Two scale hydrodynamic limit for a model of malignant tumor cells

by

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Abstract
We consider a model introduced in [10] with two species (η and ξ) of particles, representing respectively malignant and normal cells. The basic motions of the η particles are independent random walks, scaled diffusively. The ξ particles move on a slower time scale and obey an exclusion rule among themselves and with the η particles. The competition between the two species is ruled by a coupled birth and death process. We prove convergence in the hydrodynamic limit to a system of two reaction diffusion equations with measure valued initial data.

1 Introduction

We are reexamining a model for competition between malignant and normal cells introduced in [10]. The main point of the model is that there are three natural time scales. [Unfortunately in the biological applications] the fastest one is the random walk of the malignant cells; the next one is the time scale of motion of normal cells, described by an exclusion process where normal cells cannot jump over sites occupied either by normal or by malignant cells (we will refer to this for brevity as a coupled exclusion). Finally, the slowest time scale, called macroscopic, is the one described by a birth-death process which models the competition between the two species.

Calculating the local invariant measures is crucial for the behavior of the system on the macroscopic time scale. For technical reasons in [10] a stirring process was introduced to replace the coupled exclusion, not really motivated by the model itself, and, under such an assumption, the local invariant measures were found to be products of Poisson and Bernoulli measures.

Here we deal with the original model, with coupled exclusion, where the global invariant measures are not known, even if births and deaths are neglected. However, at least heuristically, the fact that on the time scale of the exclusion process the random walks of the malignant cells have already averaged out their positions, should give the stirring with
slowly varying coefficient process considered in [10]. But the usual methods of deriving hydrodynamic limits do not seem to apply here: the lack of knowledge of the invariant measures, even neglecting the birth-death process, seems to preclude the use of entropy methods. We refer to [9] for a general survey on entropy methods in hydrodynamic limits and to [11], [12] for specific applications to reaction-diffusion equations. On the other hand, the presence in the reaction terms of transcendental, non polynomial functions, makes it awkward an analysis of the BBGKY hierarchy of equations, as used in [4], [2] and [1].

So here we introduce a new method. The main point is that for our system, the energy estimate that holds for the deterministic limit, can also be obtained for mesoscopic averages of the stochastic system, obtaining $H^2$ a-priori bounds which allow to derive a substitute for the so called “two block estimate” of [8].

We also get bounds for the $L^2$ distance of our mesoscopic field variables from the deterministic solutions, derived by an explicit computation of the generator applied to the difference squared. To control the “most dangerous terms” we use homogenization techniques which play the role of the “one block estimates” in hydrodynamic limits, see again [8].

The different scales reflect also in the initial data. We suppose regularity of the malignant cells initial distribution on the macroscopic scale, while the normal cells distribution is only smooth on a smaller scale, related to the slower time scaling of their time evolution. This involves yet another homogenization process which leads to a limit coupled system of reaction diffusion equations with measure valued initial data for the normal cells.

In Section 2 we define the particles process, in Section 3 a discrete deterministic systems of reaction diffusion equations and in Section 4 mesoscopic variables. In Section 5 we discuss the choice of the initial state and in Section 6 we state our main results. In Section 7 we give a brief sketch of the proofs. In Section 8 we prove a priori bounds on exponential moments of the occupation variables and regularity in space of the mesoscopic fields. In Section 9 we prove regularity in time, while in Sections 10 and 11 we prove local ergodic theorems (one block estimates) for the normal and, respectively, malignant cells. We have shifted to some appendices the more computational parts of the proofs.

## 2 The particles model

We discretize the unit torus $\Omega$ of $\mathbb{R}^d$ by intersecting it with the lattice $\mathbb{Z}^d$, $\epsilon^{-1} \in \mathbb{N}$, and denote by $\Omega_\epsilon$ its image in stretched coordinates:

\[ \Omega_\epsilon = \{ x = (x_1, \ldots, x_d) \in \mathbb{Z}^d : 1 \leq x_i \leq \epsilon^{-1}, \; 1 \leq i \leq d \} \]  
(2.1)

As a rule we will use $x, y, z, \ldots$ for lattice sites and $r, r', \ldots$ for points in $\mathbb{R}^d$.

Particle configurations are non negative integer valued functions on $\Omega_\epsilon$ extended periodically to the whole $\mathbb{Z}^d$; in particular we consider configurations $\eta : \mathbb{Z}^d \to \mathbb{N}$ and $\xi : \mathbb{Z}^d \to \{0, 1\}$. $\eta(x)$ and $\xi(x), x \in \mathbb{Z}^d$, are thus the number of $\eta$ and $\xi$ particles at any site $y$ in $\Omega_\epsilon$ equal to $x$ modulo $\Omega_\epsilon$. $\eta$ and $\xi$ are interpreted as malignant and respectively normal cells.
The evolution is described by a Markov process whose generator $L$, defined on all functions of $(\eta, \xi)$ and whose dependence on $\epsilon$ is not made explicit, has the expression

$$L = L^{(\eta)} + L^{(\xi)}$$

$$L^{(\eta)} = \epsilon^{-2} L^{(\eta, 0)} + L^{(\eta, +)} + L^{(\eta, -)}$$

$$L^{(\xi)} = \epsilon^{-2a} L^{(\xi, 0)} + L^{(\xi, +)} + L^{(\xi, -)}$$

(2.2)

with

$$L^{(\eta, 0)} f(\eta, \xi) = \sum_{x \in \Omega_\epsilon} \sum_{|e| = 1} \eta(x) [f(\eta^{x+e}, \xi) - f(\eta, \xi)]$$

(2.3)

where, denoting by $1_x$ the configuration with only one particle at $x$, $\eta^{x+e} = \eta + 1_{x+e} - 1_x$, (same notation is used for $\xi$ configurations).

$$L^{(\xi, 0)} f(\eta, \xi) = \sum_{x \in \Omega_\epsilon} \sum_{|e| = 1} \xi(x) [1 - \xi(x+e)] 1_{\eta(x+e) = 0} [f(\eta, \xi^{x+e}) - f(\eta, \xi)]$$

(2.4)

Denoting below by $\kappa$, $\kappa'$ and $\kappa_i$ positive coefficients, the latter non decreasing functions of $i$ bounded by $\kappa_i \leq c e^{b_i}$, $b$ and $c$ positive constants, and writing $\eta^\pm = \eta \pm 1_x$, $\xi^\pm = \xi \pm 1_x$,

$$L^{(\eta, +)} f(\eta, \xi) = \sum_{x \in \Omega_\epsilon} \kappa \eta(x) [f(\eta^{x+e}, \xi) - f(\eta, \xi)]$$

$$L^{(\eta, -)} f(\eta, \xi) = \sum_{x \in \Omega_\epsilon} \eta(x) \left( \frac{\kappa}{\kappa(x)} (1 - \xi(x)) + \frac{\kappa}{\kappa(x) + 1} \xi(x) \right) [f(\eta^{x-}, \xi) - f(\eta, \xi)]$$

(2.5)

$$L^{(\xi, +)} f(\eta, \xi) = \sum_{x \in \Omega_\epsilon} \frac{\kappa'}{2d} \sum_{|e| = 1} \xi(x) [1 - \xi(x+e)] 1_{\eta(x+e) = 0} [f(\eta, \xi^{x+e, +}) - f(\eta, \xi)]$$

$$L^{(\xi, -)} f(\eta, \xi) = \sum_{x \in \Omega_\epsilon} \xi(x) \kappa_\eta(x+1) [f(\eta, \xi^{x-}) - f(\eta, \xi)]$$

(2.6)

The indices $0, +, -$ above refer respectively to displacements, births and deaths of particles, whose species is then indicated by $\eta$ and $\xi$. Notice that, as $\epsilon \to 0$, the scaling factors $\epsilon^{-2}$ and $\epsilon^{-2a}$ make displacements occur on a much faster scale than births and deaths with the $\eta$ particles moving much faster than the $\xi$ particles.

### 3 Discrete reaction-diffusion equations

The limit reaction diffusion equations that we will derive, can be approximated on the lattice $\Omega_\epsilon$, by the two equations

$$\frac{dU}{dt} = F(U, V), \quad \frac{dV}{dt} = G(U, V)$$

(3.1)
\( (U(t), V(t)) = \{(U(x, t), V(x, t)), x \in \Omega_\epsilon, t \geq 0 \} \) and where \( F = F(U, V) \) and \( G = G(U, V) \) are given by

\[
F(U, V) = \epsilon^{-2} \Delta U + \kappa U - F^{-1}_1(U) - F^{-2}_2(U)V
\]

\[
G(U, V) = e^{-U} \epsilon^{-2a} \Delta V + \kappa' V(1 - V)e^{-U} - VG^{-1}(U)
\]

with \( \Delta \) the discrete Laplacian on \( \mathbb{Z}^d \) (see (8.7), (5.3) and (8.8))

\[
F^{-1}_1(U) = e^{-U} \sum_{i \geq 0} \frac{U^i}{i!} i \kappa_i, \quad F^{-2}_2(U) = e^{-U} \sum_{i \geq 0} \frac{U^i}{i!} i [ \kappa_{i+1} - \kappa_i ]
\]

\[
G^{-1}(U) = e^{-U} \sum_{i \geq 0} \frac{U^i}{i!} \kappa_{i+1}
\]

By going to un-stretched coordinates, \( x \rightarrow \epsilon x, \Omega_\epsilon \rightarrow \Omega \cap \epsilon \mathbb{Z}^d \) and \( \epsilon^{-2} \Delta \rightarrow \Delta \) the discrete laplacian on \( \epsilon \mathbb{Z}^d \), (3.1) becomes the \( \epsilon \)-mesh discretization of the reaction diffusion system considered in [10].

Our aim is to prove closeness between (3.1) and the Markov process defined in the previous section. This will be achieved by introducing suitable functions \( (u(x, \eta), v(x, \xi)) \), \( x \in \Omega_\epsilon \), (defined in terms of convolutions of the original \( \eta \) and \( \xi \) variables) and comparing their evolution with the orbits \( (U(t), V(t)) \) of (3.1). An important step in this direction will be a comparison between the generator \( L \) and the generator \( D \) associated to (3.1), \( D \) being the transport operator with domain the space of all functions \( \psi(U, V) \) which are differentiable in all \( U(x) \) and \( V(x) \), and which is defined as

\[
D = F \frac{\partial}{\partial U} + G \frac{\partial}{\partial V} \equiv \sum_{x \in \Omega_\epsilon} F(x) \frac{\partial}{\partial U(x)} + G(x) \frac{\partial}{\partial V(x)}
\]

\( D \) transports along the orbits \( (U(t), V(t)) \) solutions of (3.1).

4 Mesoscopic variables

We denote by \( \pi_t \) the semigroup

\[
\pi_t = e^{t \Delta}
\]

with \( \Delta \) the discrete laplacian on \( \Omega_\epsilon \), so that the kernel \( \pi_t(x, y) \) is the probability that a simple random walk which jumps with intensity \( 2d \) with equal probability on its n.n. sites, reaches \( y \) at time \( t \) having started from \( x \) at time 0.

With \( \alpha < 1 \) and \( \beta < a \) positive parameters whose value will be instead specified later, we shorthand

\[
u(\eta) = p\alpha * \eta, \quad v(\xi) = q\beta * \xi, \quad p\alpha = \pi_{\epsilon^{-2+2\alpha}}, \quad q\beta = \pi_{\epsilon^{-2\alpha+2\beta}}
\]
and denote by \( u(x, \eta) \) and \( v(x, \xi) \) the values at \( x \) of \( u \) and \( v \).

We will use throughout the paper the notation:

\[
\langle f, g \rangle = \epsilon^d \sum_{x \in \Omega} f(x)g(x), \quad \|f\|^2 = \langle f, f \rangle \tag{4.3}
\]

The key for proving closeness between \((u(\eta_t), v(\xi_t))\) and \((U(t), V(t))\) will be a proof that \( L \) acts on functions of \((u, v)\) approximately as \( D \) acts on functions of \((U, V)\) (\(D\) is the transport generator defined in the previous section). We will make the statement quantitative, by studying the quantities

\[
(L + D)\|u - U\|^2, \quad (L + D)\|v - V\|^2 \tag{4.4}
\]

along the trajectories of the Markov process and the solutions of (3.1). Thus if \((u, v)\) are close to \((U, V)\) and \(L\) to \(D\), then the quantities in (4.4) will be small. The converse is (to some extent) also true, as it follows from the martingale theory. Indeed the expressions in (4.4) are the two compensators in the martingale relations

\[
\|u(\eta_t) - U(t)\|^2 - \|u(\eta_0) - U(0)\|^2 = \int_0^t (L + D)\|u(\eta_s) - U(s)\|^2 ds + M_1(t) \tag{4.5}
\]

\[
\|v(\xi_t) - V(t)\|^2 - \|v(\xi_0) - V(0)\|^2 = \int_0^t (L + D)\|v(\xi_s) - V(s)\|^2 ds + M_2(t) \tag{4.6}
\]

where \((U(s), V(s))\) solves (3.1) and \(M_1(t), M_2(t)\) are suitable martingales.

We will prove bounds for the two integrals on the r.h.s. of (4.5) and (4.6) in terms of \(\int_0^t \|u(\eta_s) - U(s)\|^2 + \|v(\xi_s) - V(s)\|^2\). We will also show that with probability going to 1 the martingales vanish in the limit as \(\epsilon \to 0\). All that will allow to reach an integral inequality in closed form for \(\|u(\eta_t) - U(t)\|^2 + \|v(\xi_t) - V(t)\|^2\) and to prove that the solution vanishes in the limit \(\epsilon \to 0\).

**Choice of \(\alpha\) and \(\beta\), assumptions on \(a\).**

We assume:

\[
\alpha < \frac{d}{d + 2}, \quad (\text{see Theorem 6}).
\]

\[
2a < (1 - \alpha)d, \quad 2a < 1, \quad \beta < a - \frac{d}{d + 2} \quad (\text{see Theorem 7}).
\]

\[
\beta < \frac{5a}{17}, \quad (\text{see Theorem 11}).
\]

5 Choice of the initial state

In this section we choose the initial law \(\mu^\epsilon\) of the process. It is now convenient to underline dependence on \(\epsilon\) and we do that by adding a superscript \(\epsilon\) when needed. The picture we
have in mind is that after a finite time $t_0$, during which the $\eta$ and the $\xi$ particles move independently of each other, the interaction is suddenly switched on with the evolution ruled, after this time, by the generator $L$. We count times from when the interaction starts, setting this time equal to 0.

We suppose that the state at time $-t_0$ is a fixed (but $\epsilon$ dependent) configuration $(\eta_{-t_0}, \xi_{-t_0})$. Due to the general class of processes that we are considering, it is convenient to assume that

$$\sup_{\epsilon > 0} \sup_{x \in \Omega} \eta_{-t_0}(x) = C < \infty \quad (5.1)$$

We do not want to make other assumptions on the configurations and we do not impose a particular behavior as $\epsilon \to 0$, which will instead determined by the same evolution. In this way our theory can cover many possible, interesting scenarios, for brevity we do not expand here the issue.

The probability $\mu^\epsilon$ at time 0 is defined as the law at time $t_0$ of the free process which starts from $(\eta_{-t_0}, \xi_{-t_0})$ and has generator $\epsilon^{-2}L(\eta,0) + \epsilon^{-2a}L(\xi,\text{st})$, where $L(\eta,0)$ is defined in (2.2) while $L(\xi,\text{st})$ is the generator of the stirring process acting on the $\xi$ variables. It is well established in the literature, see for instance [4], that $\mu^\epsilon$ is a “local equilibrium measure”, namely, to leading order in $\epsilon$, it is locally close to a product measure with the $\eta(x)$ variables close to Poisson. Moreover, the averages of the $\eta$ variables change smoothly on the scale $\epsilon^{-1}$, while the scale of the $\xi$ particles is $\epsilon^{-a}$, see Theorem 1 below, where

$$U^\epsilon_0(x) = \mu^\epsilon(u^\epsilon(x,\eta)), \quad V^\epsilon_0(x) = \mu^\epsilon(v^\epsilon(x,\xi)) \quad (5.2)$$

and where $\nabla$ is the lattice gradient:

$$e \cdot \nabla f(x) = f(x + e) - f(x), \quad |e| = 1$$

The following theorem is proved in Appendix A for the $\eta$ particles and in Appendix B for the $\xi$'s.

**Theorem 1.** $\mu^\epsilon$ is such that

$$\lim_{\epsilon \to 0} \mu^\epsilon(\|u^\epsilon(\eta) - U^\epsilon_0\|^2 + \|v^\epsilon(\xi) - V^\epsilon_0\|^2) = 0 \quad (5.4)$$

$$\sup_{\epsilon > 0} \sup_{x \in \Omega} \left| U^\epsilon_0(x) + \epsilon^{-1} \nabla U^\epsilon_0(x) \right| < \infty, \quad \sup_{\epsilon > 0} \sup_{x \in \Omega} \epsilon^{-a} |\nabla V^\epsilon_0(x)| < \infty \quad (5.5)$$

In the general setup we are considering there is no reason to expect convergence as $\epsilon \to 0$ of $U^\epsilon_0$ and $V^\epsilon_0$, but, by standard arguments, it is known that convergence is regained by going to subsequences:
Theorem 2. For any sequence $\epsilon' \to 0$ there is a subsequence $\epsilon \to 0$ so that the following holds.

- There exists a bounded, Lipschitz function $U_0(r)$, $r \in \Omega$, such that
  \[
  \lim_{\epsilon \to 0} \sup_{x \in \Omega} |U_0^\epsilon(x) - U_0(\epsilon x)| = 0 \tag{5.6}
  \]
- For each $r \in \Omega$, there is a translational invariant probability measure $\pi_r$ on the space of $[0,1]$-valued functions $V$ on $\mathbb{R}^d$ which are uniformly Lipschitz, such that for any positive integer $n$, any smooth function $F$ on $\mathbb{R}^n$ and any test functions $\phi, \phi_i, i = 1, \ldots, n$,\[
  \int F(\ldots, \phi \ast V, \ldots) \pi_r(dV) \text{ is a measurable function of } r \text{ and}
  \lim_{\epsilon \to 0} \epsilon^d \sum_{x \in \Omega} \phi(\epsilon x) F(\ldots, \epsilon^a \sum_y \phi_i(\epsilon^a y) V_0^\epsilon(x+y), \ldots) = \int_\Omega \phi(r) \int F(\ldots, \phi \ast V, \ldots) \pi_r(dV) \tag{5.7}
  \]

6 Main results

We suppose the law $\mu^\epsilon$ of $(\eta_0, \xi_0)$ (at time 0) as specified in the previous section and denote by $P^\epsilon_{\mu^\epsilon}$ and $E^\epsilon_{\mu^\epsilon}$ law and expectation of the process $(\eta_t, \xi_t)$ $t \geq 0$ with generator $L$ which starts from $\mu^\epsilon$. We will study the limit as $\epsilon \to 0$ of the process $(u^\epsilon(\eta_t), v^\epsilon(\xi_t))$ $t \geq 0$ along a subsequence convergent at time 0 in the sense of Theorem 2 ($\epsilon \to 0$ below, will always mean the limit along such a subsequence).

