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Abstract

A specific class of gauge theories is geometrically described in terms of fermions. In particular, it is shown how the geometrical frame presented naturally includes spontaneous symmetry breaking of Yang-Mills gauge theories without making use of a Higgs potential. In more physical terms, it is shown that the Yukawa coupling of fermions, together with gravity, necessarily yields a symmetry reduction provided the fermionic mass is considered as a globally well-defined concept. The structure of this symmetry breaking is shown to be compatible with the symmetry breaking that is induced by the Higgs potential of the minimal Standard Model. As a consequence, it is shown that the fermionic mass has a simple geometrical interpretation in terms of curvature and that the (semi-classical) "fermionic vacuum" determines the intrinsic geometry of space-time. We also discuss the issue of "fermion doubling" in some detail and introduce a specific projection onto the "physical sub-space" that is motivated from the Standard Model.

Keywords: Clifford Modules, Dirac Type Operators, Bundle Reduction, Spontaneous Symmetry Breaking, Fermionic Mass Operator

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1 Introduction

The aim of this paper is to put emphasis on the role of fermions in a geometrically unified description of different kinds of gauge theories as, for instance, Yang-Mills and Einstein's theory. Especially, we discuss in some detail the role of the "Yukawa coupling" of fermions with respect to the mechanism of spontaneous symmetry breaking. This may provide us with a better geometrical understanding of the relation between inertia and gravity.

Let us start out with some general remarks on the notion of "gauge symmetry". The notion of gauge symmetry, in general terms, expresses certain redundancies in the mathematical description of the interactions considered. In mathematics, by gauge theory one usually refers to gauge theories of the Yang-Mills type with the underlying geometry given by a principal G-bundle over a smooth orientable (compact) manifold endowed, in addition, with a (semi-)Riemannian structure (see, for instance, in [Ble '81], [MM '92], [MMF '95], [Nab '00] and [Trau '80]). This notion of gauge theory, however, is clearly far too restrictive when considered from a physical point of view. For instance, gravity is also usually regarded as a kind of gauge theory though it is certainly not of the Yang-Mills type. The underlying geometrical structure of gravity, regarded as a gauge theory, is that of a fiber bundle naturally associated with the frame bundle of the base manifold $\mathcal{M}$ with typical fiber given by $\mathrm{GL}(n)/\mathrm{SO}(p,q)$. Here, respectively, $\dim(\mathcal{M}) \equiv n = p+q$ equals the dimension of the oriented base manifold and $s = p - q$ is the signature. The bundle structure of the two gauge theories is obviously very different. In contrast to Yang-Mills theory, the bundle structure of gravity is fully determined (modulo diffeomorphisms) by fixing the (topology of the) base manifold and the signature s. In this sense, the bundle structure in Einstein's theory of gravity is more natural then in the Yang-Mills theory. Moreover, the mathematical notion of a local trivialization has a physical meaning in the case of gravity, however, not in Yang-Mills gauge theories (there is no "exponential map" defined in Yang-Mills theories for, in contrast to gravity, Yang-Mills connections only determine second order vector fields but no spray fields).

The respective Lagrangian densities of gravity and Yang-Mills gauge theory differ in that the former is known to be linear in the curvature of the base manifold whereas the latter is quadratic in the curvature of the bundle space. This difference is known to yield far-reaching consequences, for example, when quantization is taken into account.

But also on the purely "classical" level (i.e. gravity and electromagnetism) there are fundamental differences in these two kinds of gauge theories. For example, electromagnetism (more general, Yang-Mills gauge theories over even-dimensional base manifolds) is known to be scale invariant but not invariant with respect to the action of the diffeomorphism group (except isometries). In contrast, gravity is covariant with respect to diffeomorphisms but not scale invariant. Of course, despite these profound mathematical and physical differences there are, nonetheless, formal similarities between these two types of gauge theories. Especially, the dynamics that is defined by both theories can be expressed with respect to top-forms on the base manifold with the property of being invariant with respect to the action of their respective symmetry groups. A natural question then is whether these two fundamental kinds of gauge theories have a common geometrical root.

Of course, over the last decades there have been various attempts to geometrically unify gravity with Yang-Mills gauge theory. This holds true for string theory and, in particular, for various aspects of non-commutative geometry, see, for example, [CFG '95], [CC '97], [Con '96], [Oku '00] and the corresponding references there. The fruitful idea to consider the Higgs boson of the Standard Model as an integral part of the Yang-Mills theory goes back to fundamental works as, for example, [Con '88], [CL '90], [Coq '89] and [CEV '91]. It is well-known that this idea actually has had a tremendous impact on a vast variety of papers of the same theme (see, for instance, [FGLV '98], [KMO '99], [MO '94] and [MO '96] in the context of non-commutative geometry, or [HPS '91] and [NS '91] in the case of "super-algebras"). Basically, of all of these geometrical descriptions of gauge theory use the purely algebraic content of gauge theories of the Yang-Mills type (e.g.: the exterior differential is an nilpotent derivation and a connection is the sum of the latter and a one-form) as there starting point. However, gravity seems not to fit in this basic algebraic sight. Also, spontaneous symmetry breaking is described only in terms of (the algebraic aspects of) Yang-Mills gauge theories without using gravity. The notion of fermions only arises because in the algebraic context the exterior differential is defined in terms of specific generalizations of the notion of a Dirac operator. These purely algebraic generalizations of the latter, however, seem to have no geometrical counter part (see, for instance, the "internal Dirac operator" in the geometrical description of the Standard Model in terms of "almost commutative models", [IS '95]).

