation is solved in the sense of renormalized solutions due to DiPerna and Lions [9]. But also the result of Nouri and Poupaud does not give good information for the interface $\Gamma(t)$ since $\Omega^+(t) = \{x \in \Omega : \chi(t) = 1\}$ is only known to be a measurable set. Moreover, we note that Wagner [35] consider generalized solutions of a one-phase flow for an ideal, irrotational and incompressible fluid and that Gomez and Zolésio [11] treated a quasi-stationary two-phase flow for shear thinning fluids.

Because of the better a priori estimate in the case with surface tension, one might expect to get better results in this case. But unfortunately the additional mean curvature term causes severe problems in the construction of weak solution, which might be related to instabilities of the boundary when fingering or emulsification takes place, cf. e.g. Joseph and Renardy [13]. The only known results for generalized solutions in the case of surface tension are due to Plotnikov [20] for a two-dimensional flow of shear thickening fluids (i.e. $q > d = 2$ above) and [21] for the case of compressible fluids as well as Salvi [23] for an incompressible viscous Newtonian fluid. In Plotnikov's contributions the mean curvature term is interpreted as the first variation of a so called general varifold and it is shown that for almost all $t > 0$ the varifold is supported on a rectifiable closed curved dividing the plane into two disjoint domains $\Omega^\pm(t)$. The latter solutions can be considered as some kind of measure-valued solution and are related to the solutions constructed in the present contribution. In [23] no interpretation of the meaning of the mean curvature term for the constructed weak solution is given.

It is the purpose of this article to introduce a notion of so called measure-valued varifold solutions of the two-phase flow described above. The definitions are in the spirit of measure-valued solutions for conservation laws and the flow of non-Newtonian fluids as studied for example in [12]. Measure-valued solutions were introduced in order to model possible oscillation and concentration effects on an infinitesimal scale, which mathematically do not allow to prove the convergence of a suitable approximation scheme to a weak solution. In the present two-phase flow we have to deal with possible oscillation/concentration effects of the shear tensor $Dv(x,t)$ as well as of the boundary $\Gamma(t)$. Therefore the definition of a measure-valued varifold solution uses the Young measure generated by the shear tensors $Dv_\varepsilon(x,t)$ of an approximate sequence $(v_\varepsilon, \chi_\varepsilon)$, $\varepsilon > 0$, as well as an oriented general $(d-1)$-varifold $V(t)$ generated by the sequence of surfaces $\Gamma_\varepsilon(t)$ of the approximation. Here a generalized $(d-1)$-varifold V is simply a non-negative measure $V \in \mathcal{M}(\Omega \times \mathbb{S}^{d-1})$, which by disintegration can be represented as a non-negative measure $|V| \in \mathcal{M}(\Omega)$, corresponding to a surface measure, together with a family of probability measures V_x, $x \in \Omega$, for the normal vector of the "surface" $n \in \mathbb{S}^{d-1}$, which models possible infinitesimal oscillations of the interface.

Before we come to the precise definitions and results we make the following assumptions:

Assumption 1.1 *We assume that $\kappa > 0$ and $\Omega = \mathbb{R}^d$ or that $\kappa = 0$ and $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with Lipschitz boundary or $\Omega = \mathbb{R}^d$, $d = 2, 3$. Moreover, let $q > 1$ and let $\nu(j, s)$, $j = 0, 1$, be twice continuously differentiable for $s > 0$ such that $\nu(j, s)s^2$ is continuous at 0 and $\nu(j, s)$ satisfy*

$$c_0 s^{q-2} \leq \nu(j, s) \leq C_0 s^{q-2}, \qquad \frac{d}{ds}(\nu(j, s)s) > 0 \qquad \frac{d^2}{ds^2}(\nu(j, s)s^2) > 0 \tag{1.12}$$

for some constants $c_0, C_0 > 0$. Finally, we set $S(\theta, A) = \theta\nu(1, |A|)A + (1-\theta)\nu(0, |A|)A$ for every $A \in \mathbb{R}^{d\times d}_{sym}$, $\theta \in [0, 1]$, and $V_q(\Omega) = W^1_{q,0}(\Omega)^d \cap L^q_\sigma(\Omega)$ if Ω is a bounded domain and $V_q(\mathbb{R}^d) = \left\{ v \in \dot{W}^1_q(\mathbb{R}^d)^d : \operatorname{div} v = 0 \right\}$.

We note that the simple power law $\nu(j, s) = \nu_j s^{q-2}$ satisfies the conditions above.

Before defining generalized solutions of the two-phase flow with surface tension we need some notation: An (oriented) general varifold is a non-negative $V \in \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1})$. For such a general varifold V

$$\langle \delta V, \varphi \rangle = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (I - s \otimes s) : \nabla\varphi(x)\, dV(x, s), \qquad \varphi \in C^1_0(\mathbb{R}^d), \tag{1.13}$$

denotes its *first variation*. Moreover, let $Q := \Omega \times (0, \infty)$, $Q_t = \Omega \times (0, t)$, and let $(.,.)_M$ denote the L^2-scalar product on M.

Definition 1.2 (Measure-Valued Varifold Solutions)
Let $\kappa > 0$ and let Assumption 1.1 hold. Then $v \in L^\infty(0, \infty; L^2_\sigma(\mathbb{R}^d)) \cap L^q(0, \infty; V_q(\mathbb{R}^d))$, $\chi \in L^\infty(0, \infty; BV(\mathbb{R}^d; \{0, 1\}))$, $\mu \in L^\infty_\omega(Q; Prob(\mathbb{R}^{d\times d}_{sym}))$, and $V \in L^\infty_\omega(0, \infty; \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1}))$, $V(t) \geq 0$ for a.e. $t > 0$, is called a measure-valued varifold solution of the two-phase flow *for initial data $v_0 \in L^2_\sigma(\mathbb{R}^d)$, $\chi_0 = \chi_{\Omega_0^+}$ for a bounded domain $\Omega_0^+ \Subset \mathbb{R}^d$ of finite perimeter if*

$$\begin{aligned} &-(v, \partial_t\varphi)_Q - (v_0, \varphi(0))_{\mathbb{R}^d} - (v \otimes v, \nabla\varphi)_Q \\ &\quad + \left(\int_{\mathbb{R}^{d\times d}_{sym}} S(\chi, \lambda)\, d\mu_{x,t}(\lambda), D\varphi \right)_Q = -\kappa \int_0^T \langle \delta V(t), \varphi(t) \rangle\, dt \end{aligned} \tag{1.14}$$

for all $\varphi \in C^\infty_{(0)}(\mathbb{R}^d \times [0, \infty))^d$ with $\operatorname{div}\varphi = 0$, if

$$\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} s \cdot \psi(x)\, dV(t)(x, s) = -\int_{\mathbb{R}^d} \psi\, d\nabla\chi(t), \qquad \psi \in C_0(\mathbb{R}^d)^d, \tag{1.15}$$

$$\int_{\mathbb{R}^{d\times d}_{sym}} \lambda\, d\mu_{x,t}(\lambda) = Dv(x, t) \tag{1.16}$$

for almost all $(x,t) \in Q$*, if* χ *is the unique renormalized solution of the transport equation (1.10)-(1.11), cf. Section 2.5 below, and if* (v,χ,V,μ) *satisfies the* generalized energy inequality

$$\frac{1}{2}\|v(t)\|_2^2+\kappa\|V(t)\|_{\mathcal{M}}+\int_{Q_t}\int S(\chi,\lambda):\lambda\,d\mu_{x,\tau}\,d(x,\tau)\le\frac{1}{2}\|v_0\|_2^2+\kappa\|\nabla\chi_0\|_{\mathcal{M}} \quad (1.17)$$

for almost all $t\in(0,\infty)$.

Remark 1.3 1. If $V(t)$ is obtained from a C^1-surface $\Gamma(t)$ in the natural manner, $\langle\delta V(t),.\rangle$ coincides with the first variation of $\mathcal{H}^{d-1}\lfloor\Gamma(t)$, cf. Section 2.3 below.

2. Note that by the assumption on $\nu(j,s)$, $\lambda\mapsto S(\chi,\lambda):\lambda$, $\lambda\in\mathbb{R}^{d\times d}_{sym}$, is a strictly convex function. Therefore by the generalized Jensen inequality, cf. (2.4) below, and (1.16)

$$\int_{Q_t} S(\chi,Dv):Dv\,d(x,\tau)\le\int_{Q_t}\int S(\chi,\lambda):\lambda\,d\mu_{x,\tau}\,d(x,\tau) \quad (1.18)$$

for almost all $(x,\tau)\in Q_t$ with equality if and only if $\mu_{x,\tau}=\delta_{Dv(x,\tau)}$.

3. Let $(V_x(t),|V(t)|)$, $x\in\mathbb{R}^d$, denote the disintegration of $V(t)\in\mathcal{M}(\mathbb{R}^d\times\mathbb{S}^{d-1})$ into a non-negative measure $|V(t)|$ and a family of probability measures $V_x(t)\in\mathcal{M}(\mathbb{S}^{d-1})$ as described in Section 2.3 below. Then (1.15) implies that $|\nabla\chi(t)|(A)\le|V(t)|(A)$ for all open sets A and almost all $t\in(0,\infty)$, cf. (2.2) below. Hence $|\nabla\chi(t)|$ is absolutely continuous with respect to $|V(t)|$ and

$$\int_{\mathbb{R}^d} f(x)\,d|\nabla\chi(t)|=\int_{\mathbb{R}^d} f(x)\theta_t(x)\,d|V(t)|,\quad f\in C_0(\mathbb{R}^d),$$

for a $|V(t)|$-measurable function $\theta_t\colon\mathbb{R}^d\to[0,\infty)$ with $|\theta_t(x)|\le 1$ almost everywhere. In particular, this implies $\operatorname{supp}\nabla\chi_t\subseteq\operatorname{supp}V(t)$ and $\|\nabla\chi(t)\|_{\mathcal{M}}\le\|V(t)\|_{\mathcal{M}}$ for almost all $t\in(0,\infty)$. Hence every measure-valued varifold solution satisfies the energy inequality

$$\frac{1}{2}\|v(t)\|_2^2+\kappa\|\nabla\chi(t)\|_{\mathcal{M}}+(S(\chi,Dv),Dv)_{\Omega_t}\le\frac{1}{2}\|v_0\|_2^2+\kappa\|\nabla\chi_0\|_{\mathcal{M}} \quad (1.19)$$

for almost all $t>0$. Moreover, if $E(t)=\{x\in\mathbb{R}^d:\chi(x,t)=1\}$, $t>0$, then $E(t)$ is for almost every $t>0$ a set of finite perimeter, cf. Section 2.4 below, and (1.15) yields the relation

$$\int_{\mathbb{S}^{d-1}} s\,dV_x(t)(s)=\begin{cases}\theta_t(x)n(x) & \text{if } x\in\partial^*E_t\\ 0 & \text{else}\end{cases}$$

for $|V(t)|$-almost every $x\in\mathbb{R}^d$ and almost every $t>0$, where $n=-\frac{\nabla\chi(t)}{|\nabla\chi(t)|}$ is the exterior normal of the reduced boundary ∂^*E_t of E_t and $\chi(t)=\chi_{E_t}$. – In other words, the expectation of $V_x(t)$ is proportional to the normal n on the interface and zero inside the fluid.

4. In general, it is an open problem whether $V(t)$ is a so-called countably $(d-1)$-rectifiable varifold, which implies that $V_x(t)$ is a Dirac measure for $|V(t)|$-almost every x. Then $V(t)$ can naturally be identified with a countably $(d-1)$-rectifiable set – a "surface" – equipped with a density $\theta_t \geq 0$. So far we can only give a sufficient condition for the rectifiability of $V(t)$ in terms of a regularity condition for the pressure $p(t)$ or the first variation $\delta V(t)$. See the Appendix A below for details.

An open question is whether there are measure-valued varifold solutions such that the first variation $\langle \delta V, . \rangle$ coincides with the negative mean curvature functional associated to $\chi(t)$, which is defined below, and such that $\mu_{x,t}$ coincides with the Dirac measure $\delta_{Dv(x,t)}$ almost everywhere. If this is the case, we call it a weak solution:

Definition 1.4 (Weak Solutions)
Let (v, χ, V, μ) be a measure-valued varifold solution of the two-phase flow in the sense of Definition 1.2. Then (v, χ, V) is called a varifold solution *if $\mu_{x,t} = \delta_{Dv(x,t)}$ for almost all $(x,t) \in Q$. If (v, χ, V) is a varifold solution, then (v, χ) is called a* weak solution *of the two-phase flow if*

$$\langle \delta V(t), \psi \rangle = -\langle H_{\chi(t)}, \psi \rangle = \int_{\mathbb{R}^d} \operatorname{Tr}(P_\tau \nabla \psi)\, d|\nabla \chi(t)| \qquad \text{for all } \psi \in C_{0,\sigma}^\infty(\mathbb{R}^d)$$

and for almost all $t \in (0,\infty)$, where $P_\tau = I - \frac{\nabla\chi(t)}{|\nabla\chi(t)|} \otimes \frac{\nabla\chi(t)}{|\nabla\chi(t)|}$, cf. (2.5) below.

From the definitions one derives the following general properties of measure-valued varifold solutions:

Proposition 1.5 (Properties of Measure-Valued Varifold Solutions)
Let (v, χ, V, μ) be a measure-valued varifold solution. Then:

1. *If (v, χ, V) satisfies the* energy equality

$$\frac{1}{2}\|v(t)\|_2^2 + \kappa \|\nabla \chi(t)\|_{\mathcal{M}} + (S(\chi, Dv), Dv)_{Q_t} = \frac{1}{2}\|v_0\|_2^2 + \kappa \|\nabla \chi_0\|_{\mathcal{M}} \tag{1.20}$$

   for almost all $t \in (0,\infty)$, then (v, χ) is a weak solution. Moreover, if (1.20) *holds with $\|\nabla\chi(t)\|_{\mathcal{M}}$ replaced by $\|V(t)\|_{\mathcal{M}}$, then (v, χ, V) is a varifold solution.*

2. *If $q > d$, then $\chi \in BV(Q_T)$ for every $0 < T < \infty$.*

Our main result concerns existence of measure-valued varifold solutions with some additional properties:

THEOREM 1.6 (Existence of Measure-Valued Varifold Solutions)
Let $q > \frac{2d}{d+2}$, let $v_0 \in L_\sigma^2(\mathbb{R}^d)$, let $\Omega_0^+ \Subset \mathbb{R}^d$ be a bounded C^1-domain, and let $\chi_0 := \chi_{\Omega_0^+}$. Then there is a measure-valued varifold solution (v, χ, V, μ) of the two-phase flow as in Definition 1.2. Moreover,

1. *If $d = 2$ or $q > d$, then* $\operatorname{supp} V(t) \subseteq \overline{B_R(0)}$ *for all* $t \in [0,T]$ *for some* $R = R(T, \chi_0, v_0)$ *and arbitrary* $T > 0$.

2. *If $q > d = 2$, then (v, χ, V) is a varifold solution and* $\operatorname{supp} |V(t)| = \Gamma_t^*$ *is a compact rectifiable set and* $|V(t)| \geq \mathcal{H}^1 \lfloor \Gamma_t^*$ *for almost all* $t > 0$. *Moreover,*

$$d_H(\Gamma_{t_1}^*, \Gamma_{t_2}^*) \leq C|t_1 - t_2|^{\frac{1}{q'}} \quad \textit{for all } 0 \leq t_1, t_2 < \infty,$$

where $d_H(.,.)$ *denotes the Hausdorff distance.*

Remark 1.7 The case $q > d = 2$ was already studied by Plotnikov in [20], where a similar result is shown, but his definition of a varifold solution is different: Properties of $V(t)$ and $\operatorname{supp} |V(t)| = \Gamma_t^*$, which can be shown for $q > d = 2$, are taken as part of the definition of a varifold solution. In particular, it is needed that $\operatorname{supp} V(t)$ is a compact 1-rectifiable set separating the plane into two open set $\omega_0(t)$ and $\omega_1(t)$. Moreover, the relation (1.15) is not used. Instead $\chi(t)$ is taken as the characteristic function of the set $\omega_0(t)$ and it solves the transport equation in the weak sense. Furthermore, it is required that the space-time interface $\bigcup_{t\in[0,T]} \Gamma_t^* \times \{t\}$ has for almost every $t \in [0,T]$ and every $x \in \Gamma_t^*$ a tangent plane containing $(v, 1)$. Finally, no energy estimate is part of the definition. – See [20] for details.

Remark 1.8 We note that in the case of a Newtonian fluid, i.e. $\nu(j, |Dv|) \equiv \nu_j$. The proof of Theorem 1.6 yields a conditional existence result for weak solutions if there is no loss of area during passing to the limit in the approximation scheme, i.e. $\lim_{k\to\infty} \|\nabla \chi_k(t)\| = \|\nabla \chi(t)\|$ for almost all $t > 0$. Then the arguments in the proof of Proposition 1.5 or a convergence theorem by Reshetnjak [2, Theorem 2.39] shows that (v, χ) is a weak solution. Such kind of results are known for example for the mean curvature flow by Luckhaus and Sturzenhecker [16] and for the multi-phase Mullins-Sekerka problem by Bronsard, Garcke, and Stoth [6].