Theorem 3. Let $a$, $\alpha$ and $\beta$ verify the inequalities at the end of Section 4. Then there are a $2 \times 2$ matrix $A$ with constant, positive entries and a two vector $R^\epsilon(t)$ whose positive components depend on $(\eta_s, \xi_s)_{s \leq t}$, so that, for any $t \geq 0$,

\[
  \left( \left\| u^\epsilon(\eta_t) - U^\epsilon(t) \right\|^2 + \left\| v^\epsilon(\xi_t) - V^\epsilon(t) \right\|^2 \right) \leq \left( \left\| u^\epsilon(\eta_0) - U^\epsilon(0) \right\|^2 + \left\| v^\epsilon(\xi_0) - V^\epsilon(0) \right\|^2 \right) + \int_0^t A \left( \left\| u^\epsilon(\eta_s) - U^\epsilon(s) \right\|^2 + \left\| v^\epsilon(\xi_s) - V^\epsilon(s) \right\|^2 \right) ds + R^\epsilon(t) \tag{6.1}
  \]

\[
  \lim_{\epsilon \to 0} E^\epsilon_{\mu^\epsilon} \left( \sup_{s \leq t} R^\epsilon(s) \right) = 0 \tag{6.2}
  \]

As a corollary of Theorems 1 and 3, there is $C > 0$ so that, for any $\delta > 0$ and any $T > 0$,

\[
  \lim_{\epsilon \to 0} P^\epsilon_{\mu^\epsilon} \left( \sup_{t \leq T} \left\{ \left\| u^\epsilon(\eta_t) - U^\epsilon(t) \right\|^2 + \left\| v^\epsilon(\xi_t) - V^\epsilon(t) \right\|^2 \right\} \leq e^{CT} \delta \right) = 1 \tag{6.3}
  \]
which proves that the random fields \((u^{(\eta)}(t), v^{(\xi)}(t))\) become deterministic as \(\epsilon \to 0\) approaching the same limit behavior as \((U^{\epsilon}(t), V^{\epsilon}(t))\). The latter is described by the following, “two scale” reaction diffusion system.

**The limit evolution.** Let \(U(r, t), r \in \Omega, t \geq 0\), be a smooth, non negative, bounded function, \(U(r, 0) = U_0(r)\), \(U_0\) as in (5.6). For any \(r \in \Omega\), define the semigroup \(T_{t,r}\) on the space of Lipschitz function \(V(r^*) r^* \in \mathbb{R}^d\) with values in \([0, 1]\), by setting \(T_{t,r}(V)(r^*) = V(r^*, t)\) equal to the solution of

\[
\frac{dV(r^*, t)}{dt} = e^{-U(r, t)} \Delta V(r^*, t) + \kappa' V(r^*, t) \left[1 - V(r^*, t)\right] e^{-U(r, t)} - V(r^*, t) G^-(U(r, t)) \tag{6.4}
\]

with \(V(r^*, 0) = V(r^*)\). The dependence of \(T_{t,r}(V)\) on \(U(r, t)\) has not been made explicit. Call

\[
W(r, t) = \int T_{t,r}(V)(0) \pi_r(dV) \tag{6.5}
\]

where \(\{\pi_r(dV), r \in \Omega\}\) is the family of probabilities introduced in (5.7). Since \(T_{t,r}(V)\) has values in \([0, 1]\) and depends smoothly on \(V\), the r.h.s. of (6.5) is well defined and \(W(r, t)\) is a measurable function of \(r\). Define \(U^*(r, t), r \in \Omega, t \geq 0\), as the solution starting from \(U_0\) of

\[
\frac{dU^*}{dt} = \Delta U^* + \kappa U^* - F_1(U^*) - F_2(U^*)W \tag{6.6}
\]

We will say that \(U(r, t)\) is a fixed point of the above scheme (or just a fixed point) if, for given \(U_0\) and \(\{\pi_r(dV), r \in \Omega\}\) as above, \(U(r, t) = U^*(r, t)\) for all \(r \in \Omega\) and \(t \geq 0\). By standard arguments, details are omitted, we have:

**Theorem 4.** There is a unique fixed point \(U(r, t)\) of the above scheme. Moreover, if \(a, \alpha\) and \(\beta\) are as in Theorem 3, then for any \(t\) and \(\delta\) positive, any positive integer \(n\), any bounded, smooth function \(f\) on \(\mathbb{R}^n_+\) and any test functions \(\phi, \phi_1, .., \phi_n\),

\[
\lim_{\epsilon \to 0} \epsilon^d \sum_{x \in \Omega_n} \phi(\epsilon x) U^\epsilon(x, t) = \int_{\Omega} \phi(r) U(r, t) \, dr \tag{6.7}
\]

\[
\lim_{\epsilon \to 0} \epsilon^d \sum_{x \in \Omega_n} \phi(\epsilon x) f(\epsilon^a \sum_y \phi_i(\epsilon^a y) V^\epsilon(x + y, t), ...) = \int_{\Omega} \phi(r) \int f(\epsilon^a \sum \phi_i \ast V, ...) \pi_{r,t}(dV) \tag{6.8}
\]
7 Scheme of proof

By default, in the sequel $a$, $\alpha$ and $\beta$ satisfy the inequalities stated at the end of Section 4. The proof of Theorem 3 is based on the analysis of the two martingale relations (4.5)-(4.6). The whole art of the matter is to prove that the “two compensators”, namely the two integrals on the l.h.s. of (4.5)-(4.6), can be written as sum of terms which fall in the following two categories. The first one is made by elements which are bounded by time integrals of $\|u^\epsilon(\eta_t) - U^\epsilon(t)\|^2$ or $\|v^\epsilon(\eta_t) - V^\epsilon(t)\|^2$, multiplied by coefficients which are uniformly bounded. These terms then contribute to the integral in (6.1). All the other terms must be proved to vanish as $\epsilon \to 0$, so that they contribute to the error term $R^\epsilon(t)$ in (6.1).

A common feature to the analysis of all terms, is to control the large values of the variables $\eta$. A-priori $L^\infty$ bounds are derived in Section 8, where we show uniform in $\epsilon$ integrability of exponential moments $e^{bn(x)}$, for any $b > 0$. The result is quite standard as the process can be stochastically bounded by one having only linear births. More subtle is another bound that we use extensively in the proofs, namely that the probability that $u^\epsilon(x, \eta_t)$ exceeds a suitably large value $M = M(t)$ (but independent of $\epsilon$) vanishes as $\epsilon \to 0$. The proof of these statements is given in Appendix A, where we also recall from the literature results on independent random walks and random walks with independent branchings.

The other general ingredient, common to many of the proofs, is regularity in space of $u^\epsilon$ and $v^\epsilon$. In Section 8 we prove $H^2_1$ bounds uniform in $\epsilon$ which are obtained by mimicking the PDE proofs for the limit equations. Besides regularity in space we need also regularity in time of $u^\epsilon$. A result, maybe not optimal, but good enough for our applications, is proved in Section 9.

Such regularity estimates are the main subroutines we use to bound the “two compensators” (4.5) and (4.6). The detailed classification of all the terms which appear when computing explicitly the two compensators is reported in Appendix C. This is just some simple, but lengthy algebra, not at all deep, but necessary for the proof of Theorem 3, the compromise was to shift the computations to an appendix. Most of the terms in this expansion can be directly bounded using the boundedness and regularity estimates mentioned above, the bounds being uniform in $\epsilon$ and over compact time intervals. There are however some terms which do not fit in such an “easy class”. The origin of the problem is the typical one in the derivation of non linear hydrodynamical equations, where one needs to identify non linear microscopic observables in terms of the parameters of the limit equation: in our case we find local functions of $\eta_t$ and of $\xi_t$ and we need to express them in terms of (generally different) functions of $u^\epsilon(\eta_t)$ and $v^\epsilon(\xi_t)$, (which is easy if the functions are linear). The crucial point is that these non linear terms appear in the form of time and space averages and we will solve the problem by proving local ergodic properties of the process, reminiscent of the well known “one block estimates” in the theory of hydrodynamic limits. The “two block estimates” are here replaced by the $H^2_1$ regularity already mentioned. The one block estimates are not proved using Dirichlet forms, but closeness of the process in short times to a process with no deaths and births. The main difficulty here is that the $\xi$ process reminds of but it is not the stirring process, because the $\xi$ particles are allowed to jump only on sites where no $\eta$ particles are present. The local ergodic averages for the $\eta$ particles are easier to study, their analysis is reported in Section 10.
result for the $\xi$ particles is instead given in Section 11. The core of the proof is to show a homogenization property for which the $\xi$ particles move feeling to main order only the empirical average of the $\eta$'s. The real difficulty is to prove that such a property extends to such long times for the stirring to reach local equilibrium.

8 Boundedness and regularity in space

In this section we will prove $L^\infty$ and $H^2_1$ a priori bounds which will be extensively used in the proofs of Theorem 3. We start from the former, which are uniform bounds on the expectations of $\eta_t$ and $u^\epsilon(\eta_t)$:

**Theorem 5.** For any $b > 0$ and any $t \geq 0$

\[
\sup_{\epsilon > 0} \sup_{x \in \Omega_\epsilon} E^\epsilon_{\mu^\epsilon} \left( e^{b \eta_t(x)} \right) \leq \exp\{ [e^b - 1]e^{\kappa t} C \} \tag{8.1}
\]

with $\kappa$ as in Section 2 and $C$ as in (5.1). Moreover, for any $\tau > 0$ there are $M$ and $c$ so that

\[
\sup_{t \leq \tau} \sup_{x \in \Omega_\epsilon} P^\epsilon_{\mu^\epsilon} \left( u^\epsilon(x, \eta_t) \geq M \right) \leq ce^{(1-\alpha)d} \tag{8.2}
\]

**Proof.** Let $(\eta_t^+)_{t \geq 0}$ be the process with generator $\epsilon^{-2}L(\eta_t,0) + L(\eta_t^+)$ which starts from $\mu^\epsilon$. Then there is a coupling of $(\eta_t^+)_{t \geq 0}$ with the original process $(\eta_t, \xi_t)_{t \geq 0}$ (defined by the generator $L$ of (2.2)) such that $\eta^+_0 = \eta_0$ and $\eta^+_t \geq \eta_t$ for all $t > 0$. By an abuse of notation we still denote by $P^\epsilon_{\mu^\epsilon}$ and $E^\epsilon_{\mu^\epsilon}$, law and expectation w.r.t. the coupled process. In Appendix A it is proved that

\[
\sup_{\epsilon > 0} \sup_{x \in \Omega_\epsilon} E^\epsilon_{\mu^\epsilon} \left( e^{b \eta_t^+(x)} \right) \leq \exp\{ [e^b - 1]e^{\kappa t} C \} \tag{8.3}
\]

\[
\sup_{t \leq \tau} \sup_{x \in \Omega_\epsilon} E^\epsilon_{\mu^\epsilon} \left( \left[ u^\epsilon(x, \eta_t^+) - E^\epsilon_{\mu^\epsilon} (u^\epsilon(x, \eta_t^+)) \right]^2 \right) \leq ce^{(1-\alpha)d} \tag{8.4}
\]

Since $e^{b \eta_t^+(x)} \leq e^{b \eta_t^+(x)}$, (8.1) follows from (8.3).

Let $M = 2e^{\kappa \tau} C$, then, since $E^\epsilon_{\mu^\epsilon} (u^\epsilon(x, \eta_t^+)) \leq e^{\kappa t} C$,

\[
P^\epsilon_{\mu^\epsilon} \left( u^\epsilon(x, \eta_t^+) \geq M \right) \leq P^\epsilon_{\mu^\epsilon} \left( |u^\epsilon(x, \eta_t) - E^\epsilon_{\mu^\epsilon} (u^\epsilon(x, \eta_t^+))| \geq e^{\kappa \tau} C \right)
\]

so that

\[
P^\epsilon_{\mu^\epsilon} \left( u^\epsilon(x, \eta_t^+) \geq M \right) \leq \frac{4c}{M^2} e^{(1-\alpha)d} \tag{8.5}
\]
Since \( u^\epsilon(x, \eta_t) \leq u^\epsilon(x, \eta_t^+) \), (8.2) follows from (8.5).

We will next prove bounds on the \( H^2 \) norm of \( u^\epsilon \), which will play the role of the “two blocks estimates” in the language of hydrodynamic limit theory. Before stating definition and results, let us recall how similar bounds are obtained for the heat equation \( u_t = \Delta u \) in the unit torus \( \Omega \). The “entropy” \( \int_\Omega u^2 \, dr \) gives

\[
\int_\Omega u^2(t) \, dr - \int_\Omega u^2(0) \, dr = -2 \int_0^t \int_\Omega |\nabla u|^2 \, dr
\]

Then, supposing \( \int_\Omega u^2(0) \, dr \leq \infty \), for any \( t > 0 \),

\[
\int_0^t \int_\Omega |\nabla u|^2 \, dr < \frac{1}{2} \int_\Omega u^2(0) \, dr
\]

The proof of Theorem 6 below mimics the above argument, but let us first introduce some notation and definitions which translate to the lattice the analogous notions in the continuum. If \( f \) is a function on \( \Omega \), we write

\[
\|f\|_{H^2}^2 := \epsilon^{-2} \|\nabla f\|^2
\]

with \( \nabla f \) the lattice gradient of \( f \). We also recall that the same rules as in the continuum hold as well for the discrete gradient and laplacian. Namely, denoting by \( E_+ \) the set of unit vectors \( e \) with positive components, we have, recalling (5.3) for notation and resisting to the temptation of writing \( ( -e ) \cdot \nabla = - ( e \cdot \nabla ) \), which is false,

\[
\Delta f(x) = \sum_{e \in E_+} \left\{ ( -e ) \cdot \nabla + e \cdot \nabla \right\} f(x) = \sum_{e \in E_+} \left\{ ( -e ) \cdot \nabla \right\} \left\{ ( e \cdot \nabla ) \right\} f(x)
\]

\[
< g, e \cdot \nabla f > = < ( -e ) \cdot \nabla g, f >, \quad < g, \Delta f > = - < \nabla g, \nabla f >
\]

the last equality following from the second one in (8.7) and the first one in (8.8).

**Theorem 6.** For any \( t > 0 \), there is \( c \) so that

\[
\sup_{\epsilon > 0} E_{\mu^\epsilon} \left( \int_0^t \|u^\epsilon(\eta_s)\|^2_{H^2} \, ds \right) \leq c
\]

**Proof.** We start from the martingale relation:

\[
\|u^\epsilon(\eta_t)\|^2 = \|u^\epsilon(\eta_0)\|^2 + \int_0^t L^{(\eta)} \|u^\epsilon(\eta_s)\|^2 \, ds + M^\epsilon_t, \quad E_{\mu^\epsilon} \left( M^\epsilon_t \right) = 0
\]
After a simple computation which exploits that discrete gradient and laplacian satisfy the same relations as in the continuum, see (8.7)-(8.8),

\[ \epsilon^2 L^{(\eta,0)} ||u^\epsilon||^2 = -2 ||u^\epsilon||^2_{H_1^2} + R_1^\epsilon(\eta) \]  

(8.11)

\[ R_1^\epsilon(\eta) = 2\epsilon^d \sum_{x,z} \epsilon^{-2} |\nabla p_\alpha(x,z)|^2 \eta(z) \]  

(8.12)

\[ L^{(\eta,+)} ||u^\epsilon||^2 = 2\kappa ||u^\epsilon||^2 + R_2^\epsilon(\eta), \quad L^{(\eta,-)} ||u^\epsilon||^2 \leq R_3^\epsilon(\eta, \xi) \]  

(8.13)

\[ R_2^\epsilon(\eta) = \kappa \epsilon^d \sum_x \sum_z p_\alpha(x,z)^2 \eta(z) \]  

(8.14)

\[ R_3^\epsilon(\eta, \xi) = \epsilon^d \sum_x \sum_z p_\alpha(x,z)^2 \left[ \eta(z) \kappa_\eta(z) + \xi(z) (\kappa_\eta(z+1) - \kappa_\eta(z)) \right] \]  

(8.15)

(the seemingly random labelling of the remainder terms \( R_3^\epsilon \) is for “historical reasons”). By taking the expectation in (8.10),

\[ 2E_{\mu^\epsilon} \left( \int_0^t ||u^\epsilon(\eta_s)||^2_{H_1^2} ds \right) \leq E_{\mu^\epsilon} \left( ||u^\epsilon(\eta_0)||^2 \right) + E_{\mu^\epsilon} \left( \int_0^t R_1^\epsilon(\eta_s) + R_2^\epsilon(\eta_s) + R_3^\epsilon(\eta_s, \xi_s) + 2\kappa ||u^\epsilon(\eta_s)||^2 ds \right) \]  

(8.16)

By (8.1) there is \( c_1 = c_1(t) \), independent of \( \epsilon \), so that the r.h.s. of (8.16) is bounded by \( c_1(1 + \sum_z \epsilon^{-2} |\nabla p_\alpha(0,z)|^2 + p_\alpha(0,z)^2) \). There is \( c_2 \) so that,

\[ \sum_z \epsilon^{-2} |\nabla p_\alpha(z)|^2 \leq c_2 \epsilon^{-2+2(1-\alpha)+d(1-\alpha)} \sum_z p_\alpha(z)^2 \leq c_3 \epsilon^{d(1-\alpha)} \]  

(8.17)

which vanishes as \( \epsilon \to 0 \) because of the assumption \( \alpha < \frac{d}{d+2} \) and (8.9) is proved. ■

We have a \( H_1^2 \) bound for \( u^\epsilon \) as well, see Theorem 7 below, but we need first the following corollary of Theorem 6:

**Corollary 1.** For any \( z \in \Omega_\epsilon \) and \( t > 0 \)

\[ \int_0^t \epsilon^d \sum_{x \in \Omega_\epsilon} |u^\epsilon(x+z, \eta_s) - u^\epsilon(x, \eta_s)| \leq \sqrt{td} \epsilon |z| \left( \int_0^t ||u^\epsilon(\eta_s)||^2_{H_1^2} \right)^{1/2} \]  

(8.18)
Two scale hydrodynamic limits

Proof. For any $z$, there is “a coordinate curve” $\{y_i\}_{i=0,..,N}$ such that $y_0 = 0$, $y_N = z$, with $e_i := y_{i+1} - y_i$, $i = 0,..,N - 1$, a unit vector, and $\sum |e_i| = |z_1| + \cdots + |z_d|$, having denoted by $z_i$ the $i$-th component of $z$. Then, recalling (5.3) for notation,

$$u^\epsilon(x + z, \eta) - u^\epsilon(x, \eta) = \sum e_i \cdot \nabla u^\epsilon(x + y_i, \eta)$$

Hence

$$\epsilon^d \sum_{x \in \Omega} |u^\epsilon(x + z, \eta) - u^\epsilon(x, \eta)| \leq \sum e_i \cdot \nabla u^\epsilon(x + y_i, \eta)\]$$

$$\leq \sqrt{d} \left( \sum e_i \cdot \nabla u^\epsilon(x + y_i, \eta)^2 \right)^{1/2}$$

$$\leq \sqrt{d} |z| \|u^\epsilon(\eta_s)\|_{H^2}$$

and (8.18) follows by Cauchy-Schwartz.