In the following we shall discuss a specific class of gauge theories including Einstein's theory of gravity and (spontaneously broken) Yang-Mills theory from the point of view of fermions. The latter will be geometrically treated as certain Hermitian vector bundles over arbitrary smooth orientable manifolds of even dimension. These "fermion bundles" correspond to a global specification of a certain class of first order differential operators, called "Dirac type operators". We introduce a canonical mapping which associates with every Dirac type operator a specific top-form on the base manifold. This canonical mapping is then referred to as the "Dirac-Lagrangian" on the setup to be discussed. The Dirac-Lagrangian turns out to be equivariant with respect to bundle equivalence. In particular, it is invariant with respect to the action of the Yang-Mills and the Einstein-Hilbert gauge group. The diffeomorphism group of the base manifold is naturally included by the pull-back action. We also consider a distinguished class of Dirac type operators within this setup. The corresponding top-form associated with these Dirac type operators is shown to define a spontaneously broken gauge theory without referring to a Higgs potential. In more physical terms, it is shown that the "Yukawa coupling" together with gravity yields a symmetry reduction which is compatible with the symmetry breaking induced by a Higgs potential of the form used in the (minimal) Standard Model of Particle Physics. In fact, the latter is shown to be naturally generated by a "fluctuation of the fermionic vacuum". We will also reformulate the notion of "unitary gauge" in terms of Dirac type operators and give necessary and sufficient conditions for its global existence.

The geometrical description of gauge theories discussed in the present paper is a considerable refinement of the geometrical frame that has been introduced in [Tol '98] in the case of elliptic Dirac type operators on a smooth even-dimensional closed Riemannian Spin-manifold. In contrast to the latter we will consider in this paper the more physically appropriate case of arbitrary signature and non-compact manifolds. Also, we do not assume that "space-time" has a spin structure (please, see below). For this, however, we will focus on (globally defined) densities instead of action functionals. Accordingly, we have to demand that the densities themselves are covariant with respect to the underlying symmetry action and thus well-defined on the appropriate moduli spaces. This is achieved mainly since the densities in question are derived from evaluating a natural object (within the frame considered) with respect to specific first order differential operators. As a consequence, one ends up with densities which are linear in the curvature of the base manifold and quadratic in the curvature of the bundle space. For instance, it is shown that the total curvature of the "fermionic vacuum" decomposes into the sum of the curvature of the base manifold together with the (square of the) fermionic mass operator. Also a basic difference relative to the frame considered in loc. sit. (and subsequent papers thereof) is that all bundles, including the Higgs and the Yang-Mills bundle, are considered as specific sub-bundles of the fermion bundle (resp. of the bundle of endomorphisms of the latter). The fermion density will be considered as a specific mapping on the affine set of all Dirac type operators on a fermion bundle. Here, we also discuss the issue of the doubling of the fermionic degrees of freedom that is necessary to apply the general Bochner-Lichnerowizc-Weizenböck formula.

Finally, we want to comment on the notion of "fermions" without assuming the existence of spin structures. At least in the so-called "semi-classical approximation" of a full quantum field theory it is common to geometrically treat the "states of a fermion" as sections of a (twisted) spinor bundle over space-time. For this, of course, the topology of space-time must guarantee the existence of a spin structure (i.e. the vanishing of the second Stiefel-Whitney classes). Moreover, it is known that a non-compact Lorenzian four manifold possesses a spin structure if and only if its frame bundle is trivial. ("Geroch's Theorem", c.f. [Ger '68] and [Ger '70]). Therefore, the existence of a spin structure provides severe restrictions to the topology of space-time. However, the experiments performed to demonstrate that the double cover of the (proper orthochroneous) Lorentz group is more fundamental are purely local in nature. Also, in order to obtain a topologically non-trivial statement about the existence of spin structures, space-time has to be covered by at least three (trivializing) local charts. This, of course, rises the question of the physical sense of "locality" in this context to give the mathematical construction a physical meaning. Hence, from our point of view, the assumption of the existence of a spin structure is a purely mathematical one without a physically meaningful counterpart. In fact, in this respect the notion of "locality", as it is used in mathematics, seems physically as spurious as in the case of Yang-Mills gauge theories which do not provide any scale. Basically, this is the reason to consider in this work the more general notion of "Clifford module bundles" instead of "twisted spinor bundles" as an appropriate geometrical background. In contrast to the latter, the existence of Clifford module bundles yields no more topological restrictions on space-time than the existence of a metric itself. For instance, the bundle of Grassmann algebras severs as a natural Clifford module bundle for every space-time manifold. However, the topology of the Clifford module bundles cannot be arbitrary. The physical interpretation of the sections of Clifford module bundles in terms of the states of fermions yields restrictions to the topology of the considered Clifford module bundles (please, see below).

The paper is organized as follows: In the next section we introduce the concept of fermion bundles as a specific class of Clifford module bundles and define Dirac type gauge theories. In the third section we consider a distinguished class of such gauge theories and discuss spontaneous symmetry breaking in this context. In the fourth section we introduce the fermionic density within the presented geometrical setup and discuss the issue of fermionic doubling. In the fifth section we want to specify what we mean by a "fluctuation of the fermionic vacuum". This is done in terms of yet another class of Dirac type operators. Finally, in section six we close with an outlook. In an appendix we present a detailed proof of the explicit form of "simple type Dirac operators" of arbitrary signature, for these operators turn out to be fundamental, e.g., in our discussion of spontaneous symmetry breaking.

2 Fermion Bundles and Dirac Type Gauge Theories

In this section we introduce a specific class of Clifford module bundles which will serve as our geometrical background for gauge theories. On this background there exists a canonical mapping which permits to associate with the local data of a fermion bundle a specific top form on the base manifold. This top form turns out to be equivariant with respect to the automorphism group of the underlying geometrical structure.