Theorem 1.6 is proved by first constructing solutions to an approximate system for every $\varepsilon > 0$ and then pass to the limit $\varepsilon \to 0$ for a suitable subsequence. The approximate system is derived by replacing $\langle \delta V(t), . \rangle$ by $\langle \delta V(t), \Psi_\varepsilon . \rangle$ in (1.14) and replacing $v \cdot \nabla \chi$ by $\Psi_\varepsilon v \cdot \nabla \chi$ in (1.10), where Ψ_ε is a suitable smoothing operator. This preserves the energy estimate. Moreover, the convective term in (1.14) is smoothed suitably. Using the same approximation scheme we extend the result of Nouri and Poupaud [17] of existence of weak solution of a two-phase flow if Newtonian fluids ($q = 2$ and $\nu(j, s) = \nu_j$) to a class of non-Newtonian fluids:

THEOREM 1.9 (Existence of Weak Solution, $\kappa = 0$)
Let $d = 2, 3$, let $q \geq \frac{2d}{d+2} + 1$ or let $q = 2$ and $\nu(j, s) = \nu_j$, and let Assumption 1.1 hold. Moreover, let $v_0 \in L^2_\sigma(\Omega)$, $\chi_0 \in L^\infty(\Omega; \{0, 1\})$, $f \in L^{q'}(0, \infty; V_q(\Omega)')$. Then there are $v \in L^\infty(0, \infty; L^2_\sigma(\Omega)) \cap L^q(0, \infty; V_q(\Omega))$ and $\chi \in L^\infty(Q; \{0, 1\})$, $Q := \Omega \times (0, \infty)$, that are a weak solution of the two-phase flow without surface tension in the sense that

$$-(v, \partial_t \varphi)_Q - (v_0, \varphi(0))_\Omega - (v \otimes v, \nabla \varphi)_Q + (S(\chi, Dv), D\varphi)_Q = \langle f, \varphi \rangle \tag{1.21}$$

for all $\varphi \in C^\infty_{(0)}(\Omega \times [0,\infty))^d$ *with* $\operatorname{div}\varphi = 0$*,* χ *is the unique renormalized solution of the transport equation of (1.10)-(1.11), and (1.17) holds for almost all* $t > 0$ *with* $\kappa = 0$*.*

Remark 1.10 In the case of a two-phase flow for the *Stokes equation*, i.e. the convective term $v \cdot \nabla v$ is neglected and (1.21) is replaced by

$$-(v, \partial_t \varphi)_Q - (v_0, \varphi(0))_\Omega + (S(\chi, Du), D\varphi)_Q = \langle f, \varphi \rangle, \tag{1.22}$$

the same result as above holds for all $q > \frac{2d}{d+2}$. Comments on the prove are given in Remark 5.5 below.

The structure of the article is as follows: After studying the necessary preliminaries in Section 2, we first prove Proposition 1.5 in Section 3. Then we introduce the approximate system for the two-phase flow in Section 4 and prove existence of solutions for it. Using these solutions we pass to the limit in Section 5 and prove the Theorems 1.6 and 1.9. Finally in the appendix, we present a rectifiability criterion for the varifold in the two-phase flow, which is based on a new rectifiability result for varifolds due to Luckhaus [15].

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2 Preliminaries

2.1 Notation

The set of all symmetric $d \times d$-matrices is denoted by $\mathbb{R}^{d\times d}_{sym}$. For $A, B \in \mathbb{R}^{d\times d}_{sym}$ we denote $A : B = \operatorname{Tr}(AB)$ and $|A| = \sqrt{A : A}$, where Tr denotes the trace of matrices. Given $a \in \mathbb{R}^d$ we define $a \otimes a \in \mathbb{R}^{d\times d}_{sym}$ as the matrix with the entries $a_i a_j$, $i, j = 1, \ldots, d$.

The dual of a topological vector space V is denoted by V'. If $v \in V$ and $v' \in V'$, then $\langle v, v' \rangle \equiv \langle v, v' \rangle_{V,V'} := v'(v)$ denotes the duality product. If $A \colon V \to W$ is a continuous linear operator, $A' \colon W' \to V'$ denotes its adjoint.

For a given set $A \subset \mathbb{R}^d$, we define its ε-neighborhood A_ε, $\varepsilon > 0$, as $A_\varepsilon = \bigcup_{x\in A} B_\varepsilon(x)$. Moreover, for given compact sets $A, B \subset \mathbb{R}^d$ the Hausdorff distance is defined as

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}.$$

If $A \subset \mathbb{R}^d$ is a compact set, then $\mathcal{K}(A) = \{B \subseteq A : B \text{ closed}\}$ equipped with the Hausdorff distance is a compact metric space, cf. e.g. [7, Proposition 2.4.4.].

2.2 Function Spaces

Spaces of integrable functions: If $M \subseteq \mathbb{R}^d$ is measurable, $L^q(M)$, $1 \le q \le \infty$ denotes the usual Lebesgue-space and $\|.\|_q$ its norm. Moreover, $L^q(M;X)$ denotes its vector-valued variant of strongly measurable q-integrable functions/ essentially bounded functions, where X is a Banach space. More generally, if X is a Fréchet space, then $f \in L^q(M;X)$ if f is strongly measurable and q-integrable/essentially bounded with respect to all the semi-norms of X. For a subset $N \subset X$ we denote by $L^q(M;N)$ the set of all $f \in L^q(M;X)$ with $f(x) \in N$ for almost all $x \in M$. Furthermore, $f \in L^q_{\text{loc}}([0,\infty);X)$ if and only if $f \in L^q(0,T;X)$ for every $T > 0$. If $\Omega \subseteq \mathbb{R}^d$ is a domain, then $f \in L^q_{\text{loc}}(\overline{\Omega})$ if and only if $f \in L^q(\Omega \cap B)$ for every ball B with $B \cap \Omega \neq \emptyset$. For any measurable set $A \subset \mathbb{R}^d$, χ_A denotes its characteristic function.

Recall that, if X is a Banach space with the Radon-Nikodym property, then

$$L^q(M;X)' = L^{q'}(M;X') \qquad \text{for every } 1 \le q < \infty$$

by means of the duality product

$$\langle f, g \rangle = \int_M \langle f(x), g(x) \rangle \, dx$$

for $f \in L^q(M;X), g \in L^{q'}(M;X')$. If X is reflexive or X' is separable, then X has the Radon-Nikodym property, cf. Diestel and Uhl [8].

Moreover, recall the Lemma of Aubin-Lions: If $X_0 \hookrightarrow\hookrightarrow X_1 \hookrightarrow X_2$ are Banach spaces, $1 < p < \infty$, $1 \le q < \infty$, and $I \subset \mathbb{R}$ is a bounded interval, then

$$\left\{ v \in L^p(I;X_0) : \frac{dv}{dt} \in L^q(I;X_2) \right\} \hookrightarrow\hookrightarrow L^p(I;X_1). \tag{2.1}$$

See J.-L. Lions [14] for the case $q > 1$ and Simon [26] or Roubíček [22] for $q = 1$.

Furthermore, we note that, if $Y = X'$ is a dual space, then $L^\infty_\omega(Q;Y)$ for open $Q \subseteq \mathbb{R}^N$ is defined as all weak-$*$ measurable functions $\nu\colon Q \to Y$, i.e.,

$$x \mapsto \langle \nu_x, F(x,.) \rangle = \langle \nu_x, F(x,.) \rangle_{X',X}$$

is measurable for each $F \in L^1(Q;X)$, such that

$$\|\nu\|_{L^\infty_\omega(Q;Y)} := \text{ess sup}_{x \in Q} \|\nu_x\|_Y < \infty.$$

Sobolev and Bessel potential spaces: $W^m_q(\Omega)$, $m \in \mathbb{N}_0$, $1 \le q \le \infty$, denotes the usual L^q-Sobolev space, $W^m_{q,\text{loc}}(\overline{\Omega})$ its local version, $W^m_{q,0}(\Omega)$ the closure of $C^\infty_0(\Omega)$

in $W^m_q(\Omega)$, $W^{-m}_q(\Omega) = (W^m_{q',0}(\Omega))'$, and $f \in W^{-m}_{q,\mathrm{loc}}(\overline{\Omega})$ if $f \in W^{-m}_q(\Omega \cap B)$ for every ball $B \subset \mathbb{R}^d$. The L^2-Bessel potential spaces are denoted by $H^s(\Omega)$, $s \in \mathbb{R}$, which is defined as restriction of distributions in $H^s(\mathbb{R}^d)$ to Ω, cf. Triebel [34, Section 4.2.1]. Finally, $\dot{W}^1_q(\Omega) = \{f \in L^q_{\mathrm{loc}}(\overline{\Omega}) : \nabla f \in L^q(\Omega)\}$, normed in the obvious way, denotes the homogeneous Sobolev space of first order, where functions differing by a constant are identified.

Spaces of continuous functions: The usual spaces of continuous, Hölder continuous, k-times differentiable and smooth functions on an open or closed set A are denoted by $C(A)$, $C^\alpha(A)$, $0 < \alpha \leq 1$, $C^k(A)$, and $C^\infty(A)$, respectively, Furthermore, $C_0^\infty(\Omega) \equiv \mathcal{D}(\Omega)$ denotes the space of smooth and compactly supported functions on Ω and $C_0(\Omega)$, $C_0^k(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the corresponding norms. Moreover, if $A \subset \mathbb{R}^d$ is a set, then

$$C^\infty_{(0)}(A) = \left\{f\colon A \to \mathbb{R} : f = F|_A, F \in C_0^\infty(\mathbb{R}^d), \operatorname{supp} f \subseteq A\right\}$$

equipped with the quotient topology. If A is an open set, a subscript b as in $C^k_b(\mathbb{R}^d)$ indicates that the functions and their derivatives are required to be bounded.

Spaces of solenoidal functions: In the following $C^\infty_{0,\sigma}(\Omega)$ denotes the space of all divergence free vector fields in $C_0^\infty(\Omega)^d$ and $L^q_\sigma(\Omega)$ is its closure in the L^q-norm. The corresponding Helmholtz projection is denoted by $P_{L^q_\sigma}$ or just P_σ, cf. e.g. Simader and Sohr [25].

Finally, recall that $V_q(\mathbb{R}^d) = \{v \in \dot{W}^1_q(\mathbb{R}^d)^d : \operatorname{div} f = 0\}$ and $V_q(\Omega) = W^1_{q,0}(\Omega) \cap L^q_\sigma(\Omega)$ if Ω is a bounded domain. In both cases V_q will be normed by $\|v\|_{V_q(\Omega)} = \|Dv\|_{L^q(\Omega)}$. By Korn's inequality this norm is equivalent to the standard norms.

Spaces of measures and functions of bounded variations: These spaces are defined in the beginning of the Sections 2.3 and 2.4.

2.3 Measures, Disintegration and Young Measures

Let X be a locally compact separable metric space and let $C_0(X;\mathbb{R}^m)$ by the closure of compactly supported continuous functions $f\colon X \to \mathbb{R}^m$, $m \in \mathbb{N}$, in the supremum norm. Moreover, denote by $\mathcal{M}(X;\mathbb{R}^m)$ the space of all finite $\mathbb{R}^m$-valued Radon measures, $\mathcal{M}$(X):= $\mathcal{M}$(X;$\mathbb{R}$), and $Prob(X)$ denotes the space of all probability measure on X. Then by Riesz representation theorem $\mathcal{M}(X;\mathbb{R}^m) = C_0(X;\mathbb{R}^m)'$, cf. e.g. Ambrosio et. al. [2, Theorem 1.54]. Given $\mu \in \mathcal{M}(X;\mathbb{R}^m)$ the total variation measure is defined by

$$|\mu|(A) = \sup\left\{\sum_{k=0}^\infty |\mu(A_k)| : A_k \in \mathcal{B}(X) \text{ pairwise disjoint}, A = \bigcup_{k=0}^\infty A_k\right\}$$

for every $A \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the σ-algebra of Borel sets of X. Then by [2, Proposition 1.47]

$$|\mu|(A) = \sup\left\{\int_X f(x)\cdot d\mu(x) : f \in C_0(X;\mathbb{R}^m), \operatorname{supp} f \subset A, \|f\|_\infty \leq 1\right\} \qquad (2.2)$$

for every open set $A \subseteq X$. The restriction of a measure μ to a μ-measurable set A is denoted by $(\mu\lfloor A)(B) = \mu(A \cap B)$. Finally, the s-dimensional Hausdorff measure on $\mathbb{R}^d$, $0 \leq s \leq d$, is denoted by $\mathcal{H}^s$.

Now let $U \subset \mathbb{R}^N$, $V \subset \mathbb{R}^M$ be open sets and let $\nu \in \mathcal{M}(U \times V; \mathbb{R}^m)$. Moreover, we set $\mu(A) = |\nu|(A \times V)$. Then by the disintegration theorem, cf. [2, Theorem 2.28], there is a μ-measurable mapping $x \mapsto \nu_x$ such that $|\nu_x| \in Prob(V)$ for μ-a.e. $x \in U$ and for any $f \in L^1(U \times V, |\nu|)$

$$\begin{aligned} & f(x,.) \in L^1(V, |\nu_x|) \qquad \text{for } \mu-\text{a.e. } x \in U, \\ & x \mapsto \int_V f(x,y)\, d\nu_x(y) \in L^1(U, \mu), \\ \int_{U\times V} & f(x,y)\, d\nu(x,y) = \int_U \left(\int_V f(x,y)\, d\nu_x(y) \right) d\mu(x). \end{aligned} \tag{2.3}$$

Obviously, if $\nu \in \mathcal{M}(U \times V)$ is a non-negative measure, then $\nu_x = |\nu_x| \in Prob(V)$ for μ-a.e. $x \in U$.

We need the following version of the fundamental theorem of Young measures:

THEOREM 2.1 *Let $Q \subset \mathbb{R}^N$ be an open set and let $z_j \in L^p(Q; \mathbb{R}^m)$, $1 < p < \infty$, be a bounded sequence. Then there is a subsequence still denoted by z_j and a weak-$*$ measurable function $x \mapsto \nu_x \in Prob(\mathbb{R}^m)$ such that for every continuous $\tau\colon \mathbb{R}^m \to \mathbb{R}$ satisfying the growth condition*

$$|\tau(\xi)| \leq C(1 + |\xi|)^{p-1} \qquad \textit{for all } \xi \in \mathbb{R}^d$$

for some $C > 0$ we have

$$\tau(z^j) \rightharpoonup_{j\to\infty} \bar{\tau} \quad \textit{in } L^{p'}(Q)$$

where $\bar{\tau} = \langle \nu_x, \tau \rangle$ for almost all $x \in Q$.

Proof: The result immediately follows from Corollary 2.10 in Málek et. al. [12, Section 4.2] by choosing $q = p-1$, $r = \frac{p}{q} = p'$. Moreover, we note that the restriction to a bounded set in the latter Corollary is only needed if $1 < r < \frac{p}{q}$ as can be easily seen in the proof. ■

Finally, recall the generalized Jensen inequality: Let $g\colon \mathbb{R}^N \to \mathbb{R}$ be a strictly convex function and let μ be a probability measure on $\mathbb{R}^N$ such that Id and $|g|$ are μ-integrable. Then

$$g\left(\int x\, d\mu(x) \right) \leq \int_{\mathbb{R}^N} g(x) d\mu(x) \tag{2.4}$$

with equality if and only if μ is a Dirac measure, cf. e.g. [12, Lemma 2.27, Chapter III] and its proof.

2.4 BV-Functions and Varifolds

Let $U \subseteq \mathbb{R}^d$ be an open set. Recall that

$$\begin{aligned} BV(U) &= \{f \in L^1(U) : \nabla f \in \mathcal{M}(U;\mathbb{R}^d)\} \\ \|f\|_{BV(U)} &= \|f\|_{L^1(U)} + \|\nabla f\|_{\mathcal{M}(U;\mathbb{R}^d)}, \end{aligned}$$

where ∇f denotes the distributional derivative. Moreover, $BV(U;\{0,1\})$ denotes the set of all $\chi \in BV(U)$ such that $\chi(x) \in \{0,1\}$ for almost all $x \in U$.

Moreover, a set $E \subseteq U$ is said to have finite perimeter in U if $\chi_E \in BV(U)$. Then by the structure theorem of sets of finite perimeter $|\nabla \chi_E| = \mathcal{H}^{d-1}\lfloor\partial^* E$, where $\partial^* E$ is the so-called reduced boundary of E and

$$-\langle \nabla \chi_E, \varphi\rangle = \int_E \operatorname{div}\varphi\, dx = \int_{\partial^* E} \varphi \cdot n_E \, d\mathcal{H}^{d-1},$$

where $n_E(x) = -\frac{\nabla\chi_E}{|\nabla\chi_E|}$, cf. e.g. [2]. Note that, if E is a domain with C^1-boundary, then $\partial^* E = \partial\Omega$ and n_E coincides with the exterior unit normal.

For a set E of finite perimeter in U we define the mean curvature functional associated to $\partial^* E$ as

$$\langle H_{\partial^* E}, \varphi\rangle \equiv \langle H_{\chi_E}, \varphi\rangle := -\int_{\partial^* E} \operatorname{Tr}(P_\tau \nabla\varphi)\, d\mathcal{H}^{d-1}, \qquad \varphi \in C^1_0(\Omega)^d, \tag{2.5}$$

where $P_\tau = I - n_E(x) \otimes n_E(x)$.