\[8.19\]

Theorem 7. For any $t > 0$ there is $c$ so that

$$\sup_{\epsilon > 0} E^\epsilon_{\mu^t} \left( \int_0^t e^{-2a} \|\nabla v^\epsilon(\eta_s)\|^2 ds \right) \leq c$$

\[8.20\]

Proof. We start once again from a martingale relation:

$$\|v^\epsilon(\xi_t)\|^2 = \|v^\epsilon(\xi_0)\|^2 + \int_0^t L(\xi) \|v^\epsilon(\xi_s)\|^2 ds + M^\epsilon_t, \quad E^\epsilon_{\mu^t}(M^\epsilon_t) = 0$$

\[8.21\]

We use the following two identities (where $E_+$ is the set of unit vectors in $\mathbb{Z}^d$ with non negative components)

$$\sum_{z} \xi(z) \xi(z + e)[(-e) \cdot \nabla + e \cdot \nabla] q_{\beta}(z - x) = 0, \quad e \in E_+$$

\[8.22\]

$$\sum_{e \in E_+} \sum_{z} \xi(z)[(-e) \cdot \nabla + e \cdot \nabla] q_{\beta}(z - x) = \Delta v^\epsilon(x)$$

\[8.23\]

to get

$$e^{-2a} L(\xi_0) \|v^\epsilon\|^2 = 2e^{-2a} \langle v^\epsilon, e^{-u^\epsilon} \Delta v^\epsilon \rangle + 2S^\epsilon + 2C^\epsilon + R^\epsilon$$

\[8.24\]

where

$$S^\epsilon(\xi, \eta) = \sum_{e:|e|=1} e^d \sum_{x} v^\epsilon(x)e^{-2a} \sum_{z} [e \cdot \nabla q_{\beta}(z - x)] \xi(z)(1 - \xi(z + e))$$

$$\times \left[ 1_{\eta(z+e)=0} - e^{-u^\epsilon(z+e)} \right]$$

\[8.25\]
\[ C^\epsilon(\xi, \eta) = \epsilon^{-2a} \sum_{e:|e|=1} \epsilon^d \sum_x \sum_z \xi(z)(1 - \xi(z + e)) \left[ e^{-u^\epsilon(z+e)} - e^{-u^\epsilon(x)} \right] e \cdot \nabla q_\beta(z - x) \] (8.26)

\[ R_4^\epsilon(\xi, \eta) = \epsilon^{-2a} \sum_{e:|e|=1} \epsilon^d \sum_x \sum_z \xi(z)(1 - \xi(z + e)) \eta_{\varepsilon(z+e)=0} (e \cdot \nabla q_\beta(z - x))^2 \] (8.27)

The proof that
\[ \lim_{\epsilon \to 0} \left| \int_0^t E^\epsilon_{\mu^\epsilon} (\mathcal{S}^\epsilon(\xi_s, \eta_s)) ds \right| = 0 \] (8.28)
follows from Theorem 9 below (details are left to an appendix, Appendix D). We will use the present theorem only after Theorem 9, so that there is no circularity in our arguments.

From Corollary 1, it follows that
\[ \left| \int_0^t E^\epsilon_{\mu^\epsilon} (C^\epsilon(\xi_s, \eta_s)) ds \right| \leq c\epsilon^{1-2a} \] (8.29)

Similarly to (8.17), for a suitable constant \( c \), we have that
\[ |R_4^\epsilon(\xi, \eta)| \leq c\epsilon^{-2a+2(a-\beta)+d(a-\beta)} \] (8.30)

which also vanishes in the limit \( \epsilon \to 0 \) because of the assumption \( \beta < \frac{d}{d+2} a \). We denote by \( \chi_{M,x}(s) \) the characteristic function of \( u^\epsilon(x, \eta_s) \leq M \), and we write
\[ 2\epsilon^{-2a} \langle v^\epsilon, e^{-u^\epsilon} \Delta v^\epsilon \rangle \leq -2\epsilon^{-2a} e^{-M} \| \nabla v^\epsilon \|^2 + R_5^\epsilon + R_6^\epsilon \] (8.31)

where
\[ R_5^\epsilon = -2\epsilon^{-2a} \langle v^\epsilon, \nabla e^{-u^\epsilon} \nabla v^\epsilon \rangle, \quad R_6^\epsilon = -2\epsilon^{-2a} \langle [1 - \chi_M] \nabla v^\epsilon, e^{-u^\epsilon} \nabla v^\epsilon \rangle \] (8.32)

Since \( |\nabla e^{-f}| \leq |\nabla f| \) if \( f \geq 0 \), by (8.9), and since \( |v^\epsilon| \leq 1 \), \( |\nabla v^\epsilon| \leq 2d \),
\[ \int_0^t |R_5^\epsilon| \leq c\epsilon^{1-2a} \] (8.33)

By (8.2), we have
\[ \sup_{s \leq t} E^\epsilon_{\mu^\epsilon} (|R_6^\epsilon(\xi_s, \eta_s)|) \leq c\epsilon^{d(1-\alpha)-2a} \] (8.34)

We next observe that,
\[ L^{(\xi, \eta)} \| v^\epsilon \|^2 \leq 2\kappa \| v^\epsilon \|^2 + R_8^\epsilon, \quad L^{(\xi, \eta)} \| v^\epsilon \|^2 \leq R_9^\epsilon \] (8.35)

where
\[ R_8^\epsilon = \kappa \epsilon^d \sum_x \sum_{e:|e|=1} \sum_z q_\beta(x, z)^2 \xi(z) [1 - \xi(z + e)] \eta_{\varepsilon(z+e)=0} \] (8.36)
Then, by (8.8) 

\[ R_\theta = c^d \sum_x \sum_z q_\beta(x, z)^2 \xi(z) \kappa_\eta(z) + 1 \]  

(8.37)

Since

\[ \sum_z q_\beta(0, z)^2 \leq ce^{d(\alpha - \beta)} \]  

(8.38)

and since \( \kappa_n \leq ce^{bn} \), by (8.1), also \( R_8 \) and \( R_9 \) give a vanishing contribution. We have therefore proved that there is a positive function \( \varphi_\varepsilon(t) \to 0 \) as \( \varepsilon \to 0 \) such that

\[
\int_0^t E_{\mu^\varepsilon} \left( e^{-2aL(\xi)} \|v^\varepsilon(\xi_s)\|^2 \right) ds \leq -2e^{-2a} e^{-M} \int_0^t E_{\mu^\varepsilon} (\|\nabla v^\varepsilon(\xi_s)\|^2) \\
+ 2\kappa' \int_0^t \|v^\varepsilon(\xi_s)\|^2 ds + \varphi_\varepsilon(t) 
\]  

(8.39)

Thus from (8.21) and (8.39) we get

\[
ev^{-M} E_{\mu^\varepsilon} \left( \int_0^t e^{-2a \|\nabla v^\varepsilon(\xi_s)\|^2} ds \right) \leq E_{\mu^\varepsilon} (\|v^\varepsilon(\xi_0)\|^2) + \kappa' t + \varphi_\varepsilon(t)
\]  

(8.40)

Theorem 7 is proved.

We conclude the section with the following corollary of Theorem 6.

**Corollary 2.** For any \( t > 0 \) there is \( \bar{C} \) so that

\[
\sup_{c>0} \sup_{x \in \Omega} \sup_{s \leq t} |U^\varepsilon(x, s)| \leq \bar{C} 
\]  

(8.41)

Furthermore for any \( t > 0 \) there is \( c \) so that for any \( s \leq t \)

\[
\langle |v^\varepsilon(\xi_s) - V^\varepsilon(s)|, e^{-a} - e^{-U^\varepsilon(s)} e^{-2a \Delta V^\varepsilon(s)} \rangle \leq \frac{e^{-\bar{C}}}{10} e^{-2a} \|\nabla \{v^\varepsilon(\xi_s) - V^\varepsilon(s)\}\|^2 \\
+ c \left( \|u^\varepsilon(\eta_s) - U^\varepsilon(s)\|^2 + \|v^\varepsilon(\xi_s) - V^\varepsilon(s)\|^2 \\
+ c^{1-2a} \left\{ \|u^\varepsilon(\eta_s)\|_{H^1_t}^2 + \|U^\varepsilon(s)\|_{H^1_t}^2 \right\} \right) 
\]  

(8.42)

**Proof.** (8.41) follows from the first inequality in (5.5). From the second inequality in (5.5) it follows that the solution \( V^\varepsilon(x, t) \) of (3.1) is such that for any \( t \) there is \( C \) so that

\[
\sup_{c>0} \sup_{x \in \Omega, s \leq t} e^{-a |\nabla V^\varepsilon(x, s)|} = C < \infty 
\]  

(8.43)

Then, by (8.8)

\[
\langle |v^\varepsilon - V^\varepsilon|, e^{-a} - e^{-U^\varepsilon} e^{-2a \Delta V^\varepsilon} \rangle \leq C \left( \langle e^{-a} |\nabla \{v^\varepsilon - V^\varepsilon\}|, e^{-a} - e^{-U^\varepsilon} \rangle \\
+ \langle |v^\varepsilon - V^\varepsilon|, e^{-a} |\nabla \{e^{-u^\varepsilon} - e^{-U^\varepsilon}\}| \right) 
\]
Since $|e^{-u^\tau} - e^{-U^\tau}| \leq |u^\tau - U^\tau|$, using the inequality $|db| \leq \delta d^2 + \delta^{-1} b^2$, with $b = |u^\tau - U^\tau|$, $d = e^{-a} |\nabla \{u^\tau - V^\tau\}|$, and $\delta = e^{-C}(5C)^{-1}$, we get

$$
\langle [u^\tau - V^\tau], [e^{-u^\tau} - e^{-U^\tau}] \epsilon^{-2a} \Delta V^\tau \rangle \leq e^{-C} \frac{\epsilon^{-2a}}{10} |\nabla \{u^\tau - V^\tau\}|^2 + e^C (5C^2) |u^\tau - U^\tau|^2
$$

$$
+C \left( |u^\tau - V^\tau|^2 + \frac{\epsilon^{-2a}}{2} |\nabla u^\tau|^2 + \frac{\epsilon^{-2a}}{2} |\nabla U^\tau|^2 \right)
$$

having also used that $|\nabla e^{-f}| \leq |\nabla f|$ if $f \geq 0$. Recalling the definition of $\|f\|_{H^2_t}$ in (8.20) we then get (8.42).

9 Regularity in time

Besides regularity in space, we will also need estimates on the regularity of $u^\tau(x, \eta_t)$ as a function of the time $t$. This is needed in Section 11 and Appendix D, and the statement we will prove here is just what used in the sequel, with no aim at generality. We denote by $P^\epsilon_{\eta_t, \xi_t}$ and $E^\epsilon_{\eta_t, \xi_t}$ conditional law and expectation of the process $(\eta_t, \xi_t)_{s \geq t}$, given the state $(\eta_t, \xi_t)$ at time $t$.

**Theorem 8.** For any $\tau$ and $\gamma$ positive, with $2\gamma < (1 - \alpha)d$, there is $c$ so that for any $t \leq \tau$, $\epsilon > 0$, $x \in \Omega_e$ and any $s \in (t, e^{2\gamma}t]$

$$
E^\epsilon_{\mu^\tau_t} \left( |u^\tau(x, \eta_s) - \pi_{(s-t)\epsilon^{-2}} * u^\tau(x, \eta_t)| \right) \leq ce^{2\gamma}
$$

(9.1)

where $\pi_{(s-t)\epsilon^{-2}} * u^\tau(x, \eta_t) = \sum_y \pi_{(s-t)\epsilon^{-2}}(x, y) u^\tau(y, \eta_t)$.

**Proof.** We define an auxiliary process $(\theta_s, \xi_s)_{s \geq t}$, $\theta_s = (\eta_{F,a}^s, \eta_{F,d}^s, \eta_{b}^s)$ and the following random variables on this process,

$$
\eta_s := \eta_{F,a}^s + \eta_{b}^s, \quad \eta_{F}^s := \eta_{F,a}^s + \eta_{F,d}^s \quad (9.2)
$$

The main point of the definition will be that $(\eta_s, \xi_s)_{s \geq t}$ has the same law as our process of Section 2 (i.e. with generator $L$), while $\eta_{F}^s$ has the law of the independent process with generator $e^{-2L^{(\eta,0)}}$. The $\eta_{F,a}^s$ particles are called free and alive; $\eta_{F,d}^s$ free but dead; $\eta_{b}^s$ newly born, which already hints at the way the whole process will be defined. We set $\eta_{F,d}^t = \eta_{b}^t = 0$, so that $\eta_t = \eta_{F,a}^t$. The process $(\theta_s, \xi_s)$ at times $s \geq t$ is defined in terms of the generator $L$ which we set equal to $L = L_1 + L^{(\xi)}$, with $L^{(\xi)}$ as in Section 2 (reading
\( \eta = \eta^{(F,a)} + \eta^{(b)} \). \( L_1 \) is \( \epsilon^{-2}L_1^{(0)} + L_1^{(+)} + L_1^{(-)} \). \( L_1^{(0)} \) is the generator of independent motion for the three types of \( \eta \) particles.

\[
L_1^{(+)} f(\theta, \xi) = \sum_{x \in \Omega} \kappa \eta(x) [f(\theta + 1_{x,b}, \xi) - f(\theta, \xi)] \quad (9.3)
\]

where \( 1_{x,b}, 1_{x,(F,d)} \) and \( 1_{x,(F,a)} \) below are the configurations with only one particle at \( x \) respectively of type \( b \), \( (F,d) \) and type \( (F,a) \);

\[
L_1^{(-)} f(\theta, \xi) = \sum_{x \in \Omega} \eta(x) \left( \kappa_{\eta(x)} (1 - \xi(x)) + \kappa_{\eta(x)+1} \xi(x) \right) \\
\times \{1_{\eta^{(b)}(x)=0}[f(\theta - 1_{x,(F,a)} + 1_{x,(F,d)}, \xi) - f(\theta, \xi)] \]
\[
+ 1_{\eta^{(b)}(x)>0}[f(\theta - 1_{x,b}, \xi) - f(\theta, \xi)] \} \quad (9.4)
\]

It is then easy to check that \( (\eta_s, \xi_s) \) is our original process and \( \eta^{(F)} \) is the independent process.

By an abuse of notation we still denote by \( E_{\mu^t}^{\epsilon^2} \) the expectation relative to the process which after time \( t \) has generator \( \hat{L} \). Then

\[
E_{\mu^t}^{\epsilon^2} \left( u^\epsilon(x, \eta_s) - \pi_{(s-t)\epsilon^{-2} \ast u^\epsilon(x, \eta_t)} \right) \leq E_{\mu^t}^{\epsilon^2} \left( u^\epsilon(x, \eta^{(F)}_s) - \pi_{(s-t)\epsilon^{-2} \ast u^\epsilon(x, \eta_t)} \right) \\
+ E_{\mu^t}^{\epsilon^2} \left( u^\epsilon(x, \eta_s^{(b)}) - u^\epsilon(x, \eta^{(F,d)}_s) \right) \quad (9.5)
\]

Recalling that \( u^\epsilon(x, \eta_s^{(b)}) - u^\epsilon(x, \eta^{(F,d)}_s) = 0 \),

\[
E_{\mu^t}^{\epsilon^2} \left( u^\epsilon(x, \eta_s^{(b)}) - u^\epsilon(x, \eta^{(F,d)}_s) \right) = \int_t^\epsilon \sum_y \pi_{(s-s')\epsilon^{-2}}(x, y) \\
\times E_{\mu^t}^{\epsilon^2} \left( (L_1^{(+)} + L_1^{(-)}) \{u^\epsilon(y, \eta_s^{(b)}) - u^\epsilon(y, \eta^{(F,d)}_s) \} \right) \quad (9.6)
\]

Since

\[
(L_1^{(+)} + L_1^{(-)}) u^\epsilon(y, \eta^{(b)}) \leq \kappa u^\epsilon(y, \eta) \\
(L_1^{(+)} + L_1^{(-)}) u^\epsilon(y, \eta^{(F,d)}) \leq \sum_z p_{\alpha}(y, z) \eta(z) \kappa_{\eta(z)+1}
\]

by Theorem 5,

\[
E_{\mu^t}^{\epsilon^2} \left( u^\epsilon(x, \eta_s) - \pi_{(s-t)\epsilon^{-2} \ast u^\epsilon(x, \eta_t)} \right) \leq E_{\mu^t}^{\epsilon^2} \left( u^\epsilon(x, \eta^{(F)}_s) - \pi_{(s-t)\epsilon^{-2} \ast u^\epsilon(x, \eta_t)} \right) + c \epsilon^{2\gamma} \quad (9.7)
\]

Let

\[
\tilde{\eta}_s(y) = \eta^{(F)}_s(y) - \pi_{(s-t)\epsilon^{-2} \ast \eta_t(y)} \quad (9.8)
\]

then

\[
u^\epsilon(x, \eta^{(F)}_s) - \pi_{(s-t)\epsilon^{-2} \ast \eta_s(x)} = p_{\alpha} \ast \tilde{\eta}_s(x) \quad (9.9)
\]
and, by Cauchy-Schwartz, the square of the first term on the r.h.s. of (9.7) is bounded by
\[ E_{\mu^t}^e \left( \left| p_0 \ast \tilde{\eta}_s(x) \right|^2 \right) = E_{\mu^t}^e \left( \sum_y p_0^2(x, y) \tilde{\eta}_s(y)^2 \right) \leq c' \epsilon^{(1-\alpha)d} \] (9.10)

because \( E_{\mu^t, z}^e (\tilde{\eta}_s(y)\tilde{\eta}_s(z)) = 0 \) for \( z \neq y \) by (A.7). The last inequality in (9.10) follows from Theorem 5. By assumption, \((1 - \alpha)d > 2\gamma\) and this concludes the proof of the theorem.

10 Local ergodic theorems

In this section we will prove a local equilibrium result, close in spirit to what in the hydrodynamic literature is called the “Gibbs-Boltzmann principle”, [3], or the “one block estimate”, [8]. The question concerns time averaged quantities and the aim is to prove closeness to their equilibrium expectations with parameters determined by local empirical means (local equilibrium). This can be reduced in our case (see Appendix D) to study decay of time correlations. Decay of time correlations is then proved in Theorems 9 and 10 below respectively in the case of some bounded and unbounded local functions of \( \eta \).