2.1 Fermion Bundles, Dirac Type Operators and Connections

In this sub-section we define our notion of *fermion bundles* as a specific class of Clifford module bundles. For this let $\xi := (\mathcal{E}, \pi_{\mathcal{E}}, \mathcal{M})$ be a smooth complex vector bundle with total space $\mathcal{E}$, base manifold $\mathcal{M}$ and projection map $\pi_{\mathcal{E}} : \mathcal{E} \to \mathcal{M}$. The rank, $\mathrm{rk}(\xi) \in \mathbb{N}$, of the bundle is $\mathrm{N} \geq 1$. In what follows the base manifold is assumed to be orientable and of even dimension $n \equiv 2k$. As a topological space $\mathcal{M}$ is a para-compact and (simply-) connected Hausdorff-space. On this geometrical background we consider the following local data:

$$(\mathrm{G}, \rho_{\mathrm{F}}, D). \tag{1}$$

Here, G is a semi-simple, compact and real Lie group and $\rho_{\rm F}: {\rm G} \to {\rm SU(N_F)}$ is a unitary and faithful representation thereof. Moreover, $D: \Gamma(\xi) \to \Gamma(\xi)$ is a first order differential operator, acting on sections of the bundle ξ such that the bilinear extension $g_{\rm M}$ of the mapping $(df, dh) \mapsto {\rm tr}([D,f][D,h])/{\rm rk}(\xi)$ is non-degenerated for all smooth functions $f, g \in \mathcal{C}^\infty(\mathcal{M})$. The operator D is said to have the signature $s \in \mathbb{Z}$, provided that the quadratic form associated with the (semi-)Riemannian metric $g_{\rm M}$ has signature s. The mapping $g_{\rm M}$ corresponds to a section of the "Einstein-Hilbert bundle" $\xi_{\rm EH} := (F\mathcal{M} \times_{\rm GL(n)} {\rm GL}(n)/{\rm O}(p,q), \pi_{\rm EH}, \mathcal{M})$, with, respectively, $F\mathcal{M}$ the total space of the frame bundle $\mathcal{F}\mathcal{M}$ of the base manifold $\mathcal{M}$ and $n \equiv p+q$, $s \equiv p-q$.

Let $\tau_{\rm Cl}$ be the algebra bundle of Clifford algebras which are point-wise generated by $(\tau^*_{\rm M}, g_{\rm M})$, with $\tau^*_{\rm M}$ being the cotangent bundle of $\mathcal{M}$. By the very definition, the principal symbol of the operator D induces a Clifford (left) action $\gamma: \tau_{\rm Cl} \to {\rm End}(\xi)$ via the mapping

$$\begin{array}{ccc} \tau_{\rm Cl} \times \xi & \longrightarrow & \xi \\ (df, \mathfrak{z}) & \mapsto & [D,f]\mathfrak{z}, \end{array} \tag{2}$$

for all smooth functions $f \in \mathcal{C}^\infty(\mathcal{M})$. As a consequence, the algebra bundle of endomorphisms on ξ globally decomposes as

$${\rm End}(\xi) \simeq \tau^{\mathbb{C}}_{\rm Cl} \otimes_{\mathcal{M}} {\rm End}_{\rm Cl}(\xi). \tag{3}$$

Here, ${\rm End}_{\rm Cl}(\xi) \subset {\rm End}(\xi)$ denotes the sub-bundle of endomorphisms which super-commute with the Clifford action γ (c.f., for instance, in [ABS '64] and [BGV '96]).

Definition 2.1 *The vector bundle $\xi \equiv \xi_{\rm F}$ is called a "fermion bundle" with respect to the (local) data (1) if the structure group of ξ can be reduced to* ${\rm Spin}(p,q) \times \rho_{\rm F}({\rm G})$. *A fermion bundle is called "chiral" provided $\xi_{\rm F} = \xi^+_{\rm F} \oplus \xi^-_{\rm F}$ is $\mathbb{Z}_2-$graded with respect to some involution $\Gamma \equiv \gamma_{\rm M} \otimes \chi \in \Gamma({\rm End}(\xi_{\rm F}))$. Here, the grading involution $\gamma_{\rm M} \in \Gamma(\tau^{\mathbb{C}}_{\rm Cl})$ is defined in terms of the (semi-)Riemannian volume form $\mu_{\rm M} \in \Omega^n(\mathcal{M})$ that is induced by $g_{\rm M}$. That is, $\gamma_{\rm M} \sim \gamma(\mu_{\rm M})$. Moreover, $\xi_{\rm F}$ is called "real" if all of its odd Chern classes vanish. With respect to Γ the operator D is supposed to be odd and the representation $\rho_{\rm F}$ is assumed to be even. In this case, D is called a "Dirac type operator" and (1) a "Dirac triple".*

A fermion bundle encodes the global data of a Dirac type gauge theory. With respect to these data we consider the set $\mathcal{D}(\xi_{\rm F})$ of all Dirac type operators $D' \in \mathcal{D}(\xi_{\rm F})$

satisfying the condition $[D' - D, f] \equiv 0$ for all $f \in \mathcal{C}^\infty(\mathcal{M})$. The set $\mathcal{D}(\xi_{\rm F})$ naturally becomes an affine space with vector space $\Gamma({\rm End}^-(\xi_{\rm F}))$. In what follows we summarize the basic features of this affine space.

The affine space $\mathcal{A}(\xi_{\rm F})$ of linear connections on $\xi_{\rm F}$ has a distinguished affine subspace $\mathcal{A}_{\rm Cl}(\xi_{\rm F}) \subset \mathcal{A}(\xi_{\rm F})$ that is defined by all linear connections which are compatible with the Clifford action γ. That is, $A \in \mathcal{A}_{\rm Cl}(\xi_{\rm F})$ defines a covariant derivative ∂_A satisfying $[\,\partial_A, \gamma(a)] = \gamma(\nabla^{\rm Cl} a)$ for all sections $a \in \Gamma(\tau_{\rm Cl}^{\mathbb{C}})$ and $\nabla^{\rm Cl}$ being the covariant derivative with respect to the lifted Levi-Civita connection of $g_{\rm M}$. Accordingly, such a connection is referred to as a "Clifford connection". Hence, every $D' \in \mathcal{D}(\xi_{\rm F})$ may be written as $D' = \not{\partial}_A + \Phi$ where, respectively, $\not{\partial}_A \equiv \gamma \circ \partial_A$ is the analogue of a twisted Spin-Dirac operator in the case where $\mathcal{M}$ denotes a spin-manifold and $\Phi \equiv D - \not{\partial}_A \in \Gamma({\rm End}(\xi_{\rm F}))$. Notice, however, that in general the zero order operator Φ also depends on the Clifford connection A. Moreover, the relation between the two affine spaces $\mathcal{D}(\xi_{\rm F})$ and $\mathcal{A}(\xi_{\rm F})$ on a fermion bundle is given by the (signature independent) bijection (c.f. [Tol '98])

