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by

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Abstract

This paper is concerned with the KPP equation with strong convection. The asymptotic behavior of the minimal speed of traveling fronts is derived for shear flow convection and for cellular flow convection. In the first case a new limit problem is derived and analyzed. For cellular flows new almost optimal upper bounds are obtained in terms of the wave speed of the homogenized problem. Thereby some conjectures on the asymptotic growth of front speeds are confirmed.

Introduction

Consider the convective reaction diffusion equation for a scalar function $u(t,x)$

$$\partial_t u(t,x) = \Delta u(t,x) + Mb(x) \cdot \nabla u(t,x) + f(u(t,x)) \tag{0.1}$$

The spatial domain is a subset of $\mathbf{R}^n$, $n \geq 2$ which is translation invariant in at least one direction. Further properties and boundary conditions will be specified below. The nonlinearity $f(u)$ is of KPP type, i.e. $f \in C^1([0,1])$

$$f(0) = f(1), \quad 0 < f(u) \leq f'(0)u \text{ for } 0 < u < 1$$

The convective flow field $b(x)$ is C^1 and divergence free and $M > 0$ is the amplitude of the convection.

The convection leads to an enlargement and mixing of the reaction zone. This leads in an enhancement of the speed of reaction fronts. The main results of this paper concern the asymptotic behavior of the speed of traveling fronts for (0.1) as the amplitude M of the convection becomes large. We will consider two types of flows. First we consider shear flows which do not depend on the direction of propagation and second cellular flows which are periodic in the direction of propagation. In the case of shear flows we will show that $c(M)/M$ converges to some positive constant γ. This linear scaling behavior has first been proved in [7] and the convergence result has already been obtained in [2]. Here we give a simpler proof and identify the limit γ as the supremum of a new variational problem. As a byproduct of the proof we obtain explicit error estimates. This new variational problem allows us to draw some qualitative information about the maximal possible value of γ. The proofs use a new approach to the underlying linear selfadjoint eigenvalue problem

which determines the minimal wave speed. The advantage of our method relies on mainly explicit calculations instead of estimates as in [2].
In the case of periodic flows the corresponding linear problem is no longer selfadjoint. Therefor much less is known. The enhancement of the speed should be much smaller if there are no unbounded streamlines. In [1] it was conjectured based on some formal arguments, that the minimal speed should scale like $M^{1/4}$ for such cellular flows. This scaling is also predicted by homogenization, i.e. replacing $b(x)$ by $\frac{1}{\epsilon}b(x/\epsilon)$. It is shown in [8] and [11] that the effective diffusivity behaves as $M^{1/2}$. A simple rescaling arguments shows then $c_{hom}(M) \sim M^{1/4}$ for the homogenized wave. A lower bound of the form $M^{1/5}$ has been derived in [12] by working directly with the parabolic PDE (0.1). In [4] a characterization of the linear behavior of $c(M)$ has been obtained in terms of the existence certain first integrals of the flow $b(x)$. There it is proved by contradiction that the minimal speed cannot scale linearly with the amplitude M for cellular flows with no unbounded streamlines.
Here we will considerably improve this sublinear behavior. We derive an almost optimal upper bound of the minimal speed in terms of the homogenized wave speed which almost confirms the conjecture up to any small $\epsilon > 0$:

$$c(M) \leq K_\epsilon c_{\text{hom}}(M) M^\epsilon, \qquad \text{for all } \epsilon > 0 \tag{0.2}$$

This holds for all periodic flows, regardless if there are unbounded streamlines or not. In any case this reduces the problem of optimal upper bounds to the homogenized problem, i.e. to the determination of the effective diffusivity $d_{hom}(M)$, since with $c(0) = 2\sqrt{f'(0)}$ we have by scaling of the homogenized problem

$$c_{hom}(M) = c(0)\sqrt{d_{hom}(M)}.$$

The asymptotic scaling of $d_{hom}(M)$ has been extensively studied in [8], [11]. In particular if there are no unbounded streamlines it is shown that the effective diffusivity scales like $M^{1/2}$. This together with (0.2) almost confirms the conjecture in [1].
Our result shows that homogenization gives qualitatively correct estimates even in parameter ranges where it does not apply.
We believe that the factor M^ϵ in (0.2) can be removed.
We mention that our approach can be also used to study the dependence of the minimal speed on other parameters, e.g. small diffusivity or large reaction rates.

1 Shear Flows

Let Ω be a bounded domain in $\mathbf{R}^{n-1}$ and let $\beta \in L^\infty(\Omega), \quad \beta \neq 0$ with $\int_\Omega \beta = 0$ Consider the reaction diffusion equation in a cylindrical domain:

$$\partial_t u(t,x) = \Delta u(t,x) + M\beta(x_2,...,x_n)\partial_{x_1} u(t,x) + f(u(t,x)) \tag{1.1}$$

$$x_1 \in \mathbf{R}, \quad (x_2,...,x_n) \in \Omega$$

On $\partial\Omega$ we require Neumann boundary conditions. We change coordinates to a moving frame: $\xi = x_1 + ct$, $y_i = x_{i+1}$, $i = 1, ..., n-1$. A traveling wave with velocity c is a solution of the form $u(t,x) = U(\xi, y)$:

$$c\partial_\xi U(\xi,y) = \Delta_\Omega U(\xi,y) + \partial_{\xi\xi} U(\xi,y) + M\beta(y)\partial_\xi U(\xi,y) + f(U(\xi,y) \tag{1.2}$$

$$\xi \in \mathbf{R}, \quad y \in \Omega$$

with Neumann boundary conditions on $\partial\Omega$ and $U(-\infty, y) = 0$, $U(\infty, y) = 1$. We require $0 < U(\xi, y) < 1$ In [6] it is shown that there exist a traveling wave solution if and only if the speed c exceeds a minimal speed. A variational formula for this minimal speed $c(M)$ has been derived in [9], see also [10] for other types of nonlinearities.