**Theorem 9.** For any \( \tau > 0 \) there is \( c \) so that for any \( \epsilon > 0, x, y \in \Omega_\varepsilon, |x-y| \geq \epsilon^{-1+\alpha/2} \) and \( t \leq \tau \),
\[ \left| E_{\mu^t}^e \left( \{ 1_{\eta_{t+2\alpha}(x) = 0} - e^{-u^*(x, \eta)} \} \{ 1_{\eta_{t+2\alpha}(y) = 0} - e^{-u^*(y, \eta)} \} \right) \right| \leq c \epsilon^{2\alpha} \] (10.1)

**Proof.** The proof uses the same auxiliary process introduced in the beginning of the proof of Theorem 8, to which we refer for notation. To compute the expectation we decompose the unity by writing \( 1 = (1_{x,=} + 1_{x,>})(1_{y,=} + 1_{y,>}) \), where
\[ 1_{x,=} = 1_{\eta_{t+2\alpha}^F(x)+\eta_{t+2\alpha}^b(z)=0}, \quad 1_{x,>} = 1_{\eta_{t+2\alpha}^F(x)+\eta_{t+2\alpha}^b(z)>0} \]

Observe that
\[ \eta^F(z) = \eta(z) \Leftrightarrow 1_{z,=} = 1 \] (10.2)

Since \( \eta^F_0 = 0 \) and \( \eta^b_0 = 0 \), similarly to (9.6) we have
\[ E_{\mu^t}^e \left( 1_{z,>} \right) \leq E_{\mu^t}^e \left( \eta_{t+2\alpha}^F(z) + \eta_{t+2\alpha}^b(z) \right) \leq c \epsilon^{2\alpha} \] (10.3)

Hence
\[ \text{l.h.s. of (10.1)} \leq \left| E_{\mu^t}^e \left( \{ 1_{\eta_{t+2\alpha}^F(x) = 0} - e^{-u^*(x, \eta)} \} \{ 1_{\eta_{t+2\alpha}^F(y) = 0} - e^{-u^*(y, \eta)} \} \right) \right| \]
\[ + \left| E_{\mu^t}^e \left( \eta_{t+2\alpha}^F(x) + \eta_{t+2\alpha}^b(x) + \eta_{t+2\alpha}^F(y) + \eta_{t+2\alpha}^b(y) \right) \right| \]
\[ \leq c \epsilon^{2\alpha} \] (10.4)
By Theorem 5,
\[ P_{\eta_t}^F\left(\eta_{t+\varepsilon 2\alpha}(x) = 0\right) = \prod_z \left(1 - \pi_{\varepsilon - 2 + 2\alpha}(z, x)\right)^{\eta(z)} = \exp\left\{ \sum_z \eta(z) \log \left(1 - \pi_{\varepsilon - 2 + 2\alpha}(z, x)\right)\right\} = e^{-u^\varepsilon(x, \eta_t)} + R, \quad |R| \leq c\varepsilon^{(1-\alpha)d} \]

Using the assumption that $|x - y| \geq \varepsilon^{-1 + \alpha/2}$, we then conclude the proof of (10.1), because $(1 - \alpha)d > 2\alpha$.

---

Denoting by $P_\rho$ and $E_\rho$ law and expectation w.r.t. the product of identical Poisson measures with mean $\rho > 0$, given a function $g = g(n)$, $n \in \mathbb{N}$, we shorthand
\[ f_\varepsilon(\eta_t, \eta_{t+\varepsilon 2\alpha}(x), \eta_{t+\varepsilon 2\alpha}(y)) := 1_{u^\varepsilon(x, \eta_t) \leq M, u^\varepsilon(y, \eta_t) \leq M} \{g(\eta_{t+\varepsilon 2\alpha}(x)) - E_{u^\varepsilon(x, \eta_t)}(g)\} \times \{g(\eta_{t+\varepsilon 2\alpha}(y)) - E_{u^\varepsilon(y, \eta_t)}(g)\} \tag{10.5} \]
with $M$ as in (8.2).

**Theorem 10.** With the notation of (10.5), suppose $g(n) \leq e^{bn}$, $b > 0$, then, for any $\tau > 0$ there is $c$ so that, for any $\varepsilon > 0$ and $t \leq \tau$,
\[ \lim_{\varepsilon \to 0} \sup_{x \neq y} \left| E_{\mu^\varepsilon}^\varepsilon\left(f_\varepsilon(\eta_t, \eta_{t+\varepsilon 2\alpha}(x), \eta_{t+\varepsilon 2\alpha}(y))\right)\right| = 0 \tag{10.6} \]

**Proof.** By (10.2) and (10.3),
\[ \left| E_{\mu^\varepsilon}^\varepsilon\left(f_\varepsilon(\eta_t, \eta_{t+\varepsilon 2\alpha}(x), \eta_{t+\varepsilon 2\alpha}(y))\right) - E_{\mu^\varepsilon}^\varepsilon\left(f_\varepsilon(\eta_t, \eta_{t+\varepsilon 2\alpha}(x), \eta_{t+\varepsilon 2\alpha}(y))\right)\right| \leq E_{\mu^\varepsilon}^\varepsilon\left(f_\varepsilon^2\right)^{1/2} c\varepsilon^{\alpha} \leq c\varepsilon^{\alpha} \tag{10.7} \]
The last inequality uses first that the functions $1_{u^\varepsilon(z, \eta_t) \leq M} E_{u^\varepsilon(z, \eta_t)}(g)$, $z = x, y$, are bounded and then Theorem 8.1 (as $f_\varepsilon^2$ is exponentially bounded).

To simplify the computations, it is now convenient to expand $g$ in Poisson polynomials, we refer to Appendix A for definition and properties. We just recall here that the Poisson polynomial of order $n$, denoted by $d_n(\cdot)$ and defined in (A.1), is such that $E_n(d_n) = u^n$.

By (A.4),
\[ g(\eta(x)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} d_n(\eta(x)), \quad |a_n| \leq |e^b + 1|^n \tag{10.8} \]

For any $\omega \in \mathbb{N}^\Omega$, we call $|\omega| = \sum_{z \in \Omega_t} \omega(x)$ and define
\[ D_\omega(\eta_t) = \prod_{z \in \Omega_t} d_{\omega(z)}(\eta(z)) \tag{10.9} \]
Call
\[ g_n(\eta_{t+\varepsilon 2\alpha}(z)) = d_n(\eta_{t+\varepsilon 2\alpha}(z)) - u(z, \eta_t)^n \tag{10.10} \]
dependence on $N$ By (A.11), there is $c$ that vanishes proportionally to a positive power of the sum of terms. It is then shown in Appendix A that the contribution of the other terms $n > 0$ given any $\delta > 0$ there is $N$ so that

$$|\mathbb{E}_{\mu}^\varepsilon \left( 1_{u^w(x,\eta_t) \leq M, u^w(y,\eta_t) \leq M} g_n(\eta_{t+\varepsilon}^{(F)}(x)) g_m(\eta_{t+\varepsilon}^{(F)}(y)) \right) | \leq A_{\varepsilon,t}(n + m) + A_{\varepsilon,t}(m) M^n + A_{\varepsilon,t}(m) M^n + M^{n+m}$$

Thus, recalling (10.8), we have

$$\sup_{\varepsilon \to 0} \sup_{x \neq y, n,m \leq N} \left| \mathbb{E}_{\mu}^\varepsilon \left( 1_{u^w(x,\eta_t) \leq M, u^w(y,\eta_t) \leq M} g_n(\eta_{t+\varepsilon}^{(F)}(x)) g_m(\eta_{t+\varepsilon}^{(F)}(y)) \right) \right| = 0$$

The proof of (10.4) is reported at the end of Appendix A; it is based on a cancellation on the r.h.s. of (10.11) among the terms which appear when we limit the sum over $\omega_n$ and $\omega_n'$ to $\omega_n'$, $\omega_n'$ and $\omega_n' + \omega_m'$ being all $\leq 1$ and when we take only non diagonal terms in the expansions of $u^w(z,\eta_t^k, k = n, m, z = x, y$, (recall the definition (4.2) of $u^w$ as a sum of terms). It is then shown in Appendix A that the contribution of the other terms vanishes proportionally to a positive power of $\varepsilon$, the proportionality coefficient has a “bad dependence” on $N$, but since $N$ is held fixed as $\varepsilon \to 0$, this does not give problems. 

11 Homogenization and convergence to the Stirring Process

In this section we will prove that the process $\xi_t$ is close to the stirring process on time intervals so long for the latter to reach local equilibrium, which implies that also $\xi_t$ approaches the same local equilibrium state, see Theorem 11 below. The statement is proved in two steps. We first prove a homogenization result at “short times”. Recalling that the generator for displacements of $\xi$ particles is $\epsilon^{-2\alpha} L^{(\xi,0)}$, the $\xi$ particles start moving at times of order $\epsilon^{2\alpha}$. On such a time scale, we will prove that to leading order the $\xi$ particles move as stirring with a time dependent intensity determined by the local empirical averages of the $\eta$-particles. The result is then extended to longer times of order $\epsilon^{2\beta}$ (recall $\beta < \alpha$), when the $\xi$-particles, like the stirring, become themselves also approximately homogeneous.
The result is used in Appendix E to control a term $Q$ which is the most dangerous one among those which appear in the computation of (4.4). After some maquillage operations whose details are given in Appendix E, the space time averages involved in the expression (4.5) lead to study expectations of a measure $\nu_\epsilon^\tau$, $\tau > 0$, which is a space time average of the original $\mu^\epsilon$. $\nu_\epsilon^\tau$ is in fact defined as the probability on $\mathbb{N}^{\Omega_\epsilon} \times \{0, 1\}^{\Omega_\epsilon}$ whose expectations are

$$\nu_\epsilon^\tau(f) = e^d \sum_{x \in \Omega_\epsilon} \int_0^\tau E^\mu_\epsilon(S_x f(\eta_t, \xi_t))$$

with $S_x$ the shift by $x$ on the torus $\Omega_\epsilon$.

We fix a large constant $C$ ($C = e^M$, see Appendix E), and shorthand $T = Ce^{2\beta}, \ B = \{x : |x| \leq e^{-a+\beta/2}\}$ (11.2)

We also write $|x|$ for a subset of $\Omega_\epsilon$, $|x|$ for its cardinality and call

$$g_{\mathfrak{L}}(\xi) = \prod_{x \in \mathfrak{L}} \xi(x)$$

(11.3)

We can now state the main theorem in this section.

**Theorem 11.** Let $T$ as in (11.2), then

$$\lim_{\epsilon \to 0} \sup_{|x| = 4, \ z \in B} \left| E^\mu_\epsilon\left(g_{\mathfrak{L}}(\xi_T)\right) - \nu_\epsilon^\tau\left(\prod_{x \in \mathfrak{L}} \left(\sum_{z \in \mathfrak{L}} \pi_{T\epsilon-2\epsilon}(x, z)\xi_0(z)\right)\right)\right| = 0$$

(11.4)

The proof of the theorem is divided in five steps. From the proof it will be clear that the result extends to $|x| \leq n$, for any finite fixed positive integer $n$.

**Step 1: Stirring Process and Duality.**

We will prove (11.4) using extensively the self-duality of the stirring process. The stirring on $\{0, 1\}^{\Omega_\epsilon}$ is the process defined by the generator $L^{st}$

$$L^{st}f(x) = \sum_{z, z' : |z - z'| = 1 \text{ on } \Omega_\epsilon} [f(x^{z, z'}) - f(x)]$$

(11.5)

where the sum is over ordered pairs of nearest neighbor sites on the torus $\Omega_\epsilon$ (a bond is counted twice, in agreement with (2.4)) and $x^{z, z'}$ is defined as in Section 2, once we identify $x$ with the configuration $\xi(x) = 1$ if $z \in x$. We will also shorthand

$$p_t(x, y) = e^{tL^{st}(x, y)}$$

(11.6)

Given $\gamma \geq \beta$, we divide the time interval $[0, T]$ into a union of intervals $[t_n, t_{n+1}]$. If $\gamma = \beta$ there is only one interval, if $\gamma > \beta$ we define iteratively $t_n = t_{n-1} + e^{2\gamma}, n \leq N + 1$, where $N + 1$ is the first integer such that $t_{N+1} \geq T$ and then rename (if necessary)
$t_{N+1} = T$. The choice $\gamma = \beta$ is possible only in $d > 2$. In fact we will need in our analysis $\gamma$ such that

$$a > \gamma > a - \beta$$

(11.7)

On the other hand we have already used (as in Theorem 7) that $\beta < \frac{d}{d+2} a$ so that the condition $\gamma = \beta$ together with (11.7) imposes $2\beta > a$, hence $\frac{d}{d+2} a > \frac{a}{2}$, which is satisfied only in $d > 2$, (in $d > 2$ we then choose $\gamma = \beta$ with $\beta$ smaller but close enough to $\frac{d}{d+2} a$ so that (11.7) holds).

We will compute $E^{\epsilon}_{T^*}(g_\epsilon(\xi_T))$ by successive conditioning on $(\eta_n, \xi_n)$, $n = N, N - 1, \ldots, 1, 0$. Modulo errors which vanish in the limit $\epsilon \to 0$, at each step we always have to compute the expectation of the “same” function $g_\epsilon$, with $|y| = 4$. The origin of such a strong result is the self-duality of the stirring.

On each time interval $[t_n, t_{n+1}]$ we compare $e^{-2\alpha t} L^{(\xi,0)} g_\epsilon(\xi_t)$ with $e^{-2\alpha t} L^{st} g_\epsilon(\xi_t)$, the former acting on $\xi_t$ the latter on $\xi$, see (11.10) below. The choice of the intensity $c$ is critical. In the generator $L^{(\xi,0)}$ the intensity is proportional to $1_0(x=0)$ which by Theorem 9 can be replaced by $e^{-u^*(0, \eta_{n-1})}$. We would like to further approximate it by putting $e^{-u^*(0, \eta_{n-1})}$. We will see that the error of shifting $x$ to $0$ is under control, but for time shifts we only have Theorem 8. Thus, instead of setting $c = e^{-u^*(0, \eta_{n-1})}$ as we would have liked, we will set $c = e^{-u^*(t, \eta_{n-1}, \eta_{n-1})}$, where

$$w^*(t, t_{n-1}, \eta_{n-1}) = \sum_y \pi_{(t, t_{n-1}, \eta_{n-1})}^z(0, y) u^*(y, \eta_{n-1})$$

(11.8)

To implement the above strategy, it is convenient to rewrite

$$L^{(\xi,0)} g_\epsilon(\xi) = \sum_{z, z' : |z - z'| = 1} |\xi(z) - \xi(z')| \left( \xi(z) 1_{\eta(z)=0} + \xi(z') 1_{\eta(z)=0} \right) \left( g_\epsilon(\xi, z') - g_\epsilon(\xi) \right)$$

and telescopic sum

$$1_{\eta(z)=0} = [1_{\eta(z)=0} - (e^{-u^*(z, \eta_{n-2})}) + (e^{-u^*(z, \eta_{n-2})} - e^{-u^*(0, \eta_{n-2})})
\sum [e^{-u^*(0, \eta_{n-2})} - e^{-u^*(t, \eta_{n-1}, \eta_{n-1})}] + e^{-u^*(t, \eta_{n-1}, \eta_{n-1})}$$

(11.9)

doing the same for $1_{\eta(z')=0}$. Since

$$g_\epsilon(\xi, z') = g_\epsilon(z, z') \xi$$

we get

$$L^{(\xi,0)} g_\epsilon(\xi_t) = e^{-u^*(t, \eta_{n-1}, \eta_{n-1})} L^{st} g_\epsilon(\xi_t) + H_1(\xi_t) + \cdots + H_3(\xi_t)$$

(11.10)

where $H_i$ comes from the $i$-th term of the telescopic sum and dependence on $\xi$ and $\eta$ is not made explicit. Notice that the actual generator of the process is $e^{-2\alpha t} L^{(\xi,0)}$, so that also “the error terms” $H_i$ get multiplied by $e^{-2\alpha t}$. 


Step 2: The time interval \([t_N, t_{N+1}]\).

(11.10) is the desired duality relation, which states that modulo the errors \(H_i\) we can compute the expectation of \(g_\omega(\xi_t)\) by applying the stirring to \(g_\omega\) thought of as a function of \(\omega\). The statement will become hopefully clear below. Set

\[
A_{t_n,t_j}(\eta_t) := \int_{t_n}^{t_{n+1}} e^{-w^\nu(t,t_j ; \eta_t)} , \quad j < n
\]  

(11.11)

recalling that in the whole sequel, \(\omega^0 \subset B\) and \(|\omega^0| = 4\) (we rename by \(\omega^0\) the set \(\omega\) in (11.2) where \(B\) is also defined). We set \(2B = \{ x : |x| \leq 2 \epsilon^{-a+\beta/2} \}\). Using (11.10) and recalling (11.6), we get

\[
E^\nu_{\omega^0}(g_\omega(\xi_{N+1})) = E^\nu_{\omega^0} \left( \sum_{y \in 2B} p_{A_{t_N, t_{N-1}}(\eta_{N-1})} e^{-2a(x^0,y)} g_\omega(\xi_{t_N}) \right) + \mathcal{R}^{(1)}_N
\]  

(11.12)

\[
\mathcal{R}^{(1)}_N = \int_{t_n}^{t_{n+1}} \sum_{y} p_{(t_N-t)} e^{-2a(x^0,y)} E^\nu_{\omega^0} \left( \{ L' g_\omega(\xi_s) + \epsilon^{-2a}H_1(y,s) + \cdots + H_3(y,s) \} \right)
\]

where \(L' = L(\xi,+) + L(\xi,-)\). The remainder term \(\mathcal{R}^{(1)}_N\) will be bounded later, together with the analogous terms coming from iterating the above analysis to the successive time intervals. But we are not yet ready for the iteration, because the first term on the r.h.s. of (11.12) has not the same expression as the term we started from, since it depends on \(\eta_{N-1}\). To fix the problem we do a Taylor-Lagrange expansion to first order in the small parameter \([A_{t_N, t_{N-1}}(\eta_{N-1}) - A_{t_N, 0}(\eta_0)]\).

Shorthand

\[
b_n(\lambda) := \lambda A_{t_n, t_{n-1}}(\eta_{n-1}) + (1 - \lambda) A_{t_n, 0}(\eta_0)
\]  

(11.13)

then

\[
E^\nu_{\omega^0}(g_\omega(\xi_{t+1})) = \sum_{y} p_{A_{t_N, 0}(\eta_0)} e^{-2a(x^0,y)} E^\nu_{\omega^0}(g_\omega(\xi_{t_N})) + \mathcal{R}^{(1)}_N + \mathcal{R}^{(2)}_N
\]  

(11.14)

where

\[
\mathcal{R}^{(2)}_N = \int_0^1 E^\nu_{\omega^0} \left( \sum_{y} \left[ A_{t_N, t_{N-1}}(\eta_{N-1}) - A_{t_N, 0}(\eta_0) \right] e^{-2a}
\]

\[
\times p_{b_N(\lambda)} e^{-2a(x^0,y)} L^\nu t g_\omega(\xi_{t_N}) \right) d\lambda
\]  

(11.15)

One may wonder why not to put since from the beginning (i.e. in (11.9)) \(w'(t, 0, \eta_0)\) instead of \(w'(t, t_{N-1}, \eta_{N-1})\). The reason is that, at this stage, the replacement is just a time change in the stirring process, and its influence is smoothened by the mixing properties of the stirring, as it will become clear when bounding \(\mathcal{R}^{(2)}_N\).

Step 3: The iteration.