$$\mathcal{D}(\xi_{\rm F}) \simeq \mathcal{A}(\xi_{\rm F})/\ker(\gamma). \tag{4}$$

Therefore, to each Dirac type operator on $\xi_{\rm F}$ there corresponds an equivalence class of connections. However, each connection class has a natural representative that is constructed as follows: Firstly, on every chiral fermion bundle there is a canonical odd one-form $\Theta \in \Omega^1(\mathcal{M}, {\rm End}^-(\mathcal{E}))$ that is given by the (normalized) lifted soldering form of $\mathcal{FM}$. More precisely, let $\vartheta \in \Omega^1_{\rm eq,hor}(F\mathcal{M}, \mathbb{R}^n)$ be the soldering form on the (total space of the) frame bundle of $\mathcal{M}$. Here, the canonical identification $\Omega^*_{\rm eq,hor}(F\mathcal{M}, \mathbb{R}^n) \simeq \Omega^*(\mathcal{M}, T\mathcal{M})$ and the injection[1]

$$\Gamma(\tau_{\rm M}^* \otimes_{\mathcal{M}} \tau_{\rm M}) \xrightarrow{{\rm id}\otimes\,\flat} \Gamma(\tau_{\rm M}^* \otimes_{\mathcal{M}} \tau_{\rm M}^*) \hookrightarrow \Gamma(\tau_{\rm M}^* \otimes_{\mathcal{M}} \tau_{\rm Cl}) \xrightarrow{{\rm id}\otimes\gamma} \Gamma(\tau_{\rm M}^* \otimes_{\mathcal{M}} {\rm End}(\xi_{\rm F})) \tag{5}$$

yields $\Theta := \pm\tilde{\vartheta}/n$ with $\tilde{\vartheta} \equiv \gamma \circ \vartheta^\flat \in \Omega^1(\mathcal{M}, {\rm End}(\mathcal{E}))$. If $(X_1, \ldots, X_n)$ denotes a local frame on $\mathcal{M}$ and $(X^1, \ldots, X^n)$ its dual, then[2] $\tilde{\vartheta} \overset{loc.}{=} X^k \otimes \gamma(X_k^\flat) \otimes {\rm id}$, with the usual "musical" isomorphism $u^\flat(v) := g_{\rm M}(u, v)$ for all $u, v \in T\mathcal{M}$. The normalized soldering form Θ has the two basic properties: It is covariantly constant with respect to every Clifford connection and it induces a canonical right inverse of the Clifford action, i.e.

[1] Here, $\Omega^*_{\rm eq,hor}(FM, \mathbb{R}^n)$ denotes the "right-equivariant" and "horizontal" forms on the total space of the frame bundle of $\mathcal{M}$.

[2] Throughout the paper Einstein's summation convention is used in local formulas except where this may lead to confusions.

$\gamma \circ \mathrm{ext}_\Theta = \mathrm{id}$. Here, $\mathrm{ext}_\Theta \in \mathrm{End}(\tau^*_{\Lambda\mathrm{M}} \otimes_{\mathcal{M}} \mathrm{End}(\xi))$ denotes the operator of (point-wise) left-multiplication by Θ, and $\tau^*_{\Lambda\mathrm{M}}$ is the bundle of Grassmann algebras that is, again, generated by τ^*_{M}. Note that the linear equivalence $\tau^*_{\mathrm{Cl}} \simeq \tau^*_{\Lambda\mathrm{M}}$ is used but not explicitly indicated. Secondly, to each Dirac type operator $D' \in \mathcal{D}(\xi_{\mathrm{F}})$ there exists a correspondingly unique connection $\hat{A}'_{\mathrm{D}} \in \mathcal{A}(\xi_{\mathrm{F}})$ such that $D'^2 - \triangle'_{\mathrm{D}} \in \Gamma(\mathrm{End}(\xi_{\mathrm{F}}))$. The second order operator $\triangle'_{\mathrm{D}} := -\mathrm{tr}(\hat{\nabla}^{T^*\mathcal{M}\otimes\mathcal{E}} \circ \hat{\nabla}^{\mathcal{E}})$ is called the "Bochner-Laplacian" of D' (c.f., for example [BGV '96], [BG '90], or [Gil '95]). Here, $\hat{\nabla}^{\mathcal{E}}$ denotes the covariant derivative that corresponds to the connection $\hat{A}'_{\mathrm{D}}$. As a consequence, the covariant derivative that is defined by

$$\partial_{D'} := \hat{\nabla}^{\mathcal{E}} + \Theta \wedge (D' - \gamma \circ \hat{\nabla}^{\mathcal{E}}) \tag{6}$$

yields a connection $A'_{\mathrm{D}} \in \mathcal{A}(\xi_{\mathrm{F}})$ which clearly represents the Dirac type operator D', i.e. $D' = \gamma \circ \partial_{D'}$. We call, respectively, A'_{D} the Dirac connection associated with D' and the one-form

$$\varpi'_{\mathrm{D}} := \Theta \wedge (D' - \gamma \circ \hat{\nabla}^{\mathcal{E}}) \in \Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{E})) \tag{7}$$

the "Dirac form" associated with $D' \in \mathcal{D}(\xi_{\mathrm{F}})$. Of course, if the connections $\hat{A}'_{\mathrm{D}}$ and A'_{D} are identified with the respective connection forms $\hat{\omega}, \omega'_{\mathrm{D}} \in \Omega^1(\mathcal{E}, T\mathcal{E})$, then

$$\pi^*_{\mathcal{E}}\varpi'_{\mathrm{D}} = \omega'_{\mathrm{D}} - \hat{\omega}. \tag{8}$$