A *general* $(d-1)$*-varifold* $\widetilde{V}$ is simply a measure $\widetilde{V} \in \mathcal{M}(U \times G_{d-1})$, where G_{d-1} is the space of all unoriented $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$, cf. Simon [27]. The first variation $\delta\widetilde{V}$ of a general varifold $\widetilde{V}$ is defined as

$$\langle \delta\widetilde{V}, \psi\rangle = \int_{U\times G_{d-1}} P_T : \nabla\psi \, d\widetilde{V}(x,T) \qquad \text{for } \psi \in C^1_0(U)^d,$$

where P_T denotes the orthogonal projection onto $T \in G_{d-1}$. Note that general varifolds are unoriented and that $G_{d-1} \cong \mathbb{S}^{d-1}/\{x \equiv -x\}$. If E is a set of finite perimeter in U, then its reduced boundary can be identified with the varifold defined by

$$\langle \widetilde{V}_{\partial^* E}, \varphi\rangle = \int_{\partial^* E} \varphi(x, [n_E(x)])\, d\mathcal{H}^{d-1} \qquad \text{for all } \varphi \in C_0(U \times G_{d-1}),$$

where $[n_E(x)]$ denotes the subspace of $\mathbb{R}^{d-1}$ with normal $n_E(x)$. Then

$$\langle \delta\widetilde{V}_{\partial^* E}, \psi\rangle = -\langle H_{\partial^* E}, \psi\rangle \qquad \text{for all } \psi \in C^1_0(U)^d.$$

Hence the mean curvature functional associated to $\partial^* E$ can be obtain back from the general varifold associated to $\partial^* E$. But this is not the case for $\nabla\chi_E = -n_E\mathcal{H}^{d-1}\lfloor\partial^* E$ since general varifolds do not take orientation into account. – Therefore we define a *oriented general* $(d-1)$*-varifold* as non-negative measures $V \in \mathcal{M}(U \times \mathbb{S}^{d-1})$ as was used for example by Soner [31, Section 2.3]. By disintegration V can be written in the form

$$\langle V, \varphi \rangle = \int_U \int_{\mathbb{S}^{d-1}} \varphi(x,s)\, dV_x(s)\, d|V|(x), \qquad \varphi \in C_0(U \times \mathbb{S}^{d-1}). \tag{2.6}$$

Obviously, every oriented general varifold V induces a (unoriented) general varifold $\widetilde{V}$ by

$$\langle \widetilde{V}, \varphi \rangle = \int_{U \times \mathbb{S}^{d-1}} \varphi(x, [s])\, dV(x,s), \quad \varphi \in C_0(U \times G_{d-1}), \tag{2.7}$$

where again $[s]$ denotes the $(d-1)$-dimensional linear subspace of $\mathbb{R}^d$ with s as normal. Now, if E has finite perimeter in U, then we associate the oriented general varifold $V_{\partial^* E}$ to $\partial^* E$ defined by

$$\langle V_{\partial^* E}, \varphi \rangle = \int_{\partial^* E} \varphi(x, n_E(x))\, d\mathcal{H}^{d-1} \qquad \text{for all } \varphi \in C_0(U \times \mathbb{S}^{d-1}).$$

Note that this corresponds to the choice $|V| = \mathcal{H}^{d-1}\lfloor \partial^* E$ and $V_x = \delta_{n_E(x)}$ in (2.6).

Now we obtain $\nabla \chi_E$ back from $V = V_{\partial^* E}$ by choosing $\varphi(x,s) = s \cdot \psi(x)$ with $\psi \in C_0(U; \mathbb{R}^d)$:

$$\langle V_{\partial^* E}, \varphi \rangle = \int_{U \times \mathbb{S}^{d-1}} s \cdot \psi(x)\, dV(x,s) = \int_{\partial^* E} \psi \cdot n_E\, d\mathcal{H}^{d-1} = -\langle \nabla \chi_E, \psi \rangle.$$

Finally, let $\Gamma_0 \subseteq \mathbb{R}^d$ be the boundary of a bounded C^1-domain Ω_0^+ with exterior normal vector field n and let $X_t \colon \mathbb{R}^d \to \mathbb{R}^d$, $t > 0$, be a family of C^1-diffeomorphisms depending continuously differentiable on $t > 0$ such that $\frac{d}{dt} X_t(x) = v(X_t(x), t)$ for a sufficiently smooth vector field v. Moreover, set $\Gamma_t = X_t(\Gamma_0)$ and $\Omega_t = X_t(\Omega_0^+)$, $t > 0$. Then one calculates that

$$\frac{d}{dt} \int_{\Gamma_t} \varphi(x)\, d\mathcal{H}^{d-1}(x) = \langle \delta V_{\Gamma_t}, \varphi v(t) \rangle + \int_{\Gamma_t} n \cdot \nabla \varphi(x) n \cdot v(x,t)\, d\mathcal{H}^{d-1}(x) \tag{2.8}$$

for every $\varphi \in C_0^1(\mathbb{R}^d)$, where V_{Γ_t} denotes the associated general varifold to Γ_t defined as above and n is the exterior normal at $\Gamma_t = \partial \Omega_t$.

2.5 Transport Equation

We consider weak solutions of the transport equation

$$\partial_t \chi + v \cdot \nabla \chi = 0 \qquad \text{in } Q_T, \tag{2.9}$$

$$\chi|_{t=0} = \chi_0 \qquad \text{in } \Omega, \tag{2.10}$$

where $Q_T = \Omega \times (0,T)$, $0 < T \leq \infty$, $\Omega = \mathbb{R}^d$ or Ω is a bounded Lipschitz domain, $v \in L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma(\Omega))$, and $\chi_0 \in L^\infty(\Omega)$. Here a weak solution is a function $\chi \in L^\infty(Q)$ satisfying

$$\int_Q \chi(\partial_t \varphi + v \cdot \nabla \varphi)\, d(x,t) + \int_\Omega \chi_0 \varphi(x,0)\, dx = 0 \tag{2.11}$$

for all $\varphi \in C^\infty_{(0)}(\overline{\Omega} \times [0,T))$. Then we have

Proposition 2.2 *For every $\chi_0 \in L^\infty(\Omega)$ and $v \in L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma(\Omega))$ there is a unique weak solution of (2.9)-(2.10) with $T = \infty$. Moreover, this solution is a renormalized solution, i.e, $\beta(\chi)$ is a weak solution associated to the data $\beta(\chi_0)$ for any $\beta \in C^1(\mathbb{R})$. Furthermore, if $\chi_0 \in M$ a.e. for some finite set M, then $\chi \in M$ a.e.*

The proposition follows from Nouri, Poupaud and Demay [18, Theorem 4.1]. It essentially coincides with [17, Proposition 3.3]. These results are based on DiPerna and Lions results on weak and renormalized solutions of the transport equation, cf. [9]. In the latter work $\operatorname{div} v = 0$ or $\operatorname{div} v \in L^\infty$ is essentially used.

In order to construct approximative solutions of the two-phase flow with surface tension we use:

Lemma 2.3 *Let $\chi_0 \in BV(\mathbb{R}^d; \{0,1\})$ and let $v \in C([0,T]; C^2_b(\mathbb{R}^d)^d)$, $\operatorname{div} v = 0$, $T > 0$. Then there is a weak solution $\chi \in L^\infty(0,T; BV(\mathbb{R}^d; \{0,1\}))$ of (2.9)-(2.10). Moreover,*

$$\|\chi\|_{L^\infty(0,T;BV(\mathbb{R}^d))} \leq M\left(\|v\|_{C([0,T];C^2_b(\mathbb{R}^d))}\right)\|\chi_0\|_{BV(\mathbb{R}^d)}, \tag{2.12}$$

$$\frac{d}{dt}|\nabla\chi(t)|(\mathbb{R}^d) = -\langle H_{\chi(t)}, v(t)\rangle \qquad \text{for all } t \in (0,T) \tag{2.13}$$

for some continuous function M.

Proof: The solution χ is constructed by the usual method of characteristics. Since $v \in C([0,T]; C^1_b(\mathbb{R}^d)^d)$ for every $x_0 \in \mathbb{R}^d$ there is a unique solution $x(t;x_0) \in C^1(0,\infty;\mathbb{R}^d)$ of

$$\frac{d}{dt}x(t;x_0) = v(x(t;x_0),t), \qquad t > 0, \tag{2.14}$$

$$x(0;x_0) = x_0, \tag{2.15}$$

which are the trajectories along the vector field v. Note that, since v is globally Lipschitz the solution $x(t;x_0)$ exists for all $t \in (0,T)$. Let $X(x_0,t) := x(t;x_0)$ and let $X_t = X(.,t)$ be the flow mapping. Then $X \in C^1([0,T]\times\mathbb{R}^d)$ by the usual C^1-dependence on the initial values and $X_t\colon \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism. Now define $\chi(x,t) := \chi_0(X_t^{-1}x)$. Then $\|\chi(.,t)\|_{L^1(\mathbb{R}^d)} = \|\chi_0\|_{L^1(\mathbb{R}^d)}$ since $\det DX_t(y) = \det DX_0(y) = 1$ because of $\partial_t \det DX_t(y) = \operatorname{div} v(X_t(y,t)) = 0$. In order to estimate $\chi \in L^\infty(0,T; BV(\mathbb{R}^d;\{0,1\}))$, we use that

$$\begin{aligned}\int_\Omega \chi(x,t)\operatorname{div}\psi(x)\,dx &= \int_\Omega \chi_0(y)\operatorname{Tr}((\nabla\psi)(X_t(y)))\,dy\\ &= \int_\Omega \chi_0(y)\operatorname{Tr}(\nabla\widetilde{\psi}(y))\,dy - \int_\Omega \chi_0(y)\operatorname{Tr}(\nabla DX_t^{-T})\psi(X_t(y))\,dy,\end{aligned}$$

where $\widetilde{\psi}(y) = DX_t^{-T}\psi(X_t(y))$, $\psi \in C^1_0(\mathbb{R}^d)^d$. Hence

$$\sup_{t\in[0,T]}\left|\int_\Omega \chi(x,t)\operatorname{div}\psi(x)dx\right| \leq M\left(\|v\|_{C([0,T];C^2_b(\mathbb{R}^d))}\right)\|\chi_0\|_{BV(\mathbb{R}^d)}\|\psi\|_{C^0_b(\mathbb{R}^d)}$$

for all $\psi \in C_0^1(\mathbb{R}^d)^d$ and $t > 0$ and some continuous function M. Moreover, by standard calculations:

$$\begin{aligned}(\chi, \partial_t \varphi)_Q &= \int_0^\infty \int_\Omega \chi_0(y) \partial_t \varphi(X_t(y), t)\, dy\, dt \\ &= -(\chi_0, \varphi|_{t=0}) - \int_0^\infty \int_\Omega \chi_0(y) \nabla \varphi(X_t(y), t) \cdot v(X_t(y), t)\, dy\, dt \\ &= -(\chi_0, \varphi|_{t=0}) - (\chi, v \cdot \nabla \varphi)_Q\end{aligned}$$

for all $\varphi \in C^\infty_{(0)}([0,\infty) \times \mathbb{R}^d)$. Hence χ is a weak solution of (2.9)-(2.10).

Finally, the last identity follows from (2.8). ■

For the following we note that $C^1(\mathbb{R}^d)$ is equipped with the topology of locally uniform convergence of the functions and their first order derivatives.

Lemma 2.4 *Let $\chi_0 = \chi_{\Omega_0^+}$, where Ω_0^+ is a bounded C^1-domain. Moreover, let $u_k, u \in C([0,T]; C_b^2(\mathbb{R}^d)^d)$ such that $u_k \to_{k\to\infty} u$ in $C([0,T]; C^1(\mathbb{R}^d)^d)$. Then for any $f \in C_b^0(\mathbb{R}^d \times \mathbb{S}^{d-1})$*

$$\lim_{k\to\infty} \int_{\Gamma_{u_k}(t)} f(x, n_x)\, d\mathcal{H}^{d-1}(x) = \int_{\Gamma_u(t)} f(x, n_x)\, d\mathcal{H}^{d-1}(x) \tag{2.16}$$

uniformly in $t \in (0,T)$, where $\Gamma_w(t) = X_w(t)(\partial\Omega_0^+)$ and $X_w(t)$ the flow map obtained from (2.14)-(2.15) with $v = w$ as above. Finally, $\{\Gamma_{u_k}(t), \Gamma_u(t) : k \in \mathbb{N}, t \in [0,T]\}$ is contained in a compact set.

Proof: First of all $X_{u_k} \in C^1([0,T] \times \mathbb{R}^d)$ and $X_{u_k} \to X_u \in C^1([0,T] \times B_R(0))$, $R > 0$, by the usual C^1-dependence of solutions of ordinary differential equations on the data. Moreover, by construction $X_{u_k}(t)\colon \mathbb{R}^d \to \mathbb{R}^d$ are bijective for any $t \in [0,T]$. Hence $X_{u_k}^{-1}(t)\colon \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable and $X_{u_k}^{-1}(t) \to X_u^{-1}(t)$ in $C^1(\mathbb{R}^d)$ for any $t \in [0,T]$, Using all this, the lemma can be proved by either introducing a local parameterization of $\partial\Omega_0^+$ and using $X_{u_k}(t)$ and $X_u(t)$ to get a suitable parameterizations of $\Gamma_{u_k}(t)$ and $\Gamma_u(t)$ or one uses the continuity theorem by Reshetnjak: Since $X_{u_k}(t) \to_{k\to\infty} X_u(t)$ and $X_{u_k}^{-1}(t) \to_{k\to\infty} X_u^{-1}(t)$ in $C^1(\mathbb{R}^d)$, it is an easy exercise to show $\mathcal{H}^{d-1}(\Gamma_{u_k}(t)) \to_{k\to\infty} \mathcal{H}^{d-1}(\Gamma_u(t))$. Moreover, if $\Omega_k^+(t) = X_{u_k}(t)(\Omega_0^+), \Omega^+(t) = X_u(t)(\Omega_0^+)$, then

$$\langle \nabla \chi_{\Omega_k^+(t)}, \varphi \rangle = -\int_{\Omega_0^+} \operatorname{div} \varphi(X_{u_k}(t))\, dx \to_{k\to\infty} -\int_{\Omega_0^+} \operatorname{div} \varphi(X_u(t))\, dx = \langle \nabla \chi_{\Omega^+(t)}, \varphi \rangle$$

for all $\varphi \in C_0^1(\mathbb{R}^d)^d$. This implies $\nabla \chi_{\Omega_k^+(t)} \rightharpoonup^*_{k\to\infty} \nabla \chi_{\Omega^+(t)}$ in $\mathcal{M}(\mathbb{R}^d)$ since $\|\nabla \chi_{u_k}\|_{\mathcal{M}} = \mathcal{H}^{d-1}(\Gamma_k(t))$ are uniformly bounded and $C_0^1(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$. Therefore one can apply [2, Theorem 2.39] to the vector measures $\nabla \chi_{\Omega_k^+(t)}$ and $\nabla \chi_{\Omega^+(t)}$ to show (2.16).

Finally, the last statement is an easy consequence of the fact that $X_{u_k} \to X_u$ in $C^1([0,T]\times\mathbb{R}^d)$ and the compactness of $\partial\Omega_0^+$. ■

Lemma 2.5 *Let $u_k, u \in L^1(0,T;L^2_\sigma(\Omega))$, $k\in\mathbb{N}$, for some $T>0$ such that $u_k \to_{k\to\infty} u$ in $L^1(0,T;L^2_{\mathrm{loc}}(\overline{\Omega}))$. Moreover, let $\chi_k, \chi \in L^\infty(Q_T)$ be the solutions of (2.9)-(2.10) with $v = u_k, u$, resp., and $\chi_0 = \chi_E$ for some fixed measurable set E. Then $\chi_k \rightharpoonup^*_{k\to\infty} \chi$ in $L^\infty(Q_T)$ and $\chi_k \to_{k\to\infty} \chi$ in $L^p(Q_T)$ for every $p<\infty$.*

Proof: First of all, since $\chi_k \in L^\infty(Q_T)$ are uniformly bounded, $\chi_{k_j} \rightharpoonup^*_{j\to\infty} \widetilde{\chi_0}$ in $L^\infty(Q_T)$ for some $\widetilde{\chi_0}\in L^\infty(Q_T)$ and some suitable subsequence. Since $u_k \to_{k\to\infty} u$ in $L^1(0,T;L^2_{\mathrm{loc}}(\Omega))$, $u_k\cdot\nabla\varphi \to_{k\to\infty} u\cdot\nabla\varphi$ in $L^1(0,T;L^2(\Omega))$ for any $\varphi\in C^\infty_{(0)}(\overline{\Omega}\times[0,\infty))$. Thus $\widetilde{\chi_0}$ solves the transport equation with $v=u$. Hence

$$\|\chi_{k_j}\|^q_{L^q(Q_T)} = T|E| = \|\widetilde{\chi_0}\|^q_q$$

for every $1\le q<\infty$. Thus $\chi_{k_j} \to_{j\to\infty} \widetilde{\chi_0}$ in $L^q(Q_T)$ strongly. In particular, this implies $\widetilde{\chi_0}\in\{0,1\}$ almost everywhere. Therefore $\widetilde{\chi_0}$ coincides with the unique renormalized solution χ_0. Since this argumentation holds for any subsequence, the sequence $(\chi_k)_{k\in\mathbb{N}}$ converges itself. ■

2.6 A Convergence Result for Monotone Nonlinearities

In order to construct weak solutions in the case $\kappa=0$, we will use the following result:

THEOREM 2.6 (Swierczewska [32, Lemma A.1])
Let $E\subset\mathbb{R}^d$ be a measurable set of finite measure and let $A\colon E\times\mathbb{R}^m\times\mathbb{R}^N\to\mathbb{R}^N$ be a function such that

1. *$A(x,s,\xi)$ is a Carathéodory function w.r.t. x and (s,ξ), i.e., A is measurable w.r.t x and continuous w.r.t. (s,ξ).*

2. *$A(x,s,\xi)$ is strictly monotone w.r.t. ξ: For almost all $x\in E$ and all $s\in\mathbb{R}^m$ and $\xi_1,\xi_2\in\mathbb{R}^N$, $\xi_1\neq\xi_2$,*

$$(A(x,s,\xi_1)-A(x,s,\xi_2))\cdot(\xi_1-\xi_2)>0.$$

3. *There is some $q>1$ and $c_1,c_2>0$ such that*

$$A(x,s,\xi)\cdot\xi \ge c_1|\xi|^q, \qquad |A(x,s,\xi)|\le c_2|\xi|^{q-1}$$

for almost all $x\in E$ and all $(s,\xi)\in\mathbb{R}^m\times\mathbb{R}^N$.

Moreover, let $y_n\colon E\to\mathbb{R}^m$ *and* $z_n\colon E\to\mathbb{R}^N$ *be a sequence of measurable functions such that* $y_n\to y$ *a.e. in* E, $z_n\rightharpoonup z$ *in* $L^q(E)$ *and* $A(x,y_n,z_n)\rightharpoonup\bar{A}$ *in* $L^{q'}(E)$ *as* $n\to\infty$. *Then*

$$\limsup_{n\to\infty}\int_E A(x,y_n,z_n)\cdot z_n\,dx\le\int_E\bar{A}\cdot z\,dx$$

implies $z_n\to z$ *in measure as* $n\to\infty$.

