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by

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Abstract

We consider a model introduced in [10] with two species (η and ξ) of particles, representing respectively malignant and normal cells. The basic motions of the η particles are independent random walks, scaled diffusively. The ξ particles move on a slower time scale and obey an exclusion rule among themselves and with the η particles. The competition between the two species is ruled by a coupled birth and death process. We prove convergence in the hydrodynamic limit to a system of two reaction diffusion equations with measure valued initial data.

1 Introduction

We are reexamining a model for competition between malignant and normal cells introduced in [10]. The main point of the model is that there are three natural time scales. [Unfortunately in the biological applications] the fastest one is the random walk of the malignant cells; the next one is the time scale of motion of normal cells, described by an exclusion process where normal cells cannot jump over sites occupied either by normal or by malignant cells (we will refer to this for brevity as a coupled exclusion). Finally, the slowest time scale, called macroscopic, is the one described by a birth-death process which models the competition between the two species.

Calculating the local invariant measures is crucial for the behavior of the system on the macroscopic time scale. For technical reasons in [10] a stirring process was introduced to replace the coupled exclusion, not really motivated by the model itself, and, under such an assumption, the local invariant measures were found to be products of Poisson and Bernoulli measures.

Here we deal with the original model, with coupled exclusion, where the global invariant measures are not known, even if births and deaths are neglected. However, at least heuristically, the fact that on the time scale of the exclusion process the random walks of the malignant cells have already averaged out their positions, should give the stirring with slowly varying coefficient process considered in [10]. But the usual methods of deriving hydrodynamic limits do not seem to apply here: the lack of knowledge of the invariant measures, even neglecting the birth-death process, seems to preclude the use of entropy methods. We refer to [9] for a general survey on entropy methods in hydrodynamic limits and to [11], [12] for specific applications to reaction-diffusion equations. On the other hand, the presence in the reaction terms of transcendental, non polynomial functions, makes it awkward an analysis of the BBGKY hierarchy of equations, as used in [4], [2] and [1].

So here we introduce a new method. The main point is that for our system, the energy estimate that holds for the deterministic limit, can also be obtained for mesoscopic averages of the stochastic system, obtaining H^2_1 a-priori bounds which allow to derive a substitute for the so called "two block estimate" of [8].

We also get bounds for the L^2 distance of our mesoscopic field variables from the deterministic solutions, derived by an explicit computation of the generator applied to the difference squared. To control the "most dangerous terms" we use homogenization techniques which play the role of the "one block estimates" in hydrodynamic limits, see again [8].

The different scales reflect also in the initial data. We suppose regularity of the malignant cells initial distribution on the macroscopic scale, while the normal cells distribution is only smooth on a smaller scale, related to the slower time scaling of their time evolution. This involves yet another homogenization process which leads to a limit coupled system of reaction diffusion equations with measure valued initial data for the normal cells.

In Section 2 we define the particles process, in Section 3 a discrete deterministic systems of reaction diffusion equations and in Section 4 mesoscopic variables. In Section 5 we discuss the choice of the initial state and in Section 6 we state our main results. In Section 7 we give a brief sketch of the proofs. In Section 8 we prove a priori bounds on exponential moments of the occupation variables and regularity in space of the mesoscopic fields. In Section 9 we prove regularity in time, while in Sections 10 and 11 we prove local ergodic theorems (one block estimates) for the normal and, respectively, malignant cells. We have shifted to some appendices the more computational parts of the proofs.

2 The particles model

We discretize the unit torus Ω of $\mathbb{R}^d$ by intersecting it with the lattice $\epsilon\mathbb{Z}^d$, $\epsilon^{-1} \in \mathbb{N}$, and denote by Ω_ϵ its image in stretched coordinates:

$$\Omega_\epsilon = \left\{x = (x_1, .., x_d) \in \mathbb{Z}^d : 1 \le x_i \le \epsilon^{-1},\ 1 \le i \le d\right\} \tag{2.1}$$

As a rule we will use x, y, z.. for lattice sites and r, r'... for points in $\mathbb{R}^d$.

Particle configurations are non negative integer valued functions on Ω_ϵ extended periodically to the whole $\mathbb{Z}^d$; in particular we consider configurations $\eta : \mathbb{Z}^d \to \mathbb{N}$ and $\xi : \mathbb{Z}^d \to \{0, 1\}$. $\eta(x)$ and $\xi(x)$, $x \in \mathbb{Z}^d$, are thus the number of η and ξ particles at any site y in Ω_ϵ equal to x modulo Ω_ϵ. η and ξ are interpreted as malignant and respectively normal cells.

The evolution is described by a Markov process whose generator L, defined on all functions of (η, ξ) and whose dependence on ϵ is not made explicit, has the expression

$$\begin{aligned} L &= L^{(\eta)} + L^{(\xi)} \\ L^{(\eta)} &= \epsilon^{-2} L^{(\eta,0)} + L^{(\eta,+)} + L^{(\eta,-)} \\ L^{(\xi)} &= \epsilon^{-2a} L^{(\xi,0)} + L^{(\xi,+)} + L^{(\xi,-)} \end{aligned} \tag{2.2}$$

with

$$L^{(\eta,0)} f(\eta,\xi) = \sum_{x\in\Omega_\epsilon} \sum_{e:|e|=1} \eta(x)[f(\eta^{x,x+e},\xi) - f(\eta,\xi)] \tag{2.3}$$

where, denoting by 1_x the configuration with only one particle at x, $\eta^{x,x+e} = \eta + 1_{x+e} - 1_x$, (same notation is used for ξ configurations).

$$L^{(\xi,0)} f(\eta,\xi) = \sum_{x\in\Omega_\epsilon} \sum_{e:|e|=1} \xi(x)[1-\xi(x+e)]\mathbf{1}_{\eta(x+e)=0}[f(\eta,\xi^{x,x+e}) - f(\eta,\xi)] \tag{2.4}$$

Denoting below by κ, κ' and κ_i positive coefficients, the latter non decreasing functions of i bounded by $\kappa_i \le ce^{bi}$, b and c positive constants, and writing $\eta^{x,\pm} = \eta \pm 1_x$, $\xi^{x,\pm} = \xi \pm 1_x$,

$$\begin{aligned} L^{(\eta,+)} f(\eta,\xi) &= \sum_{x\in\Omega_\epsilon} \kappa\, \eta(x)[f(\eta^{x,+},\xi) - f(\eta,\xi)] \\ L^{(\eta,-)} f(\eta,\xi) &= \sum_{x\in\Omega_\epsilon} \eta(x)\Big(\kappa_{\eta(x)}(1-\xi(x)) + \kappa_{\eta(x)+1}\xi(x)\Big)[f(\eta^{x,-},\xi) - f(\eta,\xi)] \end{aligned} \tag{2.5}$$

$$\begin{aligned} L^{(\xi,+)} f(\eta,\xi) &= \sum_{x\in\Omega_\epsilon} \frac{\kappa'}{2d} \sum_{e:|e|=1} \xi(x)[1-\xi(x+e)]\mathbf{1}_{\eta(x+e)=0}\,[f(\eta,\xi^{x+e,+}) - f(\eta,\xi)] \\ L^{(\xi,-)} f(\eta,\xi) &= \sum_{x\in\Omega_\epsilon} \xi(x)\kappa_{\eta(x)+1}[f(\eta,\xi^{x,-}) - f(\eta,\xi)] \end{aligned} \tag{2.6}$$

The indices $0,+,-$ above refer respectively to displacements, births and deaths of particles, whose species is then indicated by η and ξ. Notice that, as $\epsilon \to 0$, the scaling factors ϵ^{-2} and ϵ^{-2a} make displacements occur on a much faster scale than births and deaths with the η particles moving much faster than the ξ particles.

3 Discrete reaction-diffusion equations

The limit reaction diffusion equations that we will derive, can be approximated on the lattice Ω_ϵ, by the two equations

$$\frac{dU}{dt} = F(U,V), \quad \frac{dV}{dt} = G(U,V) \tag{3.1}$$

$(U(t),V(t)) = \{(U(x,t),V(x,t)),\, x\in\Omega_\epsilon,\, t\ge 0\}$ and where $F=F(U,V)$ and $G=G(U,V)$ are given by

$$F(U,V) = \epsilon^{-2}\Delta U + \kappa U - F_1^-(U) - F_2^-(U)V \tag{3.2}$$

$$G(U,V) = e^{-U}\epsilon^{-2a}\Delta V + \kappa' V(1-V)e^{-U} - VG^-(U) \tag{3.3}$$

with Δ the discrete Laplacian on $\mathbb{Z}^d$ (see (8.7), (5.3) and (8.8))

$$F_1^-(U) = e^{-U}\sum_{i\ge 0}\frac{U^i}{i!}\,i\,\kappa_i, \qquad F_2^-(U) = e^{-U}\sum_{i\ge 0}\frac{U^i}{i!}\,i\,[\kappa_{i+1}-\kappa_i]$$
$$G^-(U) = e^{-U}\sum_{i\ge 0}\frac{U^i}{i!}\kappa_{i+1} \tag{3.4}$$

By going to un-stretched coordinates, $x\to\epsilon x$, $\Omega_\epsilon\to\Omega\cap\epsilon\mathbb{Z}^d$ and $\epsilon^{-2}\Delta\to\Delta$ the discrete laplacian on $\epsilon\mathbb{Z}^d$, (3.1) becomes the ϵ-mesh discretization of the reaction diffusion system considered in [10].

Our aim is to prove closeness between (3.1) and the Markov process defined in the previous section. This will be achieved by introducing suitable functions $(u(x,\eta),v(x,\xi))$, $x\in\Omega_\epsilon$, (defined in terms of convolutions of the original η and ξ variables) and comparing their evolution with the orbits $(U(t),V(t))$ of (3.1). An important step in this direction will be a comparison between the generator L and the generator D associated to (3.1), D being the transport operator with domain the space of all functions $\psi(U,V)$ which are differentiable in all $U(x)$ and $V(x)$, and which is defined as

$$D = F\frac{\partial}{\partial U} + G\frac{\partial}{\partial V} \equiv \sum_{x\in\Omega_\epsilon} F(x)\frac{\partial}{\partial U(x)} + G(x)\frac{\partial}{\partial V(x)} \tag{3.5}$$

D transports along the orbits $(U(t),V(t))$ solutions of (3.1).

4 Mesoscopic variables

We denote by π_t the semigroup

$$\pi_t = e^{t\Delta} \tag{4.1}$$

with Δ the discrete laplacian on Ω_ϵ, so that the kernel $\pi_t(x,y)$ is the probability that a simple random walk which jumps with intensity $2d$ with equal probability on its n.n. sites, reaches y at time t having started from x at time 0.

With $\alpha<1$ and $\beta<a$ positive parameters whose value will be instead specified later, we shorthand

$$u(\eta) = p_\alpha * \eta, \quad v(\xi) = q_\beta * \xi, \quad p_\alpha = \pi_{\epsilon^{-2+2\alpha}},\ q_\beta = \pi_{\epsilon^{-2a+2\beta}} \tag{4.2}$$

and denote by $u(x,\eta)$ and $v(x,\xi)$ the values at x of u and v

We will use throughout the paper the notation:

$$\langle f,g\rangle = \epsilon^d \sum_{x\in\Omega_\epsilon} f(x)g(x), \quad \|f\|^2 = \langle f,f\rangle \tag{4.3}$$

The key for proving closeness between $(u(\eta_t),v(\xi_t))$ and $(U(t),V(t))$ will be a proof that L acts on functions of (u,v) approximately as D acts on functions of (U,V) (D is the transport generator defined in the previous section). We will make the statement quantitative, by studying the quantities

$$(L+D)\|u-U\|^2, \quad (L+D)\|v-V\|^2 \tag{4.4}$$

along the trajectories of the Markov process and the solutions of (3.1). Thus if (u,v) are close to (U,V) and L to D, then the quantities in (4.4) will be small. The converse is (to some extent) also true, as it follows from the martingale theory. Indeed the expressions in (4.4) are the two *compensators* in the martingale relations

$$\|u(\eta_t)-U(t)\|^2 - \|u(\eta_0)-U(0)\|^2 = \int_0^t (L+D)\|u(\eta_s)-U(s)\|^2 ds + M_1(t) \tag{4.5}$$

$$\|v(\xi_t)-V(t)\|^2 - \|v(\xi_0)-V(0)\|^2 = \int_0^t (L+D)\|v(\xi_s)-V(s)\|^2 ds + M_2(t) \tag{4.6}$$

where $(U(s),V(s))$ solves (3.1) and $M_1(t)$, $M_2(t)$ are suitable martingales.

We will prove bounds for the two integrals on the r.h.s. of (4.5) and (4.6) in terms of $\int_0^t \|u(\eta_s)-U(s)\|^2 + \|v(\xi_s)-V(s)\|^2$. We will also show that with probability going to 1 the martingales vanish in the limit as $\epsilon\to 0$. All that will allow to reach an integral inequality in closed form for $\|u(\eta_s)-U(s)\|^2 + \|v(\xi_s)-V(s)\|^2$ and to prove that the solution vanishes in the limit $\epsilon\to 0$.

Choice of α and β, assumptions on a.

We assume:

$\alpha < \dfrac{d}{d+2}$, (see Theorem 6).

$2a < (1-\alpha)d$, $2a<1$, $\beta < a\dfrac{d}{d+2}$ (see Theorem 7).

$\beta < \dfrac{5a}{17}$, (see Theorem 11).

5 Choice of the initial state

In this section we choose the initial law μ^ϵ of the process. It is now convenient to underline dependence on ϵ and we do that by adding a superscript ϵ when needed. The picture we have in mind is that after a finite time t_0, during which the η and the ξ particles move independently of each other, the interaction is suddenly switched on with the evolution ruled, after this time, by the generator L. We count times from when the interaction starts, setting this time equal to 0.

We suppose that the state at time $-t_0$ is a fixed (but ϵ dependent) configuration $(\eta^\epsilon_{-t_0}, \xi^\epsilon_{-t_0})$. Due to the general class of processes that we are considering, it is convenient to assume that

$$\sup_{\epsilon>0}\sup_{x\in\Omega_\epsilon} \eta^\epsilon_{-t_0}(x) = C < \infty \tag{5.1}$$

We do not want to make other assumptions on the configurations and we do not impose a particular behavior as $\epsilon \to 0$, which will instead determined by the same evolution. In this way our theory can cover many possible, interesting scenarios, for brevity we do not expand here the issue.

The probability μ^ϵ at time 0 is defined as the law at time t_0 of the free process which starts from $(\eta^\epsilon_{-t_0}, \xi^\epsilon_{-t_0})$ and has generator $\epsilon^{-2}L^{(\eta,0)} + \epsilon^{-2a}L^{(\xi,\mathrm{st})}$, where $L^{(\eta,0)}$ is defined in (2.2) while $L^{(\xi,\mathrm{st})}$ is the generator of the stirring process acting on the ξ variables. It is well established in the literature, see for instance [4], that μ^ϵ is a "local equilibrium measure", namely, to leading order in ϵ, it is locally close to a product measure with the $\eta(x)$ variables close to Poisson. Moreover, the averages of the η variables change smoothly on the scale ϵ^{-1}, while the scale of the ξ particles is ϵ^{-a}, see Theorem 1 below, where

$$U^\epsilon_0(x) = \mu^\epsilon(u^\epsilon(x,\eta)), \quad V^\epsilon_0(x) = \mu^\epsilon(v^\epsilon(x,\xi)) \tag{5.2}$$

and where ∇ is the lattice gradient:

$$e\cdot\nabla f(x) = f(x+e) - f(x), \quad |e| = 1 \tag{5.3}$$

The following theorem is proved in Appendix A for the η particles and in Appendix B for the ξ's.

Theorem 1. *μ^ϵ is such that*

$$\lim_{\epsilon\to 0}\mu^\epsilon\Big(\|u^\epsilon(\eta) - U^\epsilon_0\|^2 + \|v^\epsilon(\xi) - V^\epsilon_0\|^2\Big) = 0 \tag{5.4}$$

$$\sup_{\epsilon>0}\sup_{x\in\Omega_\epsilon}\Big(U^\epsilon_0(x) + \epsilon^{-1}|\nabla U^\epsilon_0(x)|\Big) < \infty, \quad \sup_{\epsilon>0}\sup_{x\in\Omega_\epsilon}\epsilon^{-a}|\nabla V^\epsilon_0(x)| < \infty \tag{5.5}$$

In the general setup we are considering there is no reason to expect convergence as $\epsilon \to 0$ of U^ϵ_0 and V^ϵ_0, but, by standard arguments, it is known that convergence is regained by going to subsequences:

Theorem 2. *For any sequence $\epsilon' \to 0$ there is a subsequence $\epsilon \to 0$ so that the following holds.*

- *There exists a bounded, Lipschitz function $U_0(r)$, $r \in \Omega$, such that*

$$\lim_{\epsilon \to 0} \sup_{x \in \Omega_\epsilon} |U_0^\epsilon(x) - U_0(\epsilon x)| = 0 \tag{5.6}$$

- *For each $r \in \Omega$, there is a translational invariant probability measure π_r on the space of $[0,1]$-valued functions V on $\mathbb{R}^d$ which are uniformly Lipschitz, such that for any positive integer n, any smooth function F on $\mathbb{R}^n$ and any test functions ϕ, ϕ_i, $i = 1, .., n$, $\int F(\dots, \phi_i * V, \dots)\, \pi_r(dV)$ is a measurable function of r and*

$$\lim_{\epsilon \to 0} \epsilon^d \sum_{x \in \Omega_\epsilon} \phi(\epsilon x) F\big(\dots, \epsilon^{ad} \sum_y \phi_i(\epsilon^a y) V_0^\epsilon(x+y), \dots\big)$$
$$= \int_\Omega \phi(r) \int F(\dots, \phi_i * V, \dots)\, \pi_r(dV) \tag{5.7}$$

6 Main results

We suppose the law μ^ϵ of (η_0, ξ_0) (at time 0) as specified in the previous section and denote by $P^\epsilon_{\mu^\epsilon}$ and $E^\epsilon_{\mu^\epsilon}$ law and expectation of the process $(\eta_t, \xi_t)_{t \geq 0}$ with generator L which starts from μ^ϵ. We will study the limit as $\epsilon \to 0$ of the process $(u^\epsilon(\eta_t), v^\epsilon(\xi_t)_{t \geq 0}$ along a subsequence convergent at time 0 in the sense of Theorem 2 ($\epsilon \to 0$ below, will always mean the limit along such a subsequence).

Theorem 3. *Let a, α and β verify the inequalities at the end of Section 4. Then there are a 2×2 matrix A with constant, positive entries and a two vector $R^\epsilon(t)$ whose positive components depend on $(\eta_s, \xi_s)_{s \leq t}$, so that, for any $t \geq 0$,*

$$\begin{pmatrix} \|u^\epsilon(\eta_t) - U^\epsilon(t)\|^2 \\ \|v^\epsilon(\xi_t) - V^\epsilon(t)\|^2 \end{pmatrix} \leq \begin{pmatrix} \|u^\epsilon(\eta_0) - U^\epsilon(0)\|^2 \\ \|v^\epsilon(\xi_0) - V^\epsilon(0)\|^2 \end{pmatrix} + \int_0^t A \begin{pmatrix} \|u^\epsilon(\eta_s) - U^\epsilon(s)\|^2 \\ \|v^\epsilon(\xi_s) - V^\epsilon(s)\|^2 \end{pmatrix} \, ds \; + \; R^\epsilon(t) \tag{6.1}$$

$$\lim_{\epsilon \to 0} E^\epsilon_{\mu^\epsilon} \left(\sup_{s \leq t} R^\epsilon(s) \right) = 0 \tag{6.2}$$

As a corollary of Theorems 1 and 3, there is $C > 0$ so that, for any $\delta > 0$ and any $T > 0$,

$$\lim_{\epsilon \to 0} P^\epsilon_{\mu^\epsilon} \left(\sup_{t \leq T} \{ \|u^\epsilon(\eta_t) - U^\epsilon(t)\|^2 + \|v^\epsilon(\xi_t) - V^\epsilon(t)\|^2 \} \leq e^{CT} \delta \right) = 1 \tag{6.3}$$

which proves that the random fields $(u^\epsilon(\eta_t), v^\epsilon(\xi_t))$ become deterministic as $\epsilon \to 0$ approaching the same limit behavior as $(U^\epsilon(t), V^\epsilon(t))$. The latter is described by the following, "two scale" reaction diffusion system.