$$c(M) = \inf_{\lambda>0} \frac{\mu(\lambda, M)}{\lambda} \tag{1.3}$$

where μ is the principal eigenvalue of the following operator

$$L\phi = \Delta_\Omega \phi + (\lambda^2 + \lambda M\beta + f'(0))\phi \tag{1.4}$$

for $\phi \in H^2(\Omega)$ with Neumann boundary conditions. In the following ϕ denotes the unique, positive and normalized first eigenfunction of L. The proof relies on the following two facts: For smaller values of λ all solutions of (1.2) oscillate around zero, whereas for larger λ the function $\overline{U}(\xi, x) = e^{\lambda\xi}\phi(x)$ is a supersolution of (1.2) and can be used to construct front solutions. With

$$J_{\lambda,M}(\psi) = \int_\Omega |\nabla\psi|^2 - \int_\Omega (\lambda^2 + M\lambda\beta + f'(0))\psi^2 \tag{1.5}$$

we have the variational characterization of μ

$$-\mu(\lambda, M) = J_{\lambda,M}(\phi) = \inf_{|\psi|_{L^2}=1} J_{\lambda,M}(\psi) \tag{1.6}$$

Lemma 1.1
1. $\mu(\lambda, M)$ is twice differentiable w.r.t. λ and M.
2. For fixed $M > 0$ the function $\mu(\lambda, M)/\lambda$ attains it's infimum at a unique point $\lambda(M) > 0$. This function is twice differentiable.
3. The condition at the infimum $\partial_\lambda(\mu(\lambda, M)/\lambda) = 0$ is equivalent to the crucial identity:

$$\lambda(M)^2 + \int_\Omega |\nabla\phi|^2 = f'(0) \tag{1.7}$$

Proof:
1. The first eigenvalue μ of a second order elliptic operator is simple. It follows from the implicit function theorem that $\mu(\lambda, M)$ depends smoothly on M, λ. We will only need derivatives up to second order.
2. Decompose μ as

$$\mu(\lambda, M) = \lambda^2 + f'(0) + \eta(\rho) \quad \text{with } \rho = M\lambda. \tag{1.8}$$

The normalized first eigenfunction $\phi > 0$ satisfies

$$\eta\phi = \Delta_\Omega \phi + \rho\beta\phi =: \widetilde{L}\phi \tag{1.9}$$

Multiplication by ϕ implies

$$\eta = -\int_\Omega |\nabla\phi|^2 + \rho \int_\Omega \beta\phi^2 \tag{1.10}$$

Division by $\phi > 0$ results in

$$\mu - \lambda^2 - f'(0) = \eta = \frac{1}{|\Omega|}\int_\Omega |\nabla \ln \phi|^2 > 0 \tag{1.11}$$

$$\mu/\lambda > \lambda + f'(0)/\lambda \geq 2\sqrt{f'(0)}$$

Therefor μ/λ attains it's infimum at a finite $\lambda > 0$. In particular $c(M) > c(0)$ for all $M > 0$.
For later reference we also note that (1.10), (1.11) imply

$$\rho \int_\Omega \beta\phi^2 = \frac{1}{|\Omega|}\int_\Omega |\nabla \ln \phi|^2 + \int_\Omega |\nabla\phi|^2 > 0 \tag{1.12}$$

We calculate the derivatives of η up to second order. Denote differentiation w.r.t. ρ by $'$.

$$\eta'\phi + \eta\phi' = \widetilde{L}\phi' + \beta\phi \tag{1.13}$$

$$\eta''\phi + 2\eta'\phi' + \eta\phi'' = \widetilde{L}\phi'' + 2\beta\phi' \tag{1.14}$$

Since the Norm of ϕ is normalized to 1 we have $\int_\Omega \phi\phi' = 0$.
Multiplication of (1.13) by ϕ and integration by parts results in

$$\eta' = \int_\Omega \beta\phi^2 > 0 \qquad \text{by (1.12)} \tag{1.15}$$

Multiply (1.14) by ϕ and (1.13) by ϕ'. Integration gives

$$\eta'' = 2\int_\Omega \beta\phi\phi' = 2\int_\Omega \phi'(\eta - \widetilde{L})\phi' > 0 \tag{1.16}$$

For the strict inequality we use that ϕ' is not a multiple of ϕ. Otherwise $\int_\Omega \phi\phi' = 0$ implies $\phi' = 0$ and equation (1.13) gives the contradiction $\eta' = \beta$ since β is not a constant. This implies for μ:

$$\partial_\lambda \mu = 2\lambda + M\eta' \tag{1.17}$$

$$\partial_M \mu = \lambda\eta' > 0 \qquad \text{using (1.15)} \tag{1.18}$$

$$\partial_{\lambda\lambda}\mu = 2 + M^2\eta'' > 0 \tag{1.19}$$

$$\partial_{\lambda M}\mu = \eta' + M\lambda\eta'' \tag{1.20}$$

At a point where $\mu(\lambda)/\lambda$ attains it's infimum we calculate

$$0 = \partial_\lambda(\mu/\lambda) = \partial_\lambda\mu/\lambda - \mu/\lambda^2$$

and

$$\partial_{\lambda\lambda}(\mu/\lambda) = \partial_{\lambda\lambda}\mu/\lambda > 0$$

