Max-Planck-Institut
für Mathematik
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Semiclassical Limit of the Focusing Nonlinear Schrödinger Equation under Barrier Initial Data

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by

Spyridon Kamvissis

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SEMICLASSICAL LIMIT OF THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION UNDER BARRIER INITIAL DATA

Spyridon Kamvissis

ABSTRACT

We study the semiclassical behavior of the focusing nonlinear Schrödinger equation in 1+1 dimensions under discontinuous "barrier" initial data and we describe the violent oscillations arising in terms of theta functions. The construction of proofs relies on the analysis of the associated Riemann-Hilbert factorization problem.

0. INTRODUCTION

The semiclassical limit of the 1+1-dimensional, integrable nonlinear Schrödinger equation with cubic focusing nonlinearity has been the subject of recent investigations. Several numerical studies have appeared since 1998 (see [MK], [BK], [CT], [C]) and a rigorous analysis of the initial value problem under real analytic data has just appeared [KMM]. The present paper makes use of the method and results in [KMM] for the study of a very particular problem with discontinuous (barrier) data. It has been shown by A.Cohen and T.Kappeler [CK] that weak solutions exist for all time under barrier data and, even more, that the inverse scattering technique is still applicable to the integration of the problem. Here, we use the inverse scattering method of [CK] to pose an associated Riemann-Hilbert factorization problem, which we then asymptotically (as $h \to 0$) reduce to the Riemann-Hilbert factorization problem that can be explicilty solved in terms of theta functions.

The real aim of this paper is to indicate that a discontinuity in the initial data does not necessarily alter the behavior of the semiclassical focusing NLS problem. Some changes of course *have* to happen. For example, if the Euler system that appears as a formal limit of the focusing semiclassical NLS does not even admit a solution for small times, it is obvious that the genus zero ansatz cannot hold uniformly for small times.

A natural generalization of the barrier data problem is the problem of general step data. For such data, the eigenvalue density ρ^0 can no longer be analytically extended from the "spike" where the eigenvalues accumulate. So, the only way the analysis of the present paper can be immediately extended is if the contour of discontinuity of the g-function (see section 3) is forced to touch the spike at the points of non-analyticity of ρ^0. We conjecture that under this constraint, the appropriate contour still exists and the finite (or perhaps infinite) genus ansatz still applies. We plan to address the problem of general real step initial data in a future publication.

1. FOCUSING NLS WITH BARRIER DATA

We consider the nonlinear Schrödinger equation (1+1-dimensional, integrable, focusing case), on the half-line

$$
\begin{aligned}
ihu_t^h(x,t) + \frac{h^2}{2}u_{xx}^h(x,t) + |u^h(x,t)|^2 u^h(x,t) = 0,\\
u^h(x,0) = u_0(x),
\end{aligned} \tag{1}
$$

under barrier-like initial data:

$$
\begin{aligned}
u_0(x) = A,\ -1/2 < x < 1/2,\\
= 0,\ otherwise.
\end{aligned} \tag{2}
$$

Here A is a fixed positive constant and h is a small positive constant. Eventually we will take $h \to 0$. We will assume that h is staying away from the discrete set $\{\frac{2A}{(2k+1)\pi}\}, k = 0, 1, 2, ...$ For simplicity, we will actually require that

$$
h \textit{ takes values in the discrete set } \{\frac{A}{k\pi}\},\ k = 0, 1, 2, ... \tag{2a}
$$

Setting

$$
\begin{aligned}
\rho^h = |u^h|^2,\\
\mu^h = -i\frac{h}{2}(\bar{u}^h u_x^h - u^h \bar{u}_x^h),
\end{aligned}
$$

(1) is transformed to

$$
\begin{aligned}
\partial_t \rho^h + \partial_x \mu^h = 0,\\
\partial_t \mu^h + \partial_x(\frac{(\mu^h)^2}{\rho^h}) - \partial_x(\rho^h)^2/2 = \frac{h^2}{4}\partial_x(\rho^h \partial_x^2 log\rho^h).
\end{aligned}
$$

The formal limit as $h \to 0$ is the Euler system

$$
\begin{aligned}
\partial_t \rho + \partial_x \mu = 0,\\
\partial_t \mu + \partial_x(\frac{\mu^2}{\rho}) - \partial_x(\rho)^2/2 = 0.
\end{aligned} \tag{1a}
$$

The initial data become $\rho = u_0^2(x), \mu = 0$.

This initial value problem admits a weak solution for all time, as shown by A.Cohen and T.Kappeler ([CK]). Furthermore, the inverse scattering theory is still applicable.

The associated linear system is

$$h\psi_x = \begin{pmatrix} -i\lambda & u(x) \\ -u^*(x) & i\lambda \end{pmatrix} \psi, \tag{3}$$

where * denotes complex conjugation. Jost functions ϕ, ψ are defined as column vector solutions of (3) satisfying the asymptotic conditions

$$\begin{aligned} \psi &\sim \begin{pmatrix} 0 \\ e^{i\lambda x/h} \end{pmatrix}, \; as \; x \to +\infty, \\ \phi &\sim \begin{pmatrix} e^{-i\lambda x/h} \\ 0 \end{pmatrix}, \; as \; x \to -\infty. \end{aligned} \tag{4}$$

One can define the reflection and transmission coefficient as follows.

Let ψ^0 be the transpose of $(\psi_2^*, -\psi_1^*)$. Then, following [CK] define a_+, b_+ by

$$\phi = a_+\psi^0 + b_+\psi.$$

The reflection and transmission coefficients are given by

$$r_+ = b_+/a_+,$$

$$t_+ = 1/a_+.$$

Similarly, one can define coefficients r_-, t_- by normalizing the Jost functions at the opposite infinities.

From [CK], we have

(5)

$$\psi(x,\lambda) = \begin{pmatrix} 0 \\ e^{i\lambda x/h} \end{pmatrix}, \; x < -1/2,$$

$$= \begin{pmatrix} Ae^{i\lambda/h}((x-1)/h)\frac{sin[(A^2+\lambda^2)^{1/2}((x-1)/h)]}{(A^2+\lambda^2)^{1/2}((x-1)/h)} \\ e^{i\lambda/h}cos[(A^2+\lambda^2)^{1/2}((x-1)/h)] + \lambda e^{i\lambda/h}\frac{sin[(A^2+\lambda^2)^{1/2}((x-1)/h)]}{(A^2+\lambda^2)^{1/2}((x-1)/h)} \end{pmatrix}, \; -1/2 < x < 1/2,$$

$$= \begin{pmatrix} \beta(\lambda,h)e^{i\lambda x/h} \\ \alpha(\lambda,h)e^{i\lambda x/h} \end{pmatrix}, \; x > 1/2,$$

where

$$\begin{aligned} \alpha =& e^{i\lambda/h} \; (\; cos[(A^2+\lambda^2)^{1/2}/h] - i\lambda\frac{sin[(A^2+\lambda^2)^{1/2}(1/h)]}{(A^2+\lambda^2)^{1/2}} \;), \\ \beta =& - Ae^{i\lambda/h}\frac{sin[(A^2+\lambda^2)^{1/2}(1/h)]}{(A^2+\lambda^2)^{1/2}}. \end{aligned} \tag{6}$$

The coefficients a_+, b_+, r are given by

$$
\begin{aligned}
a_+ &= \frac{\alpha}{|\alpha|^2+|\beta|^2}, \\
b_+ &= \frac{\beta}{|\alpha|^2+|\beta|^2}, \qquad (7) \\
r(\lambda) = \frac{\beta}{\alpha} &= -A\frac{1}{cot[(A^2+\lambda^2)^{1/2}(1/h)](A^2+\lambda^2)^{1/2}-i\lambda}.
\end{aligned}
$$

The eigenvalues are the zeros of a_+. They lie on the imaginary interval $[-iA, iA]$ and are given by

$$
\begin{aligned}
\lambda &= i\eta, \\
tan[\frac{(A^2-\eta^2)^{1/2}}{h}] &= \frac{(A^2-\eta^2)^{1/2}}{\eta}. \qquad (8)
\end{aligned}
$$

In other words,

$$\frac{(A^2-\eta^2)^{1/2}}{h} = arctan[\frac{(A^2-\eta^2)^{1/2}}{\eta}] + k\pi,\ \ k \in \mathbb{Z}.$$

As $h \to 0$, we get an asymptotic expression for the eigenvalues $\lambda_k = i\eta_k$.

$$(A^2-\eta_k^2)^{1/2} \sim hk\pi,\ \ k \in \mathbb{Z}. \qquad (8')$$

The limiting density of eigenvalues is then given by

$$\frac{\eta}{\pi(A^2-\eta^2)^{1/2}}. \qquad (9)$$

or, in terms of $\lambda = i\eta$,

$$\rho^0(\lambda) = \frac{\lambda}{\pi(A^2+\lambda^2)^{1/2}}. \qquad (9a)$$

Here the branch is chosen such that $\rho^0(\lambda) \sim \frac{i}{\pi A}$ as $\lambda \to i\infty$.

We note here that as a consequence of the simplifying condition (2a) we have $\int_0^{iA} \rho^0(\lambda)d\lambda = -iA/\pi = ikh$, for some integer k. Hence

$$exp[\frac{\pi}{h}\int_0^{iA} \rho^0(\lambda)d\lambda] = 1. \qquad (9b)$$

This will simplify the analysis of the "local" Riemann-Hilbert problem on a cross centered at the origin; see section 5, paragraph 6. Another consequence of (2a) is that $r(0) = 0$.

It is easily seen that the associated norming constants can only take the values $-1, 1$ (by symmetry in x) and that in fact they have to oscillate between these two values (by a Sturm-Liouville oscillation argument; see [KMM]).

We end this section by defining the three Pauli matrices; we will be using them later.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. THE RIEMANN-HILBERT PROBLEM

We can now state the following Riemann-Hilbert problem, following [KMM]. Let C be a piecewise smooth loop encircling all eigenvalues in the upper half-plane and lying entirely in the upper half-plane except for the point 0. Let C^* be its conjugate, encircling the eigenvalues in the lower half-plane. Also give the following orientation:

(i) the real axis is oriented from left to right,

(ii) the loop C is oriented counterclockwise,

(iii) the loop C^* is also oriented counterclockwise.

We use the following convention: the +-side of an oriented contour is always to its left, according to the given orientation.

We also choose C so that it approaches 0 (from left and right) at a non-zero, non-straight angle with the real axis.