In the following we will apply the theorem to the case $x\in\Omega$, $s\in\mathbb{R}$, $\xi=\lambda\in\mathbb{R}^{d\times d}_{sym}$, and $A(x,s,\xi)=S(s,\lambda)$. In this case Assumption 1.1 implies the assumptions of the theorem.

3 Proof of Proposition 1.5

First of all, we note that by (1.12) $\lambda\mapsto S(l,\lambda):\lambda$, $\lambda\in\mathbb{R}^{d\times d}_{sym}$, is a strictly convex function for every $l\in[0,1]$.

First assume that $\|V(t)\|=\|\nabla\chi(t)\|$ for almost all $t\in(0,T)$. We will prove that $V_x(t)=\delta_{n(x,t)}$ for $|V(t)|$-almost every $x\in\mathbb{R}^d$ and for almost every $t\in(0,\infty)$, where $n(x,t)=-\frac{\nabla\chi(t)}{|\nabla\chi(t)|}(x)$. From (1.15) we know that

$$\int_\Omega\int_{\mathbb{S}^{d-1}}s\cdot\psi(x)\,dV_x(t)\,d|V(t)|=\int_\Omega n(x,t)\cdot\psi(x)\,d|\nabla\chi(t)|$$

for all $\psi\in C_0(\Omega)^d$. Hence by (2.2) $|\nabla\chi(t)|(A)\le|V(t)|(A)$ for every open $A\subset\Omega$. Thus $|\nabla\chi(t)|$ is absolutely continuous with respect to $|V(t)|$ and

$$|\nabla\chi(t)|(A)=\int_A\theta_t(x)\,d|V(t)|(x)$$

with some $|V(t)|$-measurable function $\theta_t\colon\mathbb{R}^d\to[0,\infty)$ and $\theta_t(x)\le 1$ for $|V(t)|$-almost all $x\in\mathbb{R}^d$. But, since $\|\nabla\chi(t)\|=|\nabla\chi(t)|(\mathbb{R}^d)=|V(t)|(\mathbb{R}^d)=\|V(t)\|$, we conclude that $\theta_t(x)=1$ almost everywhere and $|V(t)|=|\nabla\chi(t)|$ as measures. Therefore (1.15) yields

$$\int_{\mathbb{S}^{d-1}}s\,dV_x(t)(s)=n(x,t)\qquad\text{for }|V(t)|\text{-almost all }\ x\in\Omega.$$

Thus

$$\frac{1}{2}\int|s-n(x,t)|^2\,dV_x(t)(s)=1-n(x,t)\cdot\int_{\mathbb{S}^{d-1}}s\,dV_x(t)(s)=0$$

for $|V(t)|$-almost all $x\in\Omega$, which implies that $V_x(t)=\delta_{n(x,t)}$ for $|V(t)|$-almost every $x\in\mathbb{R}^d$.

If (v,χ) satisfies (1.20), then necessarily $\|\nabla\chi(t)\|_{\mathcal{M}}=\|V(t)\|_{\mathcal{M}}$ for almost all $t>0$ because of (1.17) and (1.18). Hence the first part implies that $V_x(t)=\delta_{n(x,t)}$ which yields $\delta V(t)=-H_{\chi(t)}$. Moreover, by (2.4) and (1.16)

$$S(\chi(x,t),Dv(x,t)):Dv(x,t)\le\int S(\chi(x,t),\lambda):\lambda\,d\mu_{x,t}(\lambda)$$

with equality if and only if $\mu_{x,t}$ is a Dirac measure. – Note that $\int S(\chi(x,t),\lambda) : \lambda\, d\mu_{x,t}(\lambda) < \infty$ for almost all $(x,t) \in Q$ by (1.17). – On the other hand by (1.17)-(1.20),

$$\int_{Q_t} S(\chi(x,t), Dv(x,t)) : Dv(x,t)\, d(x,t) = \int_{Q_t} S(\chi,\lambda) : \lambda\, d\mu_{x,t}(\lambda)\, d(x,t)$$

for almost all $t > 0$. Hence $\mu_{x,t}$ is a Dirac measure for almost all $(x,t) \in Q$, which implies that $\mu_{x,t} = \delta_{Dv(x,t)}$ because of (1.16). – Altogether, we have proved that (v,χ) is a weak solution. The same argumentation also shows that (v,χ,V) is a varifold solution if (1.20) holds with $\|\nabla\chi(t)\|$ replaced by $\|V(t)\|$.

Finally, if $q > d$, then (2.11) and the fact that $\chi \in L^\infty(0,\infty; BV(\mathbb{R}^d))$ yield

$$\begin{aligned}
-(\chi, \operatorname{div}_{x,t}\psi) &= \int_0^\infty \int \psi'(x,t)\, d\nabla\chi(t)\, dt + \int_Q \chi \operatorname{div}(v\psi_{d+1})\, d(x,t) \\
&= \int_0^\infty \int \psi'(x,t)\, d\nabla\chi(t)\, dt - \int_0^\infty \int v\psi_{d+1}\, d\nabla\chi(t)\, dt
\end{aligned}$$

for all $\psi = (\psi', \psi_{d+1}) \in C_0^1(Q; \mathbb{R}^{d+1})$ where $\psi_{d+1}(x,t) \in \mathbb{R}$. Moreover, since $q > d$, $L^q(0,T; V_q(\mathbb{R}^d)) \cap L^\infty(0,T; L^2_\sigma(\mathbb{R}^d)) \hookrightarrow L^q(0,T; C_0(\mathbb{R}^d))$ for each $T > 0$ and

$$\begin{aligned}
\left|\int_0^\infty \int \psi'(x,t)\, d\nabla\chi(t)\, dt\right| &\leq C(E_0,T)\|\psi\|_{C_0(Q_T)}, \\
\left|\int_0^\infty \int v\psi_{d+1}\, d\nabla\chi(t)\, dt\right| &\leq C(E_0,T)\|v\|_{L^q(0,T;C_0(\mathbb{R}^d))}\|\psi\|_{C_0(Q_T)} \\
&\leq C(E_0,T)\|\psi\|_{C_0(Q_T)}
\end{aligned}$$

if $\operatorname{supp}\psi \subseteq Q_T$ for $T > 0$, where $E_0 = \frac{1}{2}\|v_0\|_2^2 + \kappa\|\nabla\chi_0\|_{\mathcal{M}}$. This shows that $\chi \in BV(Q_T)$ for every $0 < T < \infty$.

4 Approximative Two-Phase Flow

In the following we denote $X_\kappa = BV$ if $\kappa > 0$ and $X_\kappa = L^\infty$ if $\kappa = 0$.

In order to formulate the approximation equations, let $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\psi \subseteq \overline{B_1(0)}$, $\int \psi dx = 1$ and $\psi \geq 0$. Moreover, let $\Psi_\varepsilon f = \psi_\varepsilon * f$ if $\Omega = \mathbb{R}^d$ and $\Psi_\varepsilon = P_\sigma(\psi_\varepsilon * f)$, where $\psi_\varepsilon(x) := \varepsilon^{-d}\psi(\varepsilon^{-1}x)$, $\varepsilon > 0$, f is extended by 0 to $\mathbb{R}^d$, and P_σ denotes the Helmholtz projection, cf. [25]. Then we consider the approximative two-phase flow on $(0,T)$, $T > 0$, which is $v_\varepsilon \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^q(0,T; V_q(\Omega))$ solves

$$\begin{aligned}
&-(v_\varepsilon, \partial_t\varphi)_{Q_T} - (v_0, \varphi(0))_\Omega - (\Psi_\varepsilon v_\varepsilon \otimes \psi_\varepsilon * v_\varepsilon, \nabla\psi_\varepsilon * \varphi)_{Q_T} \\
&\qquad + (S(\chi_\varepsilon, Dv_\varepsilon), D\varphi)_{Q_T} = \kappa \int_0^T \langle H_{\chi_\varepsilon(t)}, \Psi_\varepsilon\varphi(t)\rangle\, dt \qquad (4.1)
\end{aligned}$$

for all $\varphi \in C^\infty_{(0)}(\Omega \times [0,T))^d$ with $\operatorname{div}\varphi = 0$, and $\chi_\varepsilon \in L^\infty(0,T;X_\kappa(\Omega;\{0,1\}))$ is the unique renormalized solution of the transport equation

$$\partial_t\chi_\varepsilon + (\Psi_\varepsilon v_\varepsilon)\cdot\nabla\chi_\varepsilon = 0 \qquad \text{in } Q_T, \tag{4.2}$$
$$\chi_\varepsilon|_{t=0} = \chi_0 \qquad \text{in } \Omega. \tag{4.3}$$

Throughout this section we will frequently use that

$$(\Psi_\varepsilon v \otimes \psi_\varepsilon * w, \nabla\psi_\varepsilon * w)_\Omega = -(\Psi_\varepsilon v \otimes \psi_\varepsilon * w, \nabla\psi_\varepsilon * w)_\Omega = 0 \tag{4.4}$$

for all $v,w \in L^2(\Omega)^d$. This fact follows from integration by parts and from $\operatorname{div}\Psi_\varepsilon v = 0$, $n\cdot\Psi_\varepsilon|_{\partial\Omega} = 0$.

First of all, we need

Lemma 4.1 *Let Assumption 1.1 hold and let $T,\varepsilon > 0$. Then for every $v_0 \in L^2_\sigma(\Omega)$, $f \in L^{q'}(0,T;V_q(\Omega)')$, and $\chi \in L^\infty(Q_T;[0,1])$, there is a unique solution $v \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^q(0,T;V_q(\Omega))$ with $\partial_t v \in L^{q'}(0,T;V_q(\Omega)')$ solving*

$$-(v,\partial_t\varphi)_Q - (v_0,\varphi(0))_\Omega - (\Psi_\varepsilon v\otimes\psi_\varepsilon * v, \nabla\psi_\varepsilon * v)_Q + (S(\chi,Dv),D\varphi)_Q = \langle f,\varphi\rangle \tag{4.5}$$

for all $\varphi \in C^\infty_{(0)}(\Omega\times[0,T))^d$ with $\operatorname{div}\varphi = 0$. Moreover,

$$\sup_{0\le t\le T}\|v(t)\|_2^2 + \|v\|^q_{L^q(0,T;V_q)} \le C\left(\|f\|^{q'}_{L^{q'}(0,T;V_q')} + \|v_0\|_2^2\right) \tag{4.6}$$
$$\|\partial_t v\|_{L^{q'}(0,T;V_q')} \le M\left(\|f\|_{L^{q'}(0,T;V_q')}, \|v_0\|_2^2\right) \tag{4.7}$$

for some continuous function M. Finally, if $f_k, f \in L^1(0,T;L^2_\sigma(\Omega))\cap L^{q'}(0,T;V_q(\Omega)')$ and $\chi_k,\chi \in L^\infty(Q_T;[0,1])$, $k\in\mathbb{N}$, are bounded sequences such that $f_k \to_{k\to\infty} f$ in $L^1(0,T;L^2(\Omega))$ and $\chi_k \to_{k\to\infty} \chi$ in $L^p(Q_T)$ for some $1\le p\le\infty$, then $v_k \to_{k\to\infty} v \in C([0,T];L^2(\Omega))$ where v_k is the solution of (4.5) with (f,χ) replaced by (f_k,χ_k).

Proof: The proof of existence of solutions can be done by a standard Galerkin approximation using the fact that

$$\langle A(u),v\rangle := \int_\Omega \nu(\chi,|Du|)Du : Dv\,dx, \qquad u,v \in V := V_q(\Omega)$$

is a strictly monotone, coercive, hemicontinuous bounded operator $A\colon V \to V'$. More precisely:

First assume that the convective term is not present, i.e., $\Psi_\varepsilon \equiv 0$. If Ω is a bounded domain, then the lemma is a consequence of Zeidler [36, Theorem 30.A] with V as above and $H = L^2(\Omega)$. The conditions (H1)-(H6) are easily verified. If $\Omega = \mathbb{R}^d$, then $V = V_q(\mathbb{R}^d)$, $H = L^2(\mathbb{R}^d)$, $V' = V_q(\mathbb{R}^d)'$ is no longer an evolution triple. But V,H,V' still have a common dense basis and the fundamental relation

$$(u(t),v(t))_{L^2(\Omega)} - (u(0),v(0))_{L^2(\Omega)} = \int_0^t \langle u'(s),v(s)\rangle + \langle v'(s),u(s)\rangle\,ds \tag{4.8}$$

for all $0 \le t \le T$ and $u, v \in L^q(0,T;V)$ with $u', v' \in L^{q'}(0,T;V')$ still holds. Then the proof given for [36, Theorem 30.A] easily carries over.

If the convective term is present, the proof can be easily modified using the fact that

$$(\Psi_\varepsilon v \otimes \psi_\varepsilon * v, \nabla \psi_\varepsilon * v)_Q = 0$$

due to (4.4). Therefore the energy estimate for the case with convective term is the same as without. Moreover, in order to pass to the limit in the convective term during the Galerkin approximation, one simply uses that

$$X := \left\{ u \in L^q(0,T;V_q(\Omega)) : \partial_t u \in L^{q'}(0,T;V_q(\Omega)') \right\} \hookrightarrow\hookrightarrow L^1(0,T;L^2_{\mathrm{loc}}(\overline{\Omega}))$$

because of (2.1) applied to $X_0 = W^1_q(\Omega_R)$, $X_1 = L^2(\Omega_R)$, and $X_2 = W^{-1}_{q'}(\Omega_R)$, where $\Omega_R = \Omega \cap B_R(0)$ and $R > 0$ is arbitrary. This is sufficient to show that

$$\lim_{n\to\infty} (\Psi_\varepsilon v_n \otimes \psi_\varepsilon * v_n, \nabla \psi_\varepsilon * \varphi)_{Q_T} = (\Psi_\varepsilon v \otimes \psi_\varepsilon * v, \nabla \psi_\varepsilon * \varphi)_{Q_T}$$

for all $\varphi \in C^\infty_{(0)}(\Omega \times [0,T))^d$, $\operatorname{div} \varphi = 0$, if $v_n \rightharpoonup_{n\to\infty} v$ in X.

Furthermore, we note that the estimate (4.6) follows from the usual energy estimate. In order to estimate $\partial_t v$, we observe that

$$\|S(\chi, Dv)\|^{q'}_{L^{q'}(Q_T)} = \int_{Q_T} |S(\chi, Dv)|^{\frac{q}{q-1}}\, d(x,t) \le C \int_{Q_T} |Dv|^q \, d(x,t).$$

Moreover, since $\Psi_\varepsilon u = P_\sigma(\psi_\varepsilon * u)$ and since P_σ is continuous on $L^s(\Omega)^d$ for all $1 < s < \infty$, we conclude that

$$\|\Psi_\varepsilon v\|_s \le C_s \|\psi_\varepsilon v\|_s \le C_{\varepsilon,s} \|v\|_2, \qquad \|\nabla \psi_\varepsilon * \varphi\|_q \le C_{\varepsilon,q} \|\varphi\|_{V_q(\Omega)}$$

for all $2 \le s < \infty$. Therefore

$$\begin{aligned} & \left| \int_{Q_T} \Psi_\varepsilon v \otimes \psi_\varepsilon * v : \nabla \psi_\varepsilon * \varphi \, d(x,t) \right| \\ \le\ & C_s T^{\frac{1}{q'}} \left(\sup_{t\in[0,T]} \|\psi_\varepsilon * v(t)\|_s^2 \right) \|\nabla \psi_\varepsilon * \varphi\|_{L^q(Q_T)} \\ \le\ & C_{\varepsilon,s,T} \|v\|^2_{L^\infty(0,T;L^2_\sigma(\Omega))} \|\varphi\|_{L^q(0,T;V_q(\Omega))}, \end{aligned}$$

where $\frac{1}{s} = \frac{1}{2} - \frac{1}{2q}$. Using these estimates and the equation (4.5), one easily derives (4.7).

In order to prove uniqueness and the last statement, let v, w be two solutions of (4.5). Then

$$\begin{aligned} & -(v - w, \partial_t \varphi)_{Q_T} + (S(\chi, Dv) - S(\chi, Dw), D\varphi)_{Q_T} \\ =\ & (\Psi_\varepsilon (v-w) \otimes \psi_\varepsilon * v, \nabla \psi_\varepsilon * \varphi)_{Q_T} + (\Psi_\varepsilon w \otimes \psi_\varepsilon * (v-w), \nabla \psi_\varepsilon * \varphi)_{Q_T} \end{aligned}$$

for all $\varphi \in C^\infty_{(0)}(\Omega \times [0,T))^d$ with $\operatorname{div}\varphi = 0$. Choosing $\varphi = (w-v)\chi_{[0,t]}$, $t \in [0,T]$, via a standard approximation, and using (4.8) we conclude

$$\begin{aligned}\|v(t)-w(t)\|_2^2 &\leq (S(\chi,Dv)-S(\chi,Dw),Dv-Dw)_{Q_t}\\ &\leq C_\varepsilon\left(\sup_{0\leq t\leq T}(\|v(t)\|_2+\|w(t)\|_2)\right)\int_0^t \|v(s)-w(s)\|_2^2\,ds\end{aligned}$$

since $Dv \mapsto S(\chi,Dv)$ is monotone. Hence Gronwall's inequality implies $v \equiv w$.