Hence the infimum is attained at a unique point $\lambda(M)$. By the implicit function theorem $\lambda(M)$ is a twice differentiable function of M.
Finally at $\lambda(M)$ we have by (1.10), (1.15)

$$0 = \lambda\partial_\lambda\mu - \mu = \lambda^2 - f'(0) + \rho\eta'(\rho) - \eta = \lambda^2 + \int_\Omega |\nabla\phi|^2 - f'(0) \tag{1.21}$$

which shows the identity (1.7).

Now we can define the twice differentiable function

$$c(M) := \frac{\mu(\lambda(M), M)}{\lambda(M)} := \inf_{\lambda>0} \frac{\mu(\lambda, M)}{\lambda} \tag{1.22}$$

and

$$J_M(\psi) = J_{\lambda(M),M}(\psi). \tag{1.23}$$

Lemma 1.2
1. $c(M)$ satisfies

$$\frac{dc(M)}{dM} > 0, \quad \frac{d^2c(M)}{dM^2} > 0, \quad \frac{d(c(M)/M)}{dM} < 0. \tag{1.24}$$

2. $\lambda(M)$ satisfies

$$\frac{d\lambda(M)}{dM} < 0, \quad \frac{d(\lambda(M)M)}{dM} > 0. \tag{1.25}$$

Proof: In the following λ means $\lambda(M)$ and similar for μ and η.
Using $\partial_\lambda(\mu/\lambda) = 0$ at $\lambda = \lambda(M)$ we have by (1.12)

$$\frac{dc(M)}{dM} = \partial_M\mu/\lambda = \eta' = \int_\Omega \beta\phi^2 > 0$$

Differentiating $\lambda\partial_\lambda\mu - \mu = 0$ and using (1.18), (1.19), (1.20) gives

$$\lambda\partial_{\lambda\lambda}\mu\frac{d\lambda}{dM} = \partial_M\mu - \lambda\partial_{\lambda M}\mu$$

$$\frac{d\lambda}{dM} = -\frac{M\lambda\eta''}{2 + M^2\eta''} < 0$$

$$\frac{d(M\lambda)}{dM} = \frac{2\lambda}{2 + M^2\eta''} > 0$$

$$\frac{d^2c(M)}{dM^2} = (M\frac{d\lambda}{dM} + \lambda)\eta'' = \frac{2\lambda\eta''}{2 + M^2\eta''} > 0$$

Remark: The monotonicity of $c(M)$ has been proved already in [2] by different methods. The convexity of $c(M)$ and the results for $\lambda(M)$ are new.

Now we can prove the main theorem of this section on the asymptotic behavior of $c(M)/M$ and the corresponding eigenfunction of the operator L.

Theorem 1.3 The limit $\lim_{M\to\infty} \frac{c(M)}{M} = \gamma$ exists and satisfies

$$\gamma = \sup_{\psi\in D} \int_\Omega \beta\psi^2 \tag{1.26}$$

$$D = \left\{\psi \in H^1(\Omega) | \int_\Omega |\nabla\psi|^2\, dx \le f'(0), \quad \int_\Omega \psi^2\, dx = 1\right\}$$

The following estimate holds for all $M > 0$:

$$0 < \frac{c(M)}{M} - \gamma \le \frac{2\sqrt{f'(0)}}{M} \tag{1.27}$$

Furthermore we have

$$\gamma \in (0, \sup_{y\in\Omega} \beta(y)] \tag{1.28}$$

The sequence of normalized positive eigenfunctions ϕ_M of L for $\lambda = \lambda(M)$ contains a subsequence which converges strongly in $H^1(\Omega)$ to a maximizer of (1.26).

Proof: Since $c(M)/M$ is strictly decreasing, the limit $\gamma \ge 0$ exist and

$$\frac{c(M)}{M} > \gamma$$

holds. Now we identify the limit γ. By the definition of $\lambda(M)$ we have

$$\frac{c(M)}{M} = \frac{\mu(\lambda(M), M)}{\lambda(M)M} = \frac{1}{\lambda(M)M} \sup_{|\psi|^2_{L^2}=1} -J_M(\psi)$$

$$\ge \frac{1}{\lambda(M)M} \sup_{\substack{|\psi|_{L^2} = 1 \\ |\nabla\psi|^2_{L^2} \le f'(0)}} -J_M(\psi) \ge \frac{\lambda(M)}{M} + \sup_{\substack{|\psi|_{L^2} = 1 \\ |\nabla\psi|^2_{L^2} \le f'(0)}} \int_\Omega \beta\psi^2$$

Using $\lambda(M) > 0$ we obtain in the limit $M \to \infty$:

$$\gamma \ge \sup_{\substack{|\psi|_{L^2} = 1 \\ |\nabla\psi|^2_{L^2} \le f'(0)}} \int_\Omega \beta\psi^2$$