The limit evolution. Let $U(r,t)$, $r \in \Omega$, $t \geq 0$, be a smooth, non negative, bounded function, $U(r,0) = U_0(r)$, U_0 as in (5.6). For any $r \in \Omega$, define the semigroup $T_{t;r}$ on the space of Lipschitz function $V(r')$ $r' \in \mathbb{R}^d$ with values in $[0,1]$, by setting $T_{t;r}(V)(r') = V(r',t)$ equal to the solution of

$$\frac{dV(r',t)}{dt} = e^{-U(r,t)}\Delta V(r',t) + \kappa' V(r',t)\big[1 - V(r',t)\big]e^{-U(r,t)} \\ -V(r',t)G^-(U(r,t)) \tag{6.4}$$

with $V(r',0) = V(r')$. The dependence of $T_{t;r}(V)$ on $U(r,t)$ has not been made explicit. Call

$$W(r,t) = \int T_{t;r}(V)(0)\ \pi_r(dV) \tag{6.5}$$

where $\{\pi_r(dV), r \in \Omega\}$ is the family of probabilities introduced in (5.7). Since $T_{t;r}(V)$ has values in $[0,1]$ and depends smoothly on V, the r.h.s. of (6.5) is well defined and $W(r,t)$ is a measurable function of r. Define $U^*(r,t)$, $r \in \Omega$, $t \geq 0$, as the solution starting from U_0 of

$$\frac{dU^*}{dt} = \Delta U^* + \kappa U^* - F_1^-(U^*) - F_2^-(U^*)W \tag{6.6}$$

We will say that $U(r,t)$ is a fixed point of the above scheme (or just a fixed point) if, for given U_0 and $\{\pi_r(dV), r \in \Omega\}$ as above, $U(r,t) = U^*(r,t)$ for all $r \in \Omega$ and $t \geq 0$. By standard arguments, details are omitted, we have:

Theorem 4. *There is a unique fixed point $U(r,t)$ of the above scheme. Moreover, if a, α and β are as in Theorem 3, then for any t and δ positive, any positive integer n, any bounded, smooth function f on $\mathbb{R}^n_+$ and any test functions $\phi, \phi_1, .., \phi_n$,*

$$\lim_{\epsilon \to 0} \epsilon^d \sum_{x \in \Omega_\epsilon} \phi(\epsilon x) U^\epsilon(x,t) = \int_\Omega \phi(r) U(r,t)\, dr \tag{6.7}$$

$$\lim_{\epsilon \to 0} \epsilon^d \sum_{x \in \Omega_\epsilon} \phi(\epsilon x) f\big(\dots, \epsilon^{ad} \sum_y \phi_i(\epsilon^a y) V^\epsilon(x+y,t), \dots\big) \\ = \int_\Omega \phi(r) \int f(\dots, \phi_i * V, \dots)\ \pi_{r,t}(dV) \tag{6.8}$$

7 Scheme of proof

By default, in the sequel a, α and β satisfy the inequalities stated at the end of Section 4. The proof of Theorem 3 is based on the analysis of the two martingale relations (4.5)-(4.6). The whole art of the matter is to prove that the "two compensators", namely the two integrals on the l.h.s. of (4.5)-(4.6), can be written as sum of terms which fall in the following two categories. The first one is made by elements which are bounded by time integrals of $\|u^\epsilon(\eta_t) - U^\epsilon(t)\|^2$ or $\|v^\epsilon(\eta_t) - V^\epsilon(t)\|^2$, multiplied by coefficients which are uniformly bounded. These terms then contribute to the integral in (6.1). All the other terms must be proved to vanish as $\epsilon \to 0$, so that they contribute to the error term $R^\epsilon(t)$ in (6.1).

A common feature to the analysis of all terms, is to control the large values of the variables η. A-priori L^∞ bounds are derived in Section 8, where we show uniform in ϵ integrability of exponential moments $e^{b\eta_t(x)}$, for any $b > 0$. The result is quite standard as the process can be stochastically bounded by one having only linear births. More subtle is another bound that we use extensively in the proofs, namely that the probability that $u^\epsilon(x, \eta_t)$ exceeds a suitably large value $M = M(t)$ (but independent of ϵ) vanishes as $\epsilon \to 0$. The proof of these statements is given in Appendix A, where we also recall from the literature results on independent random walks and random walks with independent branchings.

The other general ingredient, common to many of the proofs, is regularity in space of u^ϵ and v^ϵ. In Section 8 we prove H_1^2 bounds uniform in ϵ which are obtained by mimicking the PDE proofs for the limit equations. Besides regularity in space we need also regularity in time of u^ϵ. A result, maybe not optimal, but good enough for our applications, is proved in Section 9.

Such regularity estimates are the main subroutines we use to bound the "two compensators" (4.5) and (4.6). The detailed classification of all the terms which appear when computing explicitly the two compensators is reported in Appendix C. This is just some simple, but lengthy algebra, not at all deep, but necessary for the proof of Theorem 3, the compromise was to shift the computations to an appendix. Most of the terms in this expansion can be directly bounded using the boundedness and regularity estimates mentioned above, the bounds being uniform in ϵ and over compact time intervals. There are however some terms which do not fit in such an "easy class". The origin of the problem is the typical one in the derivation of non linear hydrodynamical equations, where one needs to identify non linear microscopic obervables in terms of the parameters of the limit equation: in our case we find local functions of η_t and of ξ_t and we need to express them in terms of (generally different) functions of $u^\epsilon(\eta_t)$ and $v^\epsilon(\xi_t)$, (which is easy if the functions are linear). The crucial point is that these non linear terms appear in the form of time and space averages and we will solve the problem by proving local ergodic properties of the process, reminiscent of the well known "one block estimates" in the theory of hydrodynamic limits. The "two block estimates" are here replaced by the H_1^2 regularity already mentioned. The one block estimates are not proved using Dirichlet forms, but closeness of the process in short times to a process with no deaths and births. The main difficulty here is that the ξ process reminds of but it is not the stirring process, because the ξ particles are allowed to jump only on sites where no η particles are present. The local ergodic averages for the η particles are easier to study, their analysis is reported in Section 10. The result for the ξ particles is instead given in Section 11. The core of the proof is to show a homogenization property for which the ξ particles move feeling to main order only the empirical average of the η's. The real difficulty is to prove that such a property extends to such long times for the stirring to reach local equilibrium.

8 Boundedness and regularity in space

In this section we will prove L^∞ and H_1^2 a priori bounds which will be extensively used in the proofs of Theorem 3. We start from the former, which are uniform bounds on the expectations of η_t and $u^\epsilon(\eta_t)$:

Theorem 5. *For any $b > 0$ and any $t \ge 0$*

$$\sup_{\epsilon>0} \sup_{x\in\Omega_\epsilon} E^\epsilon_{\mu^\epsilon}\big(e^{b\eta_t(x)}\big) \le \exp\{[e^b - 1]e^{\kappa t}C\} \tag{8.1}$$

with κ as in Section 2 and C as in (5.1)*. Moreover, for any $\tau > 0$ there are M and c so that*

$$\sup_{t\le\tau} \sup_{x\in\Omega_\epsilon} P^\epsilon_{\mu^\epsilon}\big(u^\epsilon(x,\eta_t) \ge M\big) \le c\epsilon^{(1-\alpha)d} \tag{8.2}$$

Proof. Let $(\eta^+_t)_{t\ge0}$ be the process with generator $\epsilon^{-2}L^{(\eta,0)} + L^{(\eta,+)}$ which starts from μ^ϵ. Then there is a coupling of $(\eta^+_t)_{t\ge0}$ with the original process $(\eta_t,\xi_t)_{t\ge0}$ (defined by the generator L of (2.2)) such that $\eta^+_0 = \eta_0$ and $\eta^+_t \ge \eta_t$ for all $t > 0$. By an abuse of notation we still denote by $P^\epsilon_{\mu^\epsilon}$ and $E^\epsilon_{\mu^\epsilon}$, law and expectation w.r.t. the coupled process. In Appendix A it is proved that

$$\sup_{\epsilon>0} \sup_{x\in\Omega_\epsilon} E^\epsilon_{\mu^\epsilon}\big(e^{b\eta^+_t(x)}\big) \le \exp\{[e^b - 1]e^{\kappa t}C\} \tag{8.3}$$

$$\sup_{t\le\tau} \sup_{x\in\Omega_\epsilon} E^\epsilon_{\mu^\epsilon}\Big(\big[u^\epsilon(x,\eta^+_t) - E^\epsilon_{\mu^\epsilon}\big(u^\epsilon(x,\eta^+_t)\big)\big]^2\Big) \le c\epsilon^{(1-\alpha)d} \tag{8.4}$$

Since $e^{b\eta_t(x)} \le e^{b\eta^+_t(x)}$, (8.1) follows from (8.3).

Let $M = 2e^{\kappa\tau}C$, then, since $E^\epsilon_{\mu^\epsilon}\big(u^\epsilon(x,\eta^+_t)\big) \le e^{\kappa t}C$,

$$P^\epsilon_{\mu^\epsilon}\big(u^\epsilon(x,\eta^+_t) \ge M\big) \le P^\epsilon_{\mu^\epsilon}\big(|u^\epsilon(x,\eta_t) - E^\epsilon_{\mu^\epsilon}\big(u^\epsilon(x,\eta^+_t)\big)| \ge e^{\kappa\tau}C\big)$$

so that

$$P^\epsilon_{\mu^\epsilon}\big(u^\epsilon(x,\eta^+_t) \ge M\big) \le \frac{4c}{M^2}\epsilon^{(1-\alpha)d} \tag{8.5}$$

Since $u^\epsilon(x,\eta_t) \le u^\epsilon(x,\eta_t^+)$, (8.2) follows from (8.5). ■

We will next prove bounds on the H_1^2 norm of u^ϵ, which will play the role of the "two blocks estimates" in the language of hydrodynamic limit theory. Before stating definition and results, let us recall how similar bounds are obtained for the heat equation $u_t = \Delta u$ in the unit torus Ω. The "entropy" $\displaystyle\int_\Omega u^2\,dr$ gives

$$\int_\Omega u^2(t)\,dr - \int_\Omega u^2(0)\,dr = -2\int_0^t \int_\Omega |\nabla u|^2 dr$$

Then, supposing $\displaystyle\int_\Omega u^2(0)\,dr < \infty$, for any $t>0$,

$$\int_0^t \int_\Omega |\nabla u|^2 dr < \frac{1}{2}\int_\Omega u^2(0)\,dr$$

The proof of Theorem 6 below mimics the above argument, but let us first introduce some notation and definitions which translate to the lattice the analogous notions in the continuum. If f is a function on Ω_ϵ, we write

$$\|f\|^2_{H_1^2} := \epsilon^{-2}\|\nabla f\|^2 \tag{8.6}$$

with ∇f the lattice gradient of f. We also recall that the same rules as in the continuum hold as well for the discrete gradient and laplacian. Namely, denoting by E_+ the set of unit vectors e with positive components, we have, recalling (5.3) for notation and resisting to the temptation of writing $(-e)\cdot\nabla f = -(e\cdot\nabla f)$, which is false,

$$\Delta f(x) = \sum_{e\in E_+} [(-e)\cdot\nabla + e\cdot\nabla] f(x) = \sum_{e\in E_+} \{(-e)\cdot\nabla\}\{(e\cdot\nabla)\} f(x) \tag{8.7}$$

$$< g, e\cdot\nabla f > = < (-e)\cdot\nabla g, f >, \qquad < g, \Delta f > = - < \nabla g, \nabla f > \tag{8.8}$$

the last equality following from the second one in (8.7) and the first one in (8.8).

Theorem 6. *For any $t>0$, there is c so that*

$$\sup_{\epsilon>0} E^\epsilon_{\mu^\epsilon}\Big(\int_0^t \|u^\epsilon(\eta_s)\|^2_{H_1^2}\,ds\Big) \le c \tag{8.9}$$

Proof. We start from the martingale relation:

$$\|u^\epsilon(\eta_t)\|^2 = \|u^\epsilon(\eta_0)\|^2 + \int_0^t L^{(\eta)}\|u^\epsilon(\eta_s)\|^2 ds + M_t^\epsilon, \quad E^\epsilon_{\mu^\epsilon}\big(M_t^\epsilon\big) = 0 \tag{8.10}$$

After a simple computation which exploits that discrete gradient and laplacian satisfy the same relations as in the continuum, see (8.7)-(8.8),

$$\epsilon^{-2} L^{(\eta,0)} \|u^\epsilon\|^2 = -2\|u^\epsilon\|^2_{H^2_1} + R^\epsilon_1(\eta) \tag{8.11}$$

$$R^\epsilon_1(\eta) = 2\epsilon^d \sum_{x,z} \epsilon^{-2} |\nabla p_\alpha(x,z)|^2 \eta(z) \tag{8.12}$$

$$L^{(\eta,+)} \|u^\epsilon\|^2 = 2\kappa \|u^\epsilon\|^2 + R^\epsilon_2(\eta), \quad L^{(\eta,-)} \|u^\epsilon\|^2 \le R^\epsilon_5(\eta,\xi) \tag{8.13}$$

$$R^\epsilon_2(\eta) = \kappa \epsilon^d \sum_x \sum_z p_\alpha(x,z)^2 \eta(z) \tag{8.14}$$

$$R^\epsilon_5(\eta,\xi) = \epsilon^d \sum_x \sum_z p_\alpha(x,z)^2 \big[\eta(z)\kappa_{\eta(z)} + \xi(z)(\kappa_{\eta(z)+1} - \kappa_{\eta(z)})\big] \tag{8.15}$$

(the seemingly random labelling of the remainder terms R^ϵ_i is for "historical reasons"). By taking the expectation in (8.10),

$$\begin{aligned} 2E^\epsilon_{\mu^\epsilon}\Big(\int_0^t \|u^\epsilon(\eta_s)\|^2_{H^2_1}\, ds\Big) &\le E^\epsilon_{\mu^\epsilon}\big(\|u^\epsilon(\eta_0)\|^2\big) \\ &+ E^\epsilon_{\mu^\epsilon}\Big(\int_0^t R^\epsilon_1(\eta_s) + R^\epsilon_2(\eta_s) + R^\epsilon_5(\eta_s,\xi_s) + 2\kappa\|u^\epsilon(\eta_s)\|^2\, ds\Big) \end{aligned} \tag{8.16}$$

By (8.1) there is $c_1 = c_1(t)$, independent of ϵ, so that the r.h.s. of (8.16) is bounded by $c_1(1 + \sum_z [\epsilon^{-2}|\nabla p_\alpha(0,z)|^2 + p_\alpha(0,z)^2])$. There is c_2 so that,

$$\sum_z \epsilon^{-2} |\nabla p_\alpha(z)|^2 \le c_2 \epsilon^{-2+2(1-\alpha)+d(1-\alpha)} \qquad \sum_z p_\alpha(z)^2 \le c_3 \epsilon^{d(1-\alpha)} \tag{8.17}$$

which vanishes as $\epsilon \to 0$ because of the assumption $\alpha < \dfrac{d}{d+2}$ and (8.9) is proved. ∎

We have a H^2_1 bound for v^ϵ as well, see Theorem 7 below, but we need first the following corollary of Theorem 6:

Corollary 1. *For any $z \in \Omega_\epsilon$ and $t > 0$*

$$\int_0^t \epsilon^d \sum_{x\in\Omega_\epsilon} |u^\epsilon(x+z,\eta_s) - u^\epsilon(x,\eta_s)| \le \sqrt{td}\,\epsilon\,|z| \Big(\int_0^t \|u^\epsilon(\eta_s)\|^2_{H^2_1}\Big)^{1/2} \tag{8.18}$$

Proof. For any z, there is "a coordinate curve" $\{y_i\}_{i=0,..,N}$ such that $y_0 = 0$, $y_N = z$, with $e_i := y_{i+1} - y_i$, $i = 0, .., N-1$, a unit vector, and $\sum |e_i| = |z_1| + \cdots + |z_d|$, having denoted by z_i the i-th component of z. Then, recalling (5.3) for notation,

$$u^\epsilon(x+z,\eta) - u^\epsilon(x,\eta) = \sum_i e_i \cdot \nabla u^\epsilon(x+y_i,\eta)$$

Hence

$$\begin{aligned} \epsilon^d \sum_{x\in\Omega_\epsilon} |u^\epsilon(x+z,\eta) - u^\epsilon(x,\eta)| &\le \sum_i \epsilon^d \sum_{x\in\Omega_\epsilon} |e_i \cdot \nabla u^\epsilon(x+y_i,\eta)| \\ &\le \sqrt{d}\,\Big(\sum_i \epsilon^d \sum_{x\in\Omega_\epsilon} |e_i\cdot\nabla u^\epsilon(x+y_i,\eta)|^2\Big)^{1/2} \\ &\le \sqrt{d}\;\epsilon|z|\;\|u^\epsilon(\eta_s)\|_{H_1^2} \end{aligned} \tag{8.19}$$

and (8.18) follows by Cauchy-Schwartz. ■

Theorem 7. *For any $t > 0$ there is c so that*

$$\sup_{\epsilon>0} E^\epsilon_{\mu^\epsilon}\Big(\int_0^t \epsilon^{-2a}\|\nabla v^\epsilon(\eta_s)\|^2\,ds\Big) \le c \tag{8.20}$$

Proof. We start once again from a martingale relation:

$$\|v^\epsilon(\xi_t)\|^2 = \|v^\epsilon(\xi_0)\|^2 + \int_0^t L^{(\xi)}\|v^\epsilon(\xi_s)\|^2 ds + M_t^\epsilon, \quad E^\epsilon_{\mu^\epsilon}\big(M_t^\epsilon\big) = 0 \tag{8.21}$$

We use the following two identities (where E_+ is the set of unit vectors in $\mathbb{Z}^d$ with non negative components)

$$\sum_z \xi(z)\xi(z+e)[(-e)\cdot\nabla + e\cdot\nabla]q_\beta(z-x) = 0, \qquad e \in E_+ \tag{8.22}$$

$$\sum_{e\in E_+}\sum_z \xi(z)[(-e)\cdot\nabla + e\cdot\nabla]q_\beta(z-x) = \Delta v^\epsilon(x) \tag{8.23}$$

to get

$$\epsilon^{-2a}L^{(\xi,0)}\|v^\epsilon\|^2 = 2\epsilon^{-2a}\langle v^\epsilon, e^{-u^\epsilon}\Delta v^\epsilon\rangle + 2\mathcal{S}^\epsilon + 2\mathcal{C}^\epsilon + R_4^\epsilon \tag{8.24}$$

where

$$\begin{aligned} \mathcal{S}^\epsilon(\xi,\eta) = \sum_{e:|e|=1}\epsilon^d\sum_x v^\epsilon(x)\epsilon^{-2a}\sum_z[e\cdot\nabla q_\beta(z-x)]\xi(z)(1-\xi(z+e)) \\ \times[\mathbf{1}_{\eta(z+e)=0} - e^{-u^\epsilon(z+e)}] \end{aligned} \tag{8.25}$$

$$\mathcal{C}^\epsilon(\xi,\eta) = \epsilon^{-2a} \sum_{e:|e|=1} \epsilon^d \sum_x \sum_z \xi(z)(1-\xi(z+e))\big[e^{-u^\epsilon(z+e)} - e^{-u^\epsilon(x)}\big] e\cdot\nabla q_\beta(z-x) \tag{8.26}$$

$$R_4^\epsilon(\xi,\eta) = \epsilon^{-2a} \sum_{e:|e|=1} \epsilon^d \sum_x \sum_z \xi(z)(1-\xi(z+e))\mathbf{1}_{\eta(z+e)=0}\big(e\cdot\nabla q_\beta(z-x)\big)^2 \tag{8.27}$$

The proof that

$$\lim_{\epsilon\to 0} \Big| \int_0^t E^\epsilon_{\mu^\epsilon}\big(\mathcal{S}^\epsilon(\xi_s,\eta_s)ds\big)\Big| = 0 \tag{8.28}$$

follows from Theorem 9 below (details are left to an appendix, Appendix D). We will use the present theorem only after Theorem 9, so that there is no circularity in our arguments.