The first term on the r.h.s. of (11.14) has now the same structure as its l.h.s. so we can iterate. Calling

\[
A^{(n)}(\eta_0) := A_{t_N, 0}(\eta_0) + \cdots + A_{t_n, 0}(\eta_0), \quad n \leq N, \quad A^{(N+1)}(\eta_0) \equiv 0
\]  

(11.16)
where, analogously to (11.15),
\[
\mathcal{R}_n^{(2)} = \int_0^1 E_{\nu'} \left( \sum_y \left[ A_{t_n,t_{n-1}}(\eta_{t_n-1}) - A_{t_n,0}(\eta_{0}) \right] \epsilon^{-2a} \right.
\times p|_{A(n+1)(\eta_{0})+b_n}(\lambda) e^{-2a} (x^0, y) \left. L^s g_0(\xi_{t_n}) \right) d\lambda
\]
(11.18)

**Step 4: Bounds on the remainders.**

The following bounds hold:

- The contribution of \( \mathcal{R}_n^{(1)} \) is bounded by \( c\epsilon^{2\gamma-2a+2\gamma} \). Indeed the term \( H_3 \) is bounded using Theorem 8 by \( c\epsilon^{2\gamma-2a+2\gamma} \), while all the other terms are smaller. In fact, by (10.1) the term with \( H_1 \) is bounded proportionally to \( \epsilon^{2\gamma-2a+2\alpha} \), so that, by the choice of \( \alpha \), it is \( \leq c\epsilon^{2\gamma-2a+2\gamma} \). The term with \( H_2 \) can be written as the sum of two terms. The first one takes into account the case when all the particles remain in the ball of radius \( 2\epsilon^{-a+\beta/2} \) (i.e., twice the radius of the ball where they are initially). This term is then bounded using (8.18), proportionally to \( \epsilon^{2\gamma-2a+1-a+\beta/2} \). The second term (contributing to \( H_2 \)) covers the case when there is at least a particle which travels by \( \epsilon^{-a+\beta/2} \) in a time \( \epsilon^{-2a+2\beta} \) and it is therefore bounded by \( c \exp \{ -c' \epsilon^{-\beta} \} \). Finally, the term with \( L' g_0(\xi_s) \) is bounded by \( c\epsilon^{2\gamma} \), because, by Theorem 5, \( E_{\nu'}(\kappa_{\eta_{n-1}[1]}) \leq c \).

- The contribution of \( \mathcal{R}_n^{(2)} \) is bounded by

\[
c \epsilon^{2\gamma-2a+2\beta} [(t_N - t_n)\epsilon^{-2a}]^{-1/2-1/12}
\]
(11.19)

\( c \) a positive constant. The bound is derived in Appendix B by exploiting estimates on the stirring process known in the literature.

The total contribution of all the remainders is then bounded by
\[
c \{ \epsilon^{2\gamma-2a+2\gamma} \epsilon^{2\beta-2\gamma} + \epsilon^{-2a+2\beta} T^{-1/2-1/12} \epsilon^{2a(1/2+1/12)} \}
\]
(11.20)

with \( T = C\epsilon^{2\beta} \). The first term vanishes by (11.7), the second one if \( \beta > \frac{5a}{17} \) which is compatible with the condition \( \beta < \frac{d}{d+2} a \).

**Step 5: Conclusion.**

By (11.17)
\[
\left| E_{\nu'} \left( g_{\nu}(\xi_{t_{N+1}}) \right) - \nu' \left( \sum_y p_{A(0)(y_0)} e^{-2a} (x^0, y_0) g_0(\xi_0) \right) \right| \leq c\epsilon^{\delta}
\]
where $ce^δ$, $δ > 0$, shorthands the bound (11.20) on the remainders. We are going to prove that

$$
\nu^c_\tau\left(\epsilon^{-2a}A(0)(\eta_0) - \epsilon^{-2a+2\beta}\right) \geq 1 - ce^\beta \tag{11.21}
$$

and

$$
E_{\nu^c_\tau}\left(\int_0^\tau \left|e^{-w^c(t,0,\eta_0)} - e^{-u^c(0,\eta)}\right|\right) \leq ce^\beta \tag{11.22}
$$

Proof of (11.21)-(11.22). Recall that

$$
e^{-2a}A(0)(\eta_0) = e^{-2a} \int_0^{C_e^{2\beta}} e^{-w^c(t,0,\eta_0)}, \quad w^c(t,0,\eta_0) = \sum_y \pi_{(t-\epsilon^{2a})\epsilon^{-2}(0,y)} u^c(y,\eta_0)
$$

By (8.2),

$$\nu^c_\tau\left(e^{-u^c(0,\eta)} \leq e^{-M}\right) \leq ce^{(1-\alpha)d} \tag{11.23}
$$

so that

$$
\nu^c_\tau\left(\int_0^{C_e^{2\beta}} e^{-w^c(t,0,\eta_0)} < e^{-M/2}\right) \leq ce^{(1-\alpha)d}
$$

and (11.21) follows from (11.22), which we prove next:

$$
\nu^c_\tau\left(\left|e^{-w^c(t,0,\eta_0)} - e^{-u^c(0,\eta)}\right|\right)
$$

$$
\leq \int_0^{C_e^{2\beta}} \nu^c_\tau\left(\sum_y \pi_{(t-\epsilon^{2a})\epsilon^{-2}(0,y)} |u^c(y,\eta) - u^c(0,\eta)|\right)
$$

$$
\leq \epsilon \int_0^{C_e^{2\beta}} \sum_y \pi_{(t-\epsilon^{2a})\epsilon^{-2}(0,y)} |y| \nu^c_\tau\left(\|u^c(\eta)\| H^1_T\right)
$$

having used (8.19). Since $\sum_y \pi_{(t-\epsilon^{2a})\epsilon^{-2}(0,y)} |y| \leq \epsilon^{-1+\beta}$ and

$$
E_{\nu^c_\tau}\left(\|u^c(\eta_0)\| H^1_T\right) = \int_0^\tau E_{\nu^c_\tau}\left(\|u^c(\eta)\| H^1_T\right) \leq c\tau^{-1/2}
$$

(the last inequality follows from (8.9) and Cauchy-Schwartz), (11.22) and hence (11.21).

By (B.3), we then have

$$
\left|E_{\nu^c_\tau}\left(g_2\sum_{x\leq x} \pi_{A(0)(\eta_0)\epsilon^{-2a}(x,y)} g_2\xi_0\right)\right| \leq c[\epsilon^\delta + \epsilon^\beta + \epsilon^{-2a+2\beta}/12]
$$

Moreover, by (11.22) and (11.21),

$$
E_{\nu^c_\tau}\left(\sum_y |\pi_{A(0)(\eta_0)\epsilon^{-2a}(x,y)} - \pi_{C_e^{-2a+2\beta}e^{-u^c(0,\eta)}(x,y)}|\right) \leq \frac{ce^{-2a+3\beta}}{(Ce^{-M/2})^{2-2a+2\beta}} + ce^\beta
$$

and the theorem is proved.
A Independent and branching random walks

In this appendix we prove the statements in the text which refer to independent random walks and to independent random walks with independent branchings. We start by recalling some of the main properties of independent random walks on $\mathbb{Z}^d$.

**Poisson polynomials.** The Poisson polynomial of degree $k$, $k \geq 1$, is defined as

$$d_1(n) \equiv 1, \quad d_k(n) = n(n-1) \cdots (n-k+1), \quad k > 1 \quad (A.1)$$

We have denoted by $n$ the argument of the polynomial because we are going to restrict $d_k$ to $\mathbb{N}$. The following remarkable identities hold:

$$1_{\eta(x)=i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} d_{n+i}(\eta(x))$$  

(A.2)

$$e^{b \eta(x)} = \sum_{n=0}^{\infty} \frac{(e^b-1)^n}{n!} d_n(\eta(x))$$  

(A.3)

If $f(n)$ is exponentially bounded, say $|f(n)| \leq e^{bn}$, then

$$f(\eta(x)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} d_n(\eta(x)), \quad |a_n| \leq |e^b+1|^n$$  

(A.4)

The above can and will be applied to $\kappa_\eta(x)$ (and to $\kappa_{\eta(x)}^n$, $n \in \mathbb{N}$, as well), because $\kappa_\eta(x)$, which is the function which appears in the generator $L$ in Section 2, has been supposed to be exponentially bounded.

**Poisson multi-polynomials.** The Poisson multi-polynomial $D_{\omega}$, $\omega : \Omega \to \mathbb{N}$, is

$$D_{\omega}(\eta) = \prod_{x \in \Omega} d_{\omega(x)}(\eta(x))$$  

(A.5)

The duality relation

$$L(\eta,0) D_{\omega}(\eta) = L(\omega,0) D_{\omega}(\eta)$$  

(A.6)

holds, namely the independent generator on $D_{\omega}(\eta)$ is the same either if it acts on $\omega$ or on $\eta$. Denoting by $\pi_t^{(0)}(\cdot,\cdot)$ the kernel of $e^{tL^{(0,-)}}$, it directly follows from (A.6) that

$$\sum_{\eta'} \pi_t^{(0)}(\eta,\eta') D_{\omega}(\eta') = \sum_{\omega'} \pi_t^{(0)}(\omega,\omega') D_{\omega'}(\eta)$$  

(A.7)

$$e^{(L(\eta,0)+L')t} D_{\omega} = \sum_{\omega'} \pi_t^{(0)}(\omega,\omega') D_{\omega'} + \int_0^t \sum_{\omega'} \pi_t^{(0)}(\omega,\omega') e^{(L(\eta,0)+L')s} \{ L' D_{\omega} \}$$  

(A.8)

where $L'$ is some generator acting on functions of $(\eta,\xi)$.
Notation. We will use the following notation: $1_x$ denotes the configuration $\omega$ with only one particle at $x$; $\omega \pm \omega'$ is the configuration with $\omega(x) \pm \omega'(x)$ at $x$ (the relation with the minus sign being defined only if $\omega' \leq \omega$); $|\omega| = \sum_{x \in \Omega_\epsilon}$ Then

$$\pi^0_t(\omega, \omega') = \sum_{y_1, \ldots, y_n} \prod_{i=1}^n \pi_t(x_i, y_i) 1_{\omega'=\sum_i 1_{y_i}}$$

(A.9)

In particular $\pi^0_t = \pi_t$ when restricted to $\eta : |\eta| = 1$.

Proof of Theorem 1. We will prove here only the statements of the theorem relative to the variables $\eta$. Recalling (5.2), by (A.7),

$$U_{\epsilon 0}(x) = \mu^\epsilon(u^\epsilon(x, \eta)) = \sum_y p_\alpha(x, y) \mu^\epsilon(\eta(y)) = \sum_y p_\alpha(x, y) \sum_z \pi_{\epsilon - 2t_0}(y, z) \eta_{-t_0}(z)$$

Thus $|U_{\epsilon 0}(x)| \leq C$, if $C$ is the sup in (5.1). Moreover,

$$\epsilon^{-1} |\nabla U_{\epsilon 0}(x)| \leq \epsilon^{-1} \sup_z |\nabla \pi_{\epsilon - 2t_0}(0, z)| \leq \frac{c}{t_0}$$

which proves the first inequality in (5.5).

For future reference we also observe that the same argument used above shows that

$$\mu^\epsilon(D\omega(\eta)) = \sum_{\omega'} \pi_{t_0}^{(0)} D\omega'(\eta - t_0) \leq C|\omega|$$

(A.10)

Call $\tilde{\eta}(x) = \eta(x) - \mu^\epsilon(\eta(x))$, then

$$\mu^\epsilon(||u^\epsilon - U^\epsilon||^2) = \epsilon^d \sum_{x \in \Omega_\epsilon} \sum_{z, z'} p_\alpha(x, z) p_\alpha(x, z') \mu^\epsilon(\tilde{\eta}(z) \tilde{\eta}(z'))$$

Thushanding $\omega_z = 1_z$ and $\omega_{z, z'} = 1_z + 1_{z'}$,

$$\mu^\epsilon(\tilde{\eta}(z) \tilde{\eta}(z')) = \mu^\epsilon(D\omega_{z, z'}) - \mu^\epsilon(D\omega_z) \mu^\epsilon(D\omega_{z'}) + 1_{z = z'} \mu^\epsilon(\eta(z))$$

so that

$$\mu^\epsilon(||u^\epsilon - U^\epsilon||^2) = \epsilon^d \sum_{x \in \Omega_\epsilon} \sum_{z \in \Omega_\epsilon} p_\alpha(x, z)^2 \mu^\epsilon(\eta(z))$$

$$+ \sum_{z \in \Omega_\epsilon} \left[ \sum_{y \in \Omega_\epsilon} p_\alpha(x, y) \pi_{\epsilon - 2t_0}(y, z) \right]^2 \eta_{-t_0}(z) \leq c\epsilon^{(1-\alpha)d}$$

and the part of Theorem 1 relative to the $\eta$ variables is proved.

The proof of Theorem 5 uses the following bound:
Theorem 12. There is a constant $c > 0$ so that for any $t > 0$ and any $\omega$,

$$E_{\mu^t}^{\epsilon}(D_\omega(\eta_t)) \leq E_{\mu^t}^{\epsilon}(D_\omega(\eta^+_t)) \leq (e^{c(t + 2c\sqrt{t})})^{1\omega}(1_{|\omega| \leq e^{-1}} + |\omega|!(ee)^{|\omega|}1_{|\omega| \geq e^{-1}})$$

(A.11)

where $\eta^+_t$ is the process with generator $e^{-2L(\eta,0)} + L(\eta^+,\omega)$.

Proof. We will first prove that:

$$L^{(\eta^+,\omega)} D_\omega(\eta) = \kappa |\omega| D_\omega + \kappa \sum_{x \in \Omega} \omega(x) |\omega(x) - 1| D_{\omega-1_x}$$

(A.12)

(recall that $|\omega| = \sum_x \omega(x)$, $[\omega - 1_x](z) = \omega(z)$ for all $z \neq x$ and $[\omega - 1_x](x) = \omega(x) - 1$).

Indeed

$$L^{(\eta^+,\omega)} D_\omega(\eta) = \kappa \sum_x \left[ \prod_{y \neq x} d_{\omega(y)}(\eta(y)) \right] \eta(x) \left[d_{\omega(x)}(\eta(x) + 1) - d_{\omega(x)}(\eta(x)) \right]$$

and (A.12) follows from

$$k[d_n(k+1) - d_n(k)] = nd_n(k) + n[n-1]d_{n-1}(k)$$

By (A.8) and (A.12),

$$E_{\mu^t}^{\epsilon}(D_\omega(\eta^+_t)) = \sum_{\omega'} \pi_t^{(0)}(\omega,\omega') \mu^t(D_\omega(\eta)) + \kappa n \int_0^t \sum_{\omega'} \pi_{t-s}^{(0)}(\omega,\omega') E_{\mu^t}^{\epsilon}(D_{\omega'}(\eta^+_s))$$

$$+ \kappa \sum_{y \in \Omega} \int_0^t \sum_{\omega'} \pi_{t-s}^{(0)}(\omega,\omega') \omega'(y)[\omega'(y) - 1] E_{\mu^t}^{\epsilon}(D_{\omega'-1_y}(\eta^+_s))$$

Let $\omega = \sum_{i=1}^n 1_{x_i}$. Recalling (A.9), the last term can then be written as

$$\kappa \sum_{y \in \Omega} \left\{ \int_0^t \sum_{y_i=1} \prod_{i=1}^n \pi_{t-s}(x_i, y_i) \sum_{i=1}^n 1_{y_i=y} \sum_{y_i=1}^n 1_{y_i=1} - 1 \right\} E_{\mu^t}^{\epsilon}(D_{\sum y_i-1_y}(\eta^+_s))$$

Since $\sum_{i=1}^n 1_{y_i=y} \sum_{i=1}^n 1_{y_i=1} - 1 = \sum_{i,j \neq i} 1_{y_i=y_j}$, we get

$$\kappa \sum_{i,j \neq i} 1_{y_i=y_j} \int_0^t \prod_{i=1}^n \pi_{t-s}(x_i, y_i) E_{\mu^t}^{\epsilon}(D_{\sum_{\ell \neq j} 1_{y_\ell}}(\eta^+_s))$$

$$\leq \kappa \sup_{x,y} \pi_{t-s}(x, y) \int_0^t \prod_{\ell \neq j} \pi_{t-s}(x, y_\ell) \left\{ \sup_{x,y} \pi_{t-s}(x, y) \right\} E_{\mu^t}^{\epsilon}(D_{\sum_{\ell \neq j} 1_{y_\ell}}(\eta^+_s))$$

and since $\kappa \sup_{x,y} \pi_{t-s}(x, y) \leq e^\epsilon \sqrt{t-s}$,

$$\leq c_\epsilon(n-1) \int_0^t \frac{1}{\sqrt{t-s}} \sup_{\omega^*=n-1} \sum_{\omega^*:|\omega^*|=n-1} \pi_{t-s}^{(0)}(\omega^*, \omega') E_{\mu^t}^{\epsilon}(D_{\omega'}(\eta^+_s))$$
Since \( \mu^c(D_\omega(y_0)) \leq C|\omega| \),
\[
E^c_{\mu^c}(D_\omega(y_0^+)) \leq e^{c^n t} \left( C^n + \sum_{k=1}^{n-1} |\omega|^k [n(n-1)^2 \cdots (n-k)^2(n-k)]I_k(t)C^{n-k} \right)
\]
where
\[
I_k(t) = \int_0^t \frac{1}{\sqrt{t-s_1}} \cdots \frac{1}{\sqrt{t-s_k}} = \frac{(2\sqrt{t})^k}{k!}
\]
Calling \( B_n(\epsilon) = \max_{k \leq n}(ne)^k \),
\[
E^c_{\mu^c}(D_{\omega_n,x}(\eta_\epsilon^+)) \leq B_n(\epsilon)e^{c^n t} \sum_{k=0}^n \left( \frac{n}{k} \right) [2c\sqrt{t}]^k C^{n-k} \leq B_n(\epsilon)[e^{c^n t}(C + 2c\sqrt{t})]^n
\]
Finally, \( B_n(\epsilon) \leq 1 \) for \( n \leq \epsilon^{-1} \) and otherwise \( B_n(\epsilon) \leq e^n n! \epsilon^n \) (by the Stirling formula \( n! \leq n! \epsilon^n \)), hence (A.11).

**Proof of Theorem 5.** We complete here the proof of Theorem 5 by proving (8.3) and (8.4). By (A.3),
\[
E^\epsilon_{\mu_{\epsilon}^c}(e^{b\eta_{\epsilon}^+}(x)) = \sum_{n=0}^\infty \left( \frac{e^b - 1}{n!} \right)^n E^\epsilon_{\mu_{\epsilon}^c}(D_{\omega_{n,x}}(\eta_\epsilon^+)) (A.13)
\]
where \( \omega_{n,x} = n1_x \) is the configuration with \( n \) particles, all at \( x \). By (A.11),
\[
E^\epsilon_{\mu_{\epsilon}^c}(e^{b\eta_{\epsilon}^+}(x)) \leq \sum_{n=0}^\infty \left( \frac{e^b n!}{n!} \right)^n [e^{c^n t}(C + 2c\sqrt{t})]^n + \sum_{n \geq \epsilon^{-1}} \left( e^{b+1}[e^{c^n t}(C + 2c\sqrt{t})\epsilon]^n - 2 \exp \left( e^{b+1}[e^{c^n t}(C + 2c\sqrt{t})\epsilon]^n \right) \right)
\]
for \( \epsilon \) small enough, which proves (8.1).