Remark:

As a first order differential operator each Dirac type operator D is known to be of the (local) form: $D = \gamma^\mu(\partial_\mu + \omega_\mu)$ with the appropriate "$\gamma-$matrices" $\gamma^\mu \in \mathrm{End}(\mathbb{C}^{2^{\mathrm{k}}})$ satisfying either of the Clifford relations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \pm 2g^{\mu\nu}\mathbf{1}$, and

$$\omega_\mu \equiv \omega^{\mathrm{Cl}}_\mu \otimes \mathbf{1} + \mathbf{1} \otimes A_\mu \pm \tfrac{1}{n}\, g_{\mu\nu} \sum_{0\le k\le n} \sum_{1\le i_1<i_2\cdots<i_k\le n} \gamma^\nu\gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_k} \otimes \theta_{i_1 i_2\cdots i_k}. \tag{9}$$

Here, respectively, ω^{Cl}_μ is the component of the lifted Levi-Civita form with respect to the appropriate metric coefficients $g_{\mu\nu}$, and $A_\mu, \theta_{i_1 i_2\cdots i_k}$ are the components of locally defined differential forms of various degrees which take their values in $\rho'_{\mathrm{F}}(\mathrm{Lie(G)}) \subset \mathrm{End}(\mathbb{C}^{\mathrm{N_F}})$. Obviously, these forms determine each specific Dirac type operator $D \in \mathcal{D}(\xi_{\mathrm{F}})$ locally. More precisely, let $\{(U_\alpha, \chi_\alpha) \,|\, \alpha \in \Lambda\}$ be a family of local trivializations of the underlying vector bundle ξ, i.e. $\chi_\alpha : \pi^{-1}_{\mathcal{E}}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbb{C}^{\mathrm{N}}$. Accordingly, let $\chi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL(N},\mathbb{C})$ be the appropriate transition functions. Then, a family of first order differential operators

D_α of the form $D_\alpha = \gamma_\alpha \circ \nabla_\alpha$, with $\nabla_\alpha \equiv d + \omega_\alpha$ and ω_α defined by (9), gives rise to a Dirac type operator D on ξ provided the principal symbols γ_α define a family of Clifford mappings $\mathbb{R}^{p,q} \to \mathrm{End}(\mathbb{C}^{2^{\mathrm{k}}}) \simeq \mathbb{C} \otimes \mathrm{Cl}_{p,q}$ and the transition functions take their values in the subgroup $\mathrm{Spin}(p,q) \times \rho_{\mathrm{F}}(\mathrm{G})$ such that the family $\{(U_\alpha, D_\alpha) \,|\, \alpha \in \Lambda\}$ fulfills the compatibility condition $D_\alpha = \chi_{\alpha\beta} \circ D_\beta \circ \chi_{\alpha\beta}^{-1}$ for all $x \in U_\alpha \cap U_\beta \subset \mathcal{M}$. Hence, the notion of a Dirac triple on ξ (i.e. ξ_{F}) globalizes what is encoded in the local data specifying D. In other words, the notion of a fermion bundle simply permits globalization of the local data $(U_\alpha, D_\alpha)_{\alpha\in\Lambda}$ usually encountered in physics.

2.2 Gauge Theories of Dirac Type and their Gauge Groups

In this sub-section we show that the geometrical setup of fermion bundles permits to naturally introduce a specific class of gauge theories which we call *gauge theories of Dirac type* (GTDT). The corresponding gauge group is the automorphism group of the underlying geometrical structure. It is shown that this group decomposes into certain subgroups which can be identified with the usual Yang-Mills gauge group, the Einstein-Hilbert gauge group and the diffeomorphism group of the base manifold.

Definition 2.2 *Two fermion bundles ξ_{F} and ξ'_{F} are considered to be equivalent if $\mathrm{G} \simeq \mathrm{G}'$ and ρ_{F} is similar to ρ'_{F}. Moreover, there is a bundle isomorphism $(\alpha,\beta) : \xi \to \xi'$ (i.e. diffeomorphisms $\alpha : \mathcal{M} \to \mathcal{M}'$ and $\beta : \mathcal{E} \to \mathcal{E}'$, with β being fiber-wise linear and $\alpha \circ \pi_{\mathcal{E}} = \pi_{\mathcal{E}'} \circ \beta$) such that $D' = \beta \circ D \circ \beta^{-1}$.*

Notice that the condition $D' = \beta \circ D \circ \beta^{-1}$ actually is equivalent to $g_{\mathrm{M}'} = {\alpha^{-1}}^* g_{\mathrm{M}}$.

The presented geometrical setup permits the formulation of a class of gauge theories which are based on a "universal Lagrangian" that is covariant with respect to the action of the automorphism group

$$\mathcal{G}_{\mathrm{F}} \equiv \mathrm{Aut}(\xi_{\mathrm{F}}) := \{(\alpha,\beta) \in \mathrm{Diff}(\mathcal{M}) \times \mathrm{Aut}(\mathcal{E}) \,|\, \pi_{\mathcal{E}} \circ \beta = \alpha \circ \pi_{\mathcal{E}}\} \tag{10}$$

of the fermion bundle in question. This group may be identified with the group of right-equivariant automorphisms of the frame bundle associated with ξ_{F}. That is,

$$\mathcal{G}_{\mathrm{F}} \simeq \mathrm{Aut}_{\mathrm{eq}}(\mathcal{F}\mathcal{E}) := \{f \in \mathrm{Aut}(F\mathcal{E}) \,|\, R_g \circ f = f \circ R_g,\ g \in \mathrm{G}_{\mathrm{F}}\} \tag{11}$$

where, respectively, $\mathcal{F}\mathcal{E} \equiv (F\mathcal{E}, \mathcal{M}, \pi, \mathrm{G}_{\mathrm{F}})$ is the associated frame bundle of the fermion bundle considered, $\mathrm{G}_{\mathrm{F}} \equiv \mathrm{Spin}(p,q) \times \rho_{\mathrm{F}}(\mathrm{G})$ its structure group and R the right action of the latter on the total space $F\mathcal{E}$ of the frame bundle.