From Corollary 1, it follows that

$$\Big|\int_0^t E^\epsilon_{\mu^\epsilon}\big(\mathcal{C}^\epsilon(\xi_s,\eta_s)\big)ds\Big| \le c\epsilon^{1-2a} \tag{8.29}$$

Similarly to (8.17), for a suitable constant c, we have that

$$|R_4^\epsilon(\xi,\eta)| \le c\epsilon^{-2a+2(a-\beta)+d(a-\beta)} \tag{8.30}$$

which also vanishes in the limit $\epsilon\to 0$ because of the assumption $\beta < \dfrac{d}{d+2}\,a$. We denote by $\chi_{M,x}(s)$ the characteristic function of $u^\epsilon(x,\eta_s)\le M$, and we write

$$2\epsilon^{-2a}\langle v^\epsilon, e^{-u^\epsilon}\Delta v^\epsilon\rangle \le -2\epsilon^{-2a}e^{-M}\|\nabla v^\epsilon\|^2 + \mathcal{R}_5^\epsilon + \mathcal{R}_6^\epsilon \tag{8.31}$$

where

$$\mathcal{R}_5^\epsilon = -2\epsilon^{-2a}\langle v^\epsilon, \nabla e^{-u^\epsilon}\nabla v^\epsilon\rangle, \qquad \mathcal{R}_6^\epsilon = -2\epsilon^{-2a}\langle [1-\chi_M]\nabla v^\epsilon, e^{-u^\epsilon}\nabla v^\epsilon\rangle \tag{8.32}$$

Since $|\nabla e^{-f}|\le|\nabla f|$ if $f\ge 0$, by (8.9), and since $|v^\epsilon|\le 1$, $|\nabla v^\epsilon|\le 2d$,

$$\int_0^t |\mathcal{R}_5^\epsilon| \le c\epsilon^{1-2a} \tag{8.33}$$

By (8.2), we have

$$\sup_{s\le t} E^\epsilon_{\mu^\epsilon}\big(|\mathcal{R}_6^\epsilon(\xi_s,\eta_s)|\big) \le c\epsilon^{d(1-\alpha)-2a} \tag{8.34}$$

We next observe that,

$$L^{(\xi,+)}\|v^\epsilon\|^2 \le 2\kappa'\|v^\epsilon\|^2 + R_8^\epsilon, \qquad L^{(\xi,-)}\|v^\epsilon\|^2 \le R_9^\epsilon \tag{8.35}$$

where

$$R_8^\epsilon = \kappa'\epsilon^d \sum_x \frac{1}{2d}\sum_{e:|e|=1}\sum_z q_\beta(x,z)^2\xi(z)[1-\xi(z+e)]\mathbf{1}_{\eta(z+e)=0} \tag{8.36}$$

$$R_9^\epsilon = \epsilon^d \sum_x \sum_z q_\beta(x,z)^2 \xi(z) \kappa_{\eta(z)+1} \tag{8.37}$$

Since

$$\sum_z q_\beta(0,z)^2 \le c\epsilon^{d(a-\beta)} \tag{8.38}$$

and since $\kappa_n \le ce^{bn}$, by (8.1), also R_8 and R_9 give a vanishing contribution. We have therefore proved that there is a positive function $\varphi_\epsilon(t) \to 0$ as $\epsilon \to 0$ such that

$$\begin{aligned} \int_0^t E^\epsilon_{\mu^\epsilon}\big(\epsilon^{-2a} L^{(\xi)} \|v^\epsilon(\xi_s)\|^2\big) ds \le -2\epsilon^{-2a} e^{-M} \int_0^t E^\epsilon_{\mu^\epsilon}\big(\|\nabla v^\epsilon(\xi_s)\|^2 \\ +2\kappa' \int_0^t \|v^\epsilon(\xi_s)\|^2\big) ds + \varphi_\epsilon(t) \end{aligned} \tag{8.39}$$

Thus from (8.21) and (8.39) we get

$$e^{-M} E^\epsilon_{\mu^\epsilon}\Big(\int_0^t \epsilon^{-2a} \|\nabla v^\epsilon(\xi_s)\|^2 ds\Big) \le E^\epsilon_{\mu^\epsilon}\big(\|v^\epsilon(\xi_0)\|^2\big) + \kappa' t + \varphi_\epsilon(t) \tag{8.40}$$

Theorem 7 is proved. ∎

We conclude the section with the following corollary of Theorem 6.

Corollary 2. *For any $t > 0$ there is $\bar C$ so that*

$$\sup_{\epsilon>0} \sup_{x\in\Omega_\epsilon} \sup_{s\le t} |U^\epsilon(x,s)| \le \bar C \tag{8.41}$$

Furthermore for any $t>0$ there is c so that for any $s \le t$

$$\begin{aligned} \langle [v^\epsilon(\xi_s) - V^\epsilon(s)], [e^{-u^\epsilon(\eta_s)} - e^{-U^\epsilon(s)}]\epsilon^{-2a} \Delta V^\epsilon(s)\rangle \le \frac{e^{-\bar C}}{10} \epsilon^{-2a} \|\nabla\{v^\epsilon(\xi_s) - V^\epsilon(s)\}\|^2 \\ +c\Big(\|u^\epsilon(\eta_s) - U^\epsilon(s)\|^2 + \|v^\epsilon(\xi_s) - V^\epsilon(s)\|^2 \\ +\epsilon^{1-2a}\big\{\|u^\epsilon(\eta_s)\|^2_{H^2_1} + \|U^\epsilon(s)\|^2_{H^2_1}\big\}\Big) \end{aligned} \tag{8.42}$$

Proof. (8.41) follows from the first inequality in (5.5). From the second inequality in (5.5) it follows that the solution $V^\epsilon(x,t)$ of (3.1) is such that for any t there is C so that

$$\sup_{\epsilon>0} \sup_{x\in\Omega_\epsilon, s\le t} \epsilon^{-a} |\nabla V^\epsilon(x,s)| = C < \infty \tag{8.43}$$

Then, by (8.8)

$$\begin{aligned} \langle [v^\epsilon - V^\epsilon], [e^{-u^\epsilon} - e^{-U^\epsilon}]\epsilon^{-2a} \Delta V^\epsilon\rangle \le C\Big(\langle \epsilon^{-a} |\nabla\{v^\epsilon - V^\epsilon\}|, |e^{-u^\epsilon} - e^{-U^\epsilon}|\rangle \\ +\langle |v^\epsilon - V^\epsilon|, \epsilon^{-a} |\nabla\{e^{-u^\epsilon} - e^{-U^\epsilon}\}|\rangle\Big) \end{aligned}$$

Since $|e^{-u^\epsilon}-e^{-U^\epsilon}|\le|u^\epsilon-U^\epsilon|$, using the inequality $2|db|\le\delta d^2+\delta^{-1}b^2$, with $b=|u^\epsilon-U^\epsilon|$, $d=\epsilon^{-a}|\nabla\{v^\epsilon-V^\epsilon\}|$, and $\delta=e^{-\bar C}(5C)^{-1}$, we get

$$\langle[v^\epsilon-V^\epsilon],[e^{-u^\epsilon}-e^{-U^\epsilon}]\epsilon^{-2a}\Delta V^\epsilon\rangle\le e^{-\bar C}\frac{\epsilon^{-2a}}{10}\|\nabla\{v^\epsilon-V^\epsilon\}\|^2+e^{\bar C}(5C^2)\|u^\epsilon-U^\epsilon\|^2$$
$$+C\Big(\|v^\epsilon-V^\epsilon\|^2+\frac{\epsilon^{-2a}}{2}\|\nabla u^\epsilon\|^2+\frac{\epsilon^{-2a}}{2}\|\nabla U^\epsilon\|^2\Big)$$

having also used that $|\nabla e^{-f}|\le|\nabla f|$ if $f\ge 0$. Recalling the definition of $\|f\|_{H_1^2}$ in (8.20) we then get (8.42). ■

9 Regularity in time

Besides regularity in space, we will also need estimates on the regularity of $u^\epsilon(x,\eta_t)$ as a function of the time t. This is needed in Section 11 and Appendix D, and the statement we will prove here is just what used in the sequel, with no aim at generality. We denote by $P^\epsilon_{\eta_t,\xi_t}$ and $E^\epsilon_{\eta_t,\xi_t}$, conditional law and expectation of the process $(\eta_s,\xi_s)_{s\ge t}$, given the state (η_t,ξ_t) at time t.

Theorem 8. *For any τ and γ positive, with $2\gamma<(1-\alpha)d$, there is c so that for any $t\le\tau$, $\epsilon>0$, $x\in\Omega_\epsilon$ and any $s\in(t,\epsilon^{2\gamma}]$*

$$E^\epsilon_{\mu^\epsilon}\Big(\Big|u^\epsilon(x,\eta_s)-\pi_{(s-t)\epsilon^{-2}}*u^\epsilon(x,\eta_t)\Big|\Big)\le c\epsilon^{2\gamma}\tag{9.1}$$

where $\pi_{(s-t)\epsilon^{-2}}*u^\epsilon(x,\eta_t)=\sum_y\pi_{(s-t)\epsilon^{-2}}(x,y)u^\epsilon(y,\eta_t)$.

Proof. We define an auxiliary process $(\theta_s,\xi_s)_{s\ge t}$, $\theta_s=(\eta_s^{(F,a)},\eta_s^{(F,d)},\eta_s^{(b)})$ and the following random variables on this process,

$$\eta_s:=\eta_s^{(F,a)}+\eta_s^{(b)},\qquad \eta_s^{(F)}:=\eta_s^{(F,a)}+\eta_s^{(F,d)}\tag{9.2}$$

The main point of the definition will be that $\big(\eta_s,\xi_s\big)_{s\ge t}$ has the same law as our process of Section 2 (i.e. with generator L), while $\eta_s^{(F)}$ has the law of the independent process with generator $\epsilon^{-2}L^{(\eta,0)}$. The $\eta_s^{(F,a)}$ particles are called free and alive; $\eta_s^{(F,d)}$ free but dead; $\eta_s^{(b)}$ newly born, which already hints at the way the whole process will be defined. We set $\eta_t^{(F,d)}=\eta_t^{(b)}=0$, so that $\eta_t=\eta_t^{(F,a)}$. The process (θ_s,ξ_s) at times $s\ge t$ is defined in terms of the generator $\tilde L$ which we set equal to $\tilde L=L_1+L^{(\xi)}$, with $L^{(\xi)}$ as in Section 2 (reading

$\eta = \eta^{(F,a)} + \eta^{(b)}$). L_1 is $\epsilon^{-2} L_1^{(0)} + L_1^{(+)} + L_1^{(-)}$. $L_1^{(0)}$ is the generator of independent motion for the three types of η particles.

$$L_1^{(+)} f(\theta, \xi) = \sum_{x \in \Omega_\epsilon} \kappa\, \eta(x)\, [f(\theta + 1_{x,b}, \xi) - f(\theta, \xi)] \tag{9.3}$$

where $1_{x,b}$, $1_{x,(F,d)}$ and $1_{x,(F,a)}$ below are the configurations with only one particle at x respectively of type b, (F,d) and type (F,a);

$$\begin{aligned} L_1^{(-)} f(\theta, \xi) = &\sum_{x \in \Omega_\epsilon} \eta(x) \Big(\kappa_{\eta(x)} (1 - \xi(x)) + \kappa_{\eta(x)+1} \xi(x) \Big) \\ &\times \{ \mathbf{1}_{\eta^{(b)}(x)=0} [f(\theta - 1_{x,(F,a)} + 1_{x,(F,d)}, \xi) - f(\theta, \xi)] \\ &\quad + \mathbf{1}_{\eta^{(b)}(x)>0} [f(\theta - 1_{x,b}, \xi) - f(\theta, \xi)] \} \end{aligned} \tag{9.4}$$

It is then easy to check that (η_s, ξ_s) is our original process and $\eta^{(F)}$ is the independent process.

By an abuse of notation we still denote by $E^\epsilon_{\mu^\epsilon}$ the expectation relative to the process which after time t has generator $\tilde{L}$. Then

$$\begin{aligned} E^\epsilon_{\mu^\epsilon} \Big(\Big| u^\epsilon(x, \eta_s) - \pi_{(s-t)\epsilon^{-2}} * u^\epsilon(x, \eta_t) \Big| \Big) \le\, & E^\epsilon_{\mu^\epsilon} \Big(\Big| u^\epsilon(x, \eta_s^{(F)}) - \pi_{(s-t)\epsilon^{-2}} * u^\epsilon(x, \eta_t) \Big| \Big) \\ & + E^\epsilon_{\mu^\epsilon} \Big(u^\epsilon(x, \eta_s^{(b)}) - u^\epsilon(x, \eta_s^{(F,d)}) \Big) \end{aligned} \tag{9.5}$$

Recalling that $u^\epsilon(x, \eta_t^{(b)}) - u^\epsilon(x, \eta_t^{(F,d)}) = 0$,

$$\begin{aligned} E^\epsilon_{\mu^\epsilon} \Big(u^\epsilon(x, \eta_s^{(b)}) - u^\epsilon(x, \eta_s^{(F,d)}) \Big) = & \int_t^s \sum_y \pi_{(s-s')\epsilon^{-2}}(x, y) \\ & \times E^\epsilon_{\mu^\epsilon} \Big((L_1^{(+)} + L_1^{(-)}) \{ u^\epsilon(y, \eta_{s'}^{(b)}) - u^\epsilon(y, \eta_{s'}^{(F,d)}) \} \Big) \end{aligned} \tag{9.6}$$

Since

$$\begin{aligned} &(L_1^{(+)} + L_1^{(-)}) u^\epsilon(y, \eta^{(b)}) \le \kappa u^\epsilon(y, \eta) \\ &(L_1^{(+)} + L_1^{(-)}) u^\epsilon(y, \eta^{(F,d)}) \le \sum_z p_\alpha(y, z) \eta(z) \kappa_{\eta(z)+1} \end{aligned}$$

by Theorem 5,

$$\begin{aligned} & E^\epsilon_{\mu^\epsilon} \Big(\Big| u^\epsilon(x, \eta_s) - \pi_{(s-t)\epsilon^{-2}} * u^\epsilon(x, \eta_t) \Big| \Big) \\ & \qquad \le E^\epsilon_{\mu^\epsilon} \Big(\Big| u^\epsilon(x, \eta_s^{(F)}) - \pi_{(s-t)\epsilon^{-2}} * u^\epsilon(x, \eta_t) \Big| \Big) + c\epsilon^{2\gamma} \end{aligned} \tag{9.7}$$

Let

$$\tilde{\eta}_s(y) = \eta_s^{(F)}(y) - \pi_{(s-t)\epsilon^{-2}} * \eta_t(y) \tag{9.8}$$

then

$$u^\epsilon(x, \eta_s^{(F)}) - \pi_{(s-t)\epsilon^{-2}} * u^\epsilon(x, \eta_t) = p_\alpha * \tilde{\eta}_s(x) \tag{9.9}$$

and, by Cauchy-Schwartz, the square of the first term on the r.h.s. of (9.7) is bounded by

$$E^{\epsilon}_{\mu^\epsilon}\Big(\Big|p_\alpha * \tilde\eta_s(x)\Big|^2\Big) = E^{\epsilon}_{\mu^\epsilon}\Big(\sum_y p^2_\alpha(x,y)\tilde\eta_s(y)^2\Big) \le c'\epsilon^{(1-\alpha)d} \tag{9.10}$$

because $E^{\epsilon}_{\eta_t,\xi_t}\big(\tilde\eta_s(y)\tilde\eta_s(z)\big) = 0$ for $z \neq y$ by (A.7). The last inequality in (9.10) follows from Theorem 5. By assumption, $(1-\alpha)d > 2\gamma$ and this concludes the proof of the theorem. ∎

10 Local ergodic theorems

In this section we will prove a local equilibrium result, close in spirit to what in the hydrodynamic literature is called the "Gibbs-Boltzmann principle", [3], or the "one block estimate", [8]. The question concerns time averaged quantities and the aim is to prove closeness to their equilibrium expectations with parameters determined by local empirical means (local equilibrium). This can be reduced in our case (see Appendix D) to study decay of time correlations. Decay of time correlations is then proved in Theorems 9 and 10 below respectively in the case of some bounded and unbounded local functions of η_t.

Theorem 9. *For any $\tau > 0$ there is c so that for any $\epsilon > 0$, $x, y \in \Omega_\epsilon$, $|x-y| \ge \epsilon^{-1+\alpha/2}$ and $t \le \tau$,*

$$\Big|E^{\epsilon}_{\mu^\epsilon}\Big(\{\mathbf{1}_{\eta_{t+\epsilon^{2\alpha}}(x)=0} - e^{-u^\epsilon(x,\eta_t)}\}\{\mathbf{1}_{\eta_{t+\epsilon^{2\alpha}}(y)=0} - e^{-u^\epsilon(y,\eta_t)}\}\Big)\Big| \le c\epsilon^{2\alpha} \tag{10.1}$$

Proof. The proof uses the same auxiliary process introduced in the beginning of the proof of Theorem 8, to which we refer for notation. To compute the expectation we decompose the unity by writing $1 = (\mathbf{1}_{x,=} + \mathbf{1}_{x,>})(\mathbf{1}_{y,=} + \mathbf{1}_{y,>})$, where

$$\mathbf{1}_{z,=} = \mathbf{1}_{\eta^{(F,d)}_{t+\epsilon^{2\alpha}}(z)+\eta^{(b)}_{t+\epsilon^{2\alpha}}(z)=0}, \quad \mathbf{1}_{z,>} = \mathbf{1}_{\eta^{(F,d)}_{t+\epsilon^{2\alpha}}(z)+\eta^{(b)}_{t+\epsilon^{2\alpha}}(z)>0}$$

Observe that

$$\eta^{(F)}(z) = \eta(z) \Leftrightarrow \mathbf{1}_{z,=} = 1 \tag{10.2}$$

Since $\eta^{(F,d)}_t = 0$ and $\eta^{(b)}_t = 0$, similarly to (9.6) we have

$$E^{\epsilon}_{\mu^\epsilon}\Big(\mathbf{1}_{z,>}\Big) \le E^{\epsilon}_{\mu^\epsilon}\Big(\eta^{(F,d)}_{t+\epsilon^{2\alpha}}(z) + \eta^{(b)}_{t+\epsilon^{2\alpha}}(z)\Big) \le c\epsilon^{2\alpha} \tag{10.3}$$

Hence

$$\begin{aligned} \text{l.h.s. of (10.1)} \le & \Big|E^{\epsilon}_{\mu^\epsilon}\Big(\{\mathbf{1}_{\eta^{(F)}_{t+\epsilon^{2\alpha}}(x)=0} - e^{-u^\epsilon(x,\eta_t)}\}\{\mathbf{1}_{\eta^{(F)}_{t+\epsilon^{2\alpha}}(y)=0} - e^{-u^\epsilon(y,\eta_t)}\}\Big)\Big| \\ & + \Big|E^{\epsilon}_{\mu^\epsilon}\Big(\eta^{(F,d)}_{t+\epsilon^{2\alpha}}(x) + \eta^{(b)}_{t+\epsilon^{2\alpha}}(x) + \eta^{(F,d)}_{t+\epsilon^{2\alpha}}(y) + \eta^{(b)}_{t+\epsilon^{2\alpha}}(y)\Big)\Big| \\ \le & \; c\epsilon^{2\alpha} \end{aligned} \tag{10.4}$$

By Theorem 5,

$$P^{\epsilon}_{\eta_t}\big(\{\eta^{(F)}_{t+\epsilon^{2\alpha}}(x)=0\}\big)=\prod_z\Big(1-\pi_{\epsilon^{-2+2\alpha}}(z,x)\Big)^{\eta_t(z)}=\exp\Big\{\sum_z \eta_t(z)\log[1-\pi_{\epsilon^{-2+2\alpha}}(z,x)]\Big\}$$
$$=e^{-u^{\epsilon}(x,\eta_t)}+R,\qquad |R|\le c\epsilon^{(1-\alpha)d}$$

Using the assumption that $|x-y|\ge \epsilon^{-1+\alpha/2}$, we then conclude the proof of (10.1), because $(1-\alpha)d>2\alpha$. ∎

Denoting by $\mathcal{P}_\rho$ and $\mathcal{E}_\rho$ law and expectation w.r.t. the product of identical Poisson measures with mean $\rho>0$, given a function $g=g(n)$, $n\in\mathbb{N}$, we shorthand

$$f_\epsilon(\eta_t,\eta_{t+\epsilon^{2\alpha}}(x),\eta_{t+\epsilon^{2\alpha}}(y)):=\mathbf{1}_{u^\epsilon(x,\eta_t)\le M,u^\epsilon(y,\eta_t)\le M}\big\{g(\eta_{t+\epsilon^{2\alpha}}(x))-\mathcal{E}_{u^\epsilon(x,\eta_t)}(g)\big\}$$
$$\times\big\{g(\eta_{t+\epsilon^{2\alpha}}(y))-\mathcal{E}_{u^\epsilon(y,\eta_t)}(g)\big\}\tag{10.5}$$

with M as in (8.2).

Theorem 10. *With the notation of* (10.5)*, suppose $g(n)\le e^{bn}$, $b>0$, then, for any $\tau>0$ there is c so that, for any $\epsilon>0$ and $t\le\tau$,*

$$\lim_{\epsilon\to0}\sup_{x\neq y}\Big|E^{\epsilon}_{\mu^\epsilon}\Big(f_\epsilon(\eta_t,\eta_{t+\epsilon^{2\alpha}}(x),\eta_{t+\epsilon^{2\alpha}}(y))\Big)\Big|=0\tag{10.6}$$

Proof. By (10.2) and (10.3),

$$\Big|E^{\epsilon}_{\mu^\epsilon}\Big(f_\epsilon(\eta_t,\eta_{t+\epsilon^{2\alpha}}(x),\eta_{t+\epsilon^{2\alpha}}(y))\Big)-E^{\epsilon}_{\mu^\epsilon}\Big(f_\epsilon(\eta_t,\eta^{(F)}_{t+\epsilon^{2\alpha}}(x)\eta^{(F)}_{t+\epsilon^{2\alpha}}(y))\Big)\Big|$$
$$\le E^{\epsilon}_{\mu^\epsilon}\big(f^2_\epsilon\big)^{1/2}c\epsilon^{\alpha}\le c'\epsilon^{\alpha}\tag{10.7}$$

The last inequality uses first that the functions $\mathbf{1}_{u^\epsilon(z,\eta_t)\le M}\mathcal{E}_{u^\epsilon(z,\eta_t)}(g)$, $z=x,y$, are bounded and then Theorem 8.1 (as f^2_ϵ is exponentially bounded).