For the opposite inequality we use the crucial identity (1.7) with the eigenfunction ϕ_M at the infimum of $\frac{\mu(\lambda)}{\lambda}$. It follows in particular

$$\int_\Omega |\nabla\phi_M|^2 \le f'(0), \quad \lambda(M)^2 \le f'(0)$$

Using (1.7) we get

$$c(M)/M = \frac{\mu(\lambda(M),M)}{\lambda(M)M} = -\frac{J_M(\phi_M)}{\lambda(M)M} = \frac{2\lambda(M)}{M} + \int_\Omega \beta\phi_M^2 \tag{1.29}$$

$$\leq \frac{2\sqrt{f'(0)}}{M} + \sup_{\substack{|\psi|_{L^2}=1 \\ |\nabla\psi|_{L^2}^2 \leq f'(0)}} \int_\Omega \beta\psi^2 = \frac{2\sqrt{f'(0)}}{M} + \gamma$$

This is the error estimate (1.27). In the limit $M \to \infty$ we obtain

$$\gamma \leq \sup_{\substack{|\psi|_{L^2}^2=1 \\ |\nabla\psi|_2^2 \leq f'(0)}} \int_\Omega \beta\psi^2$$

Since ϕ_M is an admissible function from D and $\int_\Omega \beta\phi^2 > 0$ by (1.15) we obtain $\gamma > 0$. The upper estimate is trivial.
The crucial identity (1.7) implies that a subsequence of ϕ_M contains a subsequence which converges strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ to some limit ϕ. Since $\lambda(M) > 0$ decreases the identity (1.7) also shows the convergence of the H^1-norm which implies strong convergence. Equation (1.29) shows that $\int_\Omega \beta\phi^2 = \gamma$ i.e. ϕ is a maximizer in (1.26).

Remark:
In a forthcoming paper we will study the homogenization of the traveling wave problem (1.2), i.e. replace $\beta(y)$ by $\frac{1}{\epsilon}\beta(y/\epsilon)$ where $\beta(\cdot)$ is 1-periodic. There we will show the surprising result that the homogenization limit and the large convection limit commute. More precisely we have for the minimal wave speed $c(M,\epsilon)$:

$$\lim_{M\to\infty}\lim_{\epsilon\to 0}\frac{c(M,\epsilon)}{M} = \lim_{\epsilon\to 0}\lim_{M\to\infty}\frac{c(M,\epsilon)}{M} = 2\left(f'(0)\int_{(0,1)^n} |\nabla\chi(y)|^2\, dy\right)^{1/2}$$

Here χ is the 1-periodic solution of the cell problem

$$\Delta_\Omega \chi(y) = -\beta(y).$$

Next we show how γ can be calculated from the Legendre transform η^* of η, where η is defined in (1.8).

Proposition 1.4 Let $\gamma = \lim_{M\to\infty} c(M)/M$, $\sigma = \lim_{M\to\infty} \lambda(M)M$ and $\lambda^* = \lim_{M\to\infty} \lambda(M)$.
1. If $\sigma < \infty$ then $\lambda^* = 0$ and γ, σ are uniquely determined by

$$f'(0) = \eta^*(\gamma), \quad \sigma = {\eta^*}'(\gamma). \tag{1.30}$$

2. If $\sigma = \infty$ then λ^* and γ are uniquely determined by

$$f'(0) - \lambda^{*2} = \eta^*(\infty), \quad \eta^{*\prime}(\gamma) = \infty. \tag{1.31}$$

Proof: We know that $t(M) := c(M)/M$ and $\lambda(M)$ are decreasing and that $s(M) := \lambda(M)M$ is increasing. They satisfy the following equations using (1.8), (1.21):

$$ts = \lambda^2 + f'(0) + \eta(s), \quad ts = 2\lambda^2 + s\eta'(s)$$

Hence

$$s\eta'(s) - \eta(s) = f'(0) - \lambda^2 = \eta^*(t - 2\lambda^2/s)$$

We have $\lambda^2/s = \lambda/M \to 0$ since λ is bounded. This implies

$$\eta^*(\sigma) = f'(0) - \lambda^{*2} \tag{1.32}$$

With $\eta^{*\prime}(\eta'(s)) = s$ we obtain

$$\eta^{*\prime}(t - 2\lambda^2/s) = s$$

and in the limit

$$\eta^{*\prime}(\gamma) = \sigma. \tag{1.33}$$

From (1.32) and (1.33) we deduce the two cases in the proposition.

Next we study some qualitative properties of maximizers of the limit problem (1.26). Replacing ϕ by $|\phi|$ there always exists a nonnegative maximizer.

The solution of the limit problem depends on the structure of the level sets of β. For $s > 0$ denote the interior of $\beta^{-1}(s)$ by Ω_s.

Proposition 1.5
1. Let ϕ be a maximizer of (1.26). Then there exists a constant κ, s.t.

$$\kappa(\Delta_\Omega \phi + f'(0)\phi) = (\gamma - \beta)\phi \tag{1.34}$$

weakly in $H^1(\Omega)$. As above γ denotes the supremum in (1.26).
2. If ϕ is a nonnegative maximizer of (1.26), then κ is nonnegative.
3. If the first Dirichlet eigenvalue ν_s of Ω_s satisfies $\nu_s \geq f'(0)$ for all $s \in (0, \sup \beta]$ then the Lagrange multiplier κ is unique and positive. Furthermore $|\nabla\phi|^2_{L^2}$ equals $f'(0)$. In this case is the maximizer ϕ unique and has a constant sign.