THEOREM 1 (discrete version). Let $d\mu = \Sigma_k h(\delta_{\lambda_k^*} - \delta_{\lambda_k})$ be a finite sum of point measures supported at the eigenvalues of the system (3), as given by (8). Let

$$X(\lambda) = -(A^2 + \lambda^2)^{1/2}. \tag{10}$$

Letting M_+ and M_- denote the limits of M on Σ from left and right respectively, we define the Riemann-Hilbert factorization problem

$$M_+(\lambda) = M_-(\lambda)J(\lambda),$$

$$where$$

$$J(\lambda) = \begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x - 2i\lambda^2 t}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x + 2i\lambda^2 x}{h}} & 1 + |r(\lambda)|^2 \end{pmatrix}, \lambda \in \mathbb{R},$$

$$= v(\lambda), \lambda \in C, \tag{11}$$

$$= \sigma_2 v(\lambda^*)^* \sigma_2, \lambda \in C^*,$$

$$lim_{\lambda\to\infty} M(\lambda) = I,$$

where

(12)
$$v(\lambda) = \begin{pmatrix} 1 & -i\ exp(\frac{1}{h}\int log(\lambda-\eta)d\mu(\eta))exp(-\frac{1}{h}(2i\lambda x + 2i\lambda^2 t - X(\lambda))) \\ 0 & 1 \end{pmatrix},$$

and r is the reflection coefficient defined in (7):

$$r(\lambda) = -A\frac{1}{cot[(A^2+\lambda^2)^{1/2}(1/h)](A^2+\lambda^2)^{1/2} - i\lambda}. \tag{13}$$

Note that $r(\lambda)$ can have a real singularity at $\lambda = 0$ if A/h is an odd multiple of $\pi/2$. By assumption, we have excluded such values of h.

The above Riemann-Hilbert problem admits a solution and the solution of (1) can be recovered from the solution of (11) as follows.

$$u(x,t) = 2ih\ lim_{\lambda\to\infty}(\lambda M^{12}(\lambda)), \tag{14}$$

where the index 12 here denotes the (12)-entry of a matrix.

PROOF: Standard; see Chapter 2 of [KMM]. The quantity $-i\ exp(\frac{1}{h}(X(\lambda_k)))$ oscillates between -1 and 1. The function $-i\ exp(\frac{1}{h}(X(\lambda)))$ is thus an extrapolation of the norming constants.

REMARKS. 1. In [KMM] the Riemann-Hilbert problem jump involves some parameters denote by K, σ, J. Here we are simply choosing $K = -1, \sigma = 1, J = 1$. This is compatible with the discussion in [KMM] as long as we focus our attention to the case $x \geq 0$. By the obvious symmetry $x \to -x$ this is acceptable.

2. Obviously, the contour C can be deformed anywhere in the upper half-plane, as long as it passes through 0 (we only require it to be non-tangent to either the real axis or the imaginary axis at 0) and does not touch the linear segment $[0, iA]$. It will be eventually fixed by the choice of the "g-function" in the next section. Similarly for its conjugate C^*.

We note the following factorization of the jump $J(\lambda)$ on the real line.

$$\begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x+2i\lambda^2 t}{h}} & 1+|r(\lambda)|^2 \end{pmatrix} = L(\lambda)U(\lambda), \quad where \tag{15+}$$

$$L(\lambda) = \begin{pmatrix} 1 & 0 \\ r^*(\lambda^*)e^{\frac{2i\lambda x+2i\lambda^2 t}{h}} & 1 \end{pmatrix},\ U(\lambda) = \begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix}.$$

A different factorization is also possible.

$$\begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x+2i\lambda^2 t}{h}} & 1+|r(\lambda)|^2 \end{pmatrix} = S(\lambda)D(\lambda)T(\lambda),$$

where

(15-)
$$S(\lambda) = \begin{pmatrix} 1 & \frac{r(\lambda)}{1+|r(\lambda)|^2}e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix}.$$

$$D(\lambda) = \begin{pmatrix} (1+|r(\lambda)|^2)^{-1} & 0 \\ 0 & 1+|r(\lambda)|^2 \end{pmatrix},$$

$$T(\lambda) = \begin{pmatrix} 1 & 0 \\ \frac{r^*(\lambda^*)}{1+|r(\lambda)|^2}e^{\frac{2i\lambda x+2i\lambda^2 t}{h}} & 1 \end{pmatrix}.$$

Both factorizations will be useful later.

REMARK: From now on we will substitute $d\mu = (\rho^0(\eta) + (\rho^0)^*(\eta^*))d\eta$ in (12), where ρ^0 is the asymptotic density of eigenvalues given by (9a), and supported on the linear segment $[0, iA]$. In other words, we will approximate a discrete density of eigenvalues by a continuous one. This is not a trivial assumption, but it is true. We refer to Chapter 3 of [KMM] for a rigorous justification. We then restate Theorem 1 as follows.

THEOREM 1 (continuous version). Let $d\mu = (\rho^0(\eta) + (\rho^0)^*(\eta^*))d\eta$, where ρ^0 is the asymptotic density of eigenvalues given by (9a), and supported on the linear segment $[0, iA]$. Set

(10)
$$X(\lambda) = -(A^2+\lambda^2)^{1/2}.$$

Letting M_+ and M_- denote the limits of M on Σ from left and right respectively, we define the Riemann-Hilbert factorization problem

$$M_+(\lambda) = M_-(\lambda)J(\lambda),$$

where

(11)
$$J(\lambda) = \begin{pmatrix} 1 & r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}} \\ r^*(\lambda)e^{\frac{2i\lambda x+2i\lambda^2 x}{h}} & 1+|r(\lambda)|^2 \end{pmatrix}, \lambda \in \mathbb{R},$$

$$= v(\lambda), \lambda \in C,$$

$$= \sigma_2 v(\lambda^*)^*\sigma_2, \lambda \in C^*,$$

$$lim_{\lambda\to\infty}M(\lambda) = I,$$

where

(12)

$$v(\lambda) = \begin{pmatrix} 1 & -i\ exp(\frac{1}{h}\int log(\lambda-\eta)d\mu(\eta))exp(-\frac{1}{h}(2i\lambda x + 2i\lambda^2 t - X(\lambda))) \\ 0 & 1 \end{pmatrix},$$

(13) $$r(\lambda) = -A\frac{1}{cot[(A^2+\lambda^2)^{1/2}(1/h)](A^2+\lambda^2)^{1/2} - i\lambda}.$$

The above Riemann-Hilbert problem admits a solution and the solution of (1) can be asymptotically recovered from the solution of (11) as follows. As $h \to 0$,

(14') $$u(x,t) \sim 2ih\ lim_{\lambda\to\infty}(\lambda M^{12}(\lambda)),$$

where the index 12 here denotes the (12)-entry of a matrix.

PROOF: See Chapter 3 of [KMM].

3. ASYMPTOTIC ANALYSIS OF THE RIEMANN-HILBERT PROBLEM.

We begin with the following observation. Consider the reflection coefficient given by (13). For λ in the upper half-plane, at least away from the real line and the eigenvalues given by (8), we have

(16) $$r(\lambda) \sim \frac{-iA}{(A^2+\lambda^2)^{1/2}+\lambda}.$$

while for λ in the lower half-plane, at least away from the real line and the eigenvalues given by (8), we have

(16*) $$r(\lambda) \sim \frac{iA}{(A^2+\lambda^2)^{1/2}-\lambda}.$$

This means that $r(\lambda)e^{\frac{-2i\lambda x-2i\lambda^2 t}{h}}$ is exponentially decaying (growing) in the upper half-plane, at least away from the real and imaginary axes, as long as $Re(\lambda) < -x/2t$ ($Re(\lambda) > -x/2t$), while $r^*(\lambda^*)e^{\frac{2i\lambda x+2i\lambda^2 t}{h}}$ is exponentially decaying (growing) in the lower half-plane, at least away from the real and imaginary axes, as long as $Re(\lambda) < -x/2t$ ($Re(\lambda) > -x/2t$).

The above suggests that using the factorizations of the Riemann-Hilbert problem defined by (15-), (15+) and applying the obviously suggested deformations, the jump across the real line should be reduced to a diagonal matrix independent of x, t.

To do this of course, we must ensure that the solution of the Riemann-Hilbert problem is uniformly bounded (in x, t) as $h \to 0$. This is actually not true as will be seen later. What is true however, is that a judicious conjugation of the jump matrix will "deform" it to a new Riemann-Hilbert problem whose solution *is* uniformly bounded (in x, t) as $h \to 0$. At *that* point we can neglect *all* terms involving r. Furthermore, the very same conjugation, will ensure that the new Riemann-Hilbert is explicitly solvable (asymptotically).

More precisely, we introduce the change of variables

$$Q(\lambda) = M(\lambda) e^{\frac{g\sigma_3}{h}}, \tag{17}$$

where $\sigma_3 = diag(1, -1)$ (a Pauli matrix) and the complex-valued function g is constrained by the following conditions:

$$g(\lambda) \; is \; independent \; of \; h.$$

$$g(\lambda) \; is \; analytic \; for \lambda \in \mathbb{C} \setminus (C \cup C^*).$$

$$g(\lambda) \to 0 \; as \; \lambda \to \infty.$$

$$g(\lambda) \; assumes \; continuous \; boundary \; values \; from \; both \; sides \; of \; C \cup C^*,$$

$$denoted \; by \; g_+ (g_-) \; on \; the \; left \; (right) \; of \; C \cup C^*.$$

$$g(\lambda^*) + g(\lambda)^* = 0 \; for \; all \; \lambda \in \mathbb{C} \setminus (C \cup C^*).$$

These conditions of course do not define g uniquely. They will be augmented by two conditons below, that will also implicitly define an admissible contour C.

The assumptions above permit us to write g in terms of a measure ρ defined on the contour C. Indeed

$$g(\lambda) = \frac{1}{2} \int_{C \cup C^*} log(\lambda - \eta) \rho(\eta) d\eta, \tag{18}$$

for an appropriate definiton of the logarithm branch (and for $x > 0$; if $x < 0$ there is a sign change in [KMM] but of course we can restrict ourselves to the case $x > 0$ here because of the symmetry of NLS).

For $\lambda \in C$, define the functions

$$\theta(\lambda) := i(g_+(\lambda) - g_-(\lambda)),$$

$$\phi(\lambda) := \int_0^{iA} log(\lambda-\eta)\rho^0(\eta)\,d\eta + \int_{-iA}^{0} log(\lambda-\eta)\rho^0(\eta^*)^*\,d\eta \tag{18a}$$
$$+2i\lambda x + 2i\lambda^2 t + i\pi \int_\lambda^{iA} \rho^0(\eta)\,d\eta - g_+(\lambda) - g_-(\lambda).$$

Now we spell out the two important conditions which determine the contour C (not uniquely) and the function g.

$$\rho(\eta)d\eta \ is \ a \ real \ measure,$$
$$\Re(\phi(\lambda)) \le 0, \tag{18b}$$

i.e. a "measure reality" condition and what can be interpreted (see Chapter 8 of [KMM]) as a "variational inequality" condition. In fact, one can eventually show that the measure $\rho(\eta)d\eta$ has to be nonpositive: strictly zero in the "gaps" and strictly negative in the "bands"; see section 4 later.

REMARK: The function g and the contour C are determined only by the density of the eigenvalues, ρ^0. They are independent of the reflection coefficient r.