Finally, let $f_k, f, \chi_k, \chi, v_k, v$ be as in the last statement. Then

$$\begin{aligned}&-(v_k-v,\partial_t\varphi)_Q + (S(\chi_k,Dv_k)-S(\chi_k,Dv),D\varphi)_Q\\ &\quad= (S(\chi,Dv)-S(\chi_k,Dv),D\varphi)_Q + \langle f_k-f,\varphi\rangle\\ &\qquad+(\Psi_\varepsilon(v_k-v)\otimes\psi_\varepsilon * v_k,\nabla\psi_\varepsilon*\varphi)_{Q_T} + (\Psi_\varepsilon v\otimes\psi_\varepsilon*(v_k-v),\nabla\psi_\varepsilon*\varphi)_{Q_T}\end{aligned}$$

for all $\varphi \in C^\infty_{(0)}(\Omega \times [0,T))^d$ with $\operatorname{div}\varphi = 0$. Choosing $\varphi = (v_k-v)\chi_{[0,t]}$, $t \in [0,T]$, we conclude, using the boundedness of v_k, v, that

$$\begin{aligned}&\|v_k(t)-v(t)\|_2^2 + (S(\chi_k,Dv_k)-S(\chi_k,Dv),Dv_k-Dv)_{Q_t}\\ &\leq C_\varepsilon\left(\|f_k-f\|_{L^1(0,T;L^2)} + \|S(\chi,Dv)-S(\chi_k,Dv)\|_{L^{q'}(Q_T)} + \int_0^t\|v_k(s)-v(s)\|_2^2\,ds\right).\end{aligned}$$

Thus

$$\sup_{0\leq t\leq T}\|v_k(t)-v(t)\|_2^2 \leq C_{\varepsilon,T}\left(\|f_k-f\|_{L^1(0,T;L^2)} + \|S(\chi,Dv)-S(\chi_k,Dv)\|_{L^{q'}(Q_T)}\right)$$

by Gronwall's inequality. The second term can be estimated as

$$\begin{aligned}\|S(\chi,Dv)-S(\chi_k,Dv)\|_{L^{q'}(Q_T)}^{q'} &\leq C_0\int_{Q_T}|\chi-\chi_k||Dv|^q\,d(x,t)\\ \leq C_1\int_{Q_T}|\chi-\chi_k||D\varphi|^q\,d(x,t) &+ C_2\int_{Q_T}|Dv-D\varphi|^q\,d(x,t)\end{aligned}$$

for all $\varphi \in C^\infty_{(0)}(\overline{Q_T})$. Now we observe that the first term on the right-hand side converges to zero as $k \to \infty$ since $\chi_k \to_{k\to\infty} \chi$ in $L^p(Q_T)$ and the second term is arbitrarily small since $C^\infty_{(0)}(\overline{Q_T})$ is dense in $L^q(0,T;\dot W^1_q(\Omega))$. Hence $\lim_{k\to\infty}\|S(\chi,Dv)-S(\chi_k,Dv)\|_{L^{q'}(Q_T)} = 0$. Altogether this implies $\lim_{k\to\infty}\sup_{0\leq t\leq T}\|v_k(t)-v(t)\|_2^2 = 0$. ■

THEOREM 4.2 *Let Assumption 1.1 hold. Then for every $\varepsilon, T > 0$, $v_0 \in L^2_\sigma(\Omega)$, $\chi_0 \in L^\infty(\Omega;\{0,1\})$ if $\kappa = 0$ and $\chi_0 = \chi_{\Omega_0^+}$ if $\kappa > 0$, where $\Omega_0^+ \Subset \Omega$ is a bounded domain with C^1-boundary, there is a solution $v_\varepsilon \in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^q(0,T;V_q(\Omega))$, $\chi_\varepsilon \in L^\infty(0,T;X_\kappa(\Omega;\{0,1\}))$ of (4.1)-(4.3). Moreover, every solution satisfies the* energy equality

$$\begin{aligned}&\frac{1}{2}\|v_\varepsilon(t)\|_2^2 + \kappa\|\nabla\chi_\varepsilon(t)\|_{\mathcal{M}}\\ &\quad+\int_s^t\int_\Omega S(\chi_\varepsilon,Dv_\varepsilon):Dv_\varepsilon\,dx\,d\tau = \frac{1}{2}\|v_\varepsilon(s)\|_2^2 + \kappa\|\nabla\chi_\varepsilon(s)\|_{\mathcal{M}} \qquad (4.9)\end{aligned}$$

for all $s,t \in (0,T)$, $s \le t$, where $t \mapsto \frac{1}{2}\|v_\varepsilon(t)\|_2^2$ and $t \mapsto \|\nabla\chi_\varepsilon(t)\|_{\mathcal{M}}$ are absolutely continuous functions satisfying

$$\frac{d}{dt}\frac{1}{2}\|v_\varepsilon(t)\|_2^2 = \langle \partial_t v_\varepsilon(t), v_\varepsilon(t)\rangle_\Omega, \quad \frac{d}{dt}\|\nabla\chi_\varepsilon(t)\|_{\mathcal{M}} = -\langle H_{\chi_\varepsilon(t)}, \Psi_\varepsilon * v_\varepsilon(t)\rangle \text{ if } \kappa > 0$$

for almost all $t \in (0,T)$ and $\partial_t v_\varepsilon, \kappa\langle H_{\chi_\varepsilon(t)}, \psi_\varepsilon * .\rangle \in L^{q'}(0,T;V_q(\Omega)')$.

Proof: Let

$$X_1 := \left\{ u \in L^q(0,T;V_q(\Omega)) : \partial_t u \in L^{q'}(0,T;V_q(\Omega)') \right\}$$

normed in a suitable way. Moreover, let $X_0 := C([0,T];L^2_\sigma(\Omega))$ and let $X_\alpha = (X_0,X_1)_\alpha$, where $(.,.)_\alpha$ is an exact interpolation functor of type $\alpha \in (0,1)$ – f.e. the real interpolation functor, cf. Bergh and Löfström [5]. – Note that by (4.8) $X_1 \hookrightarrow X_0$. Furthermore, we note that the inclusion of X_1 into $L^1(0,T;L^2_{\mathrm{loc}}(\overline{\Omega}))$ is compact because of (2.1) applied to $W^1_q(\Omega_R)$, $L^2(\Omega_R)$, and $W^{-1}_{q'}(\Omega_R)$, where $\Omega_R = \Omega \cap B_R(0)$ and $R > 0$ is arbitrary. By [5, Theorem 3.8.1] the same holds for X_α, $\alpha \in (0,1]$, instead of X_1.

We define a mapping $F\colon X_0 \to X_1$ as follows: For given $u \in X_0$ let $\chi_u \in L^\infty(0,\infty;X_\kappa(\Omega;\{0,1\}))$ be the solution of the transport equation (2.9)-(2.10) with v in (2.9) replaced by $\Psi_\varepsilon u$. Then

$$X_0 \ni u \mapsto \chi_u \in L^p(Q_T), 1 < p < \infty, \tag{4.10}$$

is strongly continuous by Lemma 2.5. Moreover, the mapping $X_\alpha \ni u \mapsto \chi_u \in L^p(Q_T)$, $\alpha \in (0,1]$, is even compact by the following argument: If $u_k \in X_\alpha$, $k \in \mathbb{N}_0$, is a bounded sequence, then after passing to a suitable subsequence $u_k \to_{k\to\infty} u$ in $L^1(0,T;L^2_{\mathrm{loc}}(\overline{\Omega}))$ by the observations above. This implies the same statement for $\Psi_\varepsilon u_k, \Psi_\varepsilon u$. Hence $\chi_{u_k} \to \chi_u$ again by Lemma 2.5.

Now let $v = F(u)$ be the solution of (4.5) with $\chi = \chi_u$ and

$$\langle f_u, \varphi\rangle := \kappa \int_0^T \langle H_{\chi_u(t)}, \Psi_\varepsilon\varphi(t)\rangle\, dt.$$

Claim: $F\colon X_0 \to X_0$ is continuous, $F\colon X_\alpha \to X_0$, $\alpha \in (0,1]$ is compact, and $F\colon X_0 \to X_1$ is bounded.

Proof of Claim: First let $\kappa = 0$. Then $F\colon X_0 \to X_0$ is continuous because of Lemma 4.1 and (4.10). Moreover, $F\colon X_0 \to X_1$ is bounded by (4.6) and (4.7). Finally, $F\colon X_1 \to X_0$ is compact since $X_1 \ni u \mapsto \chi_0 \in L^p(Q_T)$ is compact and the mapping of χ_u to the solution $v = F(u) \in X_0$ of (4.5) with $\chi = \chi_u$ and $f = 0$ is continuous.

In the case $\kappa > 0$ it remains to prove that $X_0 \ni u \mapsto f_u \in L^{q'}(0,T;V_q(\mathbb{R}^d)')$ is bounded, that $X_0 \ni u \mapsto f_u \in L^1(0,T;L^2_\sigma(\mathbb{R}^d))$ is continuous, and that $X_\alpha \ni u \mapsto f_u \in L^1(0,T;L^2_\sigma(\mathbb{R}^d))$, $\alpha \in (0,1]$, is compact. Then the claim follows in the same way.

Firstly, we estimate f_u: Since $\Omega = \mathbb{R}^d$ if $\kappa > 0$, $\Psi_\varepsilon\varphi = \psi_\varepsilon * \varphi$ and

$$\left|\langle H_{\chi_u(t)}, \Psi_\varepsilon\varphi(t)\rangle\right| \leq C\kappa\|\nabla\chi_u(t)\|_{\mathcal{M}}\|\nabla\psi_\varepsilon * \varphi\|_{C^0_b(\mathbb{R}^d)} \leq C_\varepsilon\kappa\|\nabla\chi_u(t)\|_{\mathcal{M}}\|\varphi\|_Y,$$

where $Y = L^2_\sigma(\mathbb{R}^d)$ or $Y = V_q(\mathbb{R}^d)$ and

$$\begin{aligned} \|\nabla\chi_u\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}^d))} &\leq M\left(\|\Psi_\varepsilon u\|_{C([0,T];C^2_b(\mathbb{R}^d))}\right)\|\chi_0\|_{BV(\mathbb{R}^d)} \\ &\leq M'\left(\varepsilon, \|u\|_{C([0,T];L^2_\sigma(\mathbb{R}^d))}\right)\|\chi_0\|_{BV(\mathbb{R}^d)} \end{aligned}$$

by (2.12). Hence

$$\|f_u\|_{L^{q'}(0,T;V_q')} + \|f_u\|_{L^1(0,T;L^2_\sigma)} \leq M\left(\varepsilon, T, \|u\|_{C([0,T];L^2_\sigma(\mathbb{R}^d))}\right)\kappa\|\chi_0\|_{BV(\mathbb{R}^d)}.$$

In particular this implies

$$\|F(u)\|_{X_1} \leq M\left(\varepsilon, T, \|v_0\|_2, \|u\|_{L^\infty(0,T;L^2(\Omega))}, \kappa\|\nabla\chi_u\|_{L^\infty(0,T;\mathcal{M}(\Omega))}\right) \tag{4.11}$$

for some other continuous function M and $\kappa \geq 0$ by (4.6) and (4.7).

Now let $u_k \in X_\alpha$, $k \in \mathbb{N}$, be a bounded sequence and let $\alpha \in (0,1]$. If $\alpha = 1$, then $\Psi_\varepsilon u_k \in C([0,T]; C^1(\overline{B_R(0)}))$, $k \in \mathbb{N}$, is precompact for any $R > 0$ since $\Psi_\varepsilon u_k \in C([0,T]; C^2_b(\mathbb{R}^d))$ and $\partial_t\Psi_\varepsilon u_k = \Psi_\varepsilon\partial_t u_k \in L^{q'}(0,T; C^2_b(\mathbb{R}^d))$ are uniformly bounded. Now using again [5, Theorem 3.8.1] we conclude that $\Psi_\varepsilon u_k \in C([0,T]; C^1(\overline{B_R(0)}))$, $k \in \mathbb{N}$, is precompact if $\alpha \in (0,1)$. Therefore for a suitable subsequence $\Psi_\varepsilon u_k \to_{k\to\infty} \Psi_\varepsilon u$ in $C([0,T]; C^1(\overline{B_R(0)}))$ for any $R > 0$. Hence (2.16) implies that

$$\lim_{k\to\infty}\langle H_{\chi_{u_k}(t)}, \varphi\rangle = \langle H_{\chi_u(t)}, \varphi\rangle \qquad \text{for all } \varphi \in C^1_0(\mathbb{R}^d)^d$$

uniformly in $t \in [0,T]$. Moreover, since $\operatorname{supp}\chi_{u_k}$, $k \in \mathbb{N}$, is contained in a compact set K by Lemma 2.4 and $C^2(K) \hookrightarrow\hookrightarrow C^1(K)$,

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\sup_{\|\varphi\|_{C^2_b(\mathbb{R}^d)}\leq 1}\left|\langle H_{\chi_{u_k}(t)}, \varphi\rangle - \langle H_{\chi_u(t)}, \varphi\rangle\right| = 0$$

Therefore $f_{u_k} \to_{k\to\infty} f_u \in L^1(0,T; L^2_\sigma(\mathbb{R}^d))$ since $\Psi_\varepsilon L^2(\mathbb{R}^d) \to C^2_b(\mathbb{R}^d)$. By the same arguments it follows that $f_u \in L^1(0,T; L^2_\sigma(\mathbb{R}^d))$ depends continuously on $u \in X_0$. – This finishes the proof of the claim.

Now, since $F\colon X_\alpha \to X_1$ is bounded and $F\colon X_\alpha \to X_0$ is continuous for all $\alpha \in [0,1]$, the interpolation inequality $\|u\|_{X_\alpha} \leq \|u\|_{X_0}^{1-\alpha}\|u\|_{X_1}^\alpha$ implies that $F\colon X_\alpha \to X_\alpha$ is continuous for all $\alpha \in [0,1)$. Similarly, the boundedness of $F\colon X_\alpha \to X_1$, $\alpha \in [0,1]$, and the compactness of $F\colon X_\alpha \to X_0$, $\alpha \in (0,1]$, yields the compactness of $F\colon X_\alpha \to X_\alpha$, $\alpha \in (0,1]$, Altogether $F\colon X_\alpha \to X_\alpha$ is a completely continuous mapping for all $\alpha \in (0,1)$.

In order to prove the existence of a fixed point $v_\varepsilon = F(v_\varepsilon) \in X_\alpha$, $\alpha \in (0,1)$, we will use the Leray-Schauder principle, cf. e.g. Sohr [28, Lemma 3.1.1, Chapter II], for which it only remains to verify the following condition for a suitable $R > 0$:

$$\text{If } v = \lambda F(v) \text{ for some } v \in X_\alpha, \lambda \in [0,1], \text{ then } \|v\|_{X_\alpha} \leq R. \tag{4.12}$$

Therefore we assume that $v = \lambda F(v)$ for some $v \in X_\alpha$, $\lambda \in [0,1]$, $\alpha \in (0,1)$. If $\lambda = 0$, then obviously, $\|v\|_{X_\alpha} = 0 \leq R$ for any $R > 0$. Thus it remains to consider the case $\lambda > 0$. Set $\alpha = \lambda^{-1} \geq 1$. Then $u = \alpha v = F(v)$ solves

$$\begin{aligned} &- (u, \partial_t \varphi)_{Q_T} - (v_0, \varphi(0))_\Omega - (\Psi_\varepsilon u \otimes \psi_\varepsilon * u, \nabla \psi_\varepsilon * \varphi)_{Q_T} + (S(\chi_v, Du), D\varphi)_{Q_T} \\ &= \kappa \int_0^T \langle H_{\chi_v(t)}, \Psi_\varepsilon \varphi(t) \rangle \, dt \end{aligned}$$

for all $\varphi \in C^\infty_{(0)}(\Omega \times [0,T))^d$ with $\operatorname{div} \varphi = 0$. Hence choosing $\varphi = u\chi_{[0,t]}$ (after a standard approximation) we conclude

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \int_\Omega S(\chi_v, Du) : Du \, dx \, d\tau = \frac{1}{2}\|v_0\|_2^2 + \kappa \int_0^t \langle H_{\chi_v(\tau)}, \Psi_\varepsilon u(\tau) \rangle \, d\tau.$$

where we have used (4.4).

Now, since $\Psi_\varepsilon u(\tau) = \alpha \Psi_\varepsilon v(\tau)$, Lemma 2.3 implies that

$$\int_0^t \langle H_{\chi_v(\tau)}, \Psi_\varepsilon u(\tau) \rangle \, d\tau = \alpha \left(|\nabla \chi_0|(\Omega) - |\nabla \chi_v(t)|(\Omega) \right) \qquad \text{if } \kappa > 0.$$

Hence

$$\frac{\alpha}{2}\|v(t)\|_2^2 + \kappa |\nabla \chi_v(t)|(\Omega) + \alpha^{q-1} c_0 \int_0^t \int_\Omega |Dv|^q \, dx \, d\tau \leq \frac{1}{2}\|v_0\|_2^2 + \kappa |\nabla \chi_0|(\Omega) =: E_0$$

for all $0 \leq t \leq T$, where c_0 is the same as in (1.12). Hence using (4.11) and the latter estimate, we conclude

$$\begin{aligned} \|v\|_{X_\alpha} &\leq C\lambda \|F(v)\|_{X_1} \\ &\leq M'\left(\varepsilon, T, \|v\|_{L^\infty(0,T;L^2(\Omega))}, \kappa \|\nabla \chi_v\|_{L^\infty(0,T;\mathcal{M}(\Omega))}\right) \leq M''(\varepsilon, T, E_0) \end{aligned}$$

for some continuous functions M', M''. Hence for $R := M''(\varepsilon, E_0)$ the condition (4.12) is valid. This implies that there is fixed point $v_\varepsilon = F(v_\varepsilon) \in X_\alpha$, which is a solution of (4.1)-(4.3) by definition of F.

The remaining statements easily follow from (4.8) and (2.13). ■

5 Proofs of the Main Theorems

5.1 Approximation Sequence

Throughout this section we assume that the assumptions of Theorem 1.6 if $\kappa > 0$ and Theorem 1.9 if $\kappa = 0$ hold. Moreover, we denote by $E_0 = \frac{1}{2}\|v_0\|_2^2 + \kappa |\nabla \chi_0|(\Omega)$ the initial energy of the flow.