To simplify the computations, it is now convenient to expand g in Poisson polynomials, we refer to Appendix A for definition and properties. We just recall here that the Poisson polynomial of order n, denoted by $d_n(\cdot)$ and defined in (A.1), is such that $\mathcal{E}_u(d_n)=u^n$. By (A.4),

$$g(\eta(x))=\sum_{n=0}^{\infty}\frac{a_n}{n!}d_n(\eta(x)),\qquad |a_n|\le|e^b+1|^n\tag{10.8}$$

For any $\omega\in\mathbb{N}^{\Omega_\epsilon}$ we call $|\omega|=\sum_{x\in\Omega_\epsilon}\omega(x)$ and define

$$D_\omega(\eta)=\prod_{z\in\Omega_\epsilon}d_{\omega(z)}(\eta(z))\tag{10.9}$$

Call

$$g_n(\eta_{t+\epsilon^{2\alpha}}(z))=d_n(\eta_{t+\epsilon^{2\alpha}}(z))-u(z,\eta_t)^n\tag{10.10}$$

and denote by $E^\epsilon_{\eta_t,\xi_t}$ the conditional expectation given η_t, ξ_t at time t. By (A.7)

$$E^\epsilon_{\eta_t,\xi_t}\Big(g_n(\eta^{(F)}_{t+\epsilon^{2\alpha}}(x))g_m(\eta^{(F)}_{t+\epsilon^{2\alpha}}(y))\Big) = \sum_{\omega'_n,\omega'_m} \pi^{(0)}_{\epsilon^{-2+2\alpha}}(\omega_{n,x},\omega'_n)\pi^{(0)}_{\epsilon^{-2+2\alpha}}(\omega_{m,y},\omega'_m)\{D_{\omega'_n+\omega'_m}(\eta_t) \\ +u^\epsilon(x,\eta_t)^n u^\epsilon(y,\eta_t)^m - D_{\omega'_n}(\eta_t)u^\epsilon(y,\eta_t)^m - D_{\omega'_m}(\eta_t)u^\epsilon(x,\eta_t)^n\} \tag{10.11}$$

where $\omega_{k,z}$ is the configuration with k particles at z and $\pi^{(0)}_t(\omega,\omega')$ is the kernel of $e^{tL^{(\eta,0)}}$. By (A.11), there is $c(t)$ bounded on the compacts, so that calling $A_{\epsilon,t}(n) = c(t)^n[1 + n!\epsilon^n \mathbf{1}_{n\geq\epsilon^{-1}}]$,

$$|E^\epsilon_{\mu^\epsilon}\Big(\mathbf{1}_{u^\epsilon(x,\eta_t)\leq M, u^\epsilon(y,\eta_t)\leq M} g_n(\eta^{(F)}_{t+\epsilon^{2\alpha}}(x))g_m(\eta^{(F)}_{t+\epsilon^{2\alpha}}(y))\Big)| \\ \leq A_{\epsilon,t}(n+m) + A_{\epsilon,t}(n)M^m + A_{\epsilon,t}(m)M^n + M^{n+m} \tag{10.12}$$

Thus, recalling (10.8), given any $\delta > 0$ there is N so that

$$\Big|E^\epsilon_{\mu^\epsilon}\Big(f(\eta_t, \eta^{(F)}_{t+\epsilon^{2\alpha}}(x)\eta^{(F)}_{t+\epsilon^{2\alpha}}(y))\Big) - \sum_{n\leq N, m\leq N} \frac{a_n a_m}{n!m!} E^\epsilon_{\mu^\epsilon}\Big(\mathbf{1}_{u^\epsilon(x,\eta_t)\leq M, u^\epsilon(y,\eta_t)\leq M} \\ \times g_n(\eta^{(F)}_{t+\epsilon^{2\alpha}}(x))g_m(\eta^{(F)}_{t+\epsilon^{2\alpha}}(y))\Big)\Big| \;\leq\; \delta \tag{10.13}$$

Theorem 10 will then follow from showing that for any fixed N,

$$\lim_{\epsilon\to 0}\sup_{x\neq y}\sup_{n,m\leq N}\Big|E^\epsilon_{\mu^\epsilon}\Big(\mathbf{1}_{u^\epsilon(x,\eta_t)\leq M, u^\epsilon(y,\eta_t)\leq M} g_n(\eta^{(F)}_{t+\epsilon^{2\alpha}}(x))g_m(\eta^{(F)}_{t+\epsilon^{2\alpha}}(y))\Big)\Big| = 0 \tag{10.14}$$

The proof of (10.14) is reported at the end of Appendix A; it is based on a cancellation on the r.h.s. of (10.11) among the terms which appear when we limit the sum over ω'_n and ω'_m to ω'_n, ω'_m and $\omega'_n + \omega'_m$ being all ≤ 1 and when we take only non diagonal terms in the expansions of $u^\epsilon(z,\eta_t)^k$, $k = n, m$, $z = x, y$, (recall the definition (4.2) of u^ϵ as a sum of terms). It is then shown in Appendix A that the contribution of the other terms vanishes proportionally to a positive power of ϵ, the proportionality coefficient has a "bad dependence" on N, but since N is held fixed as $\epsilon \to 0$, this does not give problems. ■

11 Homogenization and convergence to the Stirring Process

In this section we will prove that the process ξ_t is close to the stirring process on time intervals so long for the latter to reach local equilibrium, which implies that also ξ_t approaches the same local equilibrium state, see Theorem 11 below. The statement is proved in two steps. We first prove a homogenization result at "short times". Recalling that the generator for displacements of ξ particles is $\epsilon^{-2a}L^{(\xi,0)}$, the ξ particles start moving at times of order ϵ^{2a}. On such a time scale, we will prove that to leading order the ξ particles move as stirring with a time dependent intensity determined by the local empirical averages of the η-particles. The result is then extended to longer times of order $\epsilon^{2\beta}$ (recall $\beta < a$), when the ξ-particles, like the stirring, become themselves also approximately exchangeable.

The result is used in Appendix E to control a term $\mathcal{Q}$ which is the most dangerous one among those which appear in the computation of (4.4). After some maquillage operations whose details are given in Appendix E, the space time averages involved in the expression (4.5) lead to study expectations of a measure ν^ϵ_τ, $\tau > 0$, which is a space time average of the original μ^ϵ. ν^ϵ_τ is in fact defined as the probability on $\mathbb{N}^{\Omega_\epsilon} \times \{0,1\}^{\Omega_\epsilon}$ whose expectations are

$$\nu^\epsilon_\tau(f) = \epsilon^d \sum_{x\in\Omega_\epsilon} \fint_0^\tau E^\epsilon_{\mu^\epsilon}\big(S_x f(\eta_t,\xi_t)\big) \tag{11.1}$$

with S_x the shift by x on the torus Ω_ϵ.

We fix a large constant C ($C = e^M$, see Appendix E), and shorthand

$$T = C\epsilon^{2\beta}, \quad B = \{x : |x| \le \epsilon^{-a+\beta/2}\} \tag{11.2}$$

We also write $\underline{x}$ for a subset of Ω_ϵ, $|\underline{x}|$ for its cardinality and call

$$g_{\underline{x}}(\xi) = \prod_{x\in\underline{x}} \xi(x) \tag{11.3}$$

We can now state the main theorem in this section.

Theorem 11. *Let T as in* (11.2), *then*

$$\lim_{\epsilon\to 0} \sup_{\underline{x}\subset B, |\underline{x}|=4} \Big| E^\epsilon_{\nu^\epsilon_\tau}\big(g_{\underline{x}}(\xi_T)\big) - \nu^\epsilon_\tau\Big(\prod_{x\in\underline{x}}\{\sum_z \pi_{T\epsilon^{-2a}}(x,z)\xi_0(z)\}\Big)\Big| = 0 \tag{11.4}$$

The proof of the theorem is divided in five steps. From the proof it will be clear that the result extends to $|\underline{x}| \le n$, for any finite fixed positive integer n.

Step 1: Stirring Process and Duality.
We will prove (11.4) using extensively the self-duality of the stirring process. The stirring on $\{0,1\}^{\Omega_\epsilon}$ is the process defined by the generator $L^{\rm st}$

$$L^{\rm st} f(\underline{x}) = \sum_{z,z':|z-z'|=1 \text{ on } \Omega_\epsilon} [f(\underline{x}^{z,z'}) - f(\underline{x})] \tag{11.5}$$

where the sum is over ordered pairs of nearest neighbor sites on the torus Ω_ϵ (a bond is counted twice, in agreement with (2.4)) and $\underline{x}^{z,z'}$ is defined as in Section 2, once we identify $\underline{x}$ with the configuration $\xi_{\underline{x}}(z) = 1$ iff $z \in \underline{x}$. We will also shorthand

$$p_t(\underline{x},\underline{y}) = e^{tL^{\rm st}}(\underline{x},\underline{y}) \tag{11.6}$$

Given $\gamma \ge \beta$, we divide the time interval $[0,T]$ into a union of intervals $[t_n, t_{n+1}]$. If $\gamma = \beta$ there is only one interval, if $\gamma > \beta$ we define iteratively $t_n = t_{n-1} + \epsilon^{2\gamma}$, $n \le N+1$, where $N+1$ is the first integer such that $t_{N+1} \ge T$ and then rename (if necessary)

$t_{N+1}=T$. The choice $\gamma=\beta$ is possible only in $d>2$. In fact we will need in our analysis γ such that

$$a>\gamma>a-\beta \tag{11.7}$$

On the other hand we have already used (as in Theorem 7) that $\beta<\dfrac{d}{d+2}\,a$ so that the condition $\gamma=\beta$ together with (11.7) imposes $2\beta>a$, hence $\dfrac{d}{d+2}\,a>\dfrac{a}{2}$, which is satisfied only in $d>2$, (in $d>2$ we then choose $\gamma=\beta$ with β smaller but close enough to $\dfrac{d}{d+2}\,a$ so that (11.7) holds).

We will compute $E^{\epsilon}_{\nu^{\epsilon}_{T}}\big(g_{\underline{x}}(\xi_T)\big)$ by successive conditioning on (η_{t_n},ξ_{t_n}), $n=N,N-1,..,1,0$. Modulo errors which vanish in the limit $\epsilon\to 0$, at each step we always have to compute the expectation of the "same" function $g_{\underline{y}}$, with $|\underline{y}|=4$. The origin of such a strong result is the self-duality of the stirring.

On each time interval $[t_n,t_{n+1}]$ we compare $\epsilon^{-2a}L^{(\xi,0)}g_{\underline{x}}(\xi_t)$ with $\epsilon^{-2a}c\,L^{\rm st}g_{\underline{x}}(\xi_t)$, the former acting on ξ_t the latter on $\underline{x}$, see (11.10) below. The choice of the intensity c is critical. In the generator $L^{(\xi,0)}$ the intensity is proportional to $\mathbf{1}_{\eta_t(x)=0}$ which by Theorem 9 can be replaced by $e^{-u^{\epsilon}(x,\eta_{t-\epsilon^{2\alpha}})}$. We would like to further approximate it by putting $e^{-u^{\epsilon}(0,\eta_{t_{n-1}})}$. We will see that the error of shifting x to 0 is under control, but for time shifts we only have Theorem 8. Thus, instead of setting $c=e^{-u^{\epsilon}(0,\eta_{t_{n-1}})}$ as we would have liked, we will set $c=e^{-w^{\epsilon}(t,t_{n-1},\eta_{t_{n-1}})}$, where

$$w^{\epsilon}(t,t_{n-1},\eta_{t_{n-1}})=\sum_y \pi_{(t-\epsilon^{2\alpha}-t_{n-1})\epsilon^{-2}}(0,y)u^{\epsilon}(y,\eta_{t_{n-1}}) \tag{11.8}$$

To implement the above strategy, it is convenient to rewrite

$$L^{(\xi,0)}g_{\underline{x}}(\xi)=\sum_{z,z':|z-z'|=1}|\xi(z)-\xi(z')|\Big(\xi(z)\mathbf{1}_{\eta(z')=0}+\xi(z')\mathbf{1}_{\eta(z)=0}\Big)\Big(g_{\underline{x}}(\xi^{z,z'})-g_{\underline{x}}(\xi)\Big)$$

and telescopic sum

$$\begin{aligned}\mathbf{1}_{\eta_t(z)=0}&=[\mathbf{1}_{\eta_t(z)=0}-e^{-u^{\epsilon}(z,\eta_{t-\epsilon^{2\alpha}})}]+[e^{-u^{\epsilon}(z,\eta_{t-\epsilon^{2\alpha}})}-e^{-u^{\epsilon}(0,\eta_{t-\epsilon^{2\alpha}})}]\\&\quad+[e^{-u^{\epsilon}(0,\eta_{t-\epsilon^{2\alpha}})}-e^{-w^{\epsilon}(t,t_{n-1},\eta_{t_{n-1}})}]+e^{-w^{\epsilon}(t,t_{n-1},\eta_{t_{n-1}})}\end{aligned} \tag{11.9}$$

doing the same for $\mathbf{1}_{\eta_s(z')=0}$. Since

$$g_{\underline{x}}(\xi^{z,z'})=g_{\underline{x}^{z,z'}}(\xi)$$

we get

$$L^{(\xi,0)}g_{\underline{x}}(\xi_t)=e^{-w^{\epsilon}(t,t_{n-1},\eta_{t_n})}L^{\rm st}g_{\underline{x}}(\xi_t)+H_1(\underline{x},t)+\cdots+H_3(\underline{x},t) \tag{11.10}$$

where H_i comes from the i-th term of the telescopic sum and dependence on ξ and η is not made explicit. Notice that the actual generator of the process is $\epsilon^{-2a}L^{(\xi,0)}$, so that also "the error terms" H_i get multiplied by ϵ^{-2a}.

Step 2: The time interval $[t_N, t_{N+1}]$.
(11.10) is the desired duality relation, which states that modulo the errors H_i we can compute the expectation of $g_{\underline{x}}(\xi_t)$ by applying the stirring to $g_{\underline{x}}$ thought of as a function of $\underline{x}$. The statement will become hopefully clear below. Set

$$A_{t_n,t_j}(\eta_{t_j}) := \int_{t_n}^{t_{n+1}} e^{-w^\epsilon(t,t_j,\eta_{t_j})}, \quad j < n \tag{11.11}$$

recalling that in the whole sequel, $\underline{x}^0 \subset B$ and $|\underline{x}^0| = 4$ (we rename by $\underline{x}^0$ the set $\underline{x}$ in (11.2) where B is also defined). We set $2B = \{x : |x| \le 2\epsilon^{-a+\beta/2}\}$. Using (11.10) and recalling (11.6), we get

$$E^\epsilon_{\nu^\epsilon_\tau}\big(g_{\underline{x}^0}(\xi_{t_{N+1}})\big) = E^\epsilon_{\nu^\epsilon_\tau}\Big(\sum_{\underline{y}\subset 2B} p_{A_{t_N,t_{N-1}}(\eta_{t_{N-1}})\epsilon^{-2a}}(\underline{x}^0,\underline{y}) g_{\underline{y}}(\xi_{t_N})\Big) + \mathcal{R}^{(1)}_N \tag{11.12}$$

$$\mathcal{R}^{(1)}_n = \int_{t_n}^{t_{n+1}} \sum_{\underline{y}} p_{(t_N-t)\epsilon^{-2a}}(\underline{x}^0,\underline{y}) E^\epsilon_{\nu^\epsilon_\tau}\Big(\{L' g_{\underline{y}}(\xi_s) + \epsilon^{-2a}[H_1(\underline{y},s) + \cdots + H_3(\underline{y},s)]\}\Big)$$

where $L' = L^{(\xi,+)} + L^{(\xi,-)}$. The remainder term $R^{(1)}_N$ will be bounded later, together with the analogous terms coming from iterating the above analysis to the successive time intervals. But we are not yet ready for the iteration, because the first term on the r.h.s. of (11.12) has not the same expression as the term we started from, since it depends on $\eta_{t_{N-1}}$. To fix the problem we do a Taylor-Lagrange expansion to first order in the small parameter $[A_{t_N,t_{N-1}}(\eta_{t_{N-1}}) - A_{t_N,0}(\eta_0)]$.

Shorthand

$$b_n(\lambda) := \lambda A_{t_n,t_{n-1}}(\eta_{t_{n-1}}) + (1-\lambda) A_{t_n,0}(\eta_0) \tag{11.13}$$

then

$$E^\epsilon_{\nu^\epsilon_\tau}\big(g_{\underline{x}^0}(\xi_{t_{N+1}})\big) = \sum_{\underline{y}} p_{A_{t_N,0}(\eta_0)\epsilon^{-2a}}(\underline{x}^0,\underline{y}) E^\epsilon_{\nu^\epsilon_\tau}\Big(g_{\underline{y}}(\xi_{t_N})\Big) + \mathcal{R}^{(1)}_N + \mathcal{R}^{(2)}_N \tag{11.14}$$

where

$$\begin{aligned}\mathcal{R}^{(2)}_N = \int_0^1 E^\epsilon_{\nu^\epsilon_\tau}\Big(\sum_{\underline{y}} \big[A_{t_N,t_{N-1}}(\eta_{t_{N-1}}) - A_{t_N,0}(\eta_0)\big]\epsilon^{-2a} \\ \times p_{b_N(\lambda)\epsilon^{-2a}}(\underline{x}^0,\underline{y}) L^{\mathrm{st}} g_{\underline{y}}(\xi_{t_N})\Big)\, d\lambda\end{aligned} \tag{11.15}$$

One may wonder why not to put since from the beginning (i.e. in (11.9)) $w^\epsilon(t,0,\eta_0)$ instead of $w^\epsilon(t,t_{N-1},\eta_{t_{N-1}})$. The reason is that, at this stage, the replacement is just a time change in the stirring process, and its influence is smoothened by the mixing properties of the stirring, as it will become clear when bounding $\mathcal{R}^{(2)}_n$.

Step 3: The iteration.
The first term on the r.h.s. of (11.14) has now the same structure as its l.h.s. so we can iterate. Calling

$$A^{(n)}(\eta_0) := A_{t_N,0}(\eta_0) + \cdots + A_{t_n,0}(\eta_0), \;\; n \le N, \qquad A^{(N+1n)}(\eta_0) \equiv 0 \tag{11.16}$$

we get

$$E^{\epsilon}_{\nu^{\epsilon}_{\tau}}\big(g_{\underline{x}^0}(\xi_{t_{N+1}})\big) = E^{\epsilon}_{\nu^{\epsilon}_{\tau}}\Big(\sum_{\underline{y}} p_{A^{(0)}(\eta_0)\epsilon^{-2a}}(\underline{x}^0,\underline{y})g_{\underline{y}}(\xi_0)\Big) + \sum_{i=1}^{2}\sum_{n=0}^{N}\mathcal{R}^{(i)}_n \tag{11.17}$$

where, analogously to (11.15),

$$\mathcal{R}^{(2)}_n = \int_0^1 E^{\epsilon}_{\nu^{\epsilon}_{\tau}}\Big(\sum_{\underline{y}}\big[A_{t_n,t_{n-1}}(\eta_{t_{n-1}}) - A_{t_n,0}(\eta_0)\big]\epsilon^{-2a} \times p_{[A^{(n+1)}(\eta_0)+b_n(\lambda)]\epsilon^{-2a}}(\underline{x}^0,\underline{y})L^{\mathrm{st}}g_{\underline{y}}(\xi_{t_n})\Big)\,d\lambda \tag{11.18}$$

Step 4: Bounds on the remainders.
The following bounds hold:

- The contribution of $\mathcal{R}^{(1)}_n$ is bounded by $c\epsilon^{2\gamma-2a+2\gamma}$. Indeed the term H_3 is bounded using Theorem 8 by $c\epsilon^{2\gamma-2a+2\gamma}$, while all the other terms are smaller. In fact, by (10.1) the term with H_1 is bounded proportionally to $\epsilon^{2\gamma-2a+2\alpha}$, so that, by the choice of α, it is $\le c\epsilon^{2\gamma-2a+2\gamma}$. The term with H_2 can be written as the sum of two terms. The first one takes into account the case when all the particles remain in the ball of radius $2\epsilon^{-a+\beta/2}$ (i.e. twice the radius of the ball where they are initially). This term is then bounded using (8.18), proportionally to $\epsilon^{2\gamma-2a+1-a+\beta/2}$. The second term (contributing to H_2) covers the case when there is at least a particle which travels by $e^{-a+\beta/2}$ in a time $\epsilon^{-2a+2\beta}$ and it is therefore bounded by $c\,\exp\big\{-c'\epsilon^{-\beta}\big\}$. Finally, the term with $L'g_{\underline{y}}(\xi_s)$ is bounded by $c\epsilon^{2\gamma}$, because, by Theorem 5, $E^{\epsilon}_{\nu^{\epsilon}_{\tau}}(\kappa_{\eta_s(\cdot)+1}) \le c$.