By (17) the Riemann-Hilbert problem for Q is $Q_+(\lambda) = Q_-(\lambda)v_Q(\lambda)$, where

$$v_Q(\lambda) = \begin{pmatrix} 1 & r(\lambda)e^{\frac{2g-2i\lambda x-2i\lambda^2 t}{h}} \\ r^*(\lambda^*)e^{\frac{-2g+2i\lambda x+2i\lambda^2 t}{h}} & 1+|r(\lambda)|^2 \end{pmatrix}, \ \lambda \in \mathbb{R},$$

$$v_Q(\lambda) = \begin{pmatrix} e^{\frac{g_+-g_-}{h}} & -i\ exp(\frac{1}{h}\int log(\lambda-\eta)d\mu(\eta))exp(-\frac{1}{h}(-g_+-g_-+2i\lambda x+2i\lambda^2 t - X(\lambda))) \\ 0 & e^{\frac{g_--g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C,$$

$$v_Q(\lambda) = \begin{pmatrix} e^{\frac{g_+-g_-}{h}} & 0 \\ i\ exp(\frac{-1}{h}\int log(\lambda-\eta)d\mu(\eta))exp(\frac{1}{h}(-g_+-g_-+2i\lambda x+2i\lambda^2 t + X^*(\lambda^*))) & e^{\frac{g_--g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C^*.$$

Also $lim_{\lambda\to\infty}Q(\lambda) = I$.

Now, we *will* be able to treat the terms involving the reflection coefficient r by arguing as follows. First we note that g is purely imaginary for real λ; this follows from (18)-(18b). So, write $g = i\psi$, so that $\psi(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}$. Let $\zeta(x,t,\lambda) = \psi - \lambda x - \lambda^2 t$. Clearly, $\zeta(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}$.

Next, divide the real line into a union of (finitely many) intervals, say J_k, such that for $\lambda \in interior(J_k)$, either $\frac{d\zeta}{d\lambda} > 0$ or $\frac{d\zeta}{d\lambda} < 0$. Denote by J_l^+ the intervals in which $\frac{d\zeta}{d\lambda} > 0$ and by J_l^- the intervals in which $\frac{d\zeta}{d\lambda} < 0$. Naturally, the (nonzero) endpoints of the intervals J_k are given by the condition $\frac{d\zeta}{d\lambda} = 0$. Note that ζ is real analytic in $\mathbb{R} \setminus 0$, and $\zeta \sim -\lambda^2 t$, as $\lambda \to \pm\infty$. The finiteness of the number of intervals follows from the analyticity of ζ, at least away from 0. But also near 0, we will show that $\frac{d\zeta}{d\lambda} < 0$.

We first show that when $\lambda = 0$, then $\frac{d\zeta}{d\lambda} = -\infty$. Recall that

$$g(\lambda) = \frac{1}{2} \int_{C \cup C^*} log(\lambda - \eta)\rho(\eta) d\eta,$$

where $C \cup C^*$ is oriented counterclockwise and $\rho(\eta)d\eta$ is a non-positive measure on C which is strictly negative in a non-trivial subset of C and is extended to C^* by the condition

$$\rho(\eta^*) = (\rho(\eta))^*.$$

Differentiating, one gets

$$\frac{dg(\lambda)}{d\lambda} = \frac{1}{2} \int_{C \cup C^*} (\lambda - \eta)^{-1} \rho(\eta) d\eta,$$

and at $\lambda = 0$,

$$\frac{dg(\lambda)}{d\lambda} = \frac{-1}{2} \int_{C \cup C^*} \frac{\rho(\eta)}{\eta} d\eta.$$

Making use of the symmetry with respect to complex reflection and remembering that the orientation is counterclockwise for both C and C^* one gets

$$[\frac{dg(\lambda)}{d\lambda}](\lambda = 0) = i \int_C \frac{\rho(\eta) Im(\eta)}{|\eta|^2} d\eta.$$

The integrand is strictly negative and the integral diverges because ρ is nonzero at zero. It follows that at $\lambda = 0$, $\frac{dg}{d\lambda} = -i\infty$, so $\frac{d\zeta}{d\lambda} = -\infty$.

Now, if λ is real and close but not equal to 0,

$$[\frac{dg(\lambda)}{d\lambda}] \sim i \int_C \frac{\rho(\eta) Im(\eta)}{|\lambda - \eta|^2} d\eta,$$

so $\frac{d\zeta}{d\lambda} < 0$. Hence the point $\lambda = 0$ belongs to the interior of some interval J_l^-.

Consider first the intervals J_l^+. By the Cauchy-Riemann relations we have $d(Im\zeta)/d(Im\lambda) > 0$, for $\lambda \in J_l^+$, in the positive direction perpendicular to the real line. This means that in an area of the upper half-plane, close to J_l^+, the real part of $i\zeta$ is negative, so $exp(i\zeta/h)$ is exponentially decaying. Similarly, in an area of the upper half-plane, close to J_l^-, the real part of $i\zeta$ is positive, so $exp(-i\zeta/h)$ is exponentially decaying. In the particular case of $\lambda = 0$ it is also easy to check that $exp(-i\zeta/h)$ is exponentially decaying as λ moves upwards along the positive imaginary axis.

Let us now introduce the following lens-like contours. For each interval J_l^+ consider small piecewise linear deformations upwards, say $J_l^{+,up}$, keeping the end points fixed, but otherwise lying entirely in the upper half-plane. Similarly, consider small piecewise linear deformations downwards, say $J_l^{+,dn}$, keeping the end points fixed, but otherwise lying entirely in the lower half-plane. For each interval J_l^- consider small piecewise linear deformations upwards, say $J_l^{-,up}$, keeping the end points fixed, but otherwise lying entirely in the upper half-plane. Similarly, consider small piecewise linear deformations downwards, say $J_l^{-,dn}$, keeping the end points fixed, but otherwise lying entirely in the lower half-plane. All orientations are compatible with $Re(\lambda)$ increasing. We also make sure that $J_l^{+,up}, J_l^{+,dn}, J_l^{-,up}, J_l^{-,dn}$ cut the real line at angles $\neq 0, \pi/2$. See Figure 1.

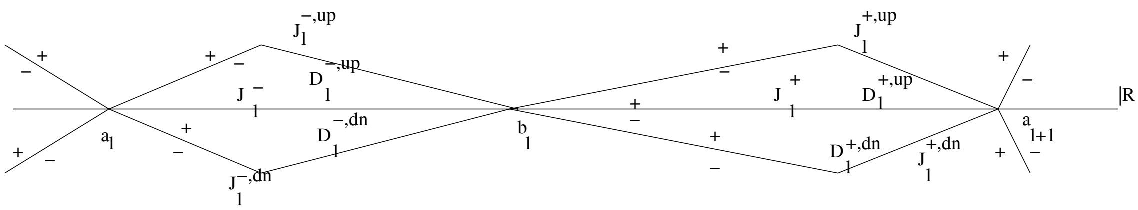

Fig.1. Part of the extended contour, with lenses

Let us now introduce the following lens-like contours. For each interval J_l^+ consider small piecewise linear deformations upwards, say $J_l^{+,up}$, keeping the end points fixed, but otherwise lying entirely in the upper half-plane. Similarly, consider small piecewise linear deformations downwards, say $J_l^{+,dn}$, keeping the end points fixed, but otherwise lying entirely in the lower half-plane. For each interval J_l^- consider small piecewise linear deformations upwards, say $J_l^{-,up}$, keeping the end points fixed, but otherwise lying entirely in the upper half-plane. Similarly, consider small piecewise linear deformations downwards, say $J_l^{-,dn}$, keeping the end points fixed, but otherwise lying entirely in the lower half-plane. All orientations are compatible with $Re(\lambda)$ increasing. We also make sure that $J_l^{+,up}, J_l^{+,dn}, J_l^{-,up}, J_l^{-,dn}$ cut the real line at angles $\neq 0, \pi/2$. See Figure 1.

Note here that one of the lens-like contours $J_l^{-,up}$ will cut the contour C while one of the lens-like contours $J_l^{-,dn}$ will cut the contour C^*.

Let the open regions bounded by J_l^+ and $J_l^{+,up}$ be denoted by $D_l^{+,up}$ and the open regions bounded by J_l^+ and $J_l^{+,dn}$ be denoted by $D_l^{+,dn}$. Similarly, let the open regions bounded by J_l^- and $J_l^{-,up}$ be denoted by $D_l^{-,up}$ and the open regions bounded by J_l^- and $J_l^{-,dn}$ be denoted by $D_l^{-,dn}$. We make sure that $Re(i\zeta) < 0$, in $D_l^{+,up} \cup J_l^{+,up} \cup D_l^{-,dn} \cup J_l^{-,dn}$. Similarly, we make sure that $Re(i\zeta) > 0$, in $D_l^{-,up} \cup J_l^{-,up} \cup D_l^{+,dn} \cup J_l^{+,dn}$.

Next, we can make use of the factorizations given in (15+), (15-). Set

$$
\begin{aligned}
Z(\lambda) = Q(\lambda),\ for\ \lambda \in \mathbb{C} \setminus \cup_l [\bar{D}_l^{+,up} \cup \bar{D}_l^{+,dn} \cup \bar{D}_l^{-,up} \cup \bar{D}_l^{-,dn}],\\
Z(\lambda) = Q(\lambda) U^{-1}(\lambda),\ for\ \lambda \in \cup_l D_l^{+,up},\\
Z(\lambda) = Q(\lambda) L(\lambda),\ for\ \lambda \in \cup_l D_l^{+,dn},\\
Z(\lambda) = Q(\lambda) T^{-1}(\lambda),\ for\ \lambda \in \cup_l D_l^{-,up},\\
Z(\lambda) = Q(\lambda) S(\lambda),\ for\ \lambda \in \cup_l D_l^{-,dn}.
\end{aligned}
\tag{19}
$$

The Riemann-Hilbert problem for Z is described by

(20)

$$Z_+(\lambda) = Z_-(\lambda) v_Z(\lambda),$$

$$v_Z(\lambda) = \begin{pmatrix} 1 & -r(\lambda) e^{\frac{2g-2i\lambda x-2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix}, \ for \ \lambda \in \cup_l J_l^{+,up},$$

$$v_Z(\lambda) = \begin{pmatrix} 1 & 0 \\ r^*(\lambda^*) e^{\frac{-2g+2i\lambda x+2i\lambda^2 t}{h}} & 1 \end{pmatrix}, \ for \ \lambda \in \cup_l J_l^{+,dn},$$

$$v_Z(\lambda) = \begin{pmatrix} 1 & 0 \\ \frac{r^*(\lambda^*)}{1+|r(\lambda)|^2} e^{\frac{-2g+2i\lambda x+2i\lambda^2 t}{h}} & 1 \end{pmatrix}, \ for \ \lambda \in \cup_l J_l^{-,up},$$

$$v_Z(\lambda) = \begin{pmatrix} 1 & \frac{r(\lambda)}{1+|r(\lambda)|^2} e^{\frac{2g-2i\lambda x-2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix}, \ for \ \lambda \in \cup_l J_l^{-,dn},$$

$$v_Z(\lambda) = \begin{pmatrix} e^{\frac{g_+-g_-}{h}} & -i\ exp(\frac{1}{h}\int log(\lambda-\eta) d\mu(\eta)) exp(-\frac{1}{h}(-g_+ - g_- + 2i\lambda x + 2i\lambda^2 t - X(\lambda))) \\ 0 & e^{\frac{g_--g_+}{h}} \end{pmatrix},$$

$$for \ \lambda \in C,$$

$$v_Z(\lambda) = \begin{pmatrix} e^{\frac{g_+-g_-}{h}} & 0 \\ i\ exp(\frac{-1}{h}\int log(\lambda-\eta) d\mu(\eta)) exp(\frac{1}{h}(-g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*))) & e^{\frac{g_--g_+}{h}} \end{pmatrix},$$

$$for \ \lambda \in C^*,$$

$$v_Z(\lambda) = \begin{pmatrix} (1+|r(\lambda)|^2)^{-1} & 0 \\ 0 & 1+|r(\lambda)|^2 \end{pmatrix}, \ for \ \lambda \in \cup_l J_l^-,$$

with the normalization $lim_{\lambda\to\infty} Z(\lambda) = I$. For a part of the extended contour, away from 0, and between the points a_l and a_{l+1}, including the small lenses, see Figure 1.