For every $\varepsilon > 0$ let $(v_\varepsilon, \chi_\varepsilon)$ be an approximative solutions due to Theorem 4.2 for $T = \frac{1}{\varepsilon}$. Because of the uniform bounds of $(v_\varepsilon, \chi_\varepsilon)$ given by the energy equality (4.9) and

$$\|S(\chi_k, Dv_k)\|_{L^{q'}(Q_T)}^{q'} = \int_{Q_T} |S(\chi_k, Dv_k)|^{\frac{q}{q-1}} \, d(x,t) \leq C \int_{Q_T} |Dv_k|^q \, d(x,t)$$

due to (1.12) there is a subsequence $(v_{\varepsilon_k}, \chi_{\varepsilon_k}) \equiv (v_k, \chi_k)$, $k \in \mathbb{N}$, such that

$$v_k \rightharpoonup_{k\to\infty} v \quad \text{in } L^q(0,\infty; V_q(\Omega)) \tag{5.1}$$

$$v_k \rightharpoonup^*_{k\to\infty} v \quad \text{in } L^\infty(0,\infty; L^2_\sigma(\Omega)) \tag{5.2}$$

$$S(\chi_k, Dv_k) \rightharpoonup_{k\to\infty} \tilde{S} \quad \text{in } L^{q'}(Q) \tag{5.3}$$

$$\chi_k \rightharpoonup^*_{k\to\infty} \chi \quad \text{in } L^\infty(Q) \tag{5.4}$$

$$\nabla\chi_k \rightharpoonup^*_{k\to\infty} \nabla\chi \quad \text{in } L^\infty(0,\infty; H^{-s}(\Omega)), s > \frac{d}{2}, \text{ if } \kappa > 0 \tag{5.5}$$

for some $v \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^q(0,\infty; V_q(\Omega))$, $\chi \in L^\infty(0,\infty; X_\kappa(\Omega))$ with $X_\kappa = BV$ if $\kappa > 0$ and $X_\kappa = L^\infty$ if $\kappa = 0$, and $\tilde{S} \in L^{q'}(Q)$. Here the functions $v_\varepsilon, \chi_\varepsilon$ are extended by 0 for $t > \frac{1}{\varepsilon}$.

5.2 Passing to the Limit in the Transport Equation and the Convective Term

We pass to the limit in the transport equation using the following lemma, which is a variant of [18, Lemma 5.1]:

Lemma 5.1 *Let $(v_k, \chi_k)_{k\in\mathbb{N}}$ be bounded in $L^q_{\text{loc}}([0,\infty); W^1_q(\Omega;\mathbb{R}^d)) \times L^\infty(Q)$, $1 < q < \infty$, such that*

$$v_k \rightharpoonup_{k\to\infty} v \quad \textit{in } L^q(0,T; W^1_q(\Omega)^d) \quad \textit{for all } T > 0, \tag{5.6}$$

$$\chi_k \rightharpoonup^*_{k\to\infty} \chi \quad \textit{in } L^\infty(Q). \tag{5.7}$$

If $(\partial_t\chi_k)_{k\in\mathbb{N}}$ is bounded in $L^{q'}(0,T; W^{-1}_{q',\text{loc}}(\overline{\Omega}))$ for any $T > 0$, then $\chi_k v_k \to_{k\to\infty} \chi v$ in $\mathcal{D}'(Q)$.

Proof: First of all, since the statement is local, it is sufficient to consider the case that Ω is a bounded domain and the case that $(0,\infty)$ is replaced by $(0,T)$, $T > 0$ arbitrary. Because of $L^{q'}(\Omega) \hookrightarrow\hookrightarrow W^{-1}_{q'}(\Omega)$ (2.1) yields

$$\chi_{k_j} \to_{j\to\infty} \chi^* \qquad \text{in } L^{q'}(0,T; W^{-1}_{q'}(\Omega))$$

for some subsequence. Since $\chi_k \rightharpoonup^*_{k\to\infty} \chi$ in $L^\infty(Q_T)$, $\chi^* = \chi$ and the full sequence χ_k converges strongly in $L^{q'}(0,T; W^{-1}_{q'}(\Omega))$. This implies that $\chi_k v_k \to \chi v$ in $\mathcal{D}'(Q_T)$. ∎

Corollary 5.2 *Let (v, χ) be as in (5.1)-(5.5). Then (v, χ) solves the transport equation (1.10)-(1.11).*

Proof: It only remains to observe that $\partial_t \chi_k$ is bounded in $L^{q'}(0, T; W^{-1}_{q',loc}(\overline{\Omega}))$ for any $T > 0$: Because of

$$(\chi_k, \partial_t \varphi)_{Q_T} + (\chi_k, v_k \cdot \nabla \varphi)_{Q_T} = 0 \quad \text{for all } \varphi \in C_0^\infty(Q_T)$$

we estimate

$$\begin{aligned} |(\chi_k, \partial_t \varphi)_{Q_T}| &\leq \|v_k\|_{L^q([0,T]\times\Omega_R)} \|\nabla\varphi\|_{L^{q'}(Q_T)} \\ &\leq C\left(\|v_k\|_{L^q(0,T;V_q(\Omega))} + T^{\frac{1}{q}} \|v_k\|_{L^\infty(0,T;L^2(\Omega))}\right) \|\nabla\varphi\|_{L^{q'}(Q_T)} \end{aligned}$$

for all $\varphi \in L^{q'}(0, T; W^1_{q'}(\Omega))$ with $\operatorname{supp} \varphi \subseteq \overline{\Omega_R}$, $\Omega_R = \Omega \cap B_R$, $R > 0$. – Note that $V_q(\mathbb{R}^d) \cap L^2_\sigma(\mathbb{R}^d) \hookrightarrow L^q(B_R(0))$ if $\Omega = \mathbb{R}^d$ and $V_q(\Omega) \hookrightarrow L^q(\Omega)$ if Ω is bounded. ■

The latter corollary and (2.11) yield

$$\|\chi\|^p_{L^p(Q_T)} = \|\chi\|_{L^1(Q_T)} = T\|\chi_0\|_{L^1(\Omega)} = \|\chi_k\|^p_{L^p(Q_T)}$$

for all $1 \leq p < \infty$ since $\int \chi(t)\, dx = \int \chi_0 \, dx$ for almost all $t > 0$. Hence

$$\chi_k \to_{k\to\infty} \chi \qquad \text{in } L^p(Q_T) \text{ for all } 1 \leq p < \infty, T > 0.$$

In particular this implies that $\chi(x, t) \in \{0, 1\}$ almost everywhere.

In order to pass to the limit in the convective term, we use the following lemma.

Lemma 5.3 *Let v_k, v be as above and let $q > \frac{2d}{d+2}$. Then*

$$v_k \to_{k\to\infty} v \qquad \text{in } L^{q'}(0, T; L^2_{\text{loc}}(\overline{\Omega})) \tag{5.8}$$

for all $T > 0$. In particular,

$$\lim_{k\to\infty} (\Psi_k v_k \otimes \psi_k * v_k, \nabla \psi_k * \varphi)_Q = (v \otimes v, \nabla\varphi)_Q$$

for all $\varphi \in C^\infty_{(0)}(\overline{Q})^d$.

Proof: First let $|\Omega| < \infty$. Since $S(\chi_k, Dv_k)$ and $\Psi_k v_k \otimes \psi_k * v_k$ are uniformly bounded in $L^{q'}(Q_T)$, $L^\infty(0, T; L^1(\Omega))$, resp., and

$$\left|\int_0^\infty \langle H_{\chi_k(t)}, \varphi(t)\rangle dt\right| \leq \sup_{0\leq t<\infty} \|\nabla\chi_k(t)\|_{\mathcal{M}(\mathbb{R}^d)} \int_0^\infty \|\varphi(t)\|_{C^1_b(\mathbb{R}^d)}\, dt \quad \text{if } \kappa > 0,$$

$\partial_t v_k$ is uniformly bounded in $L^{q'}(0, T; H^{-m}(\Omega))$ for some suitable $m \in \mathbb{N}$. Using (2.1) with $V_q(\Omega) \cap L^2(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow H^{-m}(\Omega)$ when $q > \frac{2d}{d+2}$, proves (5.8) in the case $|\Omega| < \infty$. The case $\Omega = \mathbb{R}^d$ follows from the first part applied to $\Omega' \Subset \mathbb{R}^d$.

Finally, (5.8) implies that $\psi_k * v_k \cdot \nabla\psi_k * \varphi \to v \cdot \nabla\varphi$ in $L^{q'}(0, T; L^2(\Omega))$ for all $\varphi \in C^\infty_{(0)}(\overline{Q})^d$ and $T > 0$ since ψ_k converges strongly to the identity as $k \to \infty$. Together with (5.2) this implies the last statement. ■

5.3 Case without Surface Tension

Obviously, in the case of two Newtonian fluids, i.e., $q = 2$ and $\nu(j,s) = \nu_j$ are constant, the strong convergence of χ_k and the weak convergence of Dv_k yield $\tilde{S} = S(\chi, Dv)$. For the case $q \neq 2$ and $\kappa = 0$, we use the following lemma:

Lemma 5.4 *Let $\kappa = 0$ and let $q \geq \frac{2d}{d+2} + 1$. Then*

$$S(\chi(x,t), Dv(x,t)) = \tilde{S}(x,t) \qquad \text{for almost all } (x,t) \in Q. \tag{5.9}$$

Proof: By the results so far we obtain

$$-(v, \partial_t \varphi)_Q - (v_0, \varphi(0))_\Omega - (v \otimes v, \nabla \varphi)_Q + (\tilde{S}, D\varphi)_Q = 0 \tag{5.10}$$

for all $\varphi \in C^\infty_{(0)}([0,\infty) \times \Omega)^d, \operatorname{div} \varphi = 0$. Furthermore, $q \geq \frac{2d}{d+2} + 1$ implies that for all $T > 0$

$$|(v \otimes v, \nabla \varphi)_{Q_T}| \leq C \|\nabla \varphi\|_{L^q(Q_T)} \tag{5.11}$$

for all $\varphi \in C^\infty_{(0)}([0,T) \times \Omega)^d, \operatorname{div} \varphi = 0$, because of [12, Lemma 2.44, Chapter 5]. Moreover, since $\tilde{S} \in L^{q'}(Q)$, equation (5.10) implies that $\partial_t v \in L^{q'}(0,T; V_q(\Omega)')$ for all $T > 0$. Therefore we can choose $\varphi = v|_{Q_T}$ in (5.10) and obtain

$$\frac{1}{2}\|v(T)\|_2^2 + (\tilde{S}, Dv)_{Q_T} = \frac{1}{2}\|v_0\|_2^2,$$

where we have used (4.8) and $(v \otimes v, \nabla v)_{Q_T} = 0$, cf. [12, Lemma 4.45, Chapter 5]. Moreover,

$$\frac{1}{2}\|v_k(T)\|_2^2 + (S(\chi_k, Dv_k), Dv_k)_{Q_T} = \frac{1}{2}\|v_0\|_2^2$$

and therefore

$$\begin{aligned}
\limsup_{k\to\infty} (S(\chi_k, Dv_k), Dv_k)_{Q_T} &= \frac{1}{2}\|v_0\|_2^2 - \liminf_{k\to\infty} \frac{1}{2}\|v_k(T)\|_2^2 \\
&\leq \frac{1}{2}\|v_0\|_2^2 - \frac{1}{2}\|v(T)\|_2^2 = (\tilde{S}, Dv)_{Q_T}.
\end{aligned}$$

Thus we are in the position to apply Theorem 2.6 with $A(x,s,\xi) = S(s,\xi)$, $z_k = Dv_k$, $y_k = \chi_k$ to conclude that for a suitable subsequence $\lim_{k\to\infty} S(\chi_k, Dv_k) = \tilde{S}$ in measure. Since $T > 0$ is arbitrary, this implies (5.9). ∎

Proof of Theorem 1.9: For the case $\kappa = 0$ the results obtained so far show that (v, χ) is a weak solution of (1.21) for $f = 0$ together with (1.10)-(1.11). The general case $f \in L^{q'}(0,\infty; V_q(\Omega)')$ can be proved in the same way with minor modifications. ∎

Remark 5.5 The condition $q \geq \frac{2d}{d+2} + 1$ is only needed to estimate the convective term as in (5.11). For all other parts of the proof only $q > \frac{2d}{d+2}$ is needed. Hence in the case of the Stokes equations, where the convective term $v \cdot \nabla v$ is neglected, cf. Remark 1.10, the condition $q > \frac{2d}{d+2}$ is sufficient to prove existence of weak solutions.

5.4 Case with Surface Tension: Properties of the Interface

It remains to consider the case with surface tension $\kappa > 0$. For this let $\Omega_k^+(t) = X_{k,t}(\Omega_0^+)$, where $X_{k,t} = X_{\Psi_k v_k}(t)$ is the flow map associated to (2.14)-(2.15) with $v = \Psi_k v_k$ as described above. Moreover, let $\Gamma_k(t) = \partial\Omega_k^+(t) = X_{k,t}(\partial\Omega_0^+)$ and let $\Gamma_k = \bigcup_{0\le t<\infty} \Gamma_k(t) \times \{t\}$.

First we will show that in the case $q > d$ or $d = 2$ $\Gamma_k \cap Q_T$ is contained in the compact set $\overline{B_R(0)} \times [0,T]$ for $R = R(T)$ and arbitrary $T > 0$. Then a suitable subsequence will converge in the Hausdorff distance.

Lemma 5.6 *Let $d = 2$, $\kappa > 0$. Then $\Gamma_k(t) \subseteq \overline{B_R(0)}$ for all $t \in (0,T)$ and some $R = R(T, E_0, \Omega_0^+)$.*

Proof: Since $\mathcal{H}^{d-1}(\Gamma_k(t)) \le \kappa^{-1}E_0$, obviously $\operatorname{diam}(\Omega_k^+(t)) \le \frac{E_0}{2\kappa}$. Moreover, by the transport equation

$$\int_{\Omega_k^+(t)} x\,dx = \int_{\Omega_0^+} x\,dx + \int_0^t \int_{\Omega_k^+(\tau)} v_k \cdot \mathbf{1}\,dx\,d\tau,$$

where $\mathbf{1} = (1,\dots,1)^T$, which implies

$$\left|\int_{\Omega_k^+(t)} x\,dx\right| \le C(\Omega_0^+) + t|\Omega_0^+|^{\frac12} \sup_{0\le\tau\le t} \|v(\tau)\|_2 \le C(T, E_0, \Omega_0^+)$$

for all $0 \le t \le T$ since $|\Omega_k^+(\tau)| = |\Omega_0^+|$ for all $\tau > 0$. Therefore $\overline{\Omega_k^+(t)} \subseteq \overline{B_R(0)}$ for $0 \le t \le T$ with $R = C(T, E_0, \Omega_0^+) + \frac{E_0}{2\kappa}$. ∎

In the case $q > d$, $v \in L^q(0,T; C_0(\mathbb{R}^d))$ since $V_q(\mathbb{R}^d) \cap L^2_\sigma(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$ and we can prove that $\Gamma_k(t)$ are equi-Hölder continuous in the following sense:

Lemma 5.7 *Let $q > d$. Then*

$$d_H(\Gamma_k(t_1), \Gamma_k(t_2)) \le C|t_1 - t_2|^{\frac{1}{q'}}$$

for all $0 \le t_1, t_2 \le T$, $T > 0$, where C depends only on E_0, q, T. In particular, $\Gamma_k(t) \subseteq \overline{B_R(0)}$ for all $0 \le t \le T$ for some $R = R(T, E_0, \Omega_0^+)$.

Proof: By symmetry it suffices to show that $\Gamma_k(t_1) \subseteq (\Gamma_k(t_2))_\varepsilon$ for $\varepsilon = C|t_1 - t_2|^{\frac{1}{q'}}$. Let $x_1 \in \Gamma_k(t_1)$. Then by definition of $\Gamma_k(t)$ there is a curve $x(t)$ such that $x(t_1) = x_1$ and $x'(t) = v_k(x(t), t)$ for $t > 0$. Moreover, $x_2 = x(t_2) \in \Gamma_k(t_2)$ and

$$|x_1 - x_2| \le \int_{t_1}^{t_2} |v_k(x(t), t)|\,dt \le C(E_0, T, q)|t_1 - t_2|^{\frac{1}{q'}}, \qquad 0 \le t_1 \le t_2 \le T.$$

This proves the statement. ∎

Corollary 5.8 *Let $\kappa > 0$ and let $q > d$ or let $d = 2$. Then there is a subsequence of Γ_k, $k \in \mathbb{N}$, (again denoted by Γ_k, $k \in \mathbb{N}$) and a closed set $\Gamma^* \subset \overline{Q}$ such that for every rational $T > 0$*

$$\Gamma_k^* \cap \overline{Q_T} \to_{k\to\infty} \Gamma_T^* \qquad \textit{w.r.t. } d_H$$

for some compact set $\Gamma_T^* \subseteq \overline{Q_T}$ with $\Gamma_T^* \cap Q_T = \Gamma^* \cap Q_T$.