- The contribution of $\mathcal{R}^{(2)}_n$ is bounded by

$$c\,\epsilon^{2\gamma-2a+2\beta}[(t_N-t_n)\epsilon^{-2a}]^{-1/2\,-1/12} \tag{11.19}$$

  c a positive constant. The bound is derived in Appendix B by exploiting estimates on the stirring process known in the literature.

The total contribution of all the remainders is then bounded by

$$c\,\{\epsilon^{2\gamma-2a+2\gamma}\epsilon^{2\beta-2\gamma} + \epsilon^{-2a+2\beta}T^{1/2\,-1/12}\epsilon^{2a(1/2+1/12)}\} \tag{11.20}$$

with $T = C\epsilon^{2\beta}$. The first term vanishes by (11.7), the second one if $\beta > \dfrac{5a}{17}$ which is compatible with the condition $\beta < \dfrac{d}{d+2}\,a$.

Step 5: Conclusion.
By (11.17)

$$\Big|E^{\epsilon}_{\nu^{\epsilon}_{\tau}}\big(g_{\underline{x}^0}(\xi_{t_{N+1}})\big) - \nu^{\epsilon}_{\tau}\Big(\sum_{\underline{y}} p_{A^{(0)}(\eta_0)\epsilon^{-2a}}(\underline{x}^0,\underline{y})g_{\underline{y}}(\xi_0)\Big)\Big| \le c\epsilon^{\delta}$$

where $c\epsilon^\delta$, $\delta > 0$, shorthands the bound (11.20) on the remainders. We are going to prove that

$$\nu^\epsilon_\tau\Big(\epsilon^{-2a}A^{(0)}(\eta_0) \ge \frac{Ce^{-M}}{2}\,\epsilon^{-2a+2\beta}\Big) \ge 1 - c\epsilon^\beta \tag{11.21}$$

$$E_{\nu^\epsilon_\tau}\Big(\fint_0^{C\epsilon^{2\beta}} |e^{-w^\epsilon(t,0,\eta_0)} - e^{-u^\epsilon(0,\eta)}|\Big) \le c\epsilon^\beta \tag{11.22}$$

Proof of (11.21)-(11.22). Recall that

$$\epsilon^{-2a}A^{(0)}(\eta_0) = \epsilon^{-2a}\int_0^{C\epsilon^{2\beta}} e^{-w^\epsilon(t,0,\eta_0)}, \qquad w^\epsilon(t,0,\eta_0) = \sum_y \pi_{(t-\epsilon^{2\alpha})\epsilon^{-2}}(0,y)u^\epsilon(y,\eta_0)$$

By (8.2),

$$\nu^\epsilon_\tau\big(e^{-u^\epsilon(0,\eta)} \le e^{-M}\big) \le c\epsilon^{(1-\alpha)d} \tag{11.23}$$

so that

$$\nu^\epsilon_\tau\Big(\fint_0^{C\epsilon^{2\beta}} e^{-w^\epsilon(t,0,\eta_0)} < \frac{e^{-M}}{2}\Big) \le c\epsilon^{(1-\alpha)d} + \nu^\epsilon_\tau\Big(\fint_0^{C\epsilon^{2\beta}} |e^{-w^\epsilon(t,0,\eta_0)} - e^{-u^\epsilon(0,\eta)}| \ge \frac{e^{-M}}{2}\Big) \tag{11.24}$$

and (11.21) follows from (11.22), which we prove next:

$$\begin{aligned}
&\nu^\epsilon_\tau\Big(|e^{-w^\epsilon(t,0,\eta_0)} - e^{-u^\epsilon(0,\eta)}|\Big) \\
&\qquad \le \fint_0^{C\epsilon^{2\beta}} \nu^\epsilon_\tau\Big(\sum_y \pi_{(t-\epsilon^{2\alpha})\epsilon^{-2}}(0,y)|u^\epsilon(y,\eta) - u^\epsilon(0,\eta)|\Big) \\
&\qquad \le \epsilon\{\fint_0^{C\epsilon^{2\beta}} \sum_y \pi_{(t-\epsilon^{2\alpha})\epsilon^{-2}}(0,y)|y|\}\ \nu^\epsilon_\tau\big(\|u^\epsilon(\eta)\|_{H^2_1}\big)
\end{aligned}$$

having used (8.19). Since $\sum_y \pi_{(t-\epsilon^{2\alpha})\epsilon^{-2}}(0,y)|y| \le \epsilon^{-1+\beta}$ and

$$E^\epsilon_{\nu^\epsilon_\tau}\big(\|u^\epsilon(\eta_s)\|_{H^2_1}\big) = \fint_0^\tau E^\epsilon_{\mu^\epsilon_\tau}\big(\|u^\epsilon(\eta_t)\|_{H^2_1}\big) \le c\tau^{-1/2}$$

(the last inequality follows from (8.9) and Cauchy-Schwartz), (11.22) and hence (11.21).

By (B.3), we then have

$$\Big|E^\epsilon_{\nu^\epsilon_\tau}\big(g_{\underline{x}^0}(\xi_{t_{N+1}})\big) - \nu^\epsilon_\tau\Big(\{\prod_{x\in\underline{x}}\sum_y \pi_{A^{(0)}(\eta_0)\epsilon^{-2a}}(x,y)\}g_{\underline{y}}(\xi_0)\Big)\Big| \le c[\epsilon^\delta + \epsilon^\beta + \epsilon^{(-2a+2\beta)/12}]$$

Moreover, by (11.22) and (11.21),

$$E_{\nu^\epsilon_\tau}\Big(\sum_y |\pi_{A^{(0)}(\eta_0)\epsilon^{-2a}}(x,y) - \pi_{C\epsilon^{-2a+2\beta}e^{-u^\epsilon(0,\eta)}}(x,y)|\Big) \le \frac{c\epsilon^{-2a+3\beta}}{(Ce^{-M}/2)\epsilon^{-2a+2\beta}} + c\epsilon^\beta$$

and the theorem is proved. ■

A Independent and branching random walks

In this appendix we prove the statements in the text which refer to independent random walks and to independent random walks with independent branchings. We start by recalling some of the main properties of independent random walks on $\mathbb{Z}^d$.

Poisson polynomials. The Poisson polynomial of degree k, $k \ge 1$, is defined as

$$d_1(n) \equiv 1, \quad d_k(n) = n(n-1)\cdots(n-k+1),\ k > 1 \tag{A.1}$$

We have denoted by n the argument of the polynomial because we are going to restrict d_k to $\mathbb{N}$. The following remarkable identities hold:

$$\mathbf{1}_{\eta(x)=i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} d_{n+i}(\eta(x)) \tag{A.2}$$

$$e^{b\eta(x)} = \sum_{n=0}^{\infty} \frac{(e^b-1)^n}{n!} d_n(\eta(x)) \tag{A.3}$$

If $f(n)$ is exponentially bounded, say $|f(n)| \le e^{bn}$, then

$$f(\eta(x)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} d_n(\eta(x)), \quad |a_n| \le |e^b + 1|^n \tag{A.4}$$

The above can and will be applied to $\kappa_{\eta(x)}$ (and to $\kappa^n_{\eta(x)}$, $n \in \mathbb{N}$, as well), because $\kappa_{\eta(x)}$, which is the function which appears in the generator L in Section 2, has been supposed to be exponentially bounded.

Poisson multi-polynomials. The Poisson multi-polynomial D_ω, $\omega : \Omega_\epsilon \to \mathbb{N}$, is

$$D_\omega(\eta) = \prod_{x \in \Omega_\epsilon} d_{\omega(x)}(\eta(x)) \tag{A.5}$$

The duality relation

$$L^{(\eta,0)} D_\omega(\eta) = L^{(\omega,0)} D_\omega(\eta) \tag{A.6}$$

holds, namely the independent generator on $D_\omega(\eta)$ is the same either if it acts on ω or on η. Denoting by $\pi^{(0)}_t(\cdot,\cdot)$ the kernel of $e^{tL^{(\cdot,0)}}$, it directly follows from (A.6) that

$$\sum_{\eta'} \pi^{(0)}_t(\eta,\eta') D_\omega(\eta') = \sum_{\omega'} \pi^{(0)}_t(\omega,\omega') D_{\omega'}(\eta) \tag{A.7}$$

$$e^{(L^{(\eta,0)}+L')t} D_\omega = \sum_{\omega'} \pi^{(0)}_t(\omega,\omega') D_{\omega'} + \int_0^t \sum_{\omega'} \pi^{(0)}_{t-s}(\omega,\omega') e^{(L^{(\eta,0)}+L')s}\{L' D_\omega\} \tag{A.8}$$

where L' is some generator acting on functions of (η,ξ).

Notation. We will use the following notation: 1_x denotes the configuration ω with only one particle at x; $\omega \pm \omega'$ is the configuration with $\omega(x) \pm \omega'(x)$ at x (the relation with the minus sign being defined only if $\omega' \leq \omega$); $|\omega| = \sum_{x\in\Omega_\epsilon} \omega(x)$. Then

$$\pi_t^{(0)}(\omega,\omega') = \sum_{y_1,..,y_n} \{\prod_{i=1}^n \pi_t(x_i,y_i)\}\mathbf{1}_{\omega'=\sum_i 1_{y_i}} \tag{A.9}$$

In particular $\pi_t^{(0)} = \pi_t$ when restricted to $\eta : |\eta| = 1$.

Proof of Theorem 1. We will prove here only the statements of the theorem relative to the variables η. Recalling (5.2), by (A.7),

$$U_0^\epsilon(x) = \mu^\epsilon(u^\epsilon(x,\eta)) = \sum_y p_\alpha(x,y)\mu^\epsilon\big(\eta(y)\big) = \sum_y p_\alpha(x,y)\sum_z \pi_{\epsilon^{-2}t_0}(y,z)\eta^\epsilon_{-t_0}(z)$$

Thus $|U_0^\epsilon(x)| \leq C$, if C is the sup in (5.1). Moreover,

$$\epsilon^{-1}|\nabla U_0^\epsilon(x)| \leq \epsilon^{-1}\sup_z |\nabla \pi_{\epsilon^{-2}t_0}(0,z)| \leq \frac{c}{t_0}$$

which proves the first inequality in (5.5).

For future reference we also observe that the same argument used above shows that

$$\mu^\epsilon(D_\omega(\eta)) = \sum_{\omega'} \pi_{t_0}^{(0)} D_{\omega'}(\eta_{-t_0}) \leq C^{|\omega|} \tag{A.10}$$

Call $\tilde\eta(x) = \eta(x) - \mu^\epsilon(\eta(x))$, then

$$\mu^\epsilon(\|u^\epsilon - U^\epsilon\|^2) = \epsilon^d \sum_{x\in\Omega_\epsilon}\sum_{z,z'} p_\alpha(x,z)p_\alpha(x,z')\mu^\epsilon(\tilde\eta(z)\tilde\eta(z'))$$

Shorthanding $\omega_z = 1_z$ and $\omega_{z,z'} = 1_z + 1_{z'}$,

$$\mu^\epsilon\big(\tilde\eta(z)\tilde\eta(z')\big) = \mu^\epsilon\big(D_{\omega_{z,z'}}\big) - \mu^\epsilon(D_{\omega_z})\mu^\epsilon(D_{\omega_{z'}}) + \ \mathbf{1}_{z=z'}\ \mu^\epsilon(\eta(z))$$

so that

$$\mu^\epsilon(\|u^\epsilon - U^\epsilon\|^2) = \epsilon^d \sum_{x\in\Omega_\epsilon}\sum_{z\in\Omega_\epsilon} p_\alpha(x,z)^2\mu^\epsilon(\eta(z))$$
$$+ \sum_{z\in\Omega_\epsilon}[\sum_{y\in\Omega_\epsilon} p_\alpha(x,y)\pi_{\epsilon^{-2}t_0}(y,z)]^2\eta_{-t_0}(z) \leq c\epsilon^{(1-\alpha)d}$$

and the part of Theorem 1 relative to the η variables is proved. ■

The proof of Theorem 5 uses the following bound:

Theorem 12. *There is a constant $c>0$ so that for any $t>0$ and any ω,*

$$E^{\epsilon}_{\mu^\epsilon}\big(D_\omega(\eta_t)\big)\le E^{\epsilon}_{\mu^\epsilon}\big(D_\omega(\eta^+_t)\big)\le \big(e^{\kappa t}(C+2c\sqrt{t})\big)^{|\omega|}\Big(1_{|\omega|\le\epsilon^{-1}}+|\omega|!(e\epsilon)^{|\omega|}1_{|\omega|\ge\epsilon^{-1}}\Big) \tag{A.11}$$

where η^+_t is the process with generator $\epsilon^{-2}L^{(\eta,0)}+L^{(\eta,+)}$.

Proof. We will first prove that:

$$L^{(\eta,+)}D_\omega(\eta)=\kappa|\omega|D_\omega+\kappa\sum_{x\in\Omega_\epsilon}\omega(x)[\omega(x)-1]D_{\omega-1_x} \tag{A.12}$$

(recall that $|\omega|=\sum_x\omega(x)$, $[\omega-1_x](z)=\omega(z)$ for all $z\neq x$ and $[\omega-1_x](x)=\omega(x)-1$). Indeed

$$L^{(\eta,+)}D_\omega(\eta)=\kappa\sum_x\Big[\prod_{y\neq x}d_{\omega(y)}(\eta(y))\Big]\eta(x)\big[d_{\omega(x)}(\eta(x)+1)-d_{\omega(x)}(\eta(x))\big]$$

and (A.12) follows from

$$k[d_n(k+1)-d_n(k)]=nd_n(k)+n[n-1]d_{n-1}(k)$$

By (A.8) and (A.12),

$$E^{\epsilon}_{\mu^\epsilon}\big(D_\omega(\eta^+_t)\big)=\sum_{\omega'}\pi^{(0)}_t(\omega,\omega')\mu^\epsilon\big(D_{\omega'}(\eta)\big)+\kappa n\int_0^t\sum_{\omega'}\pi^{(0)}_{t-s}(\omega,\omega')E^{\epsilon}_{\mu^\epsilon}\big(D_{\omega'}(\eta^+_s)\big)$$
$$+\kappa\sum_{y\in\Omega_\epsilon}\int_0^t\sum_{\omega'}\pi^{(0)}_{t-s}(\omega,\omega')\omega'(y)[\omega'(y)-1]E^{\epsilon}_{\mu^\epsilon}\big(D_{\omega'-1_y}(\eta^+_s)\big)$$

Let $\omega=\sum_{i=1}^n 1_{x_i}$. Recalling (A.9), the last term can then be written as

$$\kappa\sum_{y\in\Omega_\epsilon}\Big\{\int_0^t\sum_{y_1,..,y_n}\{\prod_{i=1}^n\pi_{t-s}(x_i,y_i)\}\{\sum_{i=1}^n\mathbf{1}_{y_i=y}\}\{[\sum_{i=1}^n\mathbf{1}_{y_i=y}]-1\}E^{\epsilon}_{\mu^\epsilon}\big(D_{\sum 1_{y_i}-1_y}(\eta^+_s)\big)$$

Since $\{\sum_{i=1}^n\mathbf{1}_{y_i=y}\}\{[\sum_{i=1}^n\mathbf{1}_{y_i=y}]-1\}=\sum_{i,j\neq i}\mathbf{1}_{y_i=y_j=y}$, we get

$$\kappa\sum_{i,j\neq i}\sum_{y_1,..,y_n}\mathbf{1}_{y_i=y_j}\int_0^t\{\prod_{\ell=1}^n\pi_{t-s}(x_\ell,y_\ell)\}E^{\epsilon}_{\mu^\epsilon}\big(D_{\sum_{\ell\neq j}1_{y_\ell}}(\eta^+_s)\big)$$
$$\le\kappa\sum_{i,j\neq i}\sum_{y_1,..,\not{y_j},..,y_n}\int_0^t\{\prod_{\ell\neq j}\pi_{t-s}(x_\ell,y_\ell)\}\{\sup_{x,y}\pi_{t-s}(x,y)\}E^{\epsilon}_{\mu^\epsilon}\big(D_{\sum_{\ell\neq j}1_{y_\ell}}(\eta^+_s)\big)$$

and since $\kappa\sup_{x,y}\pi_{t-s}(x,y)\le c\dfrac{\epsilon}{\sqrt{t-s}}$,

$$\le c\epsilon n(n-1)\int_0^t\frac{1}{\sqrt{t-s}}\sup_{\omega^*:|\omega^*|=n-1}\sum_{\omega':|\omega'|=n-1}\pi^{(0)}_{t-s}(\omega^*,\omega')E^{\epsilon}_{\mu^\epsilon}\big(D_{\omega'}(\eta^+_s)\big)$$

Since $\mu^\epsilon\big(D_\omega(\eta_0)\big) \le C^{|\omega|}$,

$$E^\epsilon_{\mu^\epsilon}\big(D_\omega(\eta_t^+)\big) \le e^{\kappa n t}\Big(C^n + \sum_{k=1}^{n-1}[c\epsilon]^k[n(n-1)^2\cdots(n-k+1)^2(n-k)]I_k(t)C^{n-k}\Big)$$

where

$$I_k(t) = \int_0^t \frac{1}{\sqrt{t-s_1}}\cdots\int_0^{s_{k-1}}\frac{1}{\sqrt{t-s_k}} = \frac{(2\sqrt{t})^k}{k!}$$

Calling $B_n(\epsilon) = \max_{k\le n}(n\epsilon)^k$,

$$E^\epsilon_{\mu^\epsilon}\big(D_{\omega_{n,x}}(\eta_t^+)\big) \le B_n(\epsilon)e^{\kappa n t}\sum_{k=0}^n \binom{n}{k}[2c\sqrt{t}]^k C^{n-k} \le B_n(\epsilon)[e^{\kappa t}(C+2c\sqrt{t})]^n$$

Finally, $B_n(\epsilon)\le 1$ for $n \le \epsilon^{-1}$ and otherwise $B_n(\epsilon) \le \epsilon^n n! e^n$ (by the Stirling formula $n^n \le n!e^n$), hence (A.11). ∎

Proof of Theorem 5. We complete here the proof of Theorem 5 by proving (8.3) and (8.4). By (A.3),

$$E^\epsilon_{\mu^\epsilon}\big(e^{b\eta_t^+(x)}\big) = \sum_{n=0}^\infty \frac{(e^b-1)^n}{n!}E^\epsilon_{\mu^\epsilon}\big(D_{\omega_{n,x}}(\eta_t^+)\big) \tag{A.13}$$

where $\omega_{n,x} = n1_x$ is the configuration with n particles, all at x. By (A.11),

$$\begin{aligned} E^\epsilon_{\mu^\epsilon}\big(e^{b\eta_t^+(x)}\big) &\le \sum_{n=0}^\infty \frac{e^{bn}}{n!}[e^{\kappa t}(C+2c\sqrt{t})]^n + \sum_{n\ge\epsilon^{-1}}\Big(e^{b+1}[e^{\kappa t}(C+2c\sqrt{t})\epsilon\Big)^n \\ &\le 2\exp\Big\{e^b e^{\kappa t}(C+2c\sqrt{t})\Big\} \end{aligned}$$

for ϵ small enough, which proves (8.1).