The idea for the next transformation is to get rid of the jump across the real line.

We define $\tilde{d}$ as follows. Let $\tilde{d}$ be analytic in $\mathbb{C} \setminus \cup J_l^-$, such that the normal limits from above and below the real line $d_\pm$ exist (apart from the endpoints) and such that

$$\tilde{d}_+(\lambda) = \tilde{d}_-(\lambda)(1+|r(\lambda)|^2), \ \lambda \in \cup_l interior(J_l^-),$$

$$lim_{\lambda\to\infty} d(\lambda) = 1.$$

One can even give an explicit formula for $\tilde{d}$. Denote $J_l^- = [a_l, b_l]$. Then

(21) $$\tilde{d}(\lambda) = exp\ [\Sigma_l \int_{a_l}^{b_l} log(1+|r(\zeta)|^2) \frac{d\zeta}{2\pi i(\zeta-\lambda)}], \ \lambda \in \mathbb{C} \setminus \cup J_l^-$$

where the principal branch of the logarithm is chosen. Here $\nu_l = \frac{1}{2\pi}(1+log|r(a_l)|^2)$ and $\mu_l = \frac{1}{2\pi}(1+log|r(b_l)|^2)$.

It is clear that the definitions above are not singular and indeed $\tilde{d}, \tilde{d}^{-1}$ are bounded in the complex Riemann sphere. In fact, it is easy to check that both $\tilde{d}(\lambda)$ and $[\tilde{d}(\lambda^*)^*]^{-1}$ satisfy the conditions of the scalar Riemann-Hilbert problem. Hence they must be equal, from which it easily follows that

$$\begin{aligned} |\tilde{d}_-|^2(1+|r(\lambda)|^2) = 1, \ for \ \lambda \in \cup_l J_l^-, \\ so \ |\tilde{d}_-| \leq 1, \ for \ \lambda \in \cup_l J_l^-, \\ |\tilde{d}_+| \leq 1 + sup_{\mathbb{R}} |r(\lambda)|^2 < \infty, \ for \ \lambda \in \cup_l J_l^-, \end{aligned} \tag{21a}$$

and hence, by the maximum principle, $|\tilde{d}(\lambda)| \leq 1 + sup_{\mathbb{R}} |r(\lambda)|^2 < \infty$. So $|\tilde{d}(\lambda)|$ is bounded uniformly in the complex plane. Similarly, $|\tilde{d}^{-1}(\lambda)|$ is uniformly bounded in the complex plane.

Near the points a_l the local behavior of $\tilde{d}$ off the jump contour is

$$\tilde{d} \sim d_l(\lambda - a_l)^{i\nu_l}, \tag{21b}$$

and near b_l,

$$\tilde{d} \sim f_l(\lambda - b_l)^{i\mu_l}, \tag{21c}$$

and d_l, f_l are independent of λ. In fact

(21d)

$$d_j = exp\ [\int_{-\infty}^{a_j} log(z-a_j) \frac{dlog(1+|r(z)|^2)}{2\pi i} + \Sigma_{l \neq j} \int_{a_l}^{b_l} log(1+|r(\zeta)|^2) \frac{d\zeta}{2\pi i(\zeta - a_l)}],$$
$$f_j = exp\ [-\int_{-\infty}^{b_j} log(z-b_j) \frac{dlog(1+|r(z)|^2)}{2\pi i} + \Sigma_{l \neq j} \int_{a_l}^{b_l} log(1+|r(\zeta)|^2) \frac{d\zeta}{2\pi i(\zeta - b_l)}],$$

as can be shown by integration by parts. The next transformation is then

$$\begin{aligned} Y(\lambda) = Z(\lambda)\Delta(\lambda), \\ where \\ \Delta(\lambda) = \begin{pmatrix} \tilde{d}(\lambda) & 0 \\ 0 & \tilde{d}^{-1}(\lambda) \end{pmatrix}. \end{aligned} \tag{22}$$

Then $Y_+ = Y_- v_Y$, where

(23)
$$v_Y(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & -i\ \tilde{d}^{-2}\ exp[\frac{1}{h}(\int log(\lambda-\eta)d\mu(\eta) + g_- + g_- - 2i\lambda x - 2i\lambda^2 t + X(\lambda))] \\ 0 & e^{\frac{g_- - g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C,$$
$$v_Y(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & 0 \\ i\ \tilde{d}^2\ exp[\frac{1}{h}(-\int log(\lambda-\eta)d\mu(\eta) - g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*))] & e^{\frac{g_- - g_+}{h}} \end{pmatrix},$$
$$for\ \lambda \in C^*,$$
$$v_Y(\lambda) = \begin{pmatrix} 1 & -r(\lambda)\tilde{d}^2 e^{\frac{2g - 2i\lambda x - 2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^{+,up},$$
$$v_Y(\lambda) = \begin{pmatrix} 1 & 0 \\ r^*(\lambda^*)\tilde{d}^{-2} e^{\frac{-2g + 2i\lambda x + 2i\lambda^2 t}{h}} & 1 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^{+,dn},$$
$$v_Y(\lambda) = \begin{pmatrix} 1 & 0 \\ \tilde{d}^{-2}\frac{r^*(\lambda^*)}{1+|r(\lambda)|^2} e^{\frac{-2g + 2i\lambda x + 2i\lambda^2 t}{h}} & 1 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^{-,up},$$
$$v_Y(\lambda) = \begin{pmatrix} 1 & \tilde{d}^2 \frac{r(\lambda)}{1+|r(\lambda)|^2} e^{\frac{2g - 2i\lambda x - 2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in \cup_l J_l^{-,dn}.$$

Also $lim_{\lambda \to \infty} Y(\lambda) = I$.

At this point, we still have a Riemann-Hilbert that is equivalent to the original one (11)-(12), at least accepting the discrete-to-continuous passage in Theorem 1. Now, we can finally start considering the limit $h \to 0$. Indeed, all terms involving $r(\lambda)$ and not supported on the real line can be neglected, not because r itself is small (it is not, see (16)-(16*)) but it always appears multiplied by something exponentially small.

In general jump matrices of the form $I + exponentially\ small$ can be neglected asymptotically as long as it is proved that the solution is uniformly (in x, t) bounded as $h \to 0$. We shall see that this is eventually the case.

Assuming for the moment that this is true, we can simply delete the non-real part of the contour, at least away from the endpoints of the intervals J_k. Eventually (see section 5 later) we can also delete the remaining small crosses centered at such points. We end up with a matrix valued function W, such that $W \sim Y$ near infinity, and

$$W_+(\lambda) = W_-(\lambda) v_W(\lambda), \tag{24}$$

where

(25)

$$v_W(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & -i\ \tilde{d}^{-2}\ exp(\frac{1}{h}\int log(\lambda-\eta)d\mu(\eta))exp(-\frac{1}{h}(-g_+ - g_- + 2i\lambda x + 2i\lambda^2 t - X(\lambda))) \\ 0 & e^{\frac{g_- - g_+}{h}} \end{pmatrix},$$

$$for\ \lambda \in C,$$

$$v_W(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{h}} & 0 \\ i\ \tilde{d}^2\ exp(\frac{-1}{h}\int log(\lambda-\eta)d\mu(\eta))exp(\frac{1}{h}(-g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*))) & e^{\frac{g_- - g_+}{h}} \end{pmatrix},$$

$$for\ \lambda \in C^*.$$

Also $lim_{\lambda\to\infty} W(\lambda) = I$.

We will eventually see that the solution of the Riemann-Hilbert problem for W exists and is uniformly (in x, t) bounded as $h \to 0$. This will justify neglecting the exponentially small terms in (23) according to a basic principle for deformations of Riemann-Hilbert problems (see e.g. [DZ]). The passage from (23) to (25) will then be a posteriori justified, again at least except for the remaining small crosses centered at a finite number of real points. Concerning the crosses see section 5 (Remark 5) and Appendix A.1.

After applying the above transformation $Y \to W$, we must use another lens transformation to simplify the jump across the conotur $C \cup C^*$. We will not describe this new lens transformation, since the discussion is exactly as in Chapter 4 of [KMM]. We will simply state the end result of these lens transformations and the transformation $Y \to W$. The result is the so-called outer problem. We shall show in the next section how it can be treated along the lines of Chapter 4 of [KMM].

THE OUTER PROBLEM.

We first define the analytic arcs $I_j, I_j^*, j = 1, ..., G/2$ as follows (they come in conjugate pairs). Let the points $\lambda_j,\ j = 0, ..., G$, in the open uper half-plane be the branch points of the function g. (The fact that there are $G+1$ of them is a consequence of the definition of g, according to the "finite genus ansatz"; see below.) All such points lie on the contour C and we order them as $\lambda_0, \lambda_1, ..., \lambda_G$, according to the direction given to C. The points $\lambda_0^*, \lambda_1^*, ..., \lambda_G^*$ are their complex conjugates. Then let $I_0 = [0, \lambda_0]$ be the subarc of C joining points 0 and λ_0.

Similarly, $I_j = [\lambda_{2j-1}, \lambda_{2j}],\ j = 1, ..., G/2$. The points $\lambda_j,\ j = 0, ..., G$ lie in the open uper half-plane and they are determined by a set of transcendental equations that follow directly from the definitons of g and C (cf. Remark after Lemma 5.1.5 of [KMM]). The connected components of the set $\mathbb{C} \setminus \cup_j (I_j \cup I_j^*)$ are the so-called "gaps", for example the gap Γ_1 joins λ_0 to λ_1, etc. The subarcs I_j are the "bands".

The finite genus ansatz implies that for each x, t there is a finite positive integer G such that the contour C can be divided into bands and gaps as above. In fact, it follows from the conditions defining ρ, C that the measure reality condition ($\rho(\eta)d\eta$ *real*; see (18b)) splits into a measure strict negativity condition in the bands and a measure zero condition in the gaps. Furthermore, the function $\theta(\lambda)$ of (18a) defined on C is constant on each of the gaps Γ_j, taking a value which we will denote by θ_j, while the function ϕ of (18a) is constant on each of the bands, taking the value denoted by α_j on the band I_j. For the justification of the finite ansatz under the barrier data see Appendix 2.