Therefore, the automorphism group (10) has several important sub-groups. In particular, it contains the "inner gauge group" of the fermion bundle $\xi_{\rm F}$:

$$\mathcal{G}_{\rm in} := \{(\alpha,\beta) \in \mathcal{G}_{\rm F} \,|\, \alpha := {\rm id}_{\mathcal{M}}\}, \tag{12}$$

which may be identified with the gauge group of $\mathcal{F}\mathcal{E}$. The latter contains two mutually commuting normal sub-groups $\mathcal{G}_{\rm EH}$ and $\mathcal{G}_{\rm YM}$, such that $\mathcal{G}_{\rm EH} \cap \mathcal{G}_{\rm YM} = \{e\}$. Therefore,

$$\mathcal{G}_{\rm in} \simeq \mathcal{G}_{\rm EH} \times_{\mathcal{M}} \mathcal{G}_{\rm YM}. \tag{13}$$

Here, the "Yang-Mills gauge group" $\mathcal{G}_{\rm YM}$ can be identified with the sub-group $\{(\alpha,\beta) \in {\rm Aut}(\xi_{\rm F}) \,|\, \alpha = {\rm id}_{\mathcal{M}}, \beta \in {\rm Aut}_{\rm Cl}(\mathcal{E})\}$ of the inner gauge group (13). Note that the Yang-Mills gauge group is in fact an invariant sub-group of the inner gauge group. Hence, with respect to the foregoing mentioned identification the "Einstein-Hilbert gauge group" $\mathcal{G}_{\rm EH}$ may be identified with the quotient group $\mathcal{G}_{\rm in}/\mathcal{G}_{\rm YM}$ according to the decomposition (3).

Moreover, the diffeomorphism group of the base manifold $\mathcal{M}$ has a natural non-trivial embedding into ${\rm Aut}(\xi_{\rm F})$. Indeed, if $\xi_{\rm F}$ is merely considered as a vector bundle then one gets the (trivial) embedding

$$\begin{array}{rcl} {\rm Diff}(\mathcal{M}) & \hookrightarrow & {\rm Aut}(\xi_{\rm F}) \\ \alpha & \mapsto & (\alpha, \beta := \pi_{\mathcal{E}}^*\alpha \times {\rm id}_{\mathcal{E}}). \end{array} \tag{14}$$

This embedding may actually be identified with the inclusion according to the definition (10) of the automorphism group and the identification

$$\alpha^{-1*}\mathcal{E} \equiv \{(y,\mathfrak{z}) \in \mathcal{M} \times \mathcal{E} \,|\, \pi_{\mathcal{E}}(\mathfrak{z}) = \alpha^{-1}(y)\} = \{(\pi^*\alpha(\mathfrak{z}),\mathfrak{z}) \,|\, \mathfrak{z} \in \mathcal{E}\} \simeq \mathcal{E}. \tag{15}$$

Hence, one has $\alpha^{-1*}\xi_{\rm F} = (\alpha^{-1*}\mathcal{E}, \mathcal{M}, {\rm pr}_1) = (\mathcal{E}, \mathcal{M}, \pi^*\alpha)$ which permits to replace $\beta = \pi^*\alpha \times {\rm id}_{\mathcal{E}}$ (with inverse given by ${\rm pr}_2$) simply by $\beta := {\rm id}_{\mathcal{E}}$. However, since $\xi_{\rm F}$ is a Clifford module bundle over $(\mathcal{M}, g_{\rm M})$, the embedding of ${\rm Diff}(\mathcal{M})$ into ${\rm Aut}(\xi_{\rm F})$ becomes non-trivial. In other words, there is an inner automorphism on ${\rm End}(\mathcal{E})$, induced by α, such that $\gamma' = \tilde\alpha \circ \gamma \circ \tilde\alpha^{-1}$. Here, $\gamma'|_{TM} \equiv \gamma|_{TM} \circ T\alpha^{-1}$ is the Clifford action on $\alpha^{-1*}\xi_{\rm F}$ that is defined with respect to $\alpha^{-1*}g_{\rm M}$ and $\tilde\alpha \in {\rm End}(\mathcal{E})$ an appropriate lift of α.

As a consequence, one obtains

$$\begin{array}{rcl} \mathrm{Diff}(\mathcal{M}) & \hookrightarrow & \mathrm{Aut}(\xi_{\mathrm{F}}) \\ \alpha & \mapsto & (\alpha, \beta := \tilde{\alpha}). \end{array} \tag{16}$$

We call the image of this embedding the “outer gauge group” of the fermion bundle ξ_{F}. It is denoted by $\mathcal{G}_{\mathrm{ex}}$.