We will next prove that

$$E^\epsilon_{\mu^\epsilon}\Big(|u^\epsilon(x,\eta_t^+) - E^\epsilon_{\mu^\epsilon}\big(u^\epsilon(x,\eta_t^+)\big)|^2\Big) \le ce^{2\kappa t}\epsilon^{(1-\alpha)d} \tag{A.14}$$

Proceeding as in the proof of Theorem 1,

$$\text{l.h.s. of (A.14)} = \sum_{z,z'}p_\alpha(x,z)p_\alpha(x,z')V_2(z,z',t) + \sum_z p_\alpha(x,z)^2E^\epsilon_{\mu^\epsilon}\big((\eta_t^+(z))\big)$$

$$V_2(z,z',t) = E^\epsilon_{\mu^\epsilon}\Big(D_{\omega_{z,z'}}(\eta_t^+) - E^\epsilon_{\mu^\epsilon}\big(D_{\omega_z}(\eta_t^+)\big)E^\epsilon_{\mu^\epsilon}\big(D_{\omega_{z'}}(\eta_t^+)\big)\Big)$$

Since $L^{(\eta,+)}d_2(\eta(x)) = 2\kappa[d_2(\eta(x))+\eta(x)]$,

$$\begin{aligned} \text{l.h.s. of (A.14)} \le& \sum_z p_\alpha(x,z)^2E^\epsilon_{\mu^\epsilon}\big(\eta_t^+(z)\big) + e^{2\kappa t}\sum_z \pi_{\epsilon^{-2+2\alpha}+\epsilon^{-2}(t+t_0)}(x,z)^2\eta_{-t_0}(z) \\ &+2\kappa\sum_z\int_0^t e^{2\kappa(t-s)}\pi_{\epsilon^{-2+2\alpha}+\epsilon^{-2}(t-s)}(x,z)^2E^\epsilon_{\mu^\epsilon}\big(\eta_s^+(z)\big) \end{aligned}$$

Using the inequality $E^\epsilon_{\mu^\epsilon}\big(\eta^+_t(z)\big) \le Ce^{\kappa t}$, we then get (A.14). ■

Proof of (10.14)**.** We fix ω'_n and ω'_m on the r.h.s. of (10.11) and expand the three terms $u^\epsilon(x,\eta_t)^n$, $u^\epsilon(y,\eta_t)^m$ and $u^\epsilon(x,\eta_t)^n u^\epsilon(y,\eta_t)^m$, starting from the former.

$$u^\epsilon(x,\eta_t)^n = \sum_{z_1..z_n\in X_{n,\omega'_n,\ne}} \prod_{i=1}^n p_\alpha(x,z_i)\eta_t(z_i) + \sum_{z_1..z_n\notin X_{n,\omega'_n,\ne}} \prod_{i=1}^n p_\alpha(x,z_i)\eta_t(z_i)$$

where $X_{n,\omega'_n,\ne} = \{z_1..z_n : \omega'_n(z_i)=0, i=1,..,n; z_i\ne z_j \text{ for all } i\ne j\}$. Thus

$$u^\epsilon(x,\eta_t)^n = T+S, \qquad T = \sum_{z_1..z_n\in X_{n,\omega'_m,\ne}} \{\prod_{i=1}^n p_\alpha(x,z_i)\} D_{\sum_i 1_{z_i}}(\eta_t)$$

$$S = \sum_{z_1..z_n\notin X_{n,\omega'_m,\ne}} \prod_{i=1}^n p_\alpha(x,z_i)\eta_t(z_i) \tag{A.15}$$

We next bound S:

$$S \le \sum_{i=1}^n \{ \sum_{z_i:\omega'_m(z_i)\ge 1} p_\alpha(x,z_i)\eta_t(z_i)\} \sum_{z_1..\not z_i..z_n} \prod_{i\ne j} p_\alpha(x,z_j)\eta_t(z_j)$$
$$+ \sum_{i\ne j}\{ \sum_{z:\omega'_m(z)=0} p_\alpha(x,z)^2\eta_t(z)^2\} \sum_{z_1..\not z_i..\not z_j..z_n} \prod_{\ell\ne i,j} p_\alpha(x,z_\ell)\eta_t(z_\ell)$$

Recalling that $u^\epsilon(x,\eta_t)\le M$ in (10.14),

$$S \le M^{n-1}\sum_{i=1}^n \sum_{z_i:\omega'_m(z_i)\ge 1} p_\alpha(x,z_i)\eta_t(z_i) + M^{n-2}\sum_{i\ne j}\{ \sum_{z:\omega'_m(z)=0} p_\alpha(x,z)^2\eta_t(z)^2\}$$

and since $d_k(\eta(z))\eta(z) = d_{k+1}(\eta(z)) + kd_k(\eta(z))$, $E^\epsilon_{\mu^\epsilon}\big(D_\omega(\eta_t)\big)\le c(t)^{|\omega|}$, $|\omega|\le N$,

$$E^\epsilon_{\mu^\epsilon}\Big(\mathbf{1}_{u^\epsilon(x,\eta_t)\le M} D_{\omega'_m}(\eta_t) S\Big) \le M^{n-1} n(mc\epsilon^{(1-\alpha)d})[c(t)^{m+1} + c(t)^m m]$$
$$+ M^{n-2}n(n-1)[c(t)^2+c(t)]c(t)^m$$

so that the term with S does not contribute as $\epsilon\to 0$. The term in (10.14) containing T, cf. (A.15), is

$$-\sum_{\omega'_m,\omega'_n} \mathbf{1}_{\omega'_n\le 1,\omega'_n\omega'_m\equiv 0}\, \pi^{(0)}_{\epsilon^{-2+2\alpha}}(\omega_{n,x},\omega'_n)\pi^{(0)}_{\epsilon^{-2+2\alpha}}(\omega_{m,y},\omega'_m)$$
$$\times E^\epsilon_{\mu^\epsilon}\Big(\mathbf{1}_{u^\epsilon(x,\eta_t)\le M}\mathbf{1}_{u^\epsilon(y,\eta_t)\le M} D_{\omega'_n+\omega'_m}(\eta_t)\Big) \tag{A.16}$$

By an argument similar to the one used when bounding S, the error when dropping the conditions on ω'_n is bounded by $[n(n-1)+nm]c\epsilon^{(1-\alpha)d}c(t)^{n+m}$. In conclusion the term in (10.14) containing $u^\epsilon(x,\eta_t)^n D_{\omega'_m}(\eta_t)$ is, modulo terms which vanish as $\epsilon\to 0$,

$$-\sum_{\omega'_m,\omega'_n} \pi^{(0)}_{\epsilon^{-2+2\alpha}}(\omega_{n,x},\omega'_n)\pi^{(0)}_{\epsilon^{-2+2\alpha}}(\omega_{m,y},\omega'_m)$$
$$\times E^\epsilon_{\mu^\epsilon}\Big(\mathbf{1}_{u^\epsilon(x,\eta_t)\le M}\mathbf{1}_{u^\epsilon(y,\eta_t)\le M} D_{\omega'_n+\omega'_m}(\eta_t)\Big) \tag{A.17}$$

The analogous conclusion holds for the term with $u^\epsilon(y,\eta_t)^m D_{\omega'_n}(\eta_t)$ so that the proof of (10.14) reduces to the analysis of the term

$$E^\epsilon_{\mu^\epsilon}\Big(\sum_{\omega'_n,\omega'_m}\pi^{(0)}_{\epsilon^{-2+2\alpha}}(\omega_{n,x},\omega'_n)\pi^{(0)}_{\epsilon^{-2+2\alpha}}(\omega_{m,y},\omega'_m)\{u^\epsilon(x,\eta_t)^n u^\epsilon(y,\eta_t)^m - D_{\omega'_n+\omega'_m}(\eta_t)\}\tag{A.18}$$

Proceeding as in (A.15), and calling $X_{\neq}=\{z_1..z_{n+m}:\ z_i\neq z_j \text{ for all } i\neq j\}$, we write, by an abuse of notation, $u^\epsilon(x,\eta_t)^n u^\epsilon(y,\eta_t)^m = T+S$ where

$$T=\sum_{z_1..z_{n+m}\in X_{\neq}}\{\prod_{i=1}^n p_\alpha(x,z_i)\}\{\prod_{i=n+1}^{n+m} p_\alpha(y,z_i)\}\eta_t(z_1)..\eta_t(z_{n+m})$$

$$S=\sum_{z_1..z_{n+m}\notin X_{\neq}}\{\prod_{i=1}^n p_\alpha(x,z_i)\}\{\prod_{i=n+1}^{n+m} p_\alpha(y,z_i)\}\eta_t(z_1)..\eta_t(z_{n+m})$$

The analysis is now completely analogous to the previous one, S is bounded exploiting the presence of the characteristic functions $u^\epsilon(x,\eta_t)\le M$ and $u^\epsilon(y,\eta_t)\le M$, T is equal to a $D_{\omega_{m+n}}$ Poisson multi-polynomial with $\omega_{m+n}\le 1$ and $|\omega_{m+n}|=m+n$, after which the condition that $\omega_{m+n}\le 1$ can be dropped with a vanishing error, thus proving that (A.18) vanishes as $\epsilon\to 0$ uniformly in x,y and $n,m\le N$. ∎

B Stirring process

In [6]-[7] it is proved that:

Theorem 13. *Let $p^{(\infty)}_t(\underline{x},\underline{y})$ and $\pi^{(\infty)}_t(\underline{x},\underline{y})$ denote the kernel of the semigroups of the stirring and of the independent processes in the whole $\mathbb{Z}^d$. Then, given any $n\ge 1$ (but the case $n=1$ is trivial, because $p_t(x,y)\equiv\pi_t(x,y)$), any dimension $d\ge 1$, there is c so that, for any $\underline{x}$, $|\underline{x}|=n$,*

$$\sum_{\underline{y}}\Big|p^{(\infty)}_t(\underline{x},\underline{y})-\pi^{(\infty)}_t(\underline{x},\underline{y})\Big|\le ct^{-1/12}\tag{B.1}$$

The bound (B.1) cannot hold in our case where the processes are defined on the torus Ω_ϵ, because $p_t(\underline{x},\underline{y})$ and $\pi_t(\underline{x},\underline{y})$ have different limits as $t\to\infty$: the former converges to the uniform distribution of n distinct sites in Ω_ϵ while in the latter the exclusion condition is dropped. However in our applications we consider times $t\le\tau$ (with the stirring amplified by a factor ϵ^{-2a}, thus times $\le\epsilon^{-2a}\tau$ for the process with generator $L^{\rm st}$. Since

$$\sum_{y:|y-x|>\epsilon^{-1}} p_t(x,y)\le c\frac{e^{-\epsilon^{-2}/t}}{\sqrt{t}}\tag{B.2}$$

(B.1) remains valid for the processes in Ω_ϵ provided $t \le \epsilon^{-2a}\tau$ and with c in (B.1) dependent on τ:

$$\sup_{\epsilon>0}\ \sup_{t\le\epsilon^{-2a}\tau}\ \sup_{|\underline{x}|=n} t^{1/12}\sum_{\underline{y}}\Big|p_t(\underline{x},\underline{y})-\pi_t^{(0)}(\underline{x},\underline{y})\Big| \le c_n(\tau) \tag{B.3}$$

Proof of Theorem 1. (Relative to the ξ particles). By definition

$$\mu^\epsilon(\xi(x)) = \sum_y \pi_{t_0\epsilon^{-2a}}(x,y)\xi_{-t_0}(y) \tag{B.4}$$

hence

$$\epsilon^{-a}|\nabla\mu^\epsilon(\xi(x))| \le \frac{c}{\sqrt{t_0}} \tag{B.5}$$

uniformly in ϵ and $x\in\Omega_\epsilon$, which proves the second inequality in (5.5). We have

$$\begin{aligned}\|v^\epsilon(x,\xi)-V_0^\epsilon(x)\|^2 &= \sum_y q_\beta(x,y)^2[\xi(y)-\mu^\epsilon(\xi(y))]^2\\ &\quad+\sum_{y\ne z} q_\beta(x,y)q_\beta(x,z)[\xi(y)-\mu^\epsilon(\xi(y))][\xi(z)-\mu^\epsilon(\xi(z))]\end{aligned} \tag{B.6}$$

The μ^ϵ expectation of the first term is bounded by $c\epsilon^{a-\beta}$. The expectation of the second term is computed using (B.3) and it is then bounded by $c[t_0\epsilon^{-2a}]^{-1/12}$. The second inequality in (5.4) is proved. ∎

Proof of (11.19)**.** By (9.1),

$$\epsilon^{-2a}|A_{t_N,t_{N-1}}(\eta_{t_{N-1}})-A_{t_N,0}(\eta_0)| \le c\epsilon^{2\gamma+2\beta-2a} \tag{B.7}$$

which is the first factor in (11.19). The proof of (11.19) then follows from the following theorem:

Theorem 14. *For any $n>1$ there is c so that for any $\underline{x}$, $|\underline{x}|=n$, $t\le\epsilon^{-2}\tau$,*

$$\sup_{\|\psi\|_\infty\le 1}\Big|\sum_{\underline{z}} p_t(\underline{x},\underline{z})L^{\rm st}\psi(\underline{z})\Big| \le \frac{c}{t^{1/2+1/12}} \tag{B.8}$$

where the sup is over all functions $\psi(\underline{z})$, $|\underline{z}|=n$ which are bounded by 1.

Proof. We quote from the literature, [5], the following bound: there is c so that, for any z,z', $|z-z'|=1$ and any $t\le\epsilon^{-2a}\tau$,

$$\sum_{\underline{y}}\big|p_t(\underline{x}^{(z,z')},\underline{y})-p_t(\underline{x},\underline{y})\big| \le \frac{c}{\sqrt{t}} \tag{B.9}$$

where $\underline{x}^{(z,z')}$ is the configuration obtained from $\underline{x}$ by exchanging the content of the sites z and z': thus, if they are both contained in $\underline{x}$ or both in its complement, then $\underline{x}^{(z,z')} = \underline{x}$; otherwise a particle of $\underline{x}$ is displaced from z to z' or viceversa. we need to estimate

$$\sum_{\underline{y}} p_t(\underline{x},\underline{y}) L^{\rm st}\psi(\underline{y}) = \sum_{\underline{z}} p_{t/2}(\underline{x},\underline{z}) L^{\rm st}\phi(\underline{z}), \quad \phi(\underline{z}) := \sum_{\underline{y}} p_{t/2}(\underline{z},\underline{y})\psi(\underline{y}) \tag{B.10}$$

Let ∇_i denote the gradient acting on the variable z_i and E_+ the set of all positively oriented unit vectors of $\mathbb{Z}^d$. Then

$$L^{\rm st}\phi(\underline{z}) = \sum_{i=1}^n \sum_{|e|=1} \mathbf{1}_{z_j \neq z_i+e, j\neq i} \,\{\phi(\underline{z}^{(z_i,z_i+e)}) - \phi(\underline{z})\} = T_1 + T_2 \tag{B.11}$$

$$T_1(\underline{z}) = \sum_i^n \sum_{e\in\mathcal{E}_+} \mathbf{1}_{z_j \neq z_i \pm e, j\neq i}\, [(-e)\cdot\nabla_i][e\cdot\nabla_i]\phi(\underline{z})$$

$$T_2(\underline{z}) = \sum_{i,j} \sum_{|e|=1} \mathbf{1}_{z_j = z_i - e,} \mathbf{1}_{z_k \neq z_i+e, k\neq i}\, \{\phi(\underline{z}^{(z_i,z_i+e)}) - \phi(\underline{z})\}$$

From (B.10), after "integrating by parts",

$$\sum_{\underline{z}} p_{t/2}(\underline{x},\underline{z}) T_1(\underline{z}) = \sum_i^n \sum_{e\in E_+} \sum_{\underline{z}} \{(e\cdot\nabla_i)[p_{t/2}(\underline{x},\underline{z}) \tag{B.12}$$

$$\times \mathbf{1}_{z_j\neq z_i\pm e, j\neq i}]\}\{(e\cdot\nabla_i)\phi(\underline{z})\} \tag{B.13}$$

Recalling the definition of ϕ in (B.10) and using (B.9), we bound the gradient of ϕ by $c/\sqrt{t}$. The first factor in (B.13) is instead estimated using (B.3), and gives the desired bound $ct^{-1/2-1/12}$. The term with $T_2(\underline{z})$, see (B.11), is treated similarly and (11.19) is proved. ∎

C Proof of Theorem 3. First part

In this and in the next three Appendices we will bound

$$E^{\epsilon}_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^\tau (L+D)\{\|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2\}\Big|\Big) \tag{C.1}$$

We will start by an explicit computation of $(L+D)\|u^\epsilon - U^\epsilon\|^2$ and $(L+D)\|v^\epsilon - V^\epsilon\|^2$ obtaining expressions which are sums of *Laplacian* and *Reaction* terms. The latter are those produced by the $L^{(\eta,0)}+D^{(\eta,0)}$ and $L^{(\xi,0)}+D^{(\xi,0)}$, with $D^{(\eta,0)}$ and $D^{(\xi,0)}$ the laplacian parts of D, D is defined in Section 3. The *Reaction* terms instead are produced by $L^{(\eta,+)}+D^{(\eta,+)}$ and $L^{(\xi,+)}+D^{(\xi,+)}$ for the births and by $L^{(\eta,-)}+D^{(\eta,-)}$ and $L^{(\xi,-)}+D^{(\xi,-)}$ for the deaths, having denoted by $D^{(\eta,\pm)}$ and $D^{(\xi,\pm)}$ the reaction terms in D.

A finer classification of all the terms distinguishes those denoted by $\mathcal{B}^\epsilon_i$ which are bounded proportionally to $\|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2$ and those which when inserted in

(C.1) vanish as $\epsilon \to 0$. They will be generically called remainders. Among them we call R_i^ϵ and $\mathcal{C}_i^\epsilon$ those which can be estimated using the a priori bounds of Section 8, $\mathcal{S}_i^\epsilon$ those bounded in Appendix D exploiting the results of Section 10. There are two more terms, $\mathcal{H}$ which is bounded in Remarks 1 below using Corollary 2 and $\mathcal{Q}$, studied in Appendix E using the analysis in Section 11. In Appendix F we bound the martingale terms in (4.5)-(4.6), thus completing the proof of Theorem 3.

In the sequel we will often drop ϵ from the notation.

C.1 Laplacian terms

A computation essentially similar to the one in the proof of Theorem 6 yields

$$(L^{(\eta,0)} + D^{(\eta,0)})\|u - U\|^2 = -2\|u - U\|^2_{H_1^2} + R_1^\epsilon \tag{C.2}$$

where R_1^ϵ, defined in (8.12), vanishes as $\epsilon \to 0$, see (8.17), and for this reason is called a R^ϵ term. We will not take advantage of the negative sign of the H_1^2 term $-2\|u - U\|^2_{H_1^2}$, which will be just bounded by 0.

In an analogous way,

$$\begin{aligned}(L^{(\xi,0)} + D^{(\xi,0)})\|v - V\|^2 \quad &= -2\epsilon^{-2a}\langle \nabla(v - V), e^{-U}\nabla(v - V)\rangle + 2\mathcal{H} \\ &+R_3^\epsilon + R_4^\epsilon + 2\mathcal{S}_1^\epsilon + 2\mathcal{C}_1^\epsilon\end{aligned} \tag{C.3}$$

where R_4^ϵ is defined in (8.27) and bounded in (8.30),

$$\mathcal{H}(\xi,\eta) = \epsilon^{-2a}\langle v - V, [e^{-u} - e^{-U}]\Delta v\rangle \tag{C.4}$$

$$R_3^\epsilon(\xi,\eta) = -2\epsilon^{-2a}\langle v - V, (\nabla e^{-U})\nabla(v - V)\rangle \tag{C.5}$$

$$\begin{aligned}\mathcal{S}_1^\epsilon(\xi,\eta) \quad &= \sum_{e:|e|=1} \epsilon^d \sum_x [v(x) - V(x)]\epsilon^{-2a} \sum_z e\cdot\nabla q_\beta(z - x) \\ &\qquad \times\big\{\xi(z)(1 - \xi(z+e))[\mathbf{1}_{\eta(z+e)=0} - e^{-u(z+e)}]\big\}\end{aligned} \tag{C.6}$$

$$\begin{aligned}\mathcal{C}_1^\epsilon(\xi,\eta) = \epsilon^{-2a} &\sum_{e:|e|=1} \epsilon^d \sum_x [v(x) - V(x)] \sum_z \xi(z)(1 - \xi(z+e)) \\ &\times[e^{-u(z+e)} - e^{-u(x)}]e\cdot\nabla q_\beta(x,z)\end{aligned} \tag{C.7}$$

Remarks 1. *From* (8.42) *we get that*

$$-2\epsilon^{-2a}\langle \nabla(v - V), e^{-U}\nabla(v - V)\rangle + 2\mathcal{H} \le c\Big(\|u - U\|^2 + \|v - V\|^2 + \epsilon^{1-2a}\|u\|_{H_1^2}\Big) \tag{C.8}$$

From (8.18) *and* (8.9) *it follows that*

$$E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'} \mathcal{C}_1^\epsilon(\xi_t,\eta_t)\Big|\Big) \le \epsilon^{-2a}\sum_{e,z}|e\cdot\nabla q_\beta(x,z)|\int_0^\tau \epsilon^d\sum_x |e^{-u(z+e)} - e^{-u(x)}| \le c\epsilon^{1-2a} \tag{C.9}$$

In Appendix D we prove that

$$\lim_{\epsilon\to 0} E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'}\mathcal{S}^\epsilon_1(\xi_t,\eta_t)\Big|\Big)=0$$

Finally, since $|\nabla e^{-U}|\le c\epsilon$, $|R^\epsilon_3|\le c\epsilon^{1-2a}$.