We are seeking a matrix O, which is analytic everywhere except across the contour $C \setminus \Gamma_{G/2+1}$ and its conjugate, with limits that are $L^2(C \setminus \Gamma_{G/2+1})$, converging to the identity at infinity and such that

$$
\begin{aligned}
O_+(\lambda) &= O_-(\lambda) \begin{pmatrix} 0 & i\tilde{d}^{-2} exp(-i\alpha_k/h) \\ i\tilde{d}^2 exp(i\alpha_k/h) & 0 \end{pmatrix},\ \lambda \in I_k \cup I_k^*, \\
& \qquad k = 0, 1, ..., g/2, \\
O_+(\lambda) &= O_-(\lambda) \begin{pmatrix} exp(i\theta_k/h) & 0 \\ 0 & exp(-i\theta_k/h) \end{pmatrix},\ \lambda \in \Gamma_k \cup \Gamma_k^*, \\
& \qquad k = 1, ..., G/2.
\end{aligned}
\tag{26}
$$

We here recapitulate the sequence of matrix deformations inroduced so far:

$$M(discrete) \to M(continuous) \to Q \to Z \to Y \to W \to O.$$

The first problem in the sequence, for M (in its discrete version), is equivalent to the inverse scattering problem for NLS. The last Riemann-Hilbert problem, for O, (26), will be solved explicitly via theta functions (see (33) of section 5).

4. THE RESULT

As explained in Appendix A.2, the finite genus ansatz holds for the semiclassical asymptotics under barrier initial data. This means that the x, t-plane can be divided into (possibly empty) open regions $R_G, G = 0, 2, 4, ...$ together with their boundaries, such that within each region the asymptotics of the solution of (1) with barrier data as in (2) can be given as follows.

THEOREM 2. Let x_0, t_0 lie in region R_G. The solution $u^h(x,t)$ is asymptotically described (locally) as a slowly modulated $G+1$ phase wavetrain. Setting $x = x_0 + h\hat{x}$ and $t = t_0 + h\hat{t}$, so that x_0, t_0 are "slow" variables while $\hat{x}, \hat{t}$ are "fast" variables, there exist parameters

$a, U = (U_0, U_1, ..., U_G)^T,\ k = (k_0, k_1, ..., k_G)^T,\ w = (w_0, w_1, ..., w_G)^T,\ Y = (Y_0, Y_1, ..., Y_G)^T,\ Z = (Z_0, Z_1, ..., Z_G)^T$ depending on the slow variables x_0 and t_0 and possibly h (but not $\hat{x}, \hat{t}$) such that

$$\begin{aligned} u^h(x,t) &= u^h(x_0 + h\hat{x}, t_0 + h\hat{t}) \sim a(x_0,t_0) e^{iU_0(x_0,t_0)/h} e^{i(k_0(x_0,t_0)\hat{x} - w_0(x_0,t_0)\hat{t})} \\ &\quad \cdot \tilde{d}^2(\lambda_0) \frac{\Theta(Y(x_0,t_0) + iU(x_0,t_0)/h + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}{\Theta(Z(x_0,t_0) + iU(x_0,t_0)/h + i(k(x_0,t_0)\hat{x} - w(x_0,t_0)\hat{t}))}. \end{aligned} \tag{27}$$

Unlike the analogous formula in [KMM], we allow here a dependence of the parameters on h. But of course, we can always rearrange terms to arrive at a formula like (27) where the parameters are independent of h.

All parameters can be defined in terms of an underlying Riemann surface X. The moduli of X are given by $\lambda_j,\ j = 0, ..., G$ and their complex conjugates $\lambda_j^*,\ j = 0, ..., G$. The genus of X is G. The moduli of X vary slowly with x, t, i.e. they depend on x_0, t_0 but not $\hat{x}, \hat{t}$. For the exact formulae for the parameters as well as the definition of the theta functions we present the following construction.

The Riemann surface X is constructed by cutting two copies of the complex sphere along the slits $I_0 \cup I_0^*, I_j, I_j^*, j = 1...., G$, and pasting the "top" copy to the "bottom" copy along these very slits.

We define the homology cycles $a_j, b_j,\ j = 1, ..., G$ as follows. Cycle a_1 goes around the slit $I_0 \cup I_0^*$ joining λ_0 to λ_0^*, remaining on the top sheet, oriented counterclockwise, a_2 goes through the slits I_{-1} and I_1 starting from the top sheet, also oriented counterclockwise, a_3 goes around the slits $I_{-1}, I_0 \cup I_0^*, I_1$ remaining on the top sheet, oriented counterclockwise, etc. Cycle b_1 goes through I_0 and I_1 oriented counterclockwise, cycle b_2 goes through I_{-1} and I_1, also oriented counterclockwise, cycle b_3 goes through I_{-1} and I_2, and around the slits $I_{-1}, I_0 \cup I_0^*, I_1$, oriented counterclockwise, etc.

Let

$$R(\lambda)^2 = \prod_{k=0}^{G} (\lambda - \lambda_k)(\lambda - \lambda_k^*),$$

choosing the particular branch that is cut along the bands I_k and I_k^* and such that

$$\lim_{\lambda \to \infty} \frac{R(\lambda)}{\lambda^{G+1}} = -1.$$

On X there is a complex G-dimensional linear space of holomorphic differentials, with basis elements $\nu_k(P)$ for $k = 1, \ldots, G$ that can be written in the form

$$\nu_k(P) = \frac{\sum_{j=0}^{G-1} c_{kj} \lambda(P)^j}{R_X(P)} \, d\lambda(P)\,,$$

where $R_X(P)$ is a "lifting" of the function $R(\lambda)$ from the cut plane to X: if P is on the first sheet of X then $R_X(P) = R(\lambda(P))$ and if P is on the second sheet of X then $R_X(P) = -R(\lambda(P))$. The coefficients c_{kj} are uniquely determined by the constraint that the differentials satisfy the normalization conditions:

$$\oint_{a_j} \nu_k(P) = 2\pi i \delta_{jk}.$$

From the normalized differentials, one defines a $G \times G$ matrix H (the period matrix) by the formula:

$$H_{jk} = \oint_{b_j} \nu_k(P).$$

It is a consequence of the standard theory of Riemann surfaces that H is a symmetric matrix whose real part is negative definite.

In particular, we can define the theta function

$$\Theta(w) := \sum_{n \in \mathbb{Z}^G} \exp(\frac{1}{2} n^T H n + n^T w),$$

where H is the period matrix associated to X. Since the real part of H is negative definite, the series converges.

We arbitrarily fix a base point P_0 on X. The Abel map $A : X \to Jac(X)$ is then defined componentwise as follows:

$$A_k(P; P_0) := \int_{P_0}^{P} \nu_k(P'), \; k = 1, \dots, G,$$

where P' is an integration variable.

REMARK. In [KMM] the Abel map is thought of as a map from $X \to Jac(X)$. In [KR] (Appendix A.3) it is noted that the map may be most appropriately thought as a map from an infinite sheeted Riemann surface X^{∞} with extra branch points at $\pm iA$. The bands I_j and the path of integration for the Abel actually lie in X^{∞}. For small t this distinction is not necessary.

A particularly important element of the Jacobian is the Riemann constant vector K which is defined, modulo the lattice Λ, componentwise by

$$K_k := \pi i + \frac{H_{kk}}{2} - \frac{1}{2\pi i} \sum_{\substack{j=1 \\ j \neq k}}^{G} \oint_{a_j} \left(\nu_j(P) \int_{P_0}^{P} \nu_k(P') \right),$$

where the index k varies between 1 and G.

Next, we will need to define a certain meromorphic differential on X. Let $\Omega(P)$ be holomorphic away from the points ∞_1 and ∞_2, where it has the behavior

$$\Omega(P) = dp(\lambda(P)) + \left(\frac{d\lambda(P)}{\lambda(P)^2} \right), \; P \to \infty_1,$$
$$\Omega(P) = -dp(\lambda(P)) + O\left(\frac{d\lambda(P)}{\lambda(P)^2} \right), \; P \to \infty_2,$$

and made unique by the normalization conditions

$$\oint_{a_j} \Omega(P) = 0, j = 1, \dots, G.$$

Here p is some polynomial. In the present context see section 5, equation (31) for its definition.

Let the vector $U \in \mathbb{C}^G$ be defined componentwise by

$$U_j := \oint_{b_j} \Omega(P).$$

Note that $\Omega(P)$ has no residues.

Let the vectors V_1, V_2 be defined componentwise by

$$V_{1,k} = (A_k(\lambda_{1+}^*) + A_k(\lambda_{2+}) + A_k(\lambda_{3+}^*) + \cdots + A_k(\lambda_{G+})) + A_k(\infty) + \pi i + \frac{H_{kk}}{2},$$
$$V_{2,k} = (A_k(\lambda_{1+}^*) + A_k(\lambda_{2+}) + A_k(\lambda_{3+}^*) + \cdots + A_k(\lambda_{G+})) - A_k(\infty) + \pi i + \frac{H_{kk}}{2},$$

where $k = 1, ..., G$, and the $+$ index means that the integral for A is to be taken on the first sheet of X, with base point λ_0.

Finally, let

$$a = \frac{\Theta(Z)}{\Theta(Y)} \sum_{k=0}^{G} (-1)^k \Im(\lambda_k),$$
$$k_n = \partial_x U_n, \ w_n = -\partial_t U_n, \ n = 0, \ldots, G,$$

where

$$Y = -A(\infty) - V_1, \ Z = A(\infty) - V_1,$$

and $U_0 = -(\theta_1 + \alpha_0)$ where θ_1 is the (constant in λ) value of the function θ in the gap Γ_1 and α_0 is the (constant) value of the function ϕ in the band I_0. The fact that these values are actually constants in λ follows from the conditions defining g and C.

Now, the parameters appearing in formula (27) are completely described.

We simply note here that the U_i and hence the k_i and w_i are real modulo $O(h)$. We also note that the denominator in (24) never vanishes (for any $x_0, t_0, \hat{x}, \hat{t}$).

REMARK: Because C depends only on the eigenvalue density ρ^0 and not on the reflection coefficient r, the constructions of C, the Riemann surface X, the holomorphic differential $\nu_k(P)$, the Abel map A and the theta functions are all independent of r. The only contribution of r comes from the factors $\tilde{d}^2, \tilde{d}^{-2}$ in (26) and the factor $\tilde{d}^2(\lambda_0)$ in (27). This is why our discussion in section 4 is virtually repeating verbatim the analogous discussion of section 4 in [KMM].

5. REMARKS AND PROOF OF THEOREM 2.

1. Formula (27) is locally a so-called finite gap expression. It describes violent oscillations of bounded amplitude but high frequency $O(1/h)$.

2. As in [KMM] weak limits of densities exist: $\rho = lim_{h\to 0}|u^h|^2$ and $\mu = lim_{h\to 0}\frac{-ih}{2}(\bar{u}^h u^h_x - u^h \bar{u}^h_x)$. These limits are actually strong in the genus zero region.