Proof:
We first let $\int_\Omega |\nabla\phi|^2 < f'(0)$. Then any small variation is allowed, which implies $\gamma\phi = \beta\phi$. Hence (1.34) is satisfied with $\kappa = 0$.
If $\int_\Omega |\nabla\phi|^2 = f'(0)$ The first variation at a maximizer ϕ in (1.26) gives weakly

$$\kappa(\Delta_\Omega \phi + f'(0)\phi) = (\tilde{\gamma} - \beta)\phi \tag{1.35}$$

for two Lagrange multipliers $\widetilde{\gamma}, \kappa$. then multiplication of (1.35) by ϕ and integration implies $\widetilde{\gamma} = \gamma$.
Now assume that ϕ is nonnegative. If $\kappa \neq 0$ then ϕ is a nonnegative eigenfunction of a second order operator. Hence $\phi > 0$ on Ω. Divide (1.34) by $\phi > 0$ and integrate:

$$\kappa \int_\Omega |\nabla \phi|^2/\phi^2 + \kappa f'(0) = \gamma > 0$$

This shows $\kappa > 0$.
Now assume the condition on ν_s and let ϕ be a nonnegative maximizer. The first case above cannot occur, for otherwise *support*$(\phi) \subset \Omega_\gamma$ implies

$$\nu_\gamma \leq \int_\Omega |\nabla \phi|^2 < f'(0).$$

Hence $\kappa > 0$.
Let ψ be a second maximizer ψ which may change sign. Then

$$\widetilde{\kappa}(\Delta_\Omega \psi + f'(0)\psi) = (\gamma - \beta)\psi \tag{1.36}$$

If $\kappa = \widetilde{\kappa}$ then $\psi = \phi$ since a normalized positive eigenfunction is unique. Assume $\kappa \neq \widetilde{\kappa}$. Then cross multiplication of (1.34) and (1.36) implies

$$\int_\Omega (\nabla \phi \nabla \psi - f'(0)\phi\psi) = 0 \quad \text{and} \quad \int_\Omega (\gamma - \beta)\phi\psi = 0$$

This implies that $w = \frac{t\phi - \psi}{|t\phi - \psi|_{L^2(\Omega)}}$ with $t = \sup(\psi/\phi)$ is also a nonnegative maximizer. Hence there exists $\kappa_t > 0$ with

$$\kappa_t(\Delta_\Omega w + f'(0)w) = (\gamma - \beta)w.$$

Since $w \geq 0$ and $w(x_0) = 0$ for some $x_0 \in \Omega$ the maximum principle gives $w = 0$. Now $t\phi = \psi$ implies $\psi = \pm\phi$ since ϕ and ψ have L^2-norm 1.

Remark: 1. The condition on the level sets of β means roughly that Ω_s is not too large.
2. If the condition on the level sets holds then the positive maximizer is unique and the whole sequence in theorem 1.3 converges in $H^1(\Omega)$.
3. If the condition does not hold, then maximizers are not unique. It would interesting to know which one can be obtained as limits from theorem 1.3 and if different subsequences can converge to different limits.
4. Dividing the eigenvalue problem for the operator L (1.4) by $\lambda(M)M$ one obtains the Legendre multiplier κ in (1.4) as the limit

$$\kappa = \lim_{M \to \infty} \frac{1}{\lambda(M)M} \geq 0$$

Now we characterize those functions β which give the maximal possible value for the supremum in (1.26). The result is that the level set corresponding to $\sup \beta$ is sufficiently large.

Proposition 1.6: We have $\gamma = \sup \beta =: \beta^*$ if and only if $\nu_{\beta^*} \le f'(0)$ where ν_{β^*} is the first Dirichlet eigenvalue of $-\Delta$ on Ω_{β^*}.

Proof: First suppose that $\gamma = \sup \beta$. a maximizer ϕ of (1.26) satisfies

$$\int_\Omega (\sup \beta - \beta)\phi^2 = 0$$

This implies for the support of ϕ: *support*$(\phi) \subset \Omega_{\beta^*}$. Therefor

$$\nu_{\beta^*} \le \int_\Omega |\nabla \phi|^2 \le f'(0).$$

Vice versa let $\nu_{\beta^*} \le f'(0)$ and let ψ be the first normalized Dirichlet eigenfunction on Ω_{β^*} extended to Ω by zero. Then

$$f'(0) \ge \nu_{\beta^*} = \int_\Omega |\nabla \psi|^2$$

and ψ is admissible in the variational principle for γ. Since $\beta^*\psi = \beta\psi$ this implies $\gamma = \beta^*$.