Finally, since $\mathcal{G}_{\mathrm{in}} \subset \mathcal{G}_{\mathrm{F}}$ is normal and $\mathcal{G}_{\mathrm{in}} \cap \mathcal{G}_{\mathrm{ex}} = \{e\}$, one ends up with the semi-direct decomposition of the automorphism group into the gauge and diffeomorphism group, i.e.

$$\mathcal{G}_{\mathrm{F}} = \mathcal{G}_{\mathrm{in}} \rtimes \mathcal{G}_{\mathrm{ex}}. \tag{17}$$

In fact, each $g \in \mathcal{G}_{\mathrm{F}}$ may be written as $g = g_{\mathrm{in}} g_{\mathrm{ex}} \in \mathcal{G}_{\mathrm{in}} \rtimes \mathcal{G}_{\mathrm{ex}}$ such that

$$\mathcal{G}_{\mathrm{F}} \ni gg' = (g_{\mathrm{in}} g_{\mathrm{ex}})(g'_{\mathrm{in}} g'_{\mathrm{ex}}) \equiv (g_{\mathrm{in}} g_{\mathrm{ex}} g'_{\mathrm{in}} g_{\mathrm{ex}}^{-1})(g_{\mathrm{ex}} g'_{\mathrm{ex}}) \in \mathcal{G}_{\mathrm{in}} \rtimes \mathcal{G}_{\mathrm{ex}}. \tag{18}$$

We call the automorphism group $\mathcal{G}_{\mathrm{F}} \equiv \mathrm{Aut}(\xi_{\mathrm{F}})$ the “(fermionic) gauge group” of the fermion bundle ξ_{F}.

In order to define a $\mathcal{G}_{\mathrm{F}}-$covariant theory (by which we mean that symbolically $\mathcal{L} \circ (\alpha, \beta) = \alpha^{-1*}\mathcal{L}$ where $\mathcal{L}$ is an appropriate “Lagrangian density” defining the theory) we first consider, for a given fermion bundle ξ_{F}, the canonical mapping

$$\begin{array}{rcl} V_{\mathrm{D}} : \mathcal{D}(\xi_{\mathrm{F}}) & \longrightarrow & \mathcal{C}^\infty(\mathcal{M}) \\ D' & \mapsto & \mathrm{tr}(D'^2 - \triangle'_{\mathrm{D}}) \end{array} \tag{19}$$

which is called the “Dirac potential” on ξ_{F}. Here again, the second order differential operator $\triangle'_{\mathrm{D}}$ denotes the Bochner-Laplacian that is uniquely defined with respect to D' such that $\triangle'_{\mathrm{D}} + (D'^2 - \triangle'_{\mathrm{D}})$ is the (general) Lichnerowicz decomposition of D'^2 (c.f. in [BGV ’96], [Gil ’95]).

The universal top form

$$\begin{array}{rcl} \mathcal{L}_{\mathrm{D}} : \mathcal{D}(\xi_{\mathrm{F}}) & \longrightarrow & \Omega^n(\mathcal{M}) \\ D' & \mapsto & *V_{\mathrm{D}}(D') \end{array} \tag{20}$$

is called the “Dirac-Lagrangian” on the fermion bundle ξ_{F}. This canonical mapping is universal in the sense that it is indeed covariant with respect to the action of $\mathcal{G}_{\mathrm{F}}$. In particular, it is invariant with respect to the action of the inner gauge group $\mathcal{G}_{\mathrm{in}} \subset \mathcal{G}_{\mathrm{F}}$.

Definition 2.3 *Let $\xi_{\rm F}$ be the fermion bundle with respect to the data* $({\rm G}, \rho_{\rm F}, D)$. *We call (the "bosonic part" of) the theory which is defined by the corresponding Lagrangian density $\mathcal{L}_{\rm D}(D) \in \Omega^n(\mathcal{M})$ a "gauge theory of Dirac type".*

Let again $\mathcal{A}(\xi_{\rm F})$ be the set of all linear connections on $\xi_{\rm F}$ and $\mathcal{A}_{\rm D}(\xi_{\rm F}) \subset \mathcal{A}(\xi_{\rm F})$ be the subset of all connections which yield D (i.e. $\gamma \circ \nabla^{\mathcal{E}} = D$, with $\nabla^{\mathcal{E}}$ a corresponding covariant derivative). Then, the top form $\mathcal{L}_{\rm D}(D) \in \Omega^n(\mathcal{M})$ is indeed well-defined on the moduli-space $\mathfrak{M}_{\rm D}(\xi_{\rm F}) \equiv \mathcal{A}_{\rm D}(\xi_{\rm F})/\mathcal{G}_{\rm in}$. Moreover, it transforms covariantly with respect to the (left) action of the fermionic gauge group $\mathcal{G}_{\rm F}$, i.e.

$$\mathcal{L}'_{\rm D}(\beta \circ D \circ \beta^{-1}) = (\alpha^{-1*}\mathcal{L}_{\rm D})(D). \tag{21}$$

To obtain an explicit formula for the top form $\mathcal{L}_{\rm D}(D)$ associated with a Dirac type operator D, one could use the generalized Bochner-Lichnerowicz-Weizenböck formula of $D^2 - \triangle_{\rm D} \in \Gamma({\rm End}(\xi_{\rm F}))$. As a consequence, the Dirac potential reads

$$V_{\rm D}(D) = {\rm tr}\left(\gamma(\mathcal{F}_D) + {\rm ev}_{g_{\rm M}}\left(\partial_D^{T^*\mathcal{M}\otimes{\rm End}(\mathcal{E})}\Xi + \Xi^2\right)\right). \tag{22}$$