3. Naturally, since the initial data is discontinuous and the limiting Euler system (1a) is elliptic (see [KMM]) one expects the break-time of the limiting system to be zero. This is indeed the case, as numerical experiments by H.Ceniceros and F.Tian have shown [T], or as can be shown analytically by considering the limiting Euler system directly: from the second equation of (1a) it is obvious that μ_t is infinite at $x = \pm\frac{1}{2}, t = 0$. Of course a genus zero region still exists but the first caustic (the boundary between the genus zero region and higher genus regions) touches the $t = 0$ axis of the x, t-plane at $x = \pm 1/2$.

4. The proof of the results in [KMM] makes use of the assumption that the eigenvalue density can be analytically extended in the upper half-plane with the spike where eigenvalues accumulate deleted. This *is* the case here, see formula (9a). The branch root singularity at iA does not play any role, because it is integrable. Integrals like $\int \rho^0 d\eta$ or $\int log(\lambda-\eta)\rho^0(\eta)d\eta$ can still be deformed and the Cauchy theorem holds.

5. A priori, neglecting the exponentially terms in (23) only allows us to "delete" arcs $J_l^{+,up}, J_l^{+,dn}, J_l^{-,up}, J_l^{-,dn}$ only away from the real endpoints of J_l^+, J_l^-. For entirely rigorous justifications of the deletion of the small crosses centered at each such point remaining after the deletion of the bulk of the arcs $J_l^{+,up}, J_l^{+,dn}, J_l^{-,up}, J_l^{-,dn}$, one must construct local parametrices of the Riemann-Hilbert problem and make sure they match with the solution of (25) away from the endpoints. This is a procedure that is by now standard in the literature; more details are given in the Appendix. In fact, essentially the same situation has appeared in [DZ]. The "local" Riemann-Hilbert problem can be solved via parabolic cylinder functions. (In [DZ], of course, the contribution was not negligible, since one was trying to evaluate the

term of order $O(t^{-1/2})$ of the asymptotics. Also in [DZ] the reflection coefficient is independent of the parameter $1/t$ going to zero, but this is irrelevant since the local Riemann-Hilbert problem is solved exactly via parabolic cylinder functions.)

6. As in [KMM], one needs to provide a local parametrix near the origin and then match it with the solution of the "outer problem". This can still be done, using the Fredholm theory described in [KMM]. The "cyclic" relation, that the product of limits of jumps at 0 is the identity, still holds. In view of (9b) the cyclic relation follows easily from the analogous relation in the reflectionless case. The discussion in sections (4.4.3), (4.5.1) and (4.5.2) of [KMM] can be then followed verbatim.

7. The Bohr-Sommerfeld condition (8) is not quite the same as the condition postulated in [KMM] defining the so-called soliton ensemble, which translates as

$$(A^2 + \lambda_k^2)^{1/2} = hk\pi$$

for our present problem (1)-(2). Now, one can notice that the difference between the two conditions gives rise to a uniform error of higher order $O(h^2)$. The analysis in [KMM] (Chapter 3) of estimates needed for the passage from a "discrete" Riemann-Hilbert problem to a "continuum" Riemann-Hilbert problem is not altered by this innocent modification.

8. From formulae (6)-(7) it is obvious that $\alpha(0) = 0$ if Ah is an odd multiple of $\pi/2$. So, for a particular sequence of h going to zero we have a *spectral singularity* at the real point 0. The Riemann-Hilbert problem jump becomes singular at 0. By assumption we have excluded such values of h. We plan to study the effect of a real spectral singularity on the semiclassical behavior of the focusing NLS in a later publication.

9. The proof of Theorem 2 now follows the discussion of Chapter 4 of [KMM]. One minor change is the extra factor $\tilde{d}^{\pm 2}$ appearing in the off-diagonal terms of (26). This factor can be taken care of by the auxiliary scalar Riemann-Hilbert problem (4.38) in section 4.3.1 of [KMM]. The proof goes through with only a minor change: α_k has to be substituted by $\alpha_k + 2ih\ log\tilde{d}$. Of course $\tilde{d}$ has to appear in the final formula (27). Granted that the term $h\tilde{d}$ is no more constant on bands, but then

the auxiliary scalar Riemann-Hilbert problem is still explicitly solvable.

More precisely, we introduce the scalar problem:

$$\begin{aligned} H_+(\lambda) - H_-(\lambda) &= -\theta_k,\ \lambda \in \Gamma_k \cup \Gamma_k^*,\ k = 1, \ldots, G/2\,, \\ H_+(\lambda) + H_-(\lambda) &= -(\alpha_k + 2ihlog\tilde{d}),\ \lambda \in I_k \cup I_k^*,\ k = 0, \ldots, G/2\,, \end{aligned} \tag{28}$$

not specifying any condition at infinity yet. Consider the matrix defined by

$$P(\lambda) = O(\lambda)\exp(iH(\lambda)\sigma_3/h),$$

where O is the solution of the outer Riemann-Hilbert problem (26). It is straightforward to verify that the matrix $P(\lambda)$ has the identity matrix as the jump matrix in all gaps Γ_k and Γ_k^*. Since the boundary values of $O(\lambda)$ and $H(\lambda)$ are continuous, it follows that $P(\lambda)$ is in fact analytic in the gaps. In the bands, the jump relation becomes simply

$$P_+(\lambda) = iP_-(\lambda)\sigma_1,$$

so the jump relation is the same in all bands. Next, suppose that $\beta(\lambda)$ is a scalar function analytic in the λ-plane except at the bands, where it satisfies $\beta_+(\lambda) = -i\beta_-(\lambda)$. Suppose further for the sake of concreteness that $\beta(\lambda) \to 1$ as $\lambda \to \infty$. Then, setting

$$V(\lambda) = \beta(\lambda)P(\lambda), \tag{29}$$

we see that the jump relations for $V(\lambda)$ take on the elementary form:

$$V_+(\lambda) = V_-(\lambda)\sigma_1,\ \lambda \in \cup_k(I_k \cup I_k^*). \tag{30}$$

Our purpose in reducing the jump relations to this universal constant form is that it can be explicitly solved in terms of theta functions.

But let us describe the scalar functions $H(\lambda)$ and $\beta(\lambda)$. We get

$$\beta(\lambda)^4 = \frac{\lambda - \lambda_0^*}{\lambda - \lambda_0}\prod_{k=1}^{G/2}\frac{\lambda - \lambda_{2k-1}}{\lambda - \lambda_{2k-1}^*}\cdot\frac{\lambda - \lambda_{2k}^*}{\lambda - \lambda_{2k}}\,,$$

and for $\beta(\lambda)$ we select the branch that tends to unity for large λ and that is cut along the bands I_k and I_k^*. It is easily checked that $\beta(\lambda)$ as defined here is the only function satisfying the required jump condition and normalization at infinity that has continuous boundary values (except at half of the endpoints). To find $H(\lambda)$, we introduce the function $R(\lambda)$ defined by

$$R(\lambda)^2 = \prod_{k=0}^{G} (\lambda - \lambda_k)(\lambda - \lambda_k^*),$$

choosing the particular branch that is cut along the bands I_k and I_k^* and satisfies

$$\lim_{\lambda \to \infty} \frac{R(\lambda)}{\lambda^{G+1}} = -1,$$

This defines a real function, i.e. one that satisfies $R(\lambda^*) = R(\lambda)^*$. At the bands, we have $R_+(\lambda) = -R_-(\lambda)$, while $R(\lambda)$ is analytic in the gaps. Set

$$H(\lambda) = k(\lambda)R(\lambda),$$

where

$$k(\lambda) = \frac{1}{2\pi i} \sum_{n=1}^{G/2} \theta_n \int_{\Gamma_n \cup \Gamma_n^*} \frac{d\eta}{(\lambda - \eta)R(\eta)} + \frac{1}{2\pi i} \sum_{n=0}^{G/2} \int_{I_n \cup I_n^*} \frac{(\alpha_n + 2ihlog\tilde{d}(\eta))\ d\eta}{(\lambda - \eta)R_+(\eta)}.$$

We see that $k(\lambda)$ satisfies the jump relations:

$$k_+(\lambda) - k_-(\lambda) = -\frac{\theta_n}{R(\lambda)},\ \lambda \in \Gamma_n \cup \Gamma_n^*$$
$$k_+(\lambda) - k_-(\lambda) = -\frac{\alpha_n + 2ihlog\tilde{d}(\lambda)}{R_+(\lambda)},\ \lambda \in I_n \cup I_n^*,$$

and is otherwise analytic. So H satisfies (28).

The function k blows up like $(\lambda - \lambda_n)^{-1/2}$ near each endpoint, has continuous boundary values in between the endpoints, and vanishes like $1/\lambda$ for large λ. It is the only such solution of the jump relations. The factor of $R(\lambda)$ renormalizes the singularities at the endpoints, so that, as desired, the boundary values of $H(\lambda)$ are bounded continuous functions. Near infinity, there is the asymptotic expansion:

$$\begin{aligned} H(\lambda) &= H_G\lambda^G + H_{G-1}\lambda^{G-1} + \cdots + H_1\lambda + H_0 + O(\lambda^{-1}) \\ &= p(\lambda) + O(\lambda^{-1}), \end{aligned} \tag{31}$$

where all coefficients H_j of the polynomial $p(\lambda)$ can be found explicitly by expanding $R(\lambda)$ and the Cauchy integral $k(\lambda)$ for large λ. It is easy to see from the reality of θ_j and α_j that $p(\lambda)$ is a polynomial with coefficients which are real modulo $O(h)$.

So the matrix function V defined in (29) has the following asymptotics at infinity:

$$V(\lambda)exp[-ip(\lambda)\sigma_3/h] = I + O(\lambda^{-1}). \tag{32}$$

This together with the jump relations (30) defines a Riemann-Hilbert problem for V that can be explicitly solved via theta functions. Equivalently P and O can be explicitly expressed in terms of theta functions. For example, it is now elementary to check that the solution of the outer Riemann-Hilbert problem (26) for O is given by the formulae:

$$\begin{aligned}
O_{11}(\lambda) &= \frac{b^-(\lambda)}{\beta(\lambda)} \frac{\Theta(A(\infty) - V_1)}{\Theta(A(\lambda) - V_1)} \frac{\Theta(A(\lambda) - V_1 + iU/\hbar)}{\Theta(A(\infty) - V_1 + iU/\hbar)}, \\
O_{12}(\lambda) &= \frac{b^+(\lambda)}{\beta(\lambda)} e^{2iR(\lambda)k_+(\lambda_0)/\hbar} \frac{\Theta(A(\infty) - V_1)}{\Theta(-A(\lambda) - V_1)} \frac{\Theta(-A(\lambda) - V_1 + iU/\hbar)}{\Theta(A(\infty) - V_1 + iU/\hbar)}, \\
O_{21}(\lambda) &= \frac{b^+(\lambda)}{\beta(\lambda)} e^{-2iR(\lambda)k_+(\lambda_0)/\hbar} \frac{\Theta(-A(\infty) - V_2)}{\Theta(A(\lambda) - V_2)} \frac{\Theta(fA(\lambda) - V_2 + iU/\hbar)}{\Theta(-A(\infty) - V_2 + iU/\hbar)}, \\
O_{22}(\lambda) &= \frac{b^-(\lambda)}{\beta(\lambda)} \frac{\Theta(-A(\infty) - V_2)}{\Theta(-A(\lambda) - V_2)} \frac{\Theta(-A(\lambda) - V_2 + iU/\hbar)}{\Theta(-A(\infty) - V_2 + iU/\hbar)},
\end{aligned} \tag{33}$$

where

$$b^{\pm}(\lambda) = \frac{R(\lambda) \pm (\lambda - \lambda_0^*)(\lambda - \lambda_1)\dots(\lambda - \lambda_G^*)}{2R(\lambda)}. \tag{34}$$

From the explicit solution, using formula (14) with the obvious substitution of O for M one derives formula (27). This completes the proof of Theorem 2.