2 Cellular Flows

For $x \in \mathbf{R}^2$ consider a C^1-flow field $b(x)$ which is 1-periodic in x_1, x_2. All results of this section still hold with slight modifications if one replaces the periodicity in x_2 direction by Neumann boundary conditions.
A traveling front for the convective KPP equation (0.1) in direction $k \in \mathbf{R}^2, |k| = 1$ is a positive solution of the form

$$u(t,x) = U(\xi, x), \quad \xi = k \cdot x + ct$$

such that $U(\xi, x)$ is 1-periodic w.r.t. x and satisfies

$$U(-\infty, x) = 0, \quad U(\infty, x) = 1$$

With $\widetilde{\nabla} U = k\partial_\xi + \nabla U$, $\widetilde{\Delta} = \widetilde{\nabla} \cdot \widetilde{\nabla}$ we have the following equation for a traveling front:

$$c\partial_\xi u = \widetilde{\Delta} u + Mb \cdot \widetilde{\nabla} u + f(u) \tag{2.1}$$

Existence of such fronts has been shown in [3] for all speeds above some minimal speed $c(M)$. In [5] $c(M)$ is characterized in terms of the following linear operator:

$$L\phi = \Delta\phi + 2\lambda k \cdot \nabla\phi + Mb \cdot \nabla\phi + (\lambda^2 + M\lambda k \cdot b + f'(0))\phi \tag{2.2}$$

Let $\mu(\lambda, M)$ be the principal eigenvalue of L and let $\phi > 0$ be the corresponding eigenfunction. The minimal speed is characterized by:

$$c(M) = \inf_{\lambda > 0} \frac{\mu(\lambda, M)}{\lambda}. \tag{2.3}$$

We also need the adjoint operator

$$L^*\phi^* = \Delta\phi^* - 2\lambda k \cdot \nabla\phi^* - Mb \cdot \nabla\phi^* + (\lambda^2 + f'(0) + M\lambda k \cdot b)\phi^* \qquad (2.4)$$

with first eigenfunction $\phi^*(x)$. Define new coordinates α, β such that:

$$\phi = \alpha e^{\beta - \lambda k \cdot x}, \phi^* = \alpha e^{\lambda k \cdot x - \beta}$$

where $\alpha > 0$ and $\beta - \lambda k \cdot x$ are 1-periodic. Choosing appropriate multiples of ϕ and ϕ^* we can assume:

$$\int_{(0,1)^2} \alpha^2 = 1 \quad \text{and} \quad \int_{(0,1)^2} \beta = 0.$$

The eigenvalue problem for L and its adjoint L^* transform to the coupled system:

$$\mu\alpha = \Delta\alpha + \left(|\nabla\beta|^2 + Mb \cdot \nabla\beta + f'(0)\right)\alpha =: K\alpha, \qquad (2.5)$$

$$0 = \nabla\left(\alpha^2(2\nabla\beta + Mb)\right). \qquad (2.6)$$

The advantage of these coordinates lies in the fact that only the first equation contains the eigenvalue explicitely and is selfadjoint for fixed β. Furthermore the second equation is of Poisson type for fixed α and has a divergence structure. Since $\alpha > 0$ the principal eigenvalue of L is also the principal eigenvalue of K for fixed β. Observe that by regularity theory ϕ, ϕ^* are C^2 and the same is true for α, β. Therefor all integrals below are finite.
In this chapter all occurring integrals will be over the period cell $(0,1)^2$. So will drop the indication of the domain in all integrals.
The following result parallels lemma 1.1 for shear flows.

Lemma 2.1
1. μ is strictly convex w.r.t. λ, i.e. $\partial_{\lambda\lambda}\mu(\lambda, M) > 0$. The function $\mu(\lambda, M)/\lambda$ for fixed M is minimized at a unique point $\lambda(M)$. This function is differentiable.
2. $\partial_\lambda(\mu/\lambda)(\lambda(M), M) = 0$ is equivalent to the crucial identity

$$\int |\nabla\alpha \pm \alpha\nabla\beta|^2 = \int \left(|\nabla\alpha|^2 + \alpha^2|\nabla\beta|^2\right) = f'(0) \qquad (2.7)$$

3. The minimal speed $c(M) = \mu(\lambda(M), M)/\lambda(M)$ satisfies

$$c(M) = \frac{1}{\lambda(M)} \int (2|\nabla\beta|^2 + Mb\nabla\beta)\alpha^2. \qquad (2.8)$$

4. $$\frac{d}{dM}(c(M)/M) < 0$$

Proof: We calculate the derivatives of μ. Let $V = |\nabla\beta|^2 + Mb\nabla\beta + f'(0)$ and denote differentiation w.r.t. λ by $'$.

$$\mu\alpha = \Delta\alpha + V\alpha =: K\alpha \qquad (2.9)$$

$$\mu'\alpha + \mu\alpha' = K\alpha' + V'\alpha \tag{2.10}$$

$$\mu''\alpha + 2\mu'\alpha' + \mu\alpha'' = K\alpha'' + 2V'\alpha' + V''\alpha \tag{2.11}$$

This implies

$$\mu = -\int |\nabla\alpha|^2 + \int (|\nabla\beta|^2 + Mb\nabla\beta + f'(0))\alpha^2 \tag{2.12}$$

$$\mu' = \int V'\alpha^2 = \int (2\nabla\beta + Mb)\nabla\beta'\alpha^2 = \frac{1}{\lambda}\int (2|\nabla\beta|^2 + Mb\nabla\beta)\alpha^2 \tag{2.13}$$

$$\mu'' = 2\int V'\alpha'\alpha + \int V''\alpha^2 = 2\int \alpha'(\mu - K)\alpha' + 2\int |\nabla\beta'|^2\alpha^2 \tag{2.14}$$

(2.13) follows from multiplication of (2.6) by the periodic function $\lambda\beta' - \beta$.
It follows $\mu'' \geq 0$ since μ is the principal eigenfunction of K. Equality would imply that β' is constant. This contradicts the periodicity of $\beta' - k \cdot x$.
Now divide (2.5) by α and integrate:

$$\mu = \int \frac{|\nabla\alpha|^2}{\alpha^2} + \int |\nabla\beta|^2 + f'(0) \geq \lambda^2 + f'(0) \tag{2.15}$$