C.2 The Reactions terms

- *Birth process,* η*-particles.*

$$(L^{(\eta,+)}+D^{(\eta,+)})\|u-U\|^2 \quad = \quad 2\kappa\|u-U\|^2+R^\epsilon_2 \tag{C.10}$$

where $R^\epsilon_2(\eta)$ is defined in (8.14) and bounded by $c\epsilon^{(1-\alpha)d}$, see the proof of Theorem 6.

- *Death process,* η*-particles.*

$$(L^{(\eta,-)}+D^{(\eta,-)})\|u-U\|^2=\mathcal{T}+2\langle u-U,[F^-_1(U)+F^-_2(U)V]\rangle+R^\epsilon_3-2\mathcal{S}^\epsilon_2 \tag{C.11}$$

where R^ϵ_3 is defined in (8.15), its expectation being bounded by $c\epsilon^{(1-\alpha)d}$, see again the proof of Theorem 6. Shorthanding by $\chi_{M,z}$ the characteristic function that $u(z,\eta)\le M$,

$$\begin{aligned}\mathcal{T}= \quad & -2\epsilon^d\sum_x[u(x)-U(x)]\sum_z p_\alpha(x,z)\chi_{M,z}F^-_1(u(z))\\ & -2\epsilon^d\sum_x[u(x)-U(x)]\sum_z p_\alpha(x,z)\xi(z)\chi_{M,z}F^-_2(u(z))\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}^\epsilon_2(\xi,\eta)=\epsilon^d\sum_x[u(x)-U(x)]\sum_z p_\alpha(x,z)\Big\{&\big[\eta(z)\kappa_{\eta(z)}-\chi_{M,z}F^-_1(u(z))\big]\\ &+\big[\xi(z)\eta(z)[\kappa_{\eta(z)+1}-\kappa_{\eta(z)}]-\chi_{M,z}F^-_2(u(z))\big]\Big\}\end{aligned} \tag{C.12}$$

We next write

$$\begin{aligned}\mathcal{T}=&-2\langle u-U,F^-_1(U)\rangle-2\langle u-U,F^-_2(U)V\rangle\\ &-2\epsilon^d\sum_x[u(x)-U(x)]F^-_2(U(x))\{\sum_z p_\alpha(x,z)\xi(z)-V(x)\}-2\mathcal{B}^\epsilon_1+R^\epsilon_6\end{aligned} \tag{C.13}$$

where

$$\begin{aligned}R^\epsilon_6 \quad = \quad & -2\epsilon^d\sum_x[u(x)-U(x)]\sum_z p_\alpha(x,z)\\ &\times\Big\{[F^-_1(U(z))-F^-_1(U(x))]+\xi(z)[F^-_2(U(z))-F^-_2(U(x))]\\ &\qquad -(1-\chi_{M,z})[F^-_1(U(z))+\xi(z)F^-_2(U(z))]\Big\}\end{aligned} \tag{C.14}$$

and

$$\begin{aligned}\mathcal{B}_1^\epsilon = \quad & \epsilon^d \sum_x [u(x) - U(x)] \sum_z p_\alpha(x,z)\chi_{M,z}\big\{[F_1^-(u(z)) - F_1^-(U(z))] \\ & + \xi(z)[F_2^-(u(z)) - F_2^-(U(z))]\big\}\end{aligned} \tag{C.15}$$

The first two terms on the r.h.s. of (C.13) simplify with the second term on the r.h.s. of (C.11). We then add and subtract $v(x)$ in the third term on the right hand side of (C.13), so that

$$(L^{(\eta,-)} + D^{(\eta,-)})\|u - U\|^2 = -2\mathcal{S}_2^\epsilon - 2(\mathcal{B}_0^\epsilon + \mathcal{B}_1^\epsilon) - 2\mathcal{C}_2^\epsilon + R_3^\epsilon + R_6^\epsilon + R_7^\epsilon \tag{C.16}$$

where

$$\mathcal{B}_0^\epsilon = \langle u - U, F_2^-(U)(v - V)\rangle \tag{C.17}$$

$$\mathcal{C}_2^\epsilon(\xi,\eta) =< (p_\alpha - 1) \star [(u - U)F_2^-(U)], v > \tag{C.18}$$

$$R_7^\epsilon(\xi,\eta) = \epsilon^d \sum_x [u(x) - U(x)]F_2^-(U(x)) \sum_z \big\{p_\alpha(x,z) - \sum_y p_\alpha(x,y)q_\beta(y,z)\big\}\xi(z) \tag{C.19}$$

Remarks 2. *In Appendix D we prove that*

$$\lim_{\epsilon\to 0} E_{\mu^\epsilon}^\epsilon\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'} \mathcal{S}_2^\epsilon(\xi_t,\eta_t)\Big|\Big) = 0$$

By Theorem 5,

$$\sup_{t\le\tau} E_{\mu^\epsilon}^\epsilon\big(|R_6^\epsilon(\eta_t,\xi_t)|\big) \le cM\epsilon \sum_z p_\alpha(0,z)|z| + c\epsilon^{(1-\alpha)d}$$

The expression on the r.h.s. of (C.15) *is called* $\mathcal{B}_1^\epsilon$ *because* $|\mathcal{B}_1^\epsilon| \le c_M\|u^\epsilon - U^\epsilon\|^2$, c_M *a constant which depends on* M. *Similarly,* $|\mathcal{B}_0^\epsilon| \le c_M(\|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2$, *since* $F_2^-(U^\epsilon)$ *is uniformly bounded in compact time intervals.*

From (8.18) *and* (8.9) *it follows that*

$$E_{\mu^\epsilon}^\epsilon\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'} \mathcal{C}_2^\epsilon(\xi_t,\eta_t)dt\Big|\Big) \le c\epsilon \tag{C.20}$$

Calling $t = \epsilon^{-2+2\alpha}$, $s = \epsilon^{-2a+2\beta}$, *since*

$$\sum_z |p_\alpha * q_\beta(x,z) - p_\alpha(x,z)| = \sum_z |\pi_{t+s}(x,z) - \pi_t(x,z)| \le c\frac{s}{t} = c\epsilon^{2(1-\alpha-a+\beta)}$$

the r.h.s. of (C.19) *is indeed a R-term.*

• *Birth process, ξ particles.*

$$(L^{(\xi,+)}+D^{(\xi,+)})\|v-V\|^2=\mathcal{T}_1-2\kappa'\langle v-V,V(1-V)e^{-U}\rangle+R_8^\epsilon \tag{C.21}$$

where R_8^ϵ is defined in (8.36) and

$$\mathcal{T}_1=2\kappa'\epsilon^d\sum_x[v(x)-V(x)]\frac{1}{2d}\sum_{e:|e|=1}\sum_z q_\beta(z-x)\xi(z)[1-\xi(z+e)]\mathbf{1}_{\eta(z+e)=0}$$

By adding and subtracting $e^{-u(z+e)}$ and calling

$$\begin{aligned}\mathcal{S}_3^\epsilon \;=\; & \epsilon^d\sum_x[v(x)-V(x)]\frac{1}{2d}\sum_{e:|e|=1}\sum_z q_\beta(z-x)\xi(z)[1-\xi(z+e)]\\ & \times\left[\mathbf{1}_{\eta(z+e)=0}-e^{-u(z+e)}\right]\end{aligned} \tag{C.22}$$

we get

$$\begin{aligned}\mathcal{T}_1&=2\kappa'\mathcal{S}_3^\epsilon+2\epsilon^d\sum_x[v(x)-V(x)]\frac{\kappa'}{2d}\sum_{e:|e|=1}\sum_z q_\beta(z-x)\xi(z)[1-\xi(z+e)]\Big\{e^{-U(x)}\\ &\qquad+\Big[e^{-U(z+e)}-e^{-U(x)}\Big]+\Big[e^{-u(z+e)}-e^{-U(z+e)}\Big]\Big\}\\ &=2\kappa'\mathcal{S}_3^\epsilon+\mathcal{T}_2+R_{10}^\epsilon+\mathcal{B}_2^\epsilon\end{aligned} \tag{C.23}$$

where $\mathcal{T}_2$, R_{10}^ϵ and $\mathcal{B}_2^\epsilon$ identify the corresponding terms in the previous line ($\mathcal{B}_2^\epsilon$ is explicitly written in (C.28) below). We next write

$$\begin{aligned}\mathcal{T}_2&=2\epsilon^d\sum_x[v(x)-V(x)]e^{-U(x)}\frac{\kappa'}{2d}\sum_{e:|e|=1}\sum_z q_\beta(x,z)\Big\{V(x)[1-V(x)]\\ &+\Big[v(x)[1-v(x+e)]-V(x)[1-V(x+e)]\Big]\\ &+\Big[\xi(z)[1-\xi(z+e)]-v(x)[1-v(x+e)]\Big]\Big\}\\ &=\mathcal{T}_3+\mathcal{B}_5^\epsilon+2\kappa'\mathcal{Q}^\epsilon\end{aligned} \tag{C.24}$$

where, as before, $\mathcal{T}_3$, $\mathcal{B}_5^\epsilon$ and $\mathcal{Q}^\epsilon$ identify the corresponding terms in the previous line, in particular $\mathcal{Q}^\epsilon$ is explicitly written in (C.27) below. Observe that $\mathcal{T}_3$ cancels with the first term on the right hand side of (C.21). We rewrite $\mathcal{B}_5^\epsilon$ as

$$\begin{aligned}\mathcal{B}_5^\epsilon=2\epsilon^d\sum_x[v(x)-V(x)]e^{-U(x)}\frac{\kappa'}{2d}\sum_{e:|e|=1}&\Big\{[1-v(x+e)](v(x)-V(x))]\\ &+V(x)(V(x+e)-v(x+e))\Big\}\end{aligned} \tag{C.25}$$

so that

$$(L^{(\xi,+)}+D^{(\xi,+)})\|v-V\|^2=\mathcal{B}_5^\epsilon+2\kappa'\mathcal{S}_3^\epsilon+2\kappa'\mathcal{Q}^\epsilon+\mathcal{B}_2^\epsilon+R_8^\epsilon+R_{10}^\epsilon \tag{C.26}$$

where

$$\mathcal{Q}^\epsilon = \epsilon^d \sum_x [v(x) - V(x)] e^{-U(x)} \frac{1}{2d} \sum_{e:|e|=1} \sum_z q_\beta(z-x)\{\xi(z)[1-\xi(z+e)] - v(z)[1-v(z+e)]\} \tag{C.27}$$

$$\mathcal{B}_2^\epsilon = \epsilon^d \sum_x [v(x) - V(x)] \frac{\kappa'}{2d} \sum_{e:|e|=1} \sum_z q_\beta(z-x)\xi(z)[1-\xi(z+e)][e^{-u(z+e)} - e^{-U(z+e)}] \tag{C.28}$$

Remarks 3. *In Appendix D we prove that*

$$\lim_{\epsilon\to 0} E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'} \mathcal{S}_3^\epsilon(\xi_t,\eta_t)\Big| dt\Big) = 0 \tag{C.29}$$

We further observe that $|\mathcal{B}_2^\epsilon| + |\mathcal{B}_5^\epsilon| \le c(\|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2)$. *In Appendix E we prove that*

$$\lim_{\epsilon\to 0} E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'} \mathcal{Q}^\epsilon(\xi_t,\eta_t)\Big| dt\Big) = 0$$

- *Death process, ξ-particles.*

$$\begin{aligned}(L^{(\xi,-)} + D^{(\xi,-)})\|v-V\|^2 = -2\epsilon^d \sum_x [v(x)-V(x)] \sum_z q_\beta(z-x)\xi(z)\kappa_{\eta(z)+1}\\ +2\langle v-V, VG^-(U)\rangle + R_9^\epsilon\end{aligned}$$

where R_9^ϵ is defined in (8.37). As before,

$$\begin{aligned}&2\epsilon^d \sum_x [v(x)-V(x)] \sum_z q_\beta(z-x)\xi(z)\kappa_{\eta(z)+1}\\ &\quad = 2\mathcal{S}_4 + \epsilon^d \sum_x [v(x)-V(x)] \sum_z q_\beta(z-x)\xi(z) &&\text{(C.30)}\\ &\quad\quad \times \big\{G^-(U(x)) + [G^-(U(z) - G^-(U(x))] + [G^-(u(z)) - G^-(U(z))]\big\}\\ &\quad = 2\mathcal{S}_4^\epsilon + 2\langle v-V, vG^-(U)\rangle + R_{12}^\epsilon + 2\mathcal{B}_3^\epsilon &&\text{(C.31)}\end{aligned}$$

where

$$\mathcal{S}_4 = \epsilon^d \sum_x [v(x)-V(x)] \sum_z q_\beta(z-x)\xi(z)\big\{\kappa_{\eta(z)+1} - \chi_{M,z}(\eta)G^-(u(z))\big\} \tag{C.32}$$

$$\mathcal{B}_3 = \epsilon^d \sum_x [v(x)-V(x)] \sum_z q_\beta(z-x)\xi(z)\chi_{M,z}(\eta)[G^-(u(z)) - G^-(U(z))] \tag{C.33}$$

$$R_{12}^\epsilon = 2\epsilon^d \sum_x [v(x) - V(x)] \sum_z q_\beta(z-x)\xi(z)\Big\{[G^-(U(z)) - G^-(U(x))] \\ -(1-\chi_{M,z}(\eta))G^-(U(z))\Big\} \tag{C.34}$$

We thus have

$$(L^{(\xi,-)} + D^{(\xi,-)})\|v-V\|^2 = -2\langle v-V,[v-V]G^-(U)\rangle - 2\mathcal{S}_4 - 2\mathcal{B}_3 + R_9^\epsilon - R_{12}^\epsilon \tag{C.35}$$

Remarks 4. *Since $G^-(U^\epsilon)$ is uniformly bounded in compact time intervals, there is a constant c_M (which depends on M), so that $|\mathcal{B}_3^\epsilon| \le c_M(\|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2)$. In Appendix D it is proven that the term $\mathcal{S}_4^\epsilon$ gives a vanishing contribution as in* (C.29). *Finally R_{12}^ϵ is an R term because of* (8.2).

C.3 Estimates of the remainders

From (C.2), (C.10) and (C.16) we get that

$$(L+D)\|u-U\|^2 = -2\|u-U\|_{H_1^2}^2 + 2\kappa\|u-U\|^2 - 2(\mathcal{B}_0^\epsilon + \mathcal{B}_1^\epsilon) + \mathfrak{R}_1^\epsilon \tag{C.36}$$

where

$$\mathfrak{R}_1^\epsilon(\xi,\eta) = -2\mathcal{S}_2^\epsilon - 2\mathcal{C}_2^\epsilon + \sum_{k=1,2,3,6,7} R_k^\epsilon \tag{C.37}$$

We now observe that the first term in (C.36) is non positive and that from Remarks 2 we get that given any $\tau > 0$ there are c_1 and c_2 so that

$$\Big|\int_0^\tau (L+D)\{\|u^\epsilon(\eta_t) - U^\epsilon(t)\|^2\}dt\Big| \le \int_0^\tau \{c_1\|u^\epsilon(\eta_t) - U^\epsilon(t)\|^2 + c_2\|v^\epsilon(\xi_t) - V^\epsilon(t)\|^2\} \\ + \int_0^\tau \mathfrak{R}_1^\epsilon(\xi_t,\eta_t)dt \tag{C.38}$$

$$\lim_{\epsilon\to 0} E_{\mu^\epsilon}^\epsilon\Big(\Big|\sup_{\tau'\le\tau}\int_0^{\tau'} \mathfrak{R}_1^\epsilon(\xi_t,\eta_t)\Big|\Big) = 0 \tag{C.39}$$

From (C.3), (C.26) and (C.35) we get that

$$(L+D)\|v-V\|^2 = -2\epsilon^{-2a}\langle\nabla(v-V), e^{-U}\nabla(v-V)\rangle + 2\mathcal{H} \\ -2\langle v-V,[v-V]G^-(U)\rangle + \mathcal{B}_2^\epsilon - 2\mathcal{B}_3^\epsilon + \mathcal{B}_5^\epsilon + \mathfrak{R}_2^\epsilon \tag{C.40}$$

where

$$\mathfrak{R}_2^\epsilon(\xi,\eta) = 2(\mathcal{S}_1^\epsilon + \kappa'\mathcal{S}_3^\epsilon + \mathcal{S}_4^\epsilon) + 2\mathcal{C}_1^\epsilon + 2\kappa'\mathcal{Q}^\epsilon + \sum_{k=3,4,8,9,10,12} R_k^\epsilon \tag{C.41}$$

We now observe that the third term is non negative and by Remarks 1, 3 and 4 we get given any $\tau > 0$ there are c_3 and c_4 so that

$$\Big|\int_0^\tau (L+D)\{\|v^\epsilon(\eta_t) - V^\epsilon(t)\|^2\}dt\Big| \le \int_0^\tau \{c_3\|u^\epsilon(\eta_t) - U^\epsilon(t)\|^2 + c_4\|v^\epsilon(\xi_t) - V^\epsilon(t)\|^2\} + \int_0^\tau \mathfrak{R}_2^\epsilon(\xi_t,\eta_t)dt \tag{C.42}$$

$$\lim_{\epsilon\to 0} E^\epsilon_{\mu^\epsilon}\Big(\Big|\sup_{\tau'\le\tau}\int_0^{\tau'} \mathfrak{R}_2^\epsilon(\xi_t,\eta_t)dt\Big|\Big) = 0 \tag{C.43}$$

D The $\mathcal{S}$ terms

Denote by X_i^ϵ, $i = 1, .., 4$, the terms in (4.5)-(4.6) which contain $\mathcal{S}_i^\epsilon$, (their explicit expressions will be recalled below). The most dangerous one is X_1^ϵ because it appears with a divergent multiplicative factor ϵ^{-2a}, see (D.1). We thus start from this one, although the following analysis covers, except for some coefficients which are different, the easier term $\mathcal{S}_3^\epsilon$, where the dangerous factor ϵ^{-2a} is absent.

Calling

$$a_1^\epsilon(t,y) = \sum_{x,|e|=1} [v^\epsilon(x,\xi_t) - V^\epsilon(x,t)][(-e)\cdot\nabla + e\cdot\nabla]q_\beta(y-e-x)\xi_t(y-e)(1-\xi_t(y))$$

the term of the remainder containing $\mathcal{S}_1^\epsilon$ is

$$X_1^\epsilon := \epsilon^{-2a}E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'}\epsilon^d\sum_y a_1^\epsilon(t,y)[\mathbf{1}_{\eta_t(y)=0} - e^{-u^\epsilon(y,\eta_t)}]\Big|\Big) \tag{D.1}$$

We have

$$|X_1^\epsilon| \le \epsilon^{-2a}E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'}\epsilon^d\sum_y[\mathbf{1}_{\eta_{t+\epsilon^{2\alpha}}(y)=0} - e^{-u^\epsilon(y,\eta_t)}]\}\Big|\Big) + c\epsilon^{2(\alpha-a)} \tag{D.2}$$

because the rate of change of $\xi_t(x)$ is bounded by $2d\epsilon^{-2a} + ce^{b\eta_t(x)}$ and (D.2) follows from Theorem 5. Thus

$$\begin{aligned} X_1^\epsilon &\le \epsilon^{-2a}E^\epsilon_{\mu^\epsilon}\Big(\int_0^\tau\Big|\epsilon^d\sum_y a_{t,y}[\mathbf{1}_{\eta_{t+\epsilon^{2\alpha}}(y)=0} - e^{-u^\epsilon(y,\eta_t)}]\Big|\Big) + c\epsilon^{2(\alpha-a)} \\ &\le \epsilon^{-2a}E^\epsilon_{\mu^\epsilon}\Big(\int_0^\tau\Big|\epsilon^d\sum_y a_{t,y}[\mathbf{1}_{\eta_{t+\epsilon^{2\alpha}}(y)=0} - e^{-u^\epsilon(y,\eta_t)}]\Big|^2\Big)^{1/2} + c\epsilon^{2(\alpha-a)} \\ &\le \epsilon^{-2a}\|a\|\tau\sup_{t\le\tau}\sup_{|y-y'|\ge\epsilon^{-1+\alpha/2}}|E^\epsilon_{\mu^\epsilon}\big(f_t(y)f_t(y')\big)| + c\epsilon^{2(\alpha-a)} + c'\epsilon^{\alpha d/4-2a} \end{aligned} \tag{D.3}$$

where, $\|a\|$ is the sup norm of $a_{t,y}$ and

$$f_t(y) = [\mathbf{1}_{\eta_{t+\epsilon^{2\alpha}}(y)=0} - e^{-u^\epsilon(y,\eta_t)}] \tag{D.4}$$

By Theorem 9, $|E^\epsilon_{\mu^\epsilon}\big(f_t(y)f_t(y')\big)| \le c\epsilon^{2\alpha}$ so that $X^\epsilon_1 \to 0$, thus concluding the analysis of $\mathcal{S}^\epsilon_1$ (and of $\mathcal{S}^\epsilon_3$, as well).