Proof: By the previous two lemmas, $\Gamma_k \cap \overline{Q_T}$ is contained in a compact set A_T. Hence using the compactness of the metric space $(\mathcal{K}(A), d_H)$ for compact $A \subset \mathbb{R}^N$ one easily gets a subsequence of Γ_k, $k \in \mathbb{N}$, that converges in $(\mathcal{K}(A_T), d_H)$ for every rational $T > 0$ to some compact set Γ_T^*. By the definition of d_H one easily verifies that $\Gamma_{T_1}^* \cap Q_{T_1} = \Gamma_{T_2}^* \cap Q_{T_1}$ if $0 < T_1 \leq T_2$. Using this the existence of $\Gamma^* \subseteq \overline{Q}$ is immediate. ∎

In the case $q > d$, we even obtain:

Corollary 5.9 *Let $q > d$. Then $\Gamma_k(t) \to_{k\to\infty} \Gamma_t^*$ for all $t > 0$ in Hausdorff distance, where $\Gamma_t^* = \{x \in \mathbb{R}^d : (x,t) \in \Gamma^*\}$.*

Proof: First of all, for a fixed $t > 0$ and a suitable subsequence $\Gamma_{k_j}(t) \to_{j\to\infty} \Gamma_t^{**}$ in the Hausdorff distance. We claim that $\Gamma_t^{**} = \Gamma_t^*$. The inclusion $\Gamma_t^{**} \subseteq \Gamma_t^*$ is obvious. Conversely, let $(x,t) \in \Gamma_t^*$. Then there is a sequence $(x_{k_j}, t_{k_j}) \in \Gamma_{k_j}$ such that $\lim_{j\to\infty}(x_{k_j}, t_{k_j}) = (x,t)$. But by Lemma 5.7 there are $y_j \in \Gamma_{k_j}(t)$ such that $|y_j - x_{k_j}| \leq C(E_0,T)|t - t_{k_j}|^{\frac{1}{q'}}$. Hence

$$|x - y_j| \leq |x - x_{k_j}| + |x_{k_j} - y_j| \leq |x - x_{k_j}| + C(E_0)|t - t_{k_j}|^{\frac{1}{q'}},$$

which shows that $\Gamma_{k_j}(t) \ni y_j \to x \in \Gamma_t^*$. Thus $\Gamma_t^* \subseteq \Gamma_t^{**}$. Therefore $\Gamma_t^* = \Gamma_t^{**}$ for any accumulation point Γ_t^{**} of $\Gamma_k(t)$ in the Hausdorff distance, which implies $\Gamma_k(t) \to_{k\to\infty} \Gamma_t^*$ for all $t > 0$. ∎

The latter corollary gives some compactness in time for the sequence of interfaces $\Gamma_k(t)$ if $q > d$ for $d = 2, 3$. But now there is a crucial difference between the case $d = 2$ and $d = 3$. If $d = 3$ and $t > 0$ is fixed, then the boundedness of $\mathcal{H}^{d-1}(\Gamma_k(t))$ does not imply that a limit of $\Gamma_k(t)$ in the Hausdorff distance has finite $\mathcal{H}^{d-1}$-measure. – It is easy to construct sequences of surfaces of fixed area with many "small fingers" that will converge to a set of positive Lebesgue measure. – This cannot happen in two dimension as the following lemma shows:

Lemma 5.10 *Let $\Gamma_k \subset \mathbb{R}^2$, $k \in \mathbb{N}$, be a sequence of compact Lipschitz curves and $\Gamma_k \to \Gamma^*$ in Hausdorff distance for some compact set $\Gamma^* \subset \mathbb{R}^2$. Then $\mathcal{H}^1(\Gamma^*) \leq \liminf_{k\to\infty} \mathcal{H}^1(\Gamma_k)$.*

Proof: Let $\delta > 0, q > 1$ be fixed. Then there is a $N = N(\delta, q)$ such that $d_H(\Gamma_k, \Gamma^*) \leq (1 - \frac{1}{q})\frac{\delta}{2}$ for all $k \geq N$. Moreover, for any $\varepsilon > 0$ there is some $k_\varepsilon \geq N$ such that

$L := \mathcal{H}^1(\Gamma_{k_\varepsilon}) \leq \liminf_{k\to\infty} \mathcal{H}^1(\Gamma_k) + \varepsilon$. Then there is a covering $\Gamma_{k_\varepsilon} \subseteq \bigcup_{j=1}^M \overline{B_{\frac{\delta}{2q}}(x_j)}$ such that $M \leq \frac{q}{\delta}\mathcal{H}^1(\Gamma_{k_\varepsilon}) + 1$. (Let $\gamma_k\colon [0,L] \to \mathbb{R}^d$ be a parameterization by arc-length and $t_k = k \cdot \frac{\delta}{q}$, $k = 0, \ldots, M-1$, where $M = [\frac{q}{\delta}L] + 1$, and $t_M = L$; then choose $x_j = \frac{1}{2}(\gamma_k(t_{j-1}) + \gamma_k(t_j))$.) Therefore $\Gamma^* \subseteq \bigcup_{j=1}^M \overline{B_{\delta/2}(x_j)}$ and

$$\mathcal{H}^1_\delta(\Gamma^*) \leq \delta M \leq q\mathcal{H}^1(\Gamma_{k_\varepsilon}) + \delta \leq q\left(\liminf_{k\to\infty} \mathcal{H}^1(\Gamma_k) + \varepsilon\right) + \delta,$$

where

$$\mathcal{H}^1_\delta(A) = \inf\left\{\sum_{j=1}^\infty 2r_j : A \subseteq \bigcup_{j=1}^\infty B_{r_j}(x_j), 0 < r_j \leq \delta\right\}.$$

Since $q > 1, \varepsilon > 0$ are arbitrary $\mathcal{H}^1_\delta(\Gamma^*) \leq \liminf_{k\to\infty} \mathcal{H}^1(\Gamma_k) + \delta$ for every $\delta > 0$, which proves the lemma. ■

Corollary 5.11 *Let $q > d = 2$, $\kappa > 0$. Then $\mathcal{H}^{1+q'}(\Gamma^* \cap [0,T]) < \infty$ for all $T > 0$.*

Proof: By Corollary 5.9 $\Gamma_k(t) \to_{k\to\infty} \Gamma^*_t$ in Hausdorff distance. Moreover, by Lemma 5.10 $\mathcal{H}^1(\Gamma^*_t) \leq \liminf_{k\to\infty} \mathcal{H}^1(\Gamma_k(t)) \leq \kappa^{-1}E_0$ for all $t \geq 0$. Now choose $0 = t_0 < t_1 < \ldots < t_N = T$, $T > 0$, with $|t_j - t_{j+1}| \leq \delta = \left(\frac{r}{3C}\right)^{q'}$ for $r > 0$ and $N \leq 2T\delta^{-1}$, where C is the same constant as in Lemma 5.7. Since the length of $\Gamma_k(t_j)$ is bounded by $\kappa^{-1}E_0$, there are balls $B_{\frac{r}{3}}(x_{j,l})$, $l = 1, \ldots, M_j$, $j = 1, \ldots, N$, with $M_j \leq C(E_0)r^{-1}$ covering $\Gamma_k(t_j)$. Now choose $k \in \mathbb{N}$ so large that $d_H(\Gamma_k(t_j), \Gamma^*_{t_j}) \leq \frac{r}{3}$ for $j = 1, \ldots, N$. Using Lemma 5.7 and $\Gamma_k(t) \to_{k\to\infty} \Gamma^*_t$, we conclude

$$d_H(\Gamma^*_{t_1}, \Gamma^*_{t_2}) \leq C|t_1 - t_2|^{\frac{1}{q'}}.$$

Then for $|t - t_j| \leq \delta$

$$\Gamma^*_t \subseteq \left(\Gamma^*_{t_j}\right)_{r/3} \subseteq (\Gamma_k(t_j))_{2r/3} \subseteq \bigcup_{l=1}^{M_j} B_r(x_{j,l}).$$

Thus

$$\Gamma^* \cap \overline{Q_T} \subseteq \bigcup_{j=1}^N \bigcup_{l=1}^{M_j} B_r(x_{j,l})$$

where the number of balls on the right-hand side is bounded by $Cr^{-1-q'}$. Since $r > 0$ was arbitrary, this implies that $\mathcal{H}^{1+q'}(\Gamma^*) \leq C(E_0, q)$. ■

5.5 Case with Surface Tension: Finish of the Proof

Using Corollary 5.11 we obtain:

Lemma 5.12 *Let $\kappa > 0$ and let $q > d = 2$. Then*

$$S(\chi(x,t), Dv(x,t)) = \tilde{S}(x,t) \qquad \text{for almost all } (x,t) \in Q. \tag{5.12}$$

Proof: Because of Corollary 5.11, $\mathcal{H}^3(\Gamma^* \cap [0,T]) = 0$ for all $T > 0$. Hence $\overline{B_R(0)} \times [0,T] = \bigcup_{j=1}^\infty Q_j \cup M$ with $M = \Gamma^* \cap \overline{Q_T}$ and $\mathcal{H}^3(M) = 0$, where $Q_j = (a_j, b_j) \times B_{r_j}(x_j)$ and $\overline{Q_j} \cap M = \emptyset$. Now it is sufficient to prove that (5.12) holds for all φ with $\operatorname{supp}\varphi \subset Q_j$, which shall be arbitrary but fixed in the following. Then we choose $\eta \in C_0^\infty(Q_j)$ with $\eta \equiv 1$ on $\operatorname{supp}\varphi$. Because of the convergence of Γ_k in Hausdorff distance, for every fixed $j \in \mathbb{N}$ we have $(\Gamma_k)_{\varepsilon_k} \cap Q_j = \emptyset$ for sufficiently large $k \in \mathbb{N}$. Hence $\chi_{\varepsilon_k} = l \in \{0,1\}$ is constant on Q_j for suitably large k and $w_k := P_{L^2_\sigma(\mathbb{R}^2)}(\eta v_k) \in L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^2)) \cap L^q(0,\infty; V_q(\mathbb{R}^2))$ solves

$$-(w_k, \partial_t u)_Q + (w_k|_{t=0}, u(0))_{\mathbb{R}^2} - (\Psi_k w_k \otimes w_k, \nabla u)_Q + (S(l, Dw_k), Du)_Q = \langle f_k, u\rangle$$

for all $u \in C_{(0)}^\infty([0,\infty) \times \mathbb{R}^2)^2$, $\operatorname{div} u = 0$, with a right-hand side f_k satisfying

$$f_k \to f \in L^{q'}(0,\infty; V_q(\mathbb{R}^2)') \quad \text{as } k \to \infty.$$

Moreover, $w_k \rightharpoonup_{k\to\infty} w$ in $L^q(0,\infty, V_q(\mathbb{R}^2))$ and weak-$*$ in $L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^2))$ and it can be shown by the same argument as in the case $\kappa = 0$, cf. Lemma 5.4, that $Dw_k \to Dw$ in measure. In particular this implies $\tilde{S} = S(l, Dv) = S(\chi, Dv)$ almost everywhere on $\operatorname{supp}\varphi$. Since $\varphi \in C_0^\infty(Q)$ with $\operatorname{supp}\varphi \subset Q_j$ and Q_j have been arbitrary, (5.12) follows. ■

Finally, we consider the sequence of oriented general varifolds $V_k(t)$, $t \in [0,\infty)$ associated to $\Gamma_k(t)$, i.e.,

$$\langle V_k(t), \varphi\rangle := \int_{\mathbb{R}^d} \varphi(x, n_k(x))\, d|\nabla\chi_k(t)|, \quad \varphi \in C_0(\mathbb{R}^d \times \mathbb{S}^{d-1}),$$

where $n_k(x) = -\frac{\nabla\chi_k}{|\nabla\chi_k|}$ and we set

$$\langle V_k, \varphi\rangle = \int_0^\infty \langle V_k(t), \varphi(t)\rangle\, dt \quad \text{for all } \varphi \in L^1(0,\infty; C_0(\mathbb{R}^d \times \mathbb{S}^{d-1})).$$

Hence for a suitable subsequence

$$V_k \rightharpoonup^*_{k\to\infty} V \quad \text{in } L^\infty(0,\infty; H^{-s}(\mathbb{R}^d \times \mathbb{S}^{d-1})), s > \frac{2d-1}{2} \tag{5.13}$$

for some $V \in L^\infty_\omega(0,\infty; \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1}))$ since $V_k \in L^\infty_\omega(0,\infty; \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1}))$ is uniformly bounded and $\mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1}) \hookrightarrow H^{-s}(\mathbb{R}^d \times \mathbb{S}^{d-1})$. Then by choosing the test function in the form $\varphi(x,s,t) = s \cdot \psi(x,t)$, $\psi \in C_{(0)}^\infty([0,\infty) \times \mathbb{R}^d)^d$, this implies

$$\begin{aligned} -\int_0^\infty \langle \nabla\chi_k(t), \psi(t)\rangle\, dt &= \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} s \cdot \psi(x,t)\, d\delta_{n_k(x,t)}\, d|\nabla\chi_k(t)|\, dt \\ \to_{k\to\infty} -\int_0^\infty \langle \nabla\chi(t), \psi(t)\rangle\, dt &= \int_\Omega \int_{\mathbb{S}^{d-1}} s \cdot \psi(x,t)\, dV_x(t)\, d|V(t)|\, dt, \end{aligned}$$

which shows (1.15). Similarly, choosing $\varphi(x,s,t) = (I - s\otimes s) : \nabla\psi(x,t)$ we conclude

$$-\int_0^\infty \langle H_{\chi_k(t)}, \psi(t)\rangle\, dt = \int_0^\infty \langle \delta V_k(t), \psi(t)\rangle\, dt \to_{k\to\infty} \int_0^\infty \langle \delta V(t), \psi(t)\rangle\, dt$$

for all $\psi \in C^\infty_{(0)}([0,\infty)\times\mathbb{R}^d)^d$.

Moreover, by Theorem 2.1 there is some $\mu \in L^\infty_\omega(Q; Prob(\mathbb{R}^{d\times d}_{sym}))$ such that (1.16) holds and

$$S(l, Dv_k) \rightharpoonup_{k\to\infty} \int S(l,\lambda)\, d\mu_{x,t}(\lambda) \quad \text{in } L^{q'}(Q)$$

for every $l \in [0,1]$. But this implies

$$S(\chi(x,t), Dv_k) \rightharpoonup_{k\to\infty} \int S(\chi(x,t),\lambda)\, d\mu_{x,t}(\lambda) \quad \text{in } L^{q'}(Q).$$

Moreover, $\chi_k \to_{k\to\infty} \chi$ in measure (for a suitable subsequence) and $S(\chi_k, Dv_k)$ is uniformly bounded in $L^{q'}(Q)$. Therefore

$$\lim_{k\to\infty} (S(\chi_k, Dv_k), D\varphi)_Q = \lim_{k\to\infty} (S(\chi, Dv_k), D\varphi)_Q = \left(\int S(\chi,\lambda)\, d\mu_{x,t}(\lambda), D\varphi\right)_Q$$

for each $\varphi \in C_0^\infty(Q)^d$, which proves (1.14). Hence the existence of measure-valued varifold solutions is proved.

It remains to prove the remaining properties stated in Theorem 1.6. The first statement follows from Lemma 5.6 and Lemma 5.7. The second statement is proved by first proving that for a suitable subsequence $|V_k(t)| \rightharpoonup |V(t)|$ in $\mathcal{M}(\mathbb{R}^2)$ for almost all $t > 0$ and then using an argument due to Plotnikov [20]:

Lemma 5.13 *Let $q > d$ and let $\kappa > 0$. Then there is a subsequence (again denoted by $|V_k(t)|$) such that*

$$|V_k(t)| \rightharpoonup^*_{k\to\infty} |V(t)| \quad \textit{in } \mathcal{M}(\mathbb{R}^d)$$

for almost all $t > 0$.

Proof: First, we define a measure $E_k(t)$ by

$$\langle E_k(t), \varphi\rangle := \kappa\langle |V_k(t)|, \varphi\rangle + \frac{1}{2}\int_{\mathbb{R}^d} |v_k(x,t)|^2 \varphi(x)\, dx, \quad \varphi \in C_0(\mathbb{R}^d).$$

Note that $E_k(t)$ measures approximately the kinetic energy and "surface energy" of the approximately at given time $t > 0$. We now show that $E_k(t)$ converges weak-$*$ in measure almost everywhere (for a suitable subsequence).

By (2.8) we have that

$$\begin{aligned}
\frac{d}{dt}\langle |V_k(t)|, \varphi\rangle &= \frac{d}{dt}\int_{\Gamma_k(t)} \varphi(x,t)\, d\mathcal{H}^1(x) \\
&= \langle \delta V_k(t), \varphi\Psi_k v_k(t)\rangle + \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} s\cdot\nabla\varphi(x) s\cdot \Psi_k v_k(x,t)\, dV_k(t)(x,s).
\end{aligned}$$