Here, respectively, $\mathcal{F}_D \in \Omega^2(\mathcal{M}, {\rm End}(\mathcal{E}))$ is the total curvature with respect to the Dirac connection $A_{\rm D} \in \mathcal{A}(\xi_{\rm F})$ and the one-form $\Xi \in \Omega^1(\mathcal{M}, {\rm End}(\mathcal{E}))$ measures the deviation of $A_{\rm D}$ from being a Clifford connection. With respect to a local co-frame $(X^1, \ldots, X^n)$ on $\mathcal{M}$ this one-form reads

$$\Xi \overset{loc.}{=} -\tfrac{1}{2}\, g_{li}\, X^l \otimes \gamma(X^j)\left([\partial_{D,X_j}, \gamma(X^i)] + \omega^i_{\ jk}\gamma(X^k)\right), \tag{23}$$

where $(X_1, \ldots, X_n)$ is the dual frame of $(X^1, \ldots, X^n)$ and $\omega^i_{\ jk} := X^i(\nabla^{TM}_{X_j} X_k)$ are the corresponding Levi-Civita connection coefficients with respect to $g_{\rm M}$ and the chosen frame. Again, $g_{ij} \in \mathcal{C}^\infty(U_\alpha)$ is the matrix element of $(g_{\rm M}(X^i, X^j))^{-1}$. Also, "${\rm ev}_{g_{\rm M}}$" denotes the evaluation map (contraction) with respect to the isomorphism $\tau^*_{\rm M} \simeq \tau_{\rm M}$ of the tangent and the cotangent bundle of $\mathcal{M}$ that is provided by $g_{\rm M}$ (c.f. [AT '96] and [Tol '98]).

Remark:

Let again, $\{(U_\alpha, \chi_\alpha)\,|\,\alpha \in \Lambda\}$ be a family of local trivializations of a given fermion bundle $\xi_{\rm F}$. According to (9), $D_\alpha := \chi_\alpha \circ D \circ \chi_\alpha^{-1}$ is fully determined by $(\gamma_\alpha, A_\alpha, \theta_\alpha)$. Hence, $\mathcal{A}_{\rm D}(\xi_{\rm F}) \subset \mathcal{A}(\xi_{\rm F})$ may locally be identified with the set of differential forms $\omega_\alpha \in \Omega^*(U_\alpha, {\rm End}(\mathbb{C}^{\rm N}))$ which, together with $g^{ik} \in \mathcal{C}^\infty(U_\alpha)$, determine D_α. Accordingly, the Euler-Lagrange equations

$$\mathcal{E}\mathcal{L}_{\rm D}(D) = 0 \tag{24}$$

are obtained by the first variation of the (locally defined) functional ($\Omega \subset U_\alpha$, compact)

$$\mathcal{S}[\gamma_\alpha, A_\alpha, \theta_\alpha] := \int_\Omega \mathcal{L}(\gamma_\alpha, A_\alpha, \theta_\alpha), \tag{25}$$

with $\mathcal{L}(\gamma_\alpha, A_\alpha, \theta_\alpha) \equiv (\chi_\alpha^{-1^*}\mathcal{L}_{\mathrm{D}})(D) \in \Omega^n(U_\alpha)$. Notice, however, it can easily be inferred from the local version of the Dirac potential (19) that $\mathcal{S} = \mathcal{S}[\gamma_\alpha, \theta_\alpha]$. Indeed, the local version of (22) reads

$$\begin{aligned} V(\gamma_\alpha, A_\alpha, \theta_\alpha) &\equiv (\chi_\alpha^{-1^*} V_{\mathrm{D}})(D) \\ &= \tfrac{N}{2}\, r_{\mathrm{M}} + \tfrac{1}{2}\,\mathrm{tr}\left([\gamma_\alpha^i, \gamma_\alpha^j][\theta_{\alpha,i}, \theta_{\alpha,j}]\right) \\ &+ \tfrac{1}{8}\, g_{ij}\,\mathrm{tr}\left(\gamma_\alpha^k\,[\theta_{\alpha,k}, \gamma_\alpha^i]\gamma_\alpha^l\,[\theta_{\alpha,l}, \gamma_\alpha^j]\right). \end{aligned} \tag{26}$$

The notation used is as follows:
$\theta_\alpha \equiv X^i \otimes \theta_{\alpha,i} := \pm\frac{1}{n}\, g_{\mu\nu} \sum_{0\le k\le n} \sum_{1\le i_1<i_2\cdots<i_k\le n} X^\mu \otimes \gamma_\alpha^\nu \gamma_\alpha^{i_1}\gamma_\alpha^{i_2}\cdots\gamma_\alpha^{i_k} \otimes \theta_{i_1 i_2\cdots i_k}$, *with the abbreviation* $\gamma_\alpha^i := \gamma_\alpha(X^i) \equiv \chi_\alpha \circ \gamma(X^i) \circ \chi_\alpha^{-1}$. *Moreover, $r_{\mathrm{M}} \in \mathcal{C}^\infty(\mathcal{M})$ denotes the scalar curvature of $\mathcal{M}$ with regard to g_{M}.*

It follows that Einstein's field equation of gravity is an integral part of the Euler-Lagrange equations of Dirac type gauge theories. In particular, the "energy-momentum tensor" is specified by the Dirac type operator in question (i.e. locally fixed by the one-form $\theta \in \Omega^1(U, \mathrm{End}(\mathcal{E}))$).

In the next section we discuss a specific class of Dirac type operators which is distinguished by its Lichnerowicz decomposition (c.f. [Lich '63]). Moreover, it is shown that, as a solution of the Euler-Lagrange equations, these Dirac type operators spontaneously break the gauge symmetry.

3 Simple Type Dirac Operators and Spontaneous Symmetry Breaking

In what follows we discuss a specific class of Dirac type gauge theories. The main feature of this class consists of permitting us to naturally include the notion of "spontaneous symmetry breaking" in the realm of Dirac type gauge theories. Eventually, we will show that the "Yukawa coupling" of the fermions, together with gravity, induces spontaneous symmetry breaking without use of a "Higgs potential". The inner geometry of $\mathcal{M}$ (i.e. of space-time in the case of $(n, s) = (4, \mp 2)$) in the "ground state" of the gauge theory is fully determined (up to boundary conditions) by the "fermionic masses". Here, the latter are shown to correspond to the spectrum of a certain Hermitian section of the bundle $\mathrm{End}(\xi_{\mathrm{F}})$. Because this spectrum turns out to be constant over $\mathcal{M}$ one may thus decompose the fermion bundle ξ_{F} into the Whitney sum