We will next prove that
\[
E^\epsilon_{\mu_{\epsilon}^c} \left( |u^\epsilon(x, \eta_\epsilon^+) - E^\epsilon_{\mu_{\epsilon}^c}(u^\epsilon(x, \eta_\epsilon^+))| \right)^2 \leq c\epsilon^{2c_\epsilon t \epsilon^{(1-\alpha)d}} (A.14)
\]
Proceeding as in the proof of Theorem 1,
\[
\text{l.h.s. of (A.14) = } \sum z,z' p_\alpha(x,z)p_\alpha(x,z') V_2(z,z',t) + \sum z p_\alpha(x,z)^2 E^\epsilon_{\mu^c}((\eta_\epsilon^+(z)))
\]
\[
V_2(z,z',t) = E^\epsilon_{\mu^c} \left( D_{\omega_{z,x}}(\eta_\epsilon^+) - E^\epsilon_{\mu^c}(D_{\omega_{z,x}}(\eta_\epsilon^+)) \right) E^\epsilon_{\mu^c}(D_{\omega_{z',x}}(\eta_\epsilon^+))
\]
Since \( L^{(a,+)}d_2(\eta(x)) = 2\kappa [d_2(\eta(x)) + \eta(x)] \),
\[
\text{l.h.s. of (A.14) \leq } \sum z p_\alpha(x,z)^2 E^\epsilon_{\mu^c}(\eta_\epsilon^+(z)) + e^{2c_\epsilon t} \sum z \pi_{\epsilon^{-2+2\alpha+\epsilon^{-2}(t+t_0)}(x,z)^2}\eta_{-t_0}(z)
\]
\[
+ 2\kappa \sum z \int_0^t e^{2c_\epsilon(t-s)} \pi_{\epsilon^{-2+2\alpha+\epsilon^{-2}(t-s)}(x,z)^2}\eta_{s}(z)
\]
Using the inequality $E_{\mu}^{e} \left( \eta_r^{+}(z) \right) \leq C e^{\kappa t}$, we then get (A.14).

**Proof of** (10.14). We fix $\omega'_m$ and $\omega'_m$ on the r.h.s. of (10.11) and expand the three terms $u'^{(x, \eta)}$, $u'^{(y, \eta)}$, and $u'^{(x, \eta)}$ containing $u$, starting from the former.

$$u'^{(x, \eta)} = \sum_{z_1, \ldots, z_n \in X_{n, \omega_m}, \eta \neq i} \prod_{i=1}^{n} p_\alpha(x, z_i) \eta(z_i) + \sum_{z_1, \ldots, z_n \in X_{n, \omega_m}, \eta \neq i} \prod_{i=1}^{n} p_\alpha(x, z_i) \eta(z_i)$$

where $X_{n, \omega_m, \eta} = \{ z_1, \ldots, z_n : \omega'(z_i) = 0, i = 1, \ldots, n; z_i \neq z_j \text{ for all } i \neq j \}$. Thus

$$u'^{(x, \eta)} = T + S, \quad T = \sum_{z_1, \ldots, z_n \in X_{n, \omega_m}, \eta \neq i} \prod_{i=1}^{n} p_\alpha(x, z_i) D \sum_{z_1, \ldots, z_n \in X_{n, \omega_m}, \eta \neq i} \prod_{i=1}^{n} p_\alpha(x, z_i) \eta(z_i)$$

We next bound $S$:

$$S \leq \sum_{i=1}^{n} \{ \sum_{z_1, \ldots, z_n \in X_{n, \omega_m}, \eta \neq i} \prod_{i=1}^{n} p_\alpha(x, z_i) \eta(z_i) \} \sum_{z_1, \ldots, z_n \in X_{n, \omega_m}, \eta \neq i} \prod_{i=1}^{n} p_\alpha(x, z_i) \eta(z_i)$$

Recalling that $u'^{(x, \eta)} \leq M$ in (10.14),

$$S \leq M^{n-1} \sum_{i=1}^{n} \sum_{z_1, \ldots, z_n \in X_{n, \omega_m}, \eta \neq i} \prod_{i=1}^{n} p_\alpha(x, z_i) \eta(z_i) + M^{n-2} \sum_{i \neq j} \sum_{z_1, \ldots, z_n \in X_{n, \omega_m}, \eta \neq i} \prod_{i=1}^{n} p_\alpha(x, z_i) \eta(z_i)$$

and since $d_k(\eta(z)) \eta(z) = d_k(\eta(z)) + k d_0(\eta(z))$, $E_{\mu}^{e} \left( D \omega'(\eta) \right) \leq c(t) |\omega|, |\omega| \leq N$,

$$E_{\mu}^{e} \left( 1_{u'^{(x, \eta)}} \leq M D \omega'(\eta) S \right) \leq M^{n-1} n (m c(1-\alpha) d [c(t)^{m+1} + c(t)^{m}] + M^{n-2} n (n-1) [c(t)^2 + c(t)] c(t)^m$$

so that the term with $S$ does not contribute as $\epsilon \to 0$. The term in (10.14) containing $T$, cf. (A.15), is

$$- \sum_{\omega'_m, \omega'_m} 1_{\omega'_m, \omega'_m} \prod_{\epsilon=0}^{(0)} \pi_{\epsilon-2-2\alpha}(\omega_{n,x}, \omega_{n,y}) \pi_{\epsilon-2-2\alpha}(\omega_{m,y}, \omega_{m,m})$$

$$\times E_{\mu}^{e} \left( 1_{u'^{(x, \eta)}} \leq M D \omega'(\eta) \right)$$

By an argument similar to the one used when bounding $S$, the error when dropping the conditions on $\omega'_m$ is bounded by $[n(n-1) + nm] c(1-\alpha) d [c(t)^{n+m}]$. In conclusion the term in (10.14) containing $u'^{(x, \eta)} D \omega'(\eta)$ is, modulo terms which vanish as $\epsilon \to 0$,

$$- \sum_{\omega'_m, \omega'_m} \pi_{\epsilon=0}^{(0)} \pi_{\epsilon-2-2\alpha}(\omega_{n,x}, \omega_{n,y}) \pi_{\epsilon-2-2\alpha}(\omega_{m,y}, \omega_{m,m})$$

$$\times E_{\mu}^{e} \left( 1_{u'^{(x, \eta)}} \leq M D \omega'(\eta) \right)$$

(A.16)
The analogous conclusion holds for the term with \( u'(y, \eta_t)^m D_{\omega'}(\eta_t) \) so that the proof of (10.14) reduces to the analysis of the term
\[
E_{\mu'} \left( \sum_{\omega', m} \pi_{\omega' m}^{(0)}(\omega_{n+1}, x, \omega_n') \pi_{\omega' m}^{(0)}(\omega_{n+1}, y, \omega_n') \{ u'(x, \eta_t)^{n+m} u'(y, \eta_t)^m - D_{\omega'_n + \omega'_m} (\eta_t) \} \right) \tag{A.18}
\]
Proceeding as in (A.15), and calling \( X_\neq = \{ z_1 \ldots z_{n+m} : z_i \neq z_j \text{ for all } i \neq j \} \), we write, by an abuse of notation, \( u'(x, \eta_t)^m u'(y, \eta_t)^m = T + S \) where
\[
T = \sum_{z_1 \ldots z_{n+m} \in X_\neq} \prod_{i=1}^{n+m} p_\alpha(x, z_i) \prod_{i=n+1}^m p_\alpha(y, z_i) \eta(z_1) \ldots \eta(z_{n+m})
\]
\[
S = \sum_{z_1 \ldots z_{n+m} \notin X_\neq} \prod_{i=1}^{n+m} p_\alpha(x, z_i) \prod_{i=n+1}^m p_\alpha(y, z_i) \eta(z_1) \ldots \eta(z_{n+m})
\]
The analysis is now completely analogous to the previous one, \( S \) is bounded exploiting the presence of the characteristic functions \( u'(x, \eta_t) \leq M \) and \( u'(y, \eta_t) \leq M \), \( T \) is equal to a \( D_{\omega_{m+n}} \) Poisson multi-polynomial with \( \omega_{m+n} \leq 1 \) and \( |\omega_{m+n}| = m + n \), after which the condition that \( \omega_{m+n} \leq 1 \) can be dropped with a vanishing error, thus proving that (A.18) vanishes as \( \epsilon \to 0 \) uniformly in \( x, y \) and \( n, m \leq N \).

\[\boxed{\textbf{B} \quad \text{Stirring process}}\]

In [6]-[7] it is proved that:

\[\textbf{Theorem 13.} \text{ Let } p_t^{(\infty)}(x, y) \text{ and } \pi_t^{(\infty)}(x, y) \text{ denote the kernel of the semigroups of the stirring and of the independent processes in the whole } \mathbb{Z}^d. \text{ Then, given any } n \geq 1 (\text{ but the case } n = 1 \text{ is trivial, because } p_t(x, y) \equiv \pi_t(x, y)), \text{ any dimension } d \geq 1, \text{ there is } c \text{ so that, for any } x, |x| = n,
\]
\[
\left| \sum_y \left( p_t^{(\infty)}(x, y) - \pi_t^{(\infty)}(x, y) \right) \right| \leq ct^{-1/12} \tag{B.1}
\]

The bound (B.1) cannot hold in our case where the processes are defined on the torus \( \Omega_t \), because \( p_t(x, y) \) and \( \pi_t(x, y) \) have different limits as \( t \to \infty \): the former converges to the uniform distribution of \( n \) distinct sites in \( \Omega_t \) while in the latter the exclusion condition is dropped. However in our applications we consider times \( t \leq \tau \) (with the stirring amplified by a factor \( \epsilon^{-2a} \), thus times \( \leq \epsilon^{-2a} \tau \) for the process with generator \( L^{st} \). Since
\[
\left| \sum_{y : |y-z| > \epsilon^{-1}} p_t(x, y) \right| \leq c \frac{e^{-2a}}{\sqrt{t}} \tag{B.2}
\]
\[(B.1) \text{remains valid for the processes in } \Omega \text{ provided } t \leq \epsilon^{-2a} \tau \text{ and with } c \text{ in (B.1) dependent on } \tau:\]
\[
\sup_{\epsilon > 0} \sup_{t \leq \epsilon^{-2a} \tau} \sup_{|x| = n} t^{1/12} \sum_{y} \left| p_t(x, y) - \pi^{(0)}_t(x, y) \right| \leq c_n(\tau) \tag{B.3}
\]

**Proof of Theorem 1.** (Relative to the \(\xi\) particles). By definition
\[
\mu^{\epsilon}(\xi(x)) = \sum_{y} \pi_{t_0 \epsilon^{-2a}}(x, y) \xi(x) - t_0(y) \tag{B.4}
\]
hence
\[
\epsilon^{-a} |\nabla \mu^{\epsilon}(\xi(x))| \leq \frac{c}{\sqrt{t_0}} \tag{B.5}
\]
uniformly in \(\epsilon\) and \(x \in \Omega \epsilon\), which proves the second inequality in (5.5). We have
\[
\|v^{\epsilon}(x, \xi) - V_0^{\epsilon}(x)\|^2 = \sum_{y} q_{\beta}(x, y)^2 [\xi(y) - \mu^{\epsilon}(\xi(y))]^2 + \sum_{y \neq z} q_{\beta}(x, y) q_{\beta}(x, z) [\xi(y) - \mu^{\epsilon}(\xi(y))] [\xi(z) - \mu^{\epsilon}(\xi(z))] \tag{B.6}
\]
The \(\mu^{\epsilon}\) expectation of the first term is bounded by \(c \epsilon^{-a - \beta}\). The expectation of the second term is computed using (B.3) and it is then bounded by \(c [t_0 \epsilon^{-2a}]^{-1/12}\). The second inequality in (5.4) is proved. \(\blacksquare\)

**Proof of (11.19).** By (9.1),
\[
\epsilon^{-2a} |A_{t, N, t_{N-1}}(\eta_{N-1}) - A_{t, N, 0}(\eta_0)| \leq \alpha \epsilon^{-2a} + 2 \beta - 2a \tag{B.7}
\]
which is the first factor in (11.19). The proof of (11.19) then follows from the following theorem:

**Theorem 14.** For any \(n > 1\) there is \(c\) so that for any \(x\), \(|x| = n\), \(t \leq \epsilon^{-2} \tau\),
\[
\sup_{\|\psi\|_\infty \leq 1} \left| \sum_{z} p_t(x, z) L^st \psi(z) \right| \leq \frac{c}{t^{1/2 + 1/12}} \tag{B.8}
\]
where the sup is over all functions \(\psi(z)\), \(|z| = n\) which are bounded by 1.

**Proof.** We quote from the literature, [5], the following bound: there is \(c\) so that, for any \(z, z', |z - z'| = 1\) and any \(t \leq \epsilon^{-2a} \tau\),
\[
\sum_{y} \left| p_t(x, z', y) - p_t(x, y) \right| \leq \frac{c}{\sqrt{t}} \tag{B.9}
\]
where \( x^{(z,z')} \) is the configuration obtained from \( x \) by exchanging the content of the sites \( z \) and \( z' \); thus, if they are both contained in \( x \) or both in its complement, then \( x^{(z,z')} = x \); otherwise a particle of \( x \) is displaced from \( z \) to \( z' \) or viceversa. We need to estimate

\[
\sum_{y} p_t(x,y)L^2\psi(y) = \sum_{z} p_{t/2}(x,z)L^2\phi(z), \quad \phi(z) := \sum_{y} p_{t/2}(z,y)\psi(y) \tag{B.10}
\]

Let \( \nabla_i \) denote the gradient acting on the variable \( z_i \) and \( E_+ \) the set of all positively oriented unit vectors of \( \mathbb{Z}^d \). Then

\[
L^1\phi(z) = \sum_{i=1}^{n} \sum_{e=1}^{n} 1_{z_i \neq z_i+e, j \neq i} \{ \phi(z^{(z_i z_i+e)}) - \phi(z) \} = T_1 + T_2 \tag{B.11}
\]

\[
T_1(z) = \sum_{i} \sum_{e \in E_+} 1_{z_i \neq z_i+e, j \neq i} \{(e \cdot \nabla_i)[e \cdot \nabla_i]\phi(z')
\]

\[
T_2(z) = \sum_{i,j} \sum_{|e|=1} 1_{z_j = z_i - e, z_k \neq z_i+e, k \neq i} \{ \phi(z^{(z_i z_i+e)}) - \phi(z) \}
\]

From (B.10), after “integrating by parts”,

\[
\sum_{z} p_{t/2}(x,z)T_1(z) = \sum_{i} \sum_{e \in E_+} \sum_{z} \{(e \cdot \nabla_i)p_{t/2}(x,z) \times 1_{z_i \neq z_i+e, j \neq i}\} \{(e \cdot \nabla_i)\phi(z)\} \tag{B.12}
\]

Recalling the definition of \( \phi \) in (B.10) and using (B.9), we bound the gradient of \( \phi \) by \( c/\sqrt{t} \). The first factor in (B.13) is instead estimated using (B.3), and gives the desired bound \( ct^{-1/2-1/12} \). The term with \( T_2(z) \), see (B.11), is treated similarly and (11.19) is proved.

### C Proof of Theorem 3. First part

In this and in the next three Appendices we will bound

\[
E'_\epsilon \left( \sup_{\tau \leq t} \left| \int_0^\tau (L + D)\{\|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2\} \right| \right) \tag{C.1}
\]

We will start by an explicit computation of \( (L + D)\|u^\epsilon - U^\epsilon\|^2 \) and \( (L + D)\|v^\epsilon - V^\epsilon\|^2 \) obtaining expressions which are sums of Laplacian and Reaction terms. The latter are those produced by the \( L^{(\eta,0)} + D^{(\eta,0)} \) and \( L^{(\xi,0)} + D^{(\xi,0)} \), with \( D^{(\eta,0)} \) and \( D^{(\xi,0)} \) the laplacian parts of \( D \). \( D \) is defined in Section 3. The Reaction terms instead are produced by \( L^{(\eta,\pm)} + D^{(\eta,\pm)} \) and \( L^{(\xi,\pm)} + D^{(\xi,\pm)} \) for the births and by \( L^{(\eta,-)} + D^{(\eta,-)} \) and \( L^{(\xi,-)} + D^{(\xi,-)} \) for the deaths, having denoted by \( D^{(\eta,\pm)} \) and \( D^{(\xi,\pm)} \) the reaction terms in \( D \).

A finer classification of all the terms distinguishes those denoted by \( B_i \) which are bounded proportionally to \( \|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2 \) and those which when inserted in...
(C.1) vanish as $\epsilon \to 0$. They will be generically called remainders. Among them we call $R_i^\epsilon$ and $C_i^\epsilon$ those which can be estimated using the a priori bounds of Section 8, $S_i^\epsilon$ those bounded in Appendix D exploiting the results of Section 10. There are two more terms, $\mathcal{H}$ which is bounded in Remarks 1 below using Corollary 2 and $Q$, studied in Appendix E using the analysis in Section 11. In Appendix F we bound the martingale terms in (4.5)-(4.6), thus completing the proof of Theorem 3.

In the sequel we will often drop $\epsilon$ from the notation.

### C.1 Laplacian terms

A computation essentially similar to the one in the proof of Theorem 6 yields

$$
(L^{[\eta,0]} + D^{[\eta,0]})||u - U||^2 = -2||u - U||_H^2 + R_1^\epsilon
$$

where $R_1^\epsilon$, defined in (8.12), vanishes as $\epsilon \to 0$, see (8.17), and for this reason is called a $R^\epsilon$ term. We will not take advantage of the negative sign of the $H^2$ term $-2||u - U||_H^2$, which will be just bounded by 0.

In an analogous way,

$$
(L^{[\xi,0]} + D^{[\xi,0]})||v - V||^2 = -2\epsilon^{-2\alpha} \langle \nabla (v - V), e^{-U} \nabla (v - V) \rangle + 2\mathcal{H} + R_3^\epsilon + R_1^\epsilon + 2S_1^\epsilon + 2C_1^\epsilon
$$

where $R_3^\epsilon$ is defined in (8.27) and bounded in (8.30),

$$
\mathcal{H}^{\xi,\eta} = \epsilon^{-2\alpha} \langle v - V, [e^{-u} - e^{-U}] \nabla v \rangle
$$

$$
R_3^{\xi,\eta} = -2\epsilon^{-2\alpha} \langle v - V, (\nabla e^{-U}) \nabla (v - V) \rangle
$$

$$
S_1^{\xi,\eta} = \sum_{x:|e|=1} \epsilon^d \sum_x \frac{\delta(x)}{x} - V(x) [e^{-2\alpha} \sum_z e \cdot \nabla q_\beta(z - x) \cdot \xi(z)(1 - \xi(z + e)) [1_{q(z+e)=0} - e^{-u(z+e)}]}
$$

$$
C_1^{\xi,\eta} = \epsilon^{-2\alpha} \sum_{x:|e|=1} \epsilon^d \sum_x \frac{\delta(x)}{x} - V(x) \sum_z \xi(z)(1 - \xi(z + e)) \cdot [e^{-u(z+e)} - e^{-u(x)}]) e \cdot \nabla q_\beta(x, z)
$$

**Remarks 1.** From (8.42) we get that

$$
-2\epsilon^{-2\alpha} \langle \nabla (v - V), e^{-U} \nabla (v - V) \rangle + 2\mathcal{H} \leq c \left( ||u - U||^2 + ||v - V||^2 + \epsilon^{1-2\alpha} ||u||_{H^2}^2 \right)
$$

From (8.18) and (8.9) it follows that

$$
\mathcal{E}_\mu \left( \sup_{T' \leq T} \left| \int_0^{T'} C_1^{\xi,\eta}(\xi_\tau, \eta_\tau) \right| \right) \leq \epsilon^{-2\alpha} \sum_{x, z} \left| e \cdot \nabla q_\beta(x, z) \right| \int_0^{T'} \epsilon^d \sum_x \left| e^{-u(z+e)} - e^{-u(x)} \right| \leq c \epsilon^{1-2\alpha}
$$
In Appendix D we prove that
\[
\lim_{\epsilon \to 0} E_\epsilon \left( \sup_{\tau \leq \tau'} \left| \int_0^{\tau'} S_\epsilon'(\xi, \eta) \right| \right) = 0
\]
Finally, since \( |\nabla e^{-U}| \leq c\epsilon \), \( |R_2^\epsilon| \leq c\epsilon^{1-2a} \).