Since $v_k \in L^q(0,T;C_0(\mathbb{R}^d))$, $T>0$, is uniformly bounded, the last term in the equation above is uniformly bounded in $L^{q'}(0,T;C^1(\mathbb{R}^d)')$. Moreover,

$$\langle \delta V_k(t), \varphi \Psi_k v_k\rangle = \langle \delta V_k(t), P_\sigma(\varphi\Psi_k v_k)\rangle + \langle \delta V_k(t), (I-P_\sigma)(\varphi\Psi_k v_k)\rangle,$$

where $(I-P_\sigma)(\varphi\Psi_k v_k) \in L^q(0,T;C_0^1(\mathbb{R}^d))$, $T>0$ and $\langle \delta V_k(\tau), .\rangle \in L^\infty(0,\infty;C_0^1(\mathbb{R}^d)')$ are uniformly bounded for every $\varphi \in C_0^1(\mathbb{R}^d)$. (Note that in the case $\Omega = \mathbb{R}^d$ the Helmholtz projection P_σ can be represented using classical singular integral operators.) Therefore the second term in the equation above is also uniformly bounded in $L^{q'}(0,T;C^1(\mathbb{R}^d)')$. Furthermore,

$$\langle \delta V_k(t), P_q(\varphi(\Psi_k v_k))\rangle = \langle \delta V_k(t), \Psi_k P_q(\varphi v_k))\rangle - \langle \delta V_k(t), P_q[\Psi_k,\varphi] v_k\rangle$$

where $[A,B]$ denotes the commutator of two operators. Note that P_σ and Ψ_k commute and that P_σ is a bounded operator on $C^\alpha(\mathbb{R}^d)$, $\alpha \in (0,1)$. Moreover,

$$\|[\Psi_k,\varphi]w\|_{C^{1,\alpha}(\mathbb{R}^d)} \le C\|w\|_{C^\alpha(\mathbb{R}^d)} \qquad w \in C^\alpha(\mathbb{R}^d), 0<\alpha<1,$$

uniformly in $k \in \mathbb{N}$. This implies that the second term in the equation above is uniformly bounded in $L^{q'}(0,T;C^1(\mathbb{R}^d)')$. On the other hand by (4.1)

$$\begin{aligned} -\kappa\langle \delta V_k(t), \Psi_k P_\sigma(\varphi v_k)\rangle &= -\kappa\langle H_{\Gamma_k(t)}, \Psi_k P_\sigma(\varphi v_k)\rangle \\ = \;& \frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^d} |v_k(x,t)|^2\varphi(x)\,dx + \langle (1-P_\sigma)(\varphi v_k(t)), \partial_t v_k(t)\rangle \\ & + (\Psi_k v_k \otimes \psi_k * v_k, \nabla\psi_k * P_\sigma(\varphi v_k))_{\mathbb{R}^d} + (S(\chi_k, Dv_k), DP_\sigma(\varphi v_k))_{\mathbb{R}^d}, \end{aligned}$$

where the second term vanishes and the last two terms are again uniformly bounded in $L^1(0,T;C_0^1(\mathbb{R}^d)')$. – Note that $v_k\otimes v_k \in L^\infty(0,T;L^1(\mathbb{R}^d)) \cap L^1(0,T;L^\infty(\mathbb{R}^d)) \hookrightarrow L^1(Q_T)\cap L^2(Q_T)$ and $\nabla v_k \in L^q(Q_T)$ are uniformly bounded. – Summing up, we have that

$$\frac{d}{dt}\langle E_k(t), .\rangle \in L^1(0,T;C_0^1(\mathbb{R}^d)')$$

is uniformly bounded. Hence

$$E_k \rightarrow_{k\to\infty} \tilde{E} \qquad \text{in } L^p(0,T;H^{-s}_{\mathrm{loc}}(\mathbb{R}^d)) \text{ if } s>1$$

for every $1 \le p < \infty$ by (2.1) and therefore $E_k(t) \to \tilde{E}(t)$ in $H^{-s}_{\mathrm{loc}}(\mathbb{R}^d)$ for almost all $t\in(0,T)$. On the other hand $v_k \rightarrow_{k\to\infty} v$ in $L^{q'}(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^d))$ by Lemma 5.3 and therefore $v_k(t) \to v(t)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ for almost all $t \in (0,T)$ and for a suitable subsequence. Hence

$$|V_k(t)| \rightarrow_{k\to\infty} \mu(t) \qquad \text{in } H^{-s}_{\mathrm{loc}}(\mathbb{R}^d)$$

for almost all $t\in(0,T)$. But, since $C_0^\infty(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$ and $|V_k(t)|$ is uniformly bounded in $\mathcal{M}(\mathbb{R}^d)$, we conclude

$$|V_k(t)| \rightharpoonup^*_{k\to\infty} \mu(t) \qquad \text{in } \mathcal{M}(\mathbb{R}^d)$$

for almost all $t \in (0,T)$. Finally, by (5.13)

$$|V_k| \rightharpoonup^*_{k\to\infty} |V| \qquad \text{in } L^\infty(0,\infty; H^{-s}(\mathbb{R}^d)) \text{ for } s > \frac{2d-1}{2}$$

and therefore $\mu = |V|$. ■

Lemma 5.14 *Let $q > d = 2$. Then $|V(t)|$ is supported on Γ^*_t and $|V(t)| \geq \mathcal{H}^1\lfloor\Gamma^*_t$ for almost all $t > 0$.*

Proof: First assume that Ω_0^+ is simply connected. Let $t > 0$ be such that $|V_k(t)| \rightharpoonup^*_{k\to\infty} |V(t)|$ in $\mathcal{M}(\mathbb{R}^2)$. Moreover, let $x_k \colon [0,1] \to \mathbb{R}^2$ be a parameterization of $\Gamma_{k,t}$ with respect to arclength times the total length $\mathcal{H}^1(\Gamma_{k,t})$. Then $x_k \in C^{0,1}([0,1];\mathbb{R}^2)$ is uniformly bounded since $\Gamma_{k,t} \subseteq \overline{B_R(0)}$ for some $R > 0$ and the Lipschitz constants of x_k are $\mathcal{H}^1(\Gamma_{k,t}) \leq C(E_0)$. Hence for a suitable subsequence $x_{k_j} \to_{j\to\infty} x \in C^0([0,1];\mathbb{R}^2)$ for some $x \in C^{0,1}([0,1];\mathbb{R}^2)$ and $\mathcal{H}^1(\Gamma_{k_j,t}) \to_{j\to\infty} l^*$. – Note that $\mathcal{H}^1(\Gamma_{k_j,t})$ are bounded below since they enclose $\Omega_k^+(t)$ and $|\Omega_k^+(t)| = |\Omega_0^+|$. – Then

$$\langle |V(t)|, \varphi\rangle = \lim_{j\to\infty} \mathcal{H}^1(\Gamma_{k_j,t}) \int_0^1 \varphi(x_k(s))ds = l^* \int_0^1 \varphi(x(s))ds.$$

for all $\varphi \in C_0(\mathbb{R}^2)$. Hence $\operatorname{supp}|V(t)| = x([0,1])$.

Now we prove that $\Gamma^*_t = x([0,1])$. Obviously, $x([0,1]) \subseteq \Gamma^*_t$. Conversely, if $x_0 \in \Gamma^*_t$, then $x_0 = \lim_{j\to\infty} x_{k_j}(s_j)$ for some $s_j \in [0,1]$. But then $s_j \to_{j\to\infty} s_0 \in [0,1]$ for a suitable subsequence again denoted by s_j. Hence $x_0 = \lim_{j\to\infty} x_{k_j}(s_j) = x(s_0) \in x([0,1])$. This proves the first part of the lemma.

In order to prove $|V(t)| \geq \mathcal{H}^1\lfloor\Gamma^*$ we use that

$$\langle |V(t)|, \varphi\rangle = l^* \int_0^1 \varphi(x(s))ds \geq \int_0^1 \varphi(x(s))|x'(s)|\,ds$$

since $|x'(s)| \leq l^*$ almost everywhere. Hence by the area formula

$$|V(t)|(A) \geq \int_0^1 \chi_A(x(s))|x'(s)|\,ds \geq \mathcal{H}^1\lfloor\Gamma^*_t(A)$$

for every open $A \subseteq \mathbb{R}^2$, cf. e.g. [27].

Finally, if Ω_0^+ is not simply connected, we apply the argument above to N-curves instead of one curve, where N is the number of connected components of $\partial\Omega_0^+$. ■

A Appendix: Rectifiability of the Varifold

One of the most challenging questions concerning measure-valued varifold solutions of the two-phase flow with surface tension is whether there are solutions such that the unorientated general varifold $\widetilde{V}(t)$ associated to $V(t)$ via (2.7) is a $(d-1)$-rectifiable varifold for almost all $t>0$, i.e., $\widetilde{V_x}(t)=\delta_{P(x,t)}$ and

$$\langle\widetilde{V_x}(t),\varphi\rangle=\int\varphi(x,P(x,t))\theta_t(x)\,d\mathcal{H}^{d-1}\lfloor M_t(x),\qquad\varphi\in C_0(\Omega\times G_{d-1}),$$

for some countably $(d-1)$-rectifiable set M_t and a $\mathcal{H}^{d-1}\lfloor M_t$-measurable positive function θ_t, cf. [27]. In particular, the case that $\theta_t(x)$ is a positive integer for almost all (x,t) would give a more satisfactory answer to the existence of measure-valued solutions.

As noted by Plotnikov [19], the major problem is that (1.14) gives only information of $\langle\delta V,\psi\rangle$ for $\psi\in C_0^\infty(Q;\mathbb{R}^d)$ with $\operatorname{div}\psi=0$. But in order to apply techniques from geometric measure theory it is necessary to have a good estimate of $\langle\delta V(t),\psi\rangle$ for $\psi\in C_0^\infty(Q;\mathbb{R}^d)$ with $\operatorname{div}\psi\neq 0$ or at least for suitable gradients. The following result on regularity of measure-valued varifold solutions shows that, once a $\langle\delta V,\psi\rangle$ can be estimated for all $\psi\in C_0^\infty(Q;\mathbb{R}^{d+1})$ in suitable norms and the $(d-1)$-density of $|V(t)|$ is bounded below, then $\widetilde{V}(t)$ is a $(d-1)$-rectifiable varifold. The result is based on a new rectifiability result for general varifolds due to Luckhaus [15].

THEOREM A.1 (Rectifiability) *Let (v,χ,V,μ) be a measure-valued solution due to Definition 1.2 and let $T>0$, $q>\frac{2d}{d+2}$. Assume that*

$$\limsup_{\rho\to 0}\rho^{-d+1}|V(t)|(B_\rho(x))\geq\Theta_t>0$$

for $|V(t)|$-almost all $x\in\mathbb{R}^d$ and almost all $t\in(0,T)$. If for some $s>1$

$$\langle\delta V,\cdot\rangle\in L^1(0,T;W^{-1}_{s,\mathrm{loc}}(\mathbb{R}^d))$$

or if there is some $p\in L^1(0,T;L^s_{\mathrm{loc}}(\mathbb{R}^d))$ for some $s>1$ such that

$$\begin{aligned}&(v,\partial_t\varphi)_{Q_T}+(v_0,\varphi(0))_{\mathbb{R}^d}-(v\otimes v,\nabla\varphi)_{Q_T}\\&+\left(\int S(\chi,\lambda)\,d\mu_{x,t}(\lambda),D\varphi\right)_{Q_T}-(p,\operatorname{div}\varphi)_{Q_T}=-\kappa\int_0^T\langle\delta V(t),\varphi(t)\rangle\,dt\end{aligned}\tag{A.1}$$

for all $\varphi\in C^\infty_{(0)}([0,T)\times\mathbb{R}^d;\mathbb{R}^d)$, then $\widetilde{V}(t)$ is a $(d-1)$-rectifiable varifold for almost all $t\in(0,T)$.

We note that, if $q>d=2$, then the measure-valued varifold solution due to Theorem 1.6 satisfies

$$\limsup_{\rho\to 0}\rho^{-1}|V(t)|(B_\rho(x))\geq 1$$

for $|V(t)|$-almost all x and almost all $t>0$. Hence the lower bound of the $(d-1)$-density above is satisfied in this case.

The proof of Theorem A.1 is based on the following rectifiability theorem:

THEOREM A.2 (Luckhaus [15])
Let V be a (general) $(d-1)$-dimensional varifold on a domain $\Omega \subseteq \mathbb{R}^d$ whose first variation can be represented as

$$\langle \delta V, \psi \rangle = \int (v\psi + A : \nabla\psi)\, d\mu_1, \qquad \psi \in C_0^1(\Omega;\mathbb{R}^d), \tag{A.2}$$

satisfying the estimate

$$\begin{aligned} &\rho^{-d}\int_{B_\rho(x)} |A(y)|\, d\mu_1(y) + \rho^{-(d-1)}\int_{B_\rho(x)} |v(y)|\, d\mu_1(y) \\ \leq\;& \partial_\rho F\left(\rho, \sup_{\rho<R<\operatorname{dist}(x,\partial\Omega)} R^{-(d-1)}\int_{B_R(x)} d\mu_2\right) \end{aligned}$$

for all $B_\rho(x) \subseteq \Omega$ where μ_1, μ_2 are non-negative Radon measures on Ω and $F\colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0,\infty)$ satisfies

1. *$F(0,L) = 0$, $\partial_\rho F(\rho,L) \geq 0$, $\partial_\rho^2 F(\rho,L) \leq 0$ for $\rho, L \geq 0$,*
2. *$\lim_{L\to\infty} L^{-1}g(L) = 0$ where $g(L) = \inf\left\{R^{-d+1} + F(R,L) : R > 0\right\}$.*

Moreover, assume that $\limsup_{\rho\to 0}\rho^{-d+1}\int_{B_\rho(x)} d|V| \geq \theta > 0$ *for $|V|$-almost all $x \in \Omega$. Then V is a $(d-1)$-rectifiable varifold.*

Remark A.3 We note that in the proof of Theorem A.2 the identity (A.2) is only needed if $\psi = \nabla\varphi$ is a gradient. For the convenience of the reader we repeat the first part of the proof of Theorem A.2: The monotonicity formula for

$$u(\rho,x) := \rho^{-d+1}\int \phi\left(\frac{|x-y|}{\rho}\right) d|V(t)|(y),$$

is considered, where $\phi \in C^\infty([0,\infty))$ with $\phi'(s) \leq 0$, $\phi(s) = 1$ for $s \leq \frac{1}{2}$, and $\phi(s) = 0$ for $s \geq 1$. Then

$$\begin{aligned} \partial_\rho u(\rho,x) &= -(d-1)\rho^{-d}\int \phi\left(\frac{|x-y|}{\rho}\right) d|V(t)|(y) \\ &\quad -\rho^{-d}\int \phi'\left(\frac{|x-y|}{\rho}\right)\frac{|x-y|}{\rho}\, d|V(t)|(y) \\ &= \int \operatorname{Tr}\left(P\nabla_y\left[\frac{x-y}{\rho^d}\phi\left(\frac{|x-y|}{\rho}\right)\right]\right) d|V(t)|(y) \\ &\quad +\rho^{-d}\int \left[\left|P\frac{x-y}{|x-y|}\right|^2 - 1\right]\phi'\left(\frac{|x-y|}{\rho}\right)\frac{|x-y|}{\rho}\, d|V(t)|(y) \\ &= \left\langle \delta V(t), \frac{x-y}{\rho^d}\phi\left(\frac{|x-y|}{\rho}\right)\right\rangle \\ &\quad +\rho^{-d}\int \left|(I-P)\frac{x-y}{|x-y|}\right|^2 |\phi'|\left(\frac{|x-y|}{\rho}\right)\frac{|x-y|}{\rho}\, d|V(t)|(y) \end{aligned}$$

where $\frac{x-y}{\rho^d}\phi\left(\frac{|x-y|}{\rho}\right) = -\rho^{-d+2}\nabla_y\Phi\left(\frac{|x-y|}{\rho}\right)$ for $\Phi'(s) = s^{-1}\phi(s)$ is a gradient field. Then the assumptions of the theorem are used to estimate $\left\langle \delta V(t), \frac{x-y}{\rho^d}\phi\left(\frac{|x-y|}{\rho}\right)\right\rangle$. In the rest of the proof (A.2) is not used.

Proof of Theorem A.1: First of all, since rectifiability is a local property, we replace $\mathbb{R}^d$ by $\Omega = B_R(0)$ with $R > 0$ arbitrary. Moreover, we can assume that $2s \leq q^*$, where $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n} < \frac{1}{2}$.

First we consider the case $\langle \delta V, \cdot\rangle \in L^1(0,T; W^{-1}_s(\Omega))$. Then there is some $A \in L^1(0,T;L^s(\Omega))$ such that

$$\langle \delta V, \psi\rangle = \int_{Q_T} A(x,t) : \nabla\psi(x,t)\, d(x,t) \qquad \text{for all } \psi \in L^\infty(0,T; W^1_{s',0}(\Omega)),$$

which easily follows from Hahn-Banach's theorem when identifying $W^1_{s',0}(\Omega)$ with the closed subspace $\{\nabla\psi : \psi \in W^1_{s',0}(\Omega)\} \subset L^{s'}(\Omega;\mathbb{R}^d)$. In order to apply Theorem A.2 we choose μ_1 as the d-dimensional Lebesgue measure and estimate

$$\begin{aligned} \rho^{-d}\int_{B_\rho(x)} |A(y)|\, dy &\leq \left(\rho^{-d}\int_{B_\rho(x)} dy\right)^{\frac{1}{s'}} \left(\rho^{-d}\int_{B_\rho(x)} |A(y)|^s\, dy\right)^{\frac{1}{s}} \\ &= C\rho^{-\frac{1}{s}}\left(\rho^{-d+1}\int_{B_\rho(x)} |A(y)|^s\, dy\right)^{\frac{1}{s}}. \end{aligned}$$

Hence we can choose $F(\rho, L) = C\rho^{\frac{1}{s'}}L^{\frac{1}{s}}$ for some suitable constant C since $s > 1$ and $\mu_2(M) = \int_M |A(y)|^s\, dy$. It is easy to check that $F(\rho,L)$ satisfies the condition 1 of the theorem. Moreover, choosing $\alpha = \frac{1}{ds-1}$

$$g(L) \leq C\left(L^{\alpha(d-1)} + F(L^{-\alpha}, L)\right) \leq C'\left(L^{\alpha(d-1)} + L^{-\frac{\alpha}{s'}+\frac{1}{s}}\right) = C'L^{\frac{d-1}{ds-1}},$$

where $\frac{d-1}{ds-1} < 1$ since $s > 1$. Hence $\lim_{L\to\infty} L^{-1}g(L) = 0$.

In the second case we first use (A.1) for gradients $\varphi(x,t) = \phi(t)\nabla\psi(x)$ for $\psi \in C_0^\infty(\Omega), \phi \in C_0^\infty(0,T)$, which yields

$$\begin{aligned} \left|\kappa\int_0^T \langle \delta V(t), \nabla\psi\rangle\phi(t)\, dt\right| &\leq C\Big(\|v\|^2_{L^1(0,T;L^{2s}(\Omega))}\|\phi\|_{L^\infty(0,T)}\|\nabla^2\psi\|_{L^{s'}(\Omega)} \\ &\quad + \left\|\int S(\chi,\lambda)\, d\mu_{x,t}\right\|_{L^{q'}(Q_T)}\|\phi\|_{L^\infty(0,T)}\|\nabla^2\psi\|_{L^q(\Omega)} \\ &\quad + \|p\|_{L^1(0,T;L^s(\Omega))}\|\phi\|_{L^\infty(0,T)}\|\nabla^2\psi\|_{L^{s'}(\Omega)}\Big) \\ &\leq C(T)(E_0 + \|p\|_{L^1(0,T;L^s(\Omega))})\|\phi\|_{L^\infty(0,T)}\|\nabla^2\psi\|_{L^{s'}(\Omega)} \end{aligned}$$

Hence

$$\langle \delta V, \cdot\rangle \in L^1(0,T; (G^1_{s'}(\Omega))')$$

where $G^1_{s'}(\Omega) = \{\nabla\varphi \in W^1_{s'}(\Omega) : \varphi \in L^{s'}(\Omega)\} \subset W^1_{s'}(\Omega)$. In particular, $\langle \delta V(t), \cdot\rangle \in (G^1_{s'}(\Omega))'$ for almost all $t \in (0,T)$ with $s > 1$. Now we can apply the arguments of the first part since by Remark A.3 the identity (A.2) is only needed for gradients. ■

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