Recalling (C.32),

$$X^\epsilon_4 := E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'}\epsilon^d\sum_y a^\epsilon_4(t,y)\big\{[\kappa_{\eta_t(y)+1} - \chi_{M,y}G^-(u^\epsilon(y,\eta_t)]\big\}\Big|\Big)$$
$$a^\epsilon_4(t,y) = \sum_x [v^\epsilon(x,\xi_t) - V^\epsilon(x,t)]q_\beta(x,y)\xi(y)$$

Analogously to (D.2), we get

$$\begin{aligned}|X^\epsilon_4| &\le E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'}\epsilon^d\sum_y[\kappa_{\eta_{t+\epsilon^{2\alpha}}(y)+1} - \chi_{M,y}G^-(u^\epsilon(y,\eta_t))]\Big|\Big) + c\epsilon^{2\alpha}\\ &\le E^\epsilon_{\mu^\epsilon}\Big(\int_0^{\tau}\epsilon^d\sum_y\Big|\kappa_{\eta_{t+\epsilon^{2\alpha}}(y)+1} - \chi_{M,y}G^-(u^\epsilon(y,\eta_t))\Big|\Big) + c\epsilon^{2\alpha}\\ &\le E^\epsilon_{\mu^\epsilon}\Big(\int_0^{\tau}\epsilon^d\sum_y\chi_{M,y}\Big|\kappa_{\eta_{t+\epsilon^{2\alpha}}(y)+1} - G^-(u^\epsilon(y,\eta_t))\Big|\Big) + c[\epsilon^{2\alpha} + \epsilon^{(1-\alpha)d/2}]\end{aligned}$$

having used Cauchy-Schwartz and Theorem 5 in the last inequality. Hence

$$X^\epsilon_4 \le \tau\sup_{t\le\tau}\sup_{|y-y'|\ge\epsilon^{-1+\alpha/2}}|E^\epsilon_{\mu^\epsilon}\big(g_t(y)g_t(y')\big)| + c\epsilon^{2\alpha} + c'\epsilon^{\alpha d/4-2a}$$
$$g_t(y) = \chi_{M,y}[\kappa_{\eta_{t+\epsilon^{2\alpha}}(z)+1} - G^-(u^\epsilon(z,\eta_t)] \tag{D.5}$$

Since

$$G^-(u) = \mathcal{E}_u\Big(\kappa_{\eta(0)+1}\Big), \quad \mathcal{E}_u\big(\eta(0)\big) = u \tag{D.6}$$

($\mathcal{E}_u$ is the expectation w.r.t. the Poisson law on $\mathbb{N}$ which has density u), by Theorem 10 we conclude that $X^\epsilon_4 \to 0$.

We finally consider the remainder containing $\mathcal{S}^\epsilon_2$: it consists of the sum of two terms, see (C.12), whose structures are essentially similar. For simplicity we only consider the first one. We have

$$\begin{aligned}X^\epsilon_2 := E^\epsilon_{\mu^\epsilon}\Big(\sup_{\tau'\le\tau}\Big|\int_0^{\tau'}\epsilon^d\sum_x[u^\epsilon(x,\eta_t) - U^\epsilon(x,t)]\sum_z p_\alpha(z-x)\\ \times\big[\eta_t(z)\kappa_{\eta_t(z)} - F^-_1(u^\epsilon(z,\eta_t))\big]\Big|\Big)\end{aligned} \tag{D.7}$$

where

$$F^-_1(u) = \mathcal{E}_u\Big(\eta(0)\kappa_{\eta(0)}\Big) \tag{D.8}$$

and, again,

$$X_2^\epsilon \le M\tau \sup_{t\le\tau} \sup_{|y-y'|\ge\epsilon^{-1+\alpha/2}} E^\epsilon_{\mu^\epsilon}\big(h_t(y)h_t(y')\big) + c\epsilon^{2(\alpha(7/4)} + c'\epsilon^{\alpha d/4-2a} + c''\epsilon^{(1-\alpha)d/2}$$

$$h_t(y) = \eta_{t+\epsilon^{2\alpha}}(z)\kappa_{\eta_{t+\epsilon^{2\alpha}}(z)} - F_1^-(u^\epsilon(z,\eta_t)) \tag{D.9}$$

where the error $c\epsilon^{2\alpha(7/4)}$ comes from having used Theorem 8 to express $u^\epsilon_{t+\epsilon^{2\alpha}}(\cdot)$ as a linear combination of $u^\epsilon_t(\cdot)$. We have then used Cauchy-Schwartz and Theorem 5 to bound the contribution of $u^\epsilon > M$.

E The term $\mathcal{Q}$

In this appendix we will prove that for any $\tau > 0$,

$$\lim_{\epsilon\to 0} E^\epsilon_{\mu^\epsilon}\big(\sup_{\tau'\le\tau}\big|\int_0^{\tau'} \mathcal{Q}(\xi_t)\big|\big) = 0 \tag{E.1}$$

with $\mathcal{Q}(\xi)$ defined in (C.27).

Since $|v^\epsilon(x,\xi_t) - V^\epsilon(x,t)| \le 1$, $|\mathcal{Q}(\xi_t)| \le \epsilon^d \sum_{x\in\Omega_\epsilon} S_x|f^\epsilon(\xi_t)|$, where

$$\begin{aligned} f^\epsilon(\xi) &:= \kappa'\{\sum_z \frac{1}{2d}\sum_{e:|e|=1} q_\beta(z)\xi(z)[1-\xi(z+e)]\} - v^\epsilon(0,\xi)[1-v^\epsilon(0,\xi)] \\ &= \kappa'\Big(v^\epsilon(0,\xi)^2 - \sum_z \frac{1}{2d}\sum_{e:|e|=1} q_\beta(z)\xi(z)\xi(z+e)\Big) \end{aligned} \tag{E.2}$$

and $|f^\epsilon(\xi)| \le \kappa'$. For $T > 0$,

$$\begin{aligned} E^\epsilon_{\mu^\epsilon}\big(\sup_{\tau'\le\tau}\big|\int_0^{\tau'} \mathcal{Q}(\xi_t)\big|\big) &\le E^\epsilon_{\mu^\epsilon}\big(\sup_{\tau'\le\tau}\int_0^{\tau'} |\mathcal{Q}(\xi_t)|\big) = E^\epsilon_{\mu^\epsilon}\big(\int_0^{\tau} |\mathcal{Q}(\xi_t)|\big) \le \tau\nu^\epsilon_\tau(|f^\epsilon|) \\ &\le \tau E^\epsilon_{\nu^\epsilon_\tau}\big(|f^\epsilon(\xi_T)|\big) + 2T\kappa' \end{aligned} \tag{E.3}$$

with ν^ϵ_τ defined in (11.1). It follows in fact from (11.1) that ν^ϵ_τ is invariant under space translations and that for any bounded function f

$$\Big|E^\epsilon_{\nu^\epsilon_\tau}\big(f(\eta_t,\xi_t)\big) - \nu^\epsilon_\tau\big(f(\eta,\xi)\big)\Big| \le \frac{2t}{\tau}\sup_{s\le\tau+t}\epsilon^d\sum_{x\in\Omega_\epsilon} E^\epsilon_{\mu^\epsilon}\big(|S_x f(\eta_s,\xi_s)|\big) \tag{E.4}$$

Hence the last term in (E.3), recalling that $|f^\epsilon(\xi)| \le \kappa'$.

By choosing $T = e^M\epsilon^{2\beta}$ with M as in (8.2), we get

$$\text{l.h.s. of (E.3)} \le \tau E^\epsilon_{\nu^\epsilon_\tau}\Big(\mathbf{1}_{u^\epsilon(0,\eta)<M}|f^\epsilon(\xi_T)|\Big) + 2e^M\epsilon^{2\beta} + \tau\kappa'\epsilon^{(1-\alpha)d} \tag{E.5}$$

We will then prove (E.1) by showing that

$$\lim_{\epsilon\to 0} E^\epsilon_{\nu^\epsilon_\tau}\Big(\mathbf{1}_{u^\epsilon(0,\eta)<M} f^\epsilon(\xi_T)^2\Big) = 0, \qquad T = e^M\epsilon^{2\beta} \tag{E.6}$$

We write $\underline{x}$ for a subset of Ω_ϵ, $|\underline{x}|$ for its cardinality and call

$$g_{\underline{x}}(\xi) = \prod_{x\in\underline{x}} \xi(x) \tag{E.7}$$

Then $f^\epsilon(\xi)^2 = \sum_{\underline{x}} c(\underline{x})g_{\underline{x}}(\xi)$ ($c(\underline{x})$ numerical coefficients) and

$$\Big| f^\epsilon(\xi)^2 - \sum_{\substack{\underline{x}:|\underline{x}|=4\\ \underline{x}\subset B}} c(\underline{x})g_{\underline{x}}(\xi)\Big| \le c\epsilon^{(a-\beta)d}, \quad B = \{x : |x| \le \epsilon^{-a+\beta/2}\} \tag{E.8}$$

where,

$$\sum_{\underline{x}:|\underline{x}|=4} |c(\underline{x})| \le c, \qquad \sum_{\underline{x}:|\underline{x}|<4} |c(\underline{x})| \le c\epsilon^{(a-\beta)d}, \quad \sum_{\underline{x}\not\subset B} |c(\underline{x})| \le ce^{\epsilon^{-\beta}} \tag{E.9}$$

Then (E.6) follows from (11.4). In fact

$$\sum_z \pi_{\epsilon^{-2(a-\beta)}e^{M-u^\epsilon(0,\eta_0)}}(x,z)\xi_0(z) = \sum_y \pi_{\epsilon^{-2(a-\beta)}[e^{M-u^\epsilon(0,\eta_0)}-1]}(x,y)\{\sum_z q_\beta(y,z)\xi_0(z)\}$$

$$= \sum_y \pi_{\epsilon^{-2(a-\beta)}[Me^{-u^\epsilon(0,\eta_0)}-1]}(x,y)v^\epsilon(y,\xi_0) = v^\epsilon(0,\xi_0) + R^\epsilon$$

$$R^\epsilon = \sum_y e^{\epsilon^{-2(a-\beta)}[e^{M-u^\epsilon(0,\eta_0)}-1]}(x,y)[v^\epsilon(y,\xi_0) - v^\epsilon(0,\xi_0)]$$

By Theorem 6,

$$E^\epsilon_{\nu^\epsilon_\tau}\Big(|v^\epsilon(y,\xi_0) - v^\epsilon(0,\xi_0)|\Big) \le c\epsilon^a|y| \tag{E.10}$$

so that $|R^\epsilon| \le c'\epsilon^a\epsilon^{-a+\beta}$. Then

$$\lim_{\epsilon\to 0} \sup_{\underline{x}\not\subseteq B,|\underline{x}|=4} \Big| E^\epsilon_{\nu^\epsilon_\tau}\Big(\mathbf{1}_{u^\epsilon(0,\eta)<M}\, g_{\underline{x}}(\xi_T)\Big) - E^\epsilon_{\nu^\epsilon_\tau}\Big(v^\epsilon(y,\xi_0)^4\Big)\Big| = 0$$

Hence (E.6), using (E.8)-(E.9), thus concluding the proof of (E.1).

F Proof of Theorem 3. Conclusion

With reference to (C.38) and (C.42), we call A the 2×2, matrix with entries c_1, c_2, c_3, c_4 and we define the following two dimensional vector $\mathbf{R}^\epsilon(t) = (R^\epsilon_1(t), R^\epsilon_2(t))$

$$R^\epsilon_i(t) := \int_0^t \mathfrak{R}^\epsilon_i(\xi_s,\eta_s)ds + M_i(t), \qquad i = 1,2 \tag{F.1}$$

where $M_1(t)$ and $M_2(t)$ are the mean zero martingales defined in (4.5) and (4.6) respectively. Thus (6.1) is proven. From (C.39) and (C.43) it follows that (6.2) holds for the first term on the right hand side of (F.1).

We are thus left with the proof that also the martingale terms verify (6.2). We first notice that

$$\left(\mathbb{E}_{\mu_\epsilon}\left(\sup_{s\le t}|M_i(s)|\right)\right)^2 \le \mathbb{E}_{\mu_\epsilon}\left(\sup_{s\le t} M_i(s)^2\right) \le 4\mathbb{E}^\epsilon_{\mu_\epsilon}\left(M_i(t)^2\right), \quad i=1,2 \tag{F.2}$$

and then write (see for instance Chapter 2 of [4]):

$$M_i(t)^2 - \int_0^t \gamma_i(s)ds =: N_i(t) \text{ is a martingale }, \qquad i=1,2 \tag{F.3}$$

where the *compensators* γ_i of M_i^2, $i=1,2$ are given in (F.4) below,

$$\gamma_i(s) = LX_i^2 - 2X_iLX_i, \quad i=1,2 \tag{F.4}$$

where

$$X_1(s) = \|u^\epsilon(\eta_s) - U^\epsilon(s)\|^2, \qquad X_2(s) = \|v^\epsilon(\xi_s) - V^\epsilon(s)\|^2$$

Going back to (F.3), we have for all $t>0$,

$$\mathbb{E}^\epsilon_{\mu_\epsilon}\left(M_i(t)^2\right) = \mathbb{E}^\epsilon_{\mu_\epsilon}\left(N_i(t) + \int_0^t \gamma_i(s)ds\right) \le \mathbb{E}^\epsilon_{\mu_\epsilon}\left(X_i(0)^2\right) + t\sup_{s\le t}\mathbb{E}^\epsilon_{\mu_\epsilon}\left(|\gamma_i(s)|\right) \tag{F.5}$$

Since by Theorem 1 the first term on the right hand side of (F.5) vanishes in the limit $\epsilon\to 0$, the proof of Theorem 3 is concluded by the next Lemma.

Lemma 1. *For any $t>0$,*

$$\lim_{\epsilon\to 0}\sup_{s\le t}\mathbb{E}^\epsilon_{\mu_\epsilon}\left(|\gamma_i(s)|\right) = 0 \tag{F.6}$$

Proof. We first compute γ_1 and, recalling that $L^{(\eta)}$ is the sum of three generators, we get for each of them the sum of three terms that we classify as $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ remainders.

$$L^{(\eta)}\|u-U\|^4 - 2\|u-U\|^2 L^{(\eta)}(\|u-U\|^2) = \sum_{i=1}^3 [\mathcal{A}_i + 4\mathcal{B}_i + 4\mathcal{C}_i] \tag{F.7}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= \epsilon^{-2}\epsilon^{2d}\sum_{x,y,z}\eta(z)|\nabla p_\alpha(x-z)|^2|\nabla p_\alpha(y-z)|^2\\
\mathcal{B}_1 &= \epsilon^{-2}\epsilon^{2d}\sum_{x,y,z}[u(y)-U(y)]\eta(z)\sum_{e:|e|=1}|e\cdot\nabla p_\alpha(x-z)|^2 e\cdot\nabla p_\alpha(y-z)\\
\mathcal{C}_1 &= \epsilon^{-2}\epsilon^{2d}\sum_{x,y,z}[u(y)-U(y)][u(x)-U(x)]\eta(z)\nabla p_\alpha(x-z)\cdot\nabla p_\alpha(y-z)
\end{aligned} \tag{F.8}$$

$$\mathcal{A}_2 = \kappa\epsilon^{2d}\sum_{x,y,z}\eta(z)p_\alpha(x-z)^2p_\alpha(y-z)^2$$
$$\mathcal{B}_2 = \kappa\epsilon^{2d}\sum_{x,y,z}[u(y)-U(y)]\eta(z)p_\alpha(x-z)^2p_\alpha(y-z)$$
$$\mathcal{C}_2 = \kappa\epsilon^{2d}\sum_{x,y,z}[u(y)-U(y)][u(x)-U(x)]\eta(z)p_\alpha(x-z)p_\alpha(y-z) \tag{F.9}$$

Finally, calling $\varphi(\eta,\xi;z) = \eta(z)\kappa_{\eta(z)}[1-\xi(z)] + \eta(z)\kappa_{\eta(z)+1}\xi(z)$,

$$\mathcal{A}_3 = \epsilon^{2d}\sum_{x,y,z}\varphi(\eta,\xi;z)p_\alpha(x-z)^2p_\alpha(y-z)^2$$
$$\mathcal{B}_3 = -\epsilon^{2d}\sum_{x,y,z}[u(y)-U(y)]\varphi(\eta,\xi;z)p_\alpha(x-z)^2p_\alpha(y-z)$$
$$\mathcal{C}_3 = \epsilon^{2d}\sum_{x,y,z}[u(y)-U(y)][u(x)-U(x)]\varphi(\eta,\xi;z)p_\alpha(x-z)p_\alpha(y-z) \tag{F.10}$$

By (8.1), (8.17) and (8.9) (details are omitted)

$$\lim_{\epsilon\to 0}\sum_{i=1}^{3}\sup_{s\le t}\mathbb{E}_{\mu_\epsilon}\left(\left|\mathcal{A}_i(s)+4\mathcal{B}_i(s)+4\mathcal{C}_i(s)\right|\right) = 0 \tag{F.11}$$

We next compute γ_2 and, as before, for each of the three generators we get the sum of three terms that we classify as $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$ and $\bar{\mathcal{C}}$ remainders.

$$L^{(\xi)}\|v-V\|^4 - 2\|v-V\|^2L^{(\eta)}(\|v-V\|^2) = \sum_{i=1}^{3}\left[\bar{\mathcal{A}}_i + 4\bar{\mathcal{B}}_i + 4\bar{\mathcal{C}}_i\right] \tag{F.12}$$

where, calling $\psi_e(\xi,\eta;z) = \xi(z)[1-\xi(z+e)]\mathbf{1}_{\eta(x+e)=0}$,

$$\bar{\mathcal{A}}_1 = \epsilon^{-2a}\epsilon^{2d}\sum_{x,y,z}\sum_{e:|e|=1}\psi_e(\xi,\eta;z)|\nabla q_\beta(x-z)|^2|\nabla q_\beta(y-z)|^2$$
$$\bar{\mathcal{B}}_1 = \epsilon^{-2a}\epsilon^{2d}\sum_{x,y,z}[v(y)-V(y)]\sum_{e:|e|=1}\psi_e(\xi,\eta;z)|e\cdot\nabla q_\beta(x-z)|^2 e\cdot\nabla q_\beta(y-z)$$

$$\bar{\mathcal{C}}_1 = \epsilon^{-2a}\epsilon^{2d}\sum_{x,y,z}[v(y)-V(y)][v(x)-V(x)]\psi_e(\xi,\eta;z)\nabla q_\beta(x-z)\cdot\nabla q_\beta(y-z)$$

$$\bar{\mathcal{A}}_2 = \frac{\kappa'}{2d}\epsilon^{2d}\sum_{x,y,z}\psi_e(\xi,\eta;z)q_\beta(x-z)^2q_\beta(y-z)^2$$
$$\bar{\mathcal{B}}_2 = \frac{\kappa'}{2d}\epsilon^{2d}\sum_{x,y,z}[v(y)-V(y)]\psi_e(\xi,\eta;z)q_\beta(x-z)^2q_\beta(y-z)$$
$$\bar{\mathcal{C}}_2 = \frac{\kappa'}{2d}\epsilon^{2d}\sum_{x,y,z}[v(y)-V(y)][v(x)-V(x)]\psi_e(\xi,\eta;z)q_\beta(x-z)q_\beta(y-z) \tag{F.13}$$

Finally, calling $\phi(\eta,\xi;z)=\xi(z)\kappa_{\eta(z)+1}$,

$$\begin{aligned}
\bar{\mathcal{A}}_3 &= \epsilon^{2d}\sum_{x,y,z}\phi(\eta,\xi;z)q_\beta(x-z)^2 q_\beta(y-z)^2 \\
\bar{\mathcal{B}}_3 &= -\epsilon^{2d}\sum_{x,y,z}[v(y)-V(y)]\phi(\eta,\xi;z)q_\beta(x-z)^2 q_\beta(y-z) \\
\bar{\mathcal{C}}_3 &= \epsilon^{2d}\sum_{x,y,z}[v(y)-V(y)][v(x)-V(x)]\phi(\eta,\xi;z)q_\beta(x-z)q_\beta(y-z)
\end{aligned} \tag{F.14}$$

Since the variables are bounded by 1, by (8.9) (details are again omitted)

$$\lim_{\epsilon\to 0}\sum_{i=1}^{3}\sup_{s\le t}\mathbb{E}_{\mu_\epsilon}\left(\left|\bar{\mathcal{A}}_i(s)+4\bar{\mathcal{B}}_i(s)+4\bar{\mathcal{C}}_i(s)\right|\right)=0 \tag{F.15}$$

Lemma 1 is proved. ■

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