For the inequality we used that $\beta - \lambda k \cdot x$ is periodic This shows that μ/λ attains it's infimum at some positive λ. At a minimum point we have

$$\partial_\lambda(\mu/\lambda) = 0, \quad \partial_{\lambda\lambda}(\mu/\lambda) = \partial_{\lambda\lambda}\mu/\lambda^2 > 0$$

Hence the minimum is attained at a unique $\lambda(M)$ which is a differentiable function. At the minimal point $\lambda(M)$ we have $\partial_\lambda\mu\lambda - \mu = 0$ which is by (2.12), (2.13) equivalent to (2.7). The mixed term in (2.7) vanishes, which follows by testing (2.6) with $\ln\alpha$. Using the crucial identity (2.7) we obtain from (2.12)

$$c(M) = \mu(\lambda(M), M)/\lambda(M) = \frac{1}{\lambda(M)}\int (2|\nabla\beta|^2 + Mb\nabla\beta)\alpha^2.$$

The derivative $\partial_M\mu$ is calculated as above. This gives

$$\frac{d}{dM}c(M) = \partial_M\mu(\lambda(M), M)/\lambda(M) = \frac{1}{\lambda(M)}\int \alpha^2 b \cdot \nabla\beta$$

and

$$\frac{d}{dM}(c(M)/M) = -\frac{2}{\lambda(M)M^2}\int \alpha^2|\nabla\beta|^2 < 0$$

The monotonicity of $c(M)/M$ is a new result for cellular flows. The question if $c(M)$ is increasing remains an open problem for cellular flows.

Now we can prove the main theorem of this section which shows that the minimal wave speed scales almost as the homogenized wave velocity for lage convection.
In order to formulate the theorem we briefly introduce the well known homogenization procedure. Replace the flow field $b(x)$ by the rapidly oscillating flow field $\frac{1}{\epsilon}b(x/\epsilon)$ in (2.1). Since we will not use the convergence as $\epsilon \to 0$ to the homogenized

problem we just state the formally derived limit problem. One obtains a traveling wave problem with constant coefficients:

$$cU_\xi = d_{\mathrm{hom}} U_{\xi\xi} + f(U) \tag{2.16}$$

$$U(-\infty) = 0, \quad U(\infty) = 1$$

The homogenized diffusion coefficient $d_{hom}(M)$ is calculated from a cell problem in direction k:

$$\Delta\rho + Mb \cdot \nabla\rho = 0 \tag{2.17}$$

s.t. $\rho - k \cdot x$ is 1-periodic. The effective diffusivity is defined by

$$d_{hom}(M) = \int |\nabla\rho|^2 \tag{2.18}$$

The solution of this cell problem will be used as a testfunction. It is well known and easy to see that the minimal speed for (2.16) equals

$$c_{hom}(M) = 2\sqrt{f'(0) d_{hom}(M)} \tag{2.19}$$

Theorem 2.2 Let $f(u)$ be KPP nonlinearity and let $b(x)$ be a cellular flow. Then for every $\delta > 0$ there is a constant K_δ such that the minimal speed $c(M)$ satisfies:

$$c(M) \leq K_\delta M^\delta \; c_{hom}(M) \quad \text{for all} \quad M > 1 \tag{2.20}$$

where c_{hom} is the minimal wave speed of the homogenized problem (2.16).

Proof: Let ρ be the solution of the cell problem (2.17). Multiply the equation (2.6) for β by the periodic function $\beta/\lambda - \rho$ and integrate by parts. Using (2.8) and the cell problem (2.17) we obtain:

$$\begin{aligned} c(M) &= \frac{1}{\lambda(M)} \int (2|\nabla\beta|^2 + Mb\nabla\beta)\alpha^2 = \int (2\nabla\beta + Mb)\nabla\rho\; \alpha^2 \\ &= \int (2\nabla\beta\nabla\rho - \Delta\rho)\alpha^2 = 2\int (\alpha\nabla\beta + \nabla\alpha)\nabla\rho\; \alpha \\ &\leq 2\sqrt{f'(0)} \left(\int \alpha^2 |\nabla\rho|^2\right)^{1/2} \end{aligned} \tag{2.21}$$

In the last inequality the crucial identity (2.7) is used again.
If α would be uniformly bounded in L^∞ this would give the precise rate $c_{hom}(M) = c(0)\sqrt{d_{hom}(M)}$. Here we use that the identity (2.7) implies uniform bounds in H^1 for α and therefor in any L^p for $p < \infty$ since we are in 2 dimensions.
We need an estimate for the L^q-norm of $\nabla\rho$ for q slightly larger than 2. In the following the letter K denotes different constants which are independent of M. The equation (2.17) for ρ gives

$$|\Delta\rho|_{L^2} \leq KM|\nabla\rho|_{L^2}.$$

By embedding we get for $M > 1$ and any $2 < r < \infty$

$$|\nabla\rho|_{L^r} \leq K_1(r)(|\Delta\rho|_{L^2} + |\nabla\rho|_{L^2}) \leq K_2(r)M|\nabla\rho|_{L^2}$$

Interpolating between L^r and L^2 gives

$$|\nabla\rho|_{L^q} \leq K_3(q)M^{\epsilon(q)}|\nabla\rho|_{L^2}$$

where $\epsilon(q) > 0$ tends to zero as q approaches 2. Applying Hölder's inequality with $1/p + 1/q = 2$ and q close to 2 results in

$$\left(\int \alpha^2|\nabla\rho|^2\right)^{1/2} \leq |\alpha|_{L^p}|\nabla\rho|_{L^q}.$$

Insert this into the inequality (2.21) for $c(M)$ and use that any L^p-norm of α is uniformly bounded.
The definition of the effective diffusivity (2.18) and equation (2.19) complete the proof of the theorem.