C.2 The Reactions terms

- **Birth process, \( \eta \)-particles.**

\[
(L^{(n,+)} + D^{(n,+)}) \|u - U\|^2 = 2\kappa \|u - U\|^2 + R_2^\epsilon \quad \text{(C.10)}
\]
where \( R_2^\epsilon(\eta) \) is defined in (8.14) and bounded by \( c\epsilon^{(1-a)d} \), see the proof of Theorem 6.

- **Death process, \( \eta \)-particles.**

\[
(L^{(n,-)} + D^{(n,-)}) \|u - U\|^2 = T + 2\langle u - U, [F_1^-(U) + F_2^-(U)V]\rangle + R_3^\epsilon - 2S_2^\epsilon \quad \text{(C.11)}
\]
where \( R_3^\epsilon \) is defined in (8.15), its expectation being bounded by \( c\epsilon^{(1-a)d} \), see again the proof of Theorem 6. Shorthanding by \( \chi_{M,z} \) the characteristic function that \( u(\cdot, \eta) \leq M \),

\[
T = -2\epsilon d \sum_x [u(x) - U(x)] \sum_z p_\alpha(x, z) \chi_{M,z} F_1^-(u(z))
\]
\[
-2\epsilon d \sum_x [u(x) - U(x)] \sum_z p_\alpha(x, z) \xi(z) \chi_{M,z} F_2^-(u(z))
\]
and

\[
S_2^\epsilon(\xi, \eta) = \epsilon d \sum_x [u(x) - U(x)] \sum_z p_\alpha(x, z) \left\{ \left[ \eta(z) \kappa_{\eta(z)} - \chi_{M,z} F_1^-(u(z)) \right]
\right. \\
+ \left[ \xi(z) \eta(z) [\kappa_{\eta(z)+1} - \kappa_{\eta(z)}] - \chi_{M,z} F_2^-(u(z)) \right] \right\} \quad \text{(C.12)}
\]

We next write
\[
T = -2\langle u - U, F_1^-(U) \rangle - 2\langle u - U, F_2^-(U)V \rangle
\]
\[
-2\epsilon d \sum_x [u(x) - U(x)] F_2^-(U(x)) \left\{ \sum_z p_\alpha(x, z) \xi(z) - V(x) \right\} - 2B_1^\epsilon + R_6^\epsilon \quad \text{(C.13)}
\]
where

\[
R_6^\epsilon = -2\epsilon d \sum_x [u(x) - U(x)] \sum_z p_\alpha(x, z)
\]
\[
\times \left\{ [F_1^-(U(z)) - F_1^-(U(x))] + \xi(z)[F_2^-(U(z)) - F_2^-(U(x))] \right.
\]
\[
\left. -(1 - \chi_{M,z})[F_1^-(U(z)) + \xi(z)F_2^-(U(z))] \right\} \quad \text{(C.14)}
\]
and
\[
\mathcal{B}_1^\epsilon = e^d \sum_{x} \{u(x) - U(x)\} \sum_{z} p_\alpha(x, z) \chi_{M, z}\{(F_1^-(u(z)) - F_1^-(U(z))]
+ \xi(z)[F_2^-(u(z)) - F_2^-(U(z))]\} 
\]  
(C.15)

The first two terms on the r.h.s. of (C.13) simplify with the second term on the r.h.s. of (C.11). We then add and subtract \(v(x)\) in the third term on the right hand side of (C.13), so that
\[
\left(L^{(\eta,-)} + D^{(\eta,-)}\right)\|u - U\|^2 = -2S_2^\epsilon - 2(B_0^\epsilon + B_1^\epsilon) - 2C_2^\epsilon + R_3^\epsilon + R_6^\epsilon + R_7^\epsilon
\]  
(C.16)

where
\[
B_0^\epsilon = \langle u - U, F_2^-(U)(v - V) \rangle 
\]  
(C.17)
\[
C_2^\epsilon(\xi, \eta) = \langle (u - U)F_2^-(U), v \rangle 
\]  
(C.18)
\[
R_7^\epsilon(\xi, \eta) = e^d \sum_{x} \{u(x) - U(x)\}F_2^-(U(x)) \sum_{z} \{p_\alpha(x, z) - \sum_{y} p_\alpha(x, y)q_\beta(y, z)\} \xi(z) 
\]  
(C.19)

Remarks 2. In Appendix D we prove that
\[
\lim_{\epsilon \to 0} E_{\mu^\epsilon} \left( \sup_{\tau' \leq \tau} \left| \int_{0}^{\tau'} S_2^\epsilon(\xi_t, \eta_t) dt \right| \right) = 0
\]

By Theorem 5,
\[
\sup_{t \leq \tau} E_{\mu^\epsilon} \left( \left| R_6^\epsilon(\eta_t, \xi_t) \right| \right) \leq cM \epsilon \sum_{z} p_\alpha(0, z) |z| + c\epsilon^{(1-\alpha)d}
\]

The expression on the r.h.s. of (C.15) is called \(\mathcal{B}_1^\epsilon\) because \(|\mathcal{B}_1^\epsilon| \leq c_M \|u^\epsilon - U^\epsilon\|^2\), a constant which depends on \(M\). Similarly, \(|\mathcal{B}_0^\epsilon| \leq c_M (\|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2\), since \(F_2^-(U^\epsilon)\) is uniformly bounded in compact time intervals.

From (8.18) and (8.9) it follows that
\[
E_{\mu^\epsilon} \left( \sup_{\tau' \leq \tau} \left| \int_{0}^{\tau'} C_2^\epsilon(\xi_t, \eta_t) dt \right| \right) \leq c\epsilon
\]  
(C.20)

Calling \(t = \epsilon^{-2+2\alpha}, s = \epsilon^{-2\alpha+2\beta}\), since
\[
\sum_{z} |p_\alpha \ast q_\beta(x, z) - p_\alpha(x, z)| = \sum_{z} |\pi_{t+s}(x, z) - \pi_t(x, z)| \leq c_s t = c\epsilon^{(1-\alpha-\alpha+\beta)}
\]
the r.h.s. of (C.19) is indeed a \(R\)-term.
• Birth process, $\xi$ particles.

\[
(L^{(\xi,+)} + D^{(\xi,+)}\|v - V\|^2 = T_1 - 2\kappa'\langle v - V, V(1 - V)e^{-U}\rangle + R_8' \quad \text{(C.21)}
\]

where $R_8'$ is defined in (8.36) and

\[
T_1 = 2\kappa'\epsilon^d \sum_x [v(x) - V(x)] \frac{1}{2d} \sum_{c\in\{e\}} \sum_z q_\beta(z - x)\xi(z) [1 - \xi(z + e)] 1_{\eta(z+e)=0}
\]

By adding and subtracting $e^{-u(z+e)}$ and calling

\[
S_3' = \epsilon^d \sum_x [v(x) - V(x)] \frac{1}{2d} \sum_{c\in\{e\}} \sum_z q_\beta(z - x)\xi(z) [1 - \xi(z + e)] \left[1_{\eta(z+e)=0} - e^{-u(z+e)}\right]
\]

we get

\[
T_1 = 2\kappa'S_3' + 2\epsilon^d \sum_x [v(x) - V(x)] \frac{\kappa'}{2d} \sum_{c\in\{e\}} \sum_z q_\beta(z - x)\xi(z) [1 - \xi(z + e)] \left\{e^{-U(x)} + e^{-U(z+e)} - e^{-U(x)} - e^{-U(z+e)}\right\}
\]

\[
= 2\kappa'S_3' + T_2 + R_{10}' + B_2'
\]

where $T_2$, $R_{10}'$ and $B_2'$ identify the terms in the previous line ($B_2'$ is explicitly written in (C.28) below). We next write

\[
T_2 = 2\epsilon^d \sum_x [v(x) - V(x)] e^{-U(x)} \frac{\kappa'}{2d} \sum_{c\in\{e\}} \sum_z q_\beta(x, z) \left\{V(x)[1 - V(x)] + [v(x)[1 - v(x + e)] - V(x)[1 - V(x + e)] + \xi(z)[1 - \xi(z + e)] - v(x)[1 - v(x + e)]\right\}
\]

\[
= T_3 + B_3' + 2\kappa'Q^e
\]

where, as before, $T_3$, $B_3'$ and $Q^e$ identify the corresponding terms in the previous line, in particular $Q^e$ is explicitly written in (C.27) below. Observe that $T_3$ cancels with the first term on the right hand side of (C.21). We rewrite $B_3'$ as

\[
B_3' = 2\epsilon^d \sum_x [v(x) - V(x)] e^{-U(x)} \frac{\kappa'}{2d} \sum_{c\in\{e\}} \left\{[1 - v(x + e)][v(x) - V(x)]\right\}
\]

\[
+ V(x)(V(x + e) - v(x + e)) \right\}
\]

so that

\[
(L^{(\xi,+)} + D^{(\xi,+)}\|v - V\|^2 = B_3' + 2\kappa'S_3' + 2\kappa'Q^e + B_2' + R_8' + R_{10}'
\]

\[
\text{(C.26)}
\]
where

\[ Q^\epsilon = e^d \sum_x [v(x) - V(x)] e^{-U(x)} \frac{1}{2d} \sum_{e:|e|=1} \sum_z q_\beta(z - x) \{ \xi(z)[1 - \xi(z + \epsilon)] - v(z)[1 - v(z + \epsilon)] \} \]  

(C.27)

\[ B_2^\epsilon = e^d \sum_x [v(x) - V(x)] \frac{k'_\epsilon}{2d} \sum_{e:|e|=1} \sum_z q_\beta(z - x) \xi(z)[1 - \xi(z + \epsilon)][e^{-u(z+\epsilon)} - e^{-U(z+\epsilon)}] \]  

(C.28)

Remarks 3. In Appendix D we prove that

\[ \lim_{\epsilon \to 0} E^\epsilon \left( \sup_{\tau' \leq \tau} \left| \int_0^{\tau'} S_\epsilon^\tau(\xi, \eta) \right| \right) = 0 \]  

(C.29)

We further observe that \( |B_2^\epsilon| + |B_3^\epsilon| \leq c(\|u^\epsilon - U^\epsilon\|^2 + \|v^\epsilon - V^\epsilon\|^2) \). In Appendix E we prove that

\[ \lim_{\epsilon \to 0} E^\epsilon \left( \sup_{\tau' \leq \tau} \left| \int_0^{\tau'} Q^\epsilon(\xi, \eta) \right| \right) = 0 \]

• Death process, \( \xi \)-particles.

\[(L^{(\xi, -)} + D^{(\xi, -)}) \|v - V\|^2 = -2e^d \sum_x [v(x) - V(x)] \sum_z q_\beta(z - x) \xi(z) \kappa_\eta(z) + 1 \]

\[+ 2(v - V, VG^{-}(U)) + R_9^\epsilon \]

where \( R_9^\epsilon \) is defined in (8.37). As before,

\[ 2e^d \sum_x [v(x) - V(x)] \sum_z q_\beta(z - x) \xi(z) \kappa_\eta(z) + 1 \]

\[ = 2S_4 + e^d \sum_x [v(x) - V(x)] \sum_z q_\beta(z - x) \xi(z) \]

\[ \times \{ G^{-}(U(x)) + [G^{-}(U(z)) - G^{-}(U(x))]) + [G^{-}(u(z)) - G^{-}(U(z))] \} \]

\[ = 2S_4 + 2(v - V, VG^{-}(U)) + R_{12}^\epsilon + 2B_3^\epsilon \]  

(C.31)

where

\[ S_4 = e^d \sum_x [v(x) - V(x)] \sum_z q_\beta(z - x) \xi(z) \left\{ \kappa_\eta(z) + 1 - \chi_M(x) G^{-}(u(z)) \right\} \]  

(C.32)

\[ B_3 = e^d \sum_x [v(x) - V(x)] \sum_z q_\beta(z - x) \xi(z) \chi_M(x) [G^{-}(u(z)) - G^{-}(U(z))] \]  

(C.33)
We thus have

\[ R'_{12} = 2\varepsilon^d \sum_x [v(x) - V(x)] \sum_z q_\beta (z - x) \xi (z) \left\{ \left[ G^- (U(z)) - G^- (U(x)) \right] \right\} - (1 - \chi_{M, +}(\eta)) G^- (U(z)) \]  

(C.34)

We now observe that the first term in (C.36) is nonpositive and that from Remarks 2 we get that given any \( \tau > 0 \) and from (C.10) we get that

\[ \left| \int_0^\tau (L + D) \{ \| u^\varepsilon (\eta t) - U^\varepsilon (t) \|^2 \} dt \right| \leq \int_0^\tau \left\{ c_1 \| u^\varepsilon (\eta t) - U^\varepsilon (t) \|^2 + c_2 \| v^\varepsilon (\xi t) - V^\varepsilon (t) \|^2 \right\} + \int_0^\tau R'_1 (\xi t, \eta t) dt \]  

(C.38)

We now observe that the first term in (C.36) is nonpositive and that from Remarks 2 we get that given any \( \tau > 0 \) there are \( c_1 \) and \( c_2 \) so that

\[ \lim_{\varepsilon \to 0} E^\varepsilon \left( \sup_{\varepsilon' \leq \varepsilon} \left| \int_0^\tau R'_1 (\xi t, \eta t) dt \right| \right) = 0 \]  

(C.39)

From (C.3), (C.26) and (C.35) we get that

\[ (L + D)\| v - V \|^2 = -2\varepsilon^{-2a} \langle \nabla (v - V), e^{-U} \nabla (v - V) \rangle + 2\mathcal{H} - 2\langle v - V, [v - V] G^- (U) \rangle + \mathcal{B}_3^2 - 2\mathcal{B}_3 + \mathcal{B}_5^2 + \mathcal{R}_2' \]  

(C.40)

where

\[ \mathcal{R}_2' (\xi, \eta) = 2(S'_1 + \kappa' S'_3 + S'_4) + 2C'_2 + 2\kappa' Q' + \sum_{k=3,4,8,9,10,12} R'_k \]  

(C.41)
We now observe that the third term is non negative and by Remarks 1, 3 and 4 we get given any \( \tau > 0 \) there are \( c_3 \) and \( c_4 \) so that
\[
\left| \int_0^\tau (L + D) \{ \| v'(\eta) - V'(t) \|^2 \} \, dt \right| \leq \int_0^\tau \left\{ c_3 \| v'(\eta) - U'(t) \|^2 + c_4 \| v'(\xi(t)) - V'(t) \|^2 \right\}
+ \int_0^\tau \mathcal{R}_2(\xi(t), \eta(t)) \, dt
\]
(C.42)

\[
\lim_{\epsilon \to 0} E_{\epsilon} \left( \left( \sup_{\tau' \leq \tau} \int_0^{\tau'} \mathcal{R}_2(\xi(t), \eta(t)) \, dt \right) \right) = 0
\]
(C.43)

\[ \]

D \ The \( S \) terms

Denote by \( X_i^\epsilon, i = 1, \ldots, 4 \), the terms in (4.5)-(4.6) which contain \( S_i^\epsilon \), (their explicit expressions will be recalled below). The most dangerous one is \( X_i^\epsilon \) because it appears with a divergent multiplicative factor \( \epsilon^{-2a} \), see (D.1). We thus start from this one, although the following analysis covers, except for some coefficients which are different, the easier term \( S_i^\epsilon \), where the dangerous factor \( \epsilon^{-2a} \) is absent.

Calling
\[
a_i^\epsilon(t, y) = \sum_{x, |x| = 1} [v'(x, \xi(t)) - V'(x, t)][(-e) \cdot \nabla + e \cdot \nabla]q_\beta(y - e - x)\xi(y - e)(1 - \xi(y))
\]
the term of the remainder containing \( S_i^\epsilon \) is
\[
X_i^\epsilon := \epsilon^{-2a} E_{\epsilon} \left( \sup_{\tau' \leq \tau} \int_0^{\tau'} \epsilon^d \sum_y a_i^\epsilon(t, y)[1_{\eta(y) = 0} - e^{-u'(y, \eta_y)}] \right)
\]
(D.1)

We have
\[
|X_i^\epsilon| \leq \epsilon^{-2a} E_{\epsilon} \left( \sup_{\tau' \leq \tau} \int_0^{\tau'} \epsilon^d \sum_y [1_{\eta_{1,2a}(y) = 0} - e^{-u'(y, \eta_y)}] \right) + c e^{2(\alpha-a)}
\]
(D.2)

because the rate of change of \( \xi(t) \) is bounded by \( 2de^{-2a} + ce^{bn(x)} \) and (D.2) follows from Theorem 5. Thus
\[
X_i^\epsilon \leq e^{-2a} E_{\epsilon} \left( \int_0^\tau \left\| \epsilon^d \sum_y a_{t,y}[1_{\eta_{1,2a}(y) = 0} - e^{-u'(y, \eta_y)}] \right\|^2 \right) + \epsilon e^{2(\alpha-a)}
\]
\[
\leq e^{-2a} E_{\epsilon} \left( \int_0^\tau \left\| \epsilon^d \sum_y a_{t,y}[1_{\eta_{1,2a}(y) = 0} - e^{-u'(y, \eta_y)}] \right\|^2 \right)^{1/2} + \epsilon e^{2(\alpha-a)}
\]
\[
\leq e^{-2a} \| a \| \tau \sup_{t \leq \tau} \sup_{|y - y'| \geq \epsilon^{-1/2}} |E_{\epsilon} \left( f_t(y)f_t(y') \right) | + \epsilon e^{2(\alpha-a)} + c' e^{ad/4-2a}
\]
(D.3)
where, $\|a\|$ is the sup norm of $a_{t,y}$ and

$$f_t(y) = [1_{\eta_{t+2\alpha}(y)=0} - e^{-u^*(y,\eta_t)}] \tag{D.4}$$

By Theorem 9, $|E_{\mu}^e(f_t(y)f_t(y'))| \leq c\epsilon^{2\alpha}$ so that $X'_1 \to 0$, thus concluding the analysis of $S'_1$ (and of $S'_3$, as well).