We end this section by once more recapitulating the sequence of matrix deformations used:

$$M(discrete) \to M(continuous) \to Q \to Z \to Y \to W \to O. \tag{35}$$

The first problem in the sequence, for M (in its discrete version), is equivalent to the inverse scattering problem for NLS. The last Riemann-Hilbert problem, for O, (26), was solved explicitly via theta functions. Since, as is seen, the solution O of the outer problem is uniformly bounded in x, t, with L^2 limits O_+, O_-, as $h \to 0$, so are W and Y and the (asymptotically valid) transformations from Y to O and back are a posteriori justified.

APPENDIX 1. THE CROSS PROBLEM.

The transformation from the Riemann-Hilbert problem (23) to the problem (24)-(25) requires two steps. First, the deletion of the lens contours away from the points a_l, b_l. This is immediate because the jump matrices are uniformly exponentially small perturbations of the identity.

Second, one needs to consider the small remaining crosses centered at the points a_l, b_l (see the remark of section 5). Since the jump matrices are not uniformly small there, one needs to find "local" parametrices. In other words, one needs to solve the local Riemann-Hilbert problems.

For example, after translation all problems centered at a_l look as follows: to find a matrix L which is analytic in $\mathbb{C} \setminus \Gamma$ where Γ is the cross shown in Figure A.1, centered at 0. The actual angles between the four half-lines emanating from 0 are not important as long as every half-line is in a different quadrant.

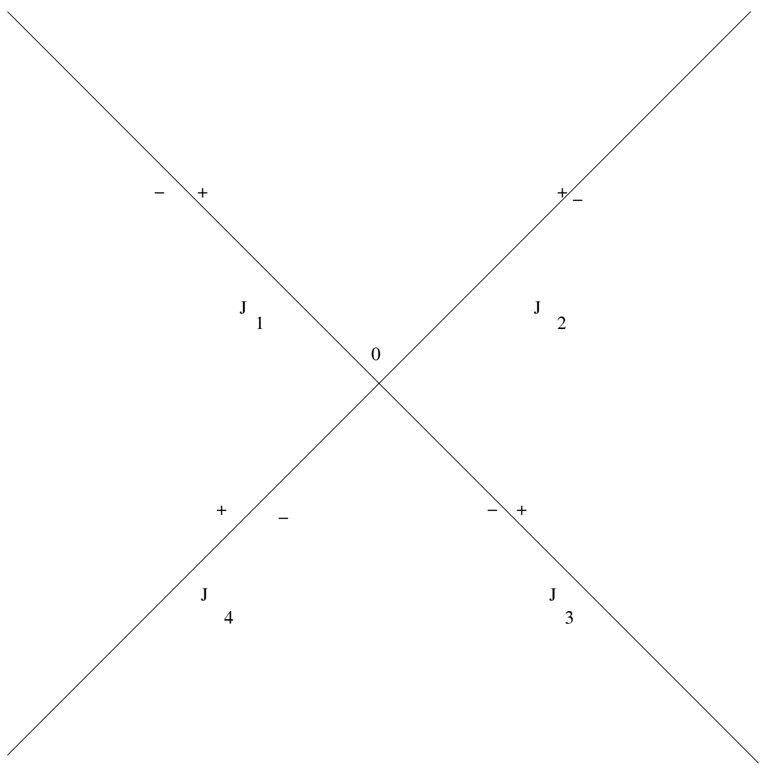

Fig.A.1. The Riemann–Hilbert problem on a cross centered at zero.

The jumps for L are

$$
\begin{aligned}
&L_+ = L_- v_L(\lambda),\ \textit{where}\\
&v_L(\lambda) = \begin{pmatrix} 1 & d_l^2 \frac{r_l}{1+|r_l|^2}\lambda^{2\nu_l i} e^{\frac{2g_l - 2i\lambda x - 2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in J_1,\\
&v_L(\lambda) = \begin{pmatrix} 1 & 0 \\ d_l^{-2} \frac{r_l^*}{1+|r_l|^2}\lambda^{-2\nu_l i} e^{\frac{-2g_l + 2i\lambda x + 2i\lambda^2 t}{h}} & 1 \end{pmatrix},\ for\ \lambda \in J_2,\\
&v_L(\lambda) = \begin{pmatrix} 1 & -d_l^2 r_l \lambda^{2\nu_l i} e^{\frac{2g_l - 2i\lambda x - 2i\lambda^2 t}{h}} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in J_3,\\
&v_L(\lambda) = \begin{pmatrix} 1 & 0 \\ d_l^{-2} r_l^* \lambda^{-2\nu_l i} e^{\frac{-2g_l + 2i\lambda x + 2i\lambda^2 t}{h}} & 1 \end{pmatrix},\ for\ \lambda \in J_4.
\end{aligned} \tag{A.1}
$$

Also $lim_{\lambda\to\infty} L(\lambda) = I$.

Here

$$\nu_l = \frac{1}{2\pi} log(1 + |r(a_l)|^2),$$

$$g_l = g(\lambda = a_l), r_l = r(a_l)$$

and d_l is defined by the local behavior of $\tilde{d}$ near a_l (see (21b)).

After a conjugation of the jump matrix by $diag(d_l e^{\frac{g_l + i\frac{x^2}{4t}}{h}}, d_l^{-1} e^{\frac{-g_l - i\frac{x^2}{4t}}{h}})$, a further translation $\lambda \to \lambda - x/2$ (to complete the square) and a rescaling $\lambda \to \lambda(\frac{t}{h})^{1/2}$, we end up with a problem with jumps on a rescaled cross which is no more small. Extending the cross to infinity (by setting the jump equal to the identity on the extension) we have a new Riemann-Hilbert problem which can be approximated by the following:

$$
\begin{aligned}
&\Psi_+ = \Psi_- v_\Psi, where\ v_\Psi(\lambda) =\\
&\begin{pmatrix} 1 & \frac{r_l}{1+|r_l|^2}\lambda^{2\nu_l i} e^{-2i\lambda^2} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in \tilde{J}_1,\\
&\begin{pmatrix} 1 & 0 \\ \frac{r_l^*}{1+|r_l|^2}\lambda^{-2\nu_l i} e^{2i\lambda^2} & 1 \end{pmatrix},\ for\ \lambda \in \tilde{J}_2,\\
&\begin{pmatrix} 1 & -r_l \lambda^{2\nu_l i} e^{-2i\lambda^2} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in \tilde{J}_3,\\
&\begin{pmatrix} 1 & 0 \\ r_l^* \lambda^{-2\nu_l i} e^{2i\lambda^2} & 1 \end{pmatrix},\ for\ \lambda \in \tilde{J}_4,\\
&lim_{\lambda\to\infty} S(\lambda) = I,
\end{aligned} \tag{A.2}
$$

where $\tilde{J}_i$ is the extension of $J_i,\ i = 1, 2, 3, 4$.

The last Riemann-Hilbert problem can be explicilty solved via parabolic cylinder functions (see e.g. [DZ]). Indeed, let $D_a(\lambda)$ denote the standard parabolic cylinder function. Then $D_a(e^{-3i\pi/4}\lambda)$ and $D_a(-e^{-3i\pi/4}\lambda)$ solve the ODE

$$\frac{d^2D}{d\lambda^2} + (\frac{1}{2} - \frac{\lambda^2}{4} + a)D = 0,$$

where $a = i\nu_l$. Let

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \tag{A.3}$$

where

$$\begin{aligned} S_{11} &= e^{-3\pi\nu_l/4} D_a(e^{-3i\pi/4}\lambda), \\ S_{12} &= \frac{r_l^*\Gamma(a)}{(2\pi)^{1/2}e^{-i\pi/4}} e^{3\pi\nu_l/4}[\frac{d}{d\lambda}D_{-a}(e^{-i\pi/4}\lambda) - \frac{i\lambda}{2}D_{-a}(e^{-i\pi/4}\lambda)], \\ S_{21} &= \frac{r_l\Gamma(-a)}{(2\pi)^{1/2}e^{i\pi/4}} e^{-\pi\nu_l/4}[\frac{d}{d\lambda}D_a(e^{-3i\pi/4}\lambda) + \frac{i\lambda}{2}D_a(e^{-3i\pi/4}\lambda)], \\ S_{22} &= e^{\pi\nu_l/4} D_a(e^{-i\pi/4}\lambda), \\ & for\ Im\lambda > 0, \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} S_{11} &= e^{\pi\nu_l/4} D_a(e^{i\pi/4}\lambda), \\ S_{12} &= \frac{r_l^*\Gamma(a)}{(2\pi)^{1/2}e^{-i\pi/4}} e^{-\pi\nu_l/4}[\frac{d}{d\lambda}D_{-a}(e^{3i\pi/4}\lambda) - \frac{i\lambda}{2}D_{-a}(e^{3i\pi/4}\lambda)], \\ S_{21} &= \frac{r_l\Gamma(-a)}{(2\pi)^{1/2}e^{i\pi/4}} e^{3\pi\nu_l/2}[\frac{d}{d\lambda}D_a(e^{i\pi/4}\lambda) + \frac{i\lambda}{2}D_a(e^{i\pi/4}\lambda)], \\ S_{22} &= e^{-3\pi\nu_l/4} D_a(e^{3i\pi/4}\lambda), \\ & for\ Im\lambda < 0. \end{aligned} \tag{A.5}$$

Then it is possible to check that

$$\begin{gathered} S_+ = S_- v_S, where\ S(\lambda) = \\ \begin{pmatrix} 1 & \frac{r_l}{1+|r_l|^2} \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in \tilde{J}_1, \\ \begin{pmatrix} 1 & 0 \\ \frac{r_l^*}{1+|r_l|^2} & 1 \end{pmatrix},\ for\ \lambda \in \tilde{J}_2, \\ \begin{pmatrix} 1 & -r_l \\ 0 & 1 \end{pmatrix},\ for\ \lambda \in \tilde{J}_3, \\ \begin{pmatrix} 1 & 0 \\ r_l^* & 1 \end{pmatrix},\ for\ \lambda \in \tilde{J}_4, \\ lim_{\lambda\to\infty} S(\lambda) = I. \end{gathered} \tag{A.6}$$

Now, setting

$$S = \Psi \lambda^{i\nu_l \sigma_3} e^{-i\lambda^2/4\ \sigma_3}. \tag{A.7}$$

and using the well known asymptotics for the parabolic cylinder function at infinity

$$\begin{gathered}
D_a(\lambda) = \lambda^a e^{-\lambda^2/4}(1+O(\lambda^{-2})),\ |arg\lambda| < \frac{3\pi}{4}, \\
D_a(\lambda) = \lambda^a e^{-\lambda^2/4}(1+O(\lambda^{-2})) - \frac{(2\pi)^{1/2}}{\Gamma(-a)} e^{i\pi a}\lambda^{-a-1}e^{\lambda^2/4}(1+O(\lambda^{-2})),\ \frac{\pi}{4} < arg\lambda < \frac{5\pi}{4}, \\
D_a(\lambda) = \lambda^a e^{-\lambda^2/4}(1+O(\lambda^{-2})) - \frac{(2\pi)^{1/2}}{\Gamma(-a)} e^{-i\pi a}\lambda^{-a-1}e^{\lambda^2/4}(1+O(\lambda^{-2})),\ \frac{-5\pi}{4} < arg\lambda < \frac{-\pi}{4},
\end{gathered} \tag{A.8}$$

it is immediate to check that Ψ solves (A.2).