Remark: Observe that the estimate (2.21) for $c(M)$ becomes an identity for $M = 0$ since then $\alpha = 1$ and $\rho = k \cdot x$. If α would be bounded uniformly in L^∞, which is 'almost' true, the right hand side in (2.21) is bounded by a fixed multiple of the wave speed in the homogenized medium. So homogenization gives qualitatively a useful estimate for all amplitudes $M > 0$.

Special Cases

Assume that all streamlines are bounded. It is shown in [8] and [11] that the effective diffusivity is estimated for $M > 1$ by

$$d_{hom}(M) \leq KM^{1/2}$$

This gives the
Proposition 2.3 For all $\epsilon > 0$ there exists a constant K_δ such that

$$c^*(M) \leq K_\delta M^{1/4+\delta}$$

The standard example for this case is

$$b(y) = \nabla^\perp H(y), \quad H(y) = \sin(2\pi y_1)\sin(2\pi y_2)$$

If one replaces $H(y)$ by $H(y)|H(y)|^{m-1}$ for $m \geq 1$ the effective diffusivity scales like $M^{1/(m+1)}$, see [11]. This results in the upper bound for the minimal speed:

$$c(M) \leq K_\delta M^{1/(m+1)+\delta}.$$

Using the results on the effective diffusivity in [8] for more general periodic flows with open channels we obtain the corresponding results for the minimal wave speed.

Asymptotics for small flows

For completeness we include a new asymptotic expansion of the minimal speed for small amplitudes M. For shear flows this expansion has been derived in [10]. The expansion for small amplitudes is a regular perturbation problem and can be obtained by Taylor expansion.

Theorem 2.4 Let b be a cellular flow. The minimal speed $c(M)$ has for small M the following expansion

$$c(M) = 2\sqrt{f'(0)} + \frac{M^2}{\sqrt{f'(0)}} \int |\nabla\psi|^2 + o(M^2) \tag{2.22}$$

where ψ is the periodic solution of

$$0 = \Delta\psi + \sqrt{f'(0)}(2k \cdot \nabla\psi + k \cdot b) \tag{2.23}$$

Proof: We first calculate the expansion of $\mu = \mu(\lambda, M)$ for small M. Denote differentiation w.r.t. M by $'$.

$$\mu\phi = \Delta\phi + Mb\nabla\phi + 2\lambda k \cdot \nabla\phi + (\lambda^2 + f'(0) + M\lambda k \cdot b)\phi =: L\phi$$

$$\mu'\phi + \mu\phi' = L\phi' + b\nabla\phi + \lambda k \cdot b\phi$$

$$\mu''\phi + 2\mu'\phi' + \mu\phi'' = L\phi'' + 2b\nabla\phi' + 2\lambda k \cdot b\phi'$$

At $M = 0$ we obtain $\phi = 1$, $\mu(\lambda, 0) = \lambda^2 + f'(0)$. With $\psi := \phi'$ at $M = 0$ we get

$$\mu'(\lambda, 0) = \Delta\psi + 2\lambda k \cdot \nabla\psi + \lambda k \cdot b$$

Integration gives equation (2.23) for ψ and $\mu'(\lambda, 0) = 0$. Using equation (2.23) for ψ we have

$$\mu''(\lambda, 0) = 2\lambda \int k \cdot b\psi = 2\int |\nabla\psi|^2 > 0$$

Differentiating $0 = \lambda(M)\mu_\lambda(\lambda(M), M) - \mu(\lambda(M), M)$ at $M = 0$ gives

$$0 = \lambda\mu_{\lambda\lambda}\lambda_M + \lambda\mu_{\lambda M} - \mu_M = \lambda\mu_{\lambda\lambda}\lambda_M$$

Hence $\lambda_M(0) = 0$. Now $c(M) = \mu(\lambda(M), M)/\lambda(M)$ implies

$$c_M(M) = \mu_M(\lambda(M), M)/\lambda(M)$$

$$c_{MM}(0) = \mu_{MM}(\lambda(0), 0)/\lambda(0) + \lambda_M(0)\left(\mu_{M\lambda}(\lambda(0), 0)/\lambda(0) - \mu_M/\lambda(0)^2\right)$$

$$= \mu_{MM}(\lambda(0), 0)/\lambda(0) = \frac{2}{\lambda(0)}\int |\nabla\psi|^2$$

Since $c_M(0) = 0$, $\lambda(0) = \sqrt{f'(0)}$ and $c(0) = 2\sqrt{f'(0)}$ we obtain the expansion (2.22).

Remark: Equation (2.23) can be solved explicity by Fourier series.
Formula (2.22) shows that $c(M)$ is increasing for small M. We conjecture that this holds for all $M > 0$.
For shear flows the expansion reduces to the formula in [10].

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