Recalling (C.32),

$$X^e_4 := E_{\mu}^e \left( \sup_{\tau' \leq \tau} \left| \int_{0}^{\tau'} \epsilon d y \sum_{y} \{ \kappa_{\eta_{t+2\alpha}(y)+1} - \chi_{M,y}G^-(u^*(y,\eta_t)) \} \right| \right)$$

$$a^*_2(t,y) = \sum_{x} [v^*(x,\xi_t) - V^*(x,t)]q_{\beta}(x,y)\xi(y)$$

Analogously to (D.2), we get

$$|X'_4| \leq E_{\mu}^e \left( \sup_{\tau' \leq \tau} \left| \int_{0}^{\tau'} \epsilon d y \sum_{y} \{ \kappa_{\eta_{t+2\alpha}(y)+1} - \chi_{M,y}G^-(u^*(y,\eta_t)) \} \right| \right) + c\epsilon^{2\alpha}$$

$$\leq E_{\mu}^e \left( \int_{0}^{\tau} \epsilon d y \sum_{y} \left\{ \kappa_{\eta_{t+2\alpha}(y)+1} - \chi_{M,y}G^-(u^*(y,\eta_t)) \right\} \right) + c\epsilon^{2\alpha}$$

$$\leq E_{\mu}^e \left( \int_{0}^{\tau} \epsilon d y \sum_{y} \chi_{M,y} \left\{ \kappa_{\eta_{t+2\alpha}(y)+1} - G^-(u^*(y,\eta_t)) \right\} \right) + c[\epsilon^{2\alpha} + \epsilon^{(1-\alpha)d/2}]$$

having used Cauchy-Schwartz and Theorem 5 in the last inequality. Hence

$$X'_4 \leq \tau \sup_{t \leq \tau} \sup_{|y-y'| \geq \epsilon^{-1+\alpha/2}} |E_{\mu}^e(g_t(y)g_t(y'))| + c\epsilon^{2\alpha} + c\epsilon^{ed/4-2\alpha}$$

$$g_t(y) = \chi_{M,y} [\kappa_{\eta_{t+2\alpha}(z)+1} - G^-(u^*(z,\eta_t))] \tag{D.5}$$

Since

$$G^-(u) = E_u(\kappa_{\eta(0)+1}), \quad E_u(\eta(0)) = u \tag{D.6}$$

($E_u$ is the expectation w.r.t. the Poisson law on $\mathbb{N}$ which has density $u$), by Theorem 10 we conclude that $X'_4 \to 0$.

We finally consider the remainder containing $S'_2$; it consists of the sum of two terms, see (C.12), whose structures are essentially similar. For simplicity we only consider the first one. We have

$$X^e_2 := E_{\mu}^e \left( \sup_{\tau' \leq \tau} \left| \int_{0}^{\tau'} \epsilon d x \sum_{x} [u^*(x,\eta_t) - U^*(x,t)] \sum_{z} p_{\alpha}(z - x) \times [\eta_t(z)\kappa_{\eta_t(z)} - F^-_1(u^*(z,\eta_t)) \right| \right) \tag{D.7}$$

where

$$F^-_1(u) = E_u(\eta(0)\kappa_{\eta(0)}) \tag{D.8}$$
and, again,
\[X_2^\epsilon \leq M \tau \sup_{t \leq \tau} \sup_{|y - y'| \geq \epsilon^{-1 + \alpha/2}} E_{\mu}^\epsilon (h_i(y)h_i(y')) + c\epsilon^{2(\alpha/7 + \alpha/4)} + c'e^{\alpha d/4 - 2\alpha} + c''\epsilon^{(1 - \alpha)d/2}
\]
\[h_i(y) = \eta_{t + 2\alpha}(z)\kappa_\eta_{t + 2\alpha}(z) - F_1(u^\epsilon(z, \eta_t))
\]
where the error \(c\epsilon^{2\alpha(7/4)}\) comes from having used Theorem 8 to express \(u^\epsilon_{t + \epsilon^{2\alpha}}(\cdot)\) as a linear combination of \(u^\epsilon_i(\cdot)\). We have then used Cauchy-Schwartz and Theorem 5 to bound the contribution of \(u^\epsilon(\cdot)\).

### E The term \(Q\)

In this appendix we will prove that for any \(\tau > 0\),
\[
\lim_{\epsilon \to 0} E_{\mu}^\epsilon (\sup_{\tau' \leq \tau} \int_0^{\tau'} Q(\xi_t)) = 0
\]
with \(Q(\xi)\) defined in (C.27).

Since \(|v^\epsilon(x, \xi) - V^\epsilon(x, t)| \leq 1, |Q(\xi_t)| \leq \epsilon^d \sum_{x \in \Omega_\epsilon} S_x|f^\epsilon(\xi)|\), where
\[
f^\epsilon(\xi) := \kappa' \left( \sum_z \frac{1}{2d} \sum_{c:|e|=1} q_{c}(z)\xi(z)|1 - \xi(z + e)| - v^\epsilon(0, \xi)|1 - v^\epsilon(0, \xi)| \right)
\]
and \(|f^\epsilon(\xi)| \leq \kappa'\). For \(T > 0\),
\[
E_{\mu}^\epsilon (\sup_{\tau' \leq \tau} \int_0^{\tau'} Q(\xi_t)) \leq E_{\mu}^\epsilon (\sup_{\tau' \leq \tau} \int_0^{\tau'} |Q(\xi_t)|) = E_{\mu}^\epsilon (\int_0^{\tau} |Q(\xi_t)|) \leq \tau \nu_{\epsilon}^\epsilon (|f^\epsilon|)
\]
\[
\leq \tau E_{\mu}^\epsilon (|f^\epsilon(\xi_T)|) + 2T\kappa'
\]
with \(\nu_{\epsilon}^\epsilon\) defined in (11.1). It follows in fact from (11.1) that \(\nu_{\epsilon}^\epsilon\) is invariant under space translations and that for any bounded function \(f\)
\[
\left| \nu_{\epsilon}^\epsilon (f(\eta_t, \xi_t)) - \nu_{\epsilon}^\epsilon (f(\eta_t, \xi_0)) \right| \leq \frac{2T}{\tau} \sup_{s \leq \tau + t} \epsilon^d \sum_{x \in \Omega_\epsilon} E_{\mu}^\epsilon (|S_x f(\eta_s, \xi_s)|)
\]
Hence the last term in (E.3), recalling that \(|f^\epsilon(\xi)| \leq \kappa'\).

By choosing \(T = e^{M\epsilon^{2\beta}}\) with \(M\) as in (8.2), we get
\[
\text{l.h.s. of (E.3)} \leq \tau E_{\mu}^\epsilon \left( 1_{u^\epsilon(0, \eta) < M} |f^\epsilon(\xi_T)| \right) + 2e^{M\epsilon^{2\beta}} + \tau \kappa' \epsilon^{(1 - \alpha)d}
\]
We will then prove (E.1) by showing that
\[
\lim_{\epsilon \to 0} E_{\mu}^\epsilon \left( 1_{u^\epsilon(0, \eta) < M} f^\epsilon(\xi_T)^2 \right) = 0, \quad T = e^{M\epsilon^{2\beta}}
\]
We write $x$ for a subset of $\Omega$, $|x|$ for its cardinality and call
\[ g_x(\xi) = \prod_{x \in x} \xi(x) \] (E.7)
Then $f^x(\xi)^2 = \sum_{x} c(x)g_x(\xi)$ (c(x) numerical coefficients) and
\[ |f^x(\xi)^2 - \sum_{x} c(x)g_x(\xi)| \leq c_\epsilon^{(a-\beta)d}, \quad B = \{ x : |x| \leq \epsilon^{-a+\beta/2} \} \] (E.8)
where,
\[ \sum_{x \in x = 4} |c(x)| \leq c, \quad \sum_{x \in x < 4} |c(x)| \leq c_\epsilon^{(a-\beta)d}, \quad \sum_{x \not\in B} |c(x)| \leq c_\epsilon^{-\beta} \] (E.9)
Then (E.6) follows from (11.4). In fact,
\[
\sum_{x} \pi_{e^{-2(a-\beta)}e^{-u^*(0,\eta)}}(x, \bar{z})\xi_0(\bar{z}) = \sum_{x} \pi_{e^{-2(a-\beta)}e^{-u^*(0,\eta)-1}}(x, y)\{\sum_{z} q_\beta(y, \bar{z})\xi_0(\bar{z})\}
= \sum_{y} \pi_{e^{-2(a-\beta)}e^{-u^*(0,\eta)-1}}(x, y)v^\epsilon(y, \xi_0) = v^\epsilon(0, \xi_0) + R^\epsilon
\]
\[ R^\epsilon = \sum_{y} e^{e^{-2(a-\beta)}e^{-u^*(0,\eta)-1}}(x, y)[v^\epsilon(y, \xi_0) - v^\epsilon(0, \xi_0)] \]
By Theorem 6,
\[ E_{\epsilon^2}(v^\epsilon(y, \xi_0) - v^\epsilon(0, \xi_0)) \leq c_\epsilon|y| \] (E.10)
so that $|R^\epsilon| \leq c_\epsilon e^{a-\beta}$. Then
\[ \lim_{\epsilon \to 0} \sup_{x \in x \in B, |x| = 4} \left| E_{\epsilon^2}(1_{v^\epsilon(0,\eta) < M}g_x(x)) - E_{\epsilon^2}(v^\epsilon(y, \xi_0)) \right| = 0 \]
Hence (E.6), using (E.8)-(E.9), thus concluding the proof of (E.1).

**F Proof of Theorem 3. Conclusion**

With reference to (C.38) and (C.42), we call $A$ the $2 \times 2$, matrix with entries $c_1, c_2, c_3, c_4$ and we define the following two dimensional vector $R^\epsilon(t) = (R^\epsilon_1(t), R^\epsilon_2(t))$
\[ R^\epsilon_i(t) := \int_0^t \Phi^\epsilon_i(\xi_s, \eta_s)ds + M_i(t), \quad i = 1, 2 \] (F.1)
where $M_1(t)$ and $M_2(t)$ are the mean zero martingales defined in (4.5) and (4.6) respectively. Thus (6.1) is proven. From (C.39) and (C.43) it follows that (6.2) holds for the first term on the right hand side of (F.1).
We are thus left with the proof that also the martingale terms verify (6.2). We first notice that
\[
\left( \mathbb{E}_{\mu} \left( \sup_{s \leq t} |M_i(s)| \right) \right)^2 \leq \mathbb{E}_{\mu_i} \left( \sup_{s \leq t} M_i(s)^2 \right) \leq 4\mathbb{E}_{\mu_i} \left( M_i(t)^2 \right), \quad i = 1, 2 \quad (F.2)
\]
and then write (see for instance Chapter 2 of [4]):
\[
M_i(t)^2 - \int_0^t \gamma_i(s) ds =: N_i(t) \text{ is a martingale}, \quad i = 1, 2
\quad (F.3)
\]
where the compensators \( \gamma_i \) of \( M_i^2 \), \( i = 1, 2 \) are given in (F.4) below,
\[
\gamma_i(s) = LX_i^2 - 2X_iLX_i, \quad i = 1, 2
\quad (F.4)
\]
where
\[
X_1(s) = ||u^e(\eta_s) - U^e(s)||^2, \quad X_2(s) = ||v^e(\xi_s) - V^e(s)||^2
\]
Going back to (F.3), we have for all \( t > 0 \),
\[
\mathbb{E}_{\mu_i}(M_i(t)^2) = \mathbb{E}_{\mu_i}(N_i(t) + \int_0^t \gamma_i(s) ds) \leq \mathbb{E}_{\mu_i}(X_i(0)^2) + t \sup_{s \leq t} \mathbb{E}_{\mu_i}(|\gamma_i(s)|) \quad (F.5)
\]
Since by Theorem 1 the first term on the right hand side of (F.5) vanishes in the limit \( \epsilon \to 0 \), the proof of Theorem 3 is concluded by the next Lemma.

**Lemma 1.** For any \( t > 0 \),
\[
\lim_{\epsilon \to 0} \sup_{s \leq t} \mathbb{E}_{\mu_i}(|\gamma_i(s)|) = 0 \quad (F.6)
\]

**Proof.** We first compute \( \gamma_1 \) and, recalling that \( L^{(\eta)} \) is the sum of three generators, we get for each of them the sum of three terms that we classify as \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) remainders,
\[
L^{(\eta)}||u - U||^4 - 2||u - U||^2 L^{(\eta)}(||u - U||^2) = \sum_{i=1}^{3} [\mathcal{A}_i + 4\mathcal{B}_i + 4\mathcal{C}_i] \quad (F.7)
\]
where
\[
\mathcal{A}_1 = \epsilon^{-2} e^{-2d} \sum_{x,y,z} \eta(z) |\nabla p_\alpha(x - z)|^2 |\nabla p_\alpha(y - z)|^2
\]
\[
\mathcal{B}_1 = \epsilon^{-2} e^{-2d} \sum_{x,y,z} [u(y) - U(y)] \eta(z) \sum_{e:|e|=1} |e \cdot \nabla p_\alpha(x - z)|^2 e \cdot \nabla p_\alpha(y - z)
\]
\[
\mathcal{C}_1 = \epsilon^{-2} e^{-2d} \sum_{x,y,z} [u(y) - U(y)][u(x) - U(x)] \eta(z) \nabla p_\alpha(x - z) \cdot \nabla p_\alpha(y - z) \quad (F.8)
\]
\[ \mathcal{A}_2 = \kappa \varepsilon^{2d} \sum_{x,y,z} \eta(z) p_{\alpha}(x - z)^2 p_{\alpha}(y - z)^2 \]
\[ \mathcal{B}_2 = \kappa \varepsilon^{2d} \sum_{x,y,z} [u(y) - U(y)] \eta(z) p_{\alpha}(x - z)^2 p_{\alpha}(y - z) \]
\[ \mathcal{C}_2 = \kappa \varepsilon^{2d} \sum_{x,y,z} [u(y) - U(y)] [u(x) - U(x)] \eta(z) p_{\alpha}(x - z) p_{\alpha}(y - z) \]  \hspace{1cm} (F.9)

Finally, calling \( \phi(\eta; \xi; z) = \eta(z) \kappa \eta(z) [1 - \xi(z)] + \eta(z) \kappa \eta(z) + \xi(z) \),
\[ \mathcal{A}_3 = \varepsilon^{2d} \sum_{x,y,z} \phi(\eta; \xi; z) p_{\alpha}(x - z)^2 p_{\alpha}(y - z)^2 \]
\[ \mathcal{B}_3 = -\varepsilon^{2d} \sum_{x,y,z} [u(y) - U(y)] \phi(\eta; \xi; z) p_{\alpha}(x - z)^2 p_{\alpha}(y - z) \]
\[ \mathcal{C}_3 = \varepsilon^{2d} \sum_{x,y,z} [u(y) - U(y)] [u(x) - U(x)] \phi(\eta; \xi; z) p_{\alpha}(x - z) p_{\alpha}(y - z) \]  \hspace{1cm} (F.10)

By (8.1), (8.17) and (8.9) (details are omitted)
\[ \lim_{\varepsilon \to 0} \sum_{i=1}^{3} \sup_{s \leq t} \mathbb{E}_{\mu_s} \left( |\mathcal{A}_i(s) + 4\mathcal{B}_i(s) + 4\mathcal{C}_i(s)| \right) = 0 \]  \hspace{1cm} (F.11)

We next compute \( \gamma_2 \) and, as before, for each of the three generators we get the sum of three terms that we classify as \( \tilde{\mathcal{A}} \), \( \tilde{\mathcal{B}} \) and \( \tilde{\mathcal{C}} \) remainders.
\[ L(\xi) \|v - V\|^4 - 2\|v - V\|^2 L(\eta) (\|v - V\|^2) = \sum_{i=1}^{3} [\tilde{\mathcal{A}}_i + 4\tilde{\mathcal{B}}_i + 4\tilde{\mathcal{C}}_i] \]  \hspace{1cm} (F.12)

where, calling \( \psi_e(\xi; \eta; z) = \xi(z) [1 - \xi(z + e)] \mathbf{1}_{y(x+e)=0} \),
\[ \tilde{\mathcal{A}}_1 = \varepsilon^{-2a} \varepsilon^{2d} \sum_{x,y,z} \sum_{e:|e|=1} \psi_e(\xi; \eta; z) |\nabla q_{\beta}(x - z)|^2 |\nabla q_{\beta}(y - z)|^2 \]
\[ \tilde{\mathcal{B}}_1 = \varepsilon^{-2a} \varepsilon^{2d} \sum_{x,y,z} [v(y) - V(y)] \sum_{e:|e|=1} \psi_e(\xi; \eta; z) |e \cdot \nabla q_{\beta}(x - z)|^2 e \cdot \nabla q_{\beta}(y - z) \]
\[ \tilde{\mathcal{C}}_1 = \varepsilon^{-2a} \varepsilon^{2d} \sum_{x,y,z} [v(y) - V(y)] [v(x) - V(x)] \psi_e(\xi; \eta; z) \nabla q_{\beta}(x - z) \cdot \nabla q_{\beta}(y - z) \]

\[ \tilde{\mathcal{A}}_2 = \frac{\kappa'}{2d} \varepsilon^{2d} \sum_{x,y,z} \psi_e(\xi; \eta; z) q_{\beta}(x - z)^2 q_{\beta}(y - z)^2 \]
\[ \tilde{\mathcal{B}}_2 = \frac{\kappa'}{2d} \varepsilon^{2d} \sum_{x,y,z} [v(y) - V(y)] \psi_e(\xi; \eta; z) q_{\beta}(x - z)^2 q_{\beta}(y - z) \]
\[ \tilde{\mathcal{C}}_2 = \frac{\kappa'}{2d} \varepsilon^{2d} \sum_{x,y,z} [v(y) - V(y)] [v(x) - V(x)] \psi_e(\xi; \eta; z) q_{\beta}(x - z) q_{\beta}(y - z) \]  \hspace{1cm} (F.13)
Finally, calling $\phi(\eta, \xi; z) = \xi(z)\kappa_{q(z)+1}$,

$$\bar{A}_3 = \epsilon^{2d} \sum_{x, y, z} \phi(\eta, \xi; z)q_\beta(x - z)^2 q_\beta(y - z)^2$$
$$\bar{B}_3 = -\epsilon^{2d} \sum_{x, y, z} [v(y) - V(y)]\phi(\eta, \xi; z)q_\beta(x - z)^2 q_\beta(y - z)$$
$$\bar{C}_3 = \epsilon^{2d} \sum_{x, y, z} [v(y) - V(y)][v(x) - V(x)]\phi(\eta, \xi; z)q_\beta(x - z)q_\beta(y - z)$$

(F.14)

Since the variables are bounded by 1, by (8.9) (details are again omitted)

$$\lim_{\epsilon \to 0} \sum_{i=1}^{3} \sup_{s \leq t} \mathbb{E}_{\mu_s} \left( |\bar{A}_i(s) + 4\bar{B}_i(s) + 4\bar{C}_i(s)| \right) = 0$$

(F.15)

Lemma 1 is proved.

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