It can be easily verified that the back-rescaled local version near a_l, b_l matches with the solution of the outer problem (as $h \to 0$). Thus the issue of the small crosses is settled.

APPENDIX 2. THE VARIATIONAL PROBLEM AND THE FINITE GENUS ANSATZ.

The function g defined by (18) and the conditions stated before (18) is crucial for the asymptotic analysis of the Riemann-Hilbert problem (11). As stated in [KMM] and [KR] the existence of such a function follows from the existence and regularity of a solution to a variational problem. In this section we pose the variational problem and we state the results of [KR] on existence. We also show that a variation of the proofs of [KR] guarantees the validity of the finite genus ansatz for the barrier data problem.

Let $\mathbb{H} = \{z : Imz > 0\}$, be the complex upper-half plane and $\bar{\mathbb{H}} = \{z : Imz \geq 0\} \cup \{\infty\}$ be the closure of $\mathbb{H}$. Let also $\mathbb{K} = \{z : Imz > 0\} \setminus \{z : Rez = 0, 0 < Imz \leq A\}$, where A is a positive constant. In the closure of this space, $\bar{\mathbb{K}}$, we consider the points ix_+ and ix_-, where $0 \leq x < A$ as distinct. In other words, we cut a slit in the upper half-plane along the segment $(0, iA)$ and distinguish between the two sides of the slit. The point infinity belongs to $\bar{\mathbb{K}}$, but not $\mathbb{K}$. Define $G(z; \eta)$

to be the Green's function for the upper half-plane

$$G(z;\eta) = log\frac{|z-\eta^*|}{|z-\eta|}$$

and let $d\mu^0(\eta)$ be the nonnegative measure $-\rho^0 d\eta$ on the segment $[0, iA]$ oriented from 0 to iA. Recall that ρ^0 is the denisty of eigenvalues given by (9a). The star denotes complex conjugation. Let the "external field" ϕ be defined by

$$\phi(z) = -\int G(z;\eta)d\mu^0(\eta) - Re(i\pi\int_z^{iA}\rho^0 d\eta + 2i(zx + z^2 t)), \tag{A.9}$$

where, without loss of generality $x > 0$.

Let $\mathbb{M}$ be the set of all positive Borel measures on $\bar{\mathbb{K}}$, such that both the free energy

$$E(\mu) = \int\int G(x,y)d\mu(x)d\mu(y), \ \ \mu \in \mathbb{M}$$

and $\int \phi d\mu$ are finite. Also, let

$$V^\mu(z) = \int G(z,x)d\mu(x), \ \ \mu \in \mathbb{M}.$$

be the Green's potential of the measure μ.

The weighted energy of the field ϕ is

$$E_\phi(\mu) = E(\mu) + 2\int \phi d\mu,$$

for any $\mu \in \mathbb{M}$.

Now, given any curve F in $\bar{\mathbb{K}}$, the equilibrium measure λ^F supported in F is defined by

$$E_\phi(\lambda^F) = min_{\mu\in M(F)} E_\phi(\mu),$$

where $M(F)$ is the set of measures in $\mathbb{M}$ which are supported in F, provided such a measure exists.

The finite gap ansatz is equivalent to the existence of a so-called S-curve joining the points 0_+ and 0_- and lying entirely in $\bar{\mathbb{K}}$. By S-curve we mean an oriented curve F such that the equilibrium measure λ^F exists, its support consists of a finite union of analytic arcs and at any interior point of $supp\mu$ the so called S-property is satisfied

$$\frac{d}{dn_+}(\phi + V^{\lambda^F}) = \frac{d}{dn_-}(\phi + V^{\lambda^F}), \tag{A.10}$$

In the next section we will see that there is a C such that

$$E_\phi(\lambda^C) = max_{contoursF} E_\phi(\lambda^F) = max_{contoursF} min_{\mu \in M(F)} E_\phi(\mu), \tag{A.11}$$

and that the existence of an S-curve follows from the existence of a contour C maximizing the equilibrium measure.

EXISTENCE THEOREM [KR]. For the external field given by (A.9), there exists a continuum $F \in \mathbb{F}$ such that the equilibrium measure λ^F exists and

$$E_\phi[F](= E_\phi(\lambda^F)) = max_{F \in \mathbb{F}} min_{\mu \in M(F)} E_\phi(\mu). \tag{A.12}$$

PROOF: See Theorem 4 in [KR].

For our particular problem we also have

REGULARITY THEOREM [KR]. The continuum F is in fact an S-curve.

PROOF: The proof follows as in [KR]. But there are some changes here. The density of eigenvalues given by (9a) does not satsify all conditions (1) set in [KR]. In particular it is not true that $Im[\rho^0(z)] > 0, \; for \; z \in (0, iA] \cup \mathbb{R}^+$. Rather $Im[\rho^0(z)] = 0$ on the real line. So a small amendment of the regularity proof is needed.

The point of the assumption that $Im[\rho^0(z)] > 0, \; for \; z \in (0, iA] \cup \mathbb{R}^+$ is to ensure that the continuum F does not touch the negative real line, except of course at 0_- and possibly infinity. In our case, we can argue as follows.

Note that the field ϕ is exactly zero on the real line. So any connected subset of the continuum F that belongs in the real line is automatically and trivially an S-curve. Because of the analyticity properties of the field, the real line can be divided into a finite number of intervals J_k such that in the interior of each J_k either

$$\frac{d\phi}{dImz} > 0,$$

or

$$\frac{d\phi}{dImz} \leq 0.$$

In the first case, one can see that the continuum F cannot touch the real line except of course at the endpoints of J_k. This is because for any configuration that involves a continuum including other points on the real line, we can find a configuration with no other points on the real line, by pushing measures up away from the real axis, which has greater (unweighted *and* weighted) energy. It is crucial here that if $u \in \mathbb{R}$ then $G(u,v) = 0$, while if both u, v are off the real line $G(u,v) > 0$.

In the second case, J_k is trivially an S-curve.

So F consists of a finite union of arcs: some of them are trivially S-curves and some of them do not touch the real line except at their endpoints, so small variations staying in the upper half-plane and keeping the endpoints fixed can be taken.

To pursue the proof of regularity one neeeds the following identity.

THEOREM [KR]. Let F be the maximizing continuum of and λ^F be the equilibrium measure. Let μ be the extension of λ^F to the lower complex plane via $\mu(z^*) = -\mu(z)$. Then

$$\begin{aligned} Re(\int \frac{d\mu(u)}{u-z} + V'(z))^2 = Re(V'(z))^2 - 2Re\int \frac{V'(z)-V'(u)}{z-u} d\mu(u) \\ +Re[\frac{1}{z^2}\int 2(u+z)V'(u)\ d\mu(u)]. \end{aligned} \tag{A.13}$$

Here V is the logarithmic potential of μ.

PROOF: By taking variations with respect to the equilibrium measure [KR].

From (A.13) it is easy to see that the support of the equilibrium measure of the maximizing continuum is characterized by

$$Re\int^z (R_\mu)^{1/2} dz = 0, \tag{A.14}$$

where

$$\begin{aligned} R_\mu(z) = (V'(z))^2 - 2\int_{supp\mu} \frac{V'(z)-V'(u)}{z-u} d\mu(u) \\ +\frac{1}{z^2}(\int_{supp\mu} 2(u+z)V'(u)\ d\mu(u)). \end{aligned}$$

Since $R_\mu(z)$ is a function analytic in $\mathbb{K}$, the locus defined by (A.14) is a union of arcs with endpoints at zeros of R_μ. Further analysis actually shows that R_μ has finitely many zeros.

(A.10) also follows easily from (A.13); see [KR]. Alternatively, see Chapter 8 of [KMM].

REMARK. As noted in Appendix A.3 of [KR], the finite gap ansatz must be interpreted in a way not made explicit in [KMM]. In Theorem 2, the contour of the Abel map should more appropriately be allowed to lie in the infinite-sheeted Riemann surface with branch points $\pm iA$, still joining points 0_+ and 0_-. For small t this extension is not necessary.

BIBLIOGRAPHY

[BK] J.C.Bronski, J.N.Kutz, Numerical Simulation of the Semi-Classical Limit of the Focusing Nonlinear Schrödinger Equation, Physics Letters A, v.254, no.6, 1999, pp.325-336.

[C] H.Ceniceros, A Semi-Implicit Moving Mesh Method for the Focusing Nonlinear Schrödinger Equation, Commun. Pure Appl. Anal. v.1, no. 1, 2002, pp.1–18.

[CT] H.Ceniceros, F.R.Tian, A Numerical Study of the Semi-Classical Limit of the Focusing Nonlinear Schrödinger Equation, Phys. Lett. A, v.306, no.1, 2002, pp.25–34.

[T] Fei-Ran Tian, private communication.

[DZ] P.Deift, X.Zhou, A Steepest Descent Method for Oscillatory Riemann-Hilbert Problems, Annals of Mathematics, 2nd Series, Vol. 137, No. 2., 1993, pp. 295-368.

[CK] A.Cohen, T.Kappeler, Solutions to the Cubic Schrödinger Equation by the Inverse Scattering Method, SIAM Journal of Mathematical Analysis, v.23, n.4, 1992, pp.900-922.

[KMM] S.Kamvissis, K.McLaughlin, P.Miller, Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation, nlin.SI/0012034, 2000; Annals of Mathematics Studies, v.154, Princeton University Press, 2003.

[KR] S.Kamvissis, E.Rakhmanov, Existence and Regularity for an Energy Maximization Problem in Two Dimensions, MPI-MIS 67/2004;

http://www.mis.mpg.de/preprints/2004/prepr2004_67.html

[MK] P.Miller, S.Kamvissis, On the Semiclassical Limit of the Focusing Nonlinear Schrdinger Equation, Phys. Lett. A, v.247, no. 1-2, 1998, pp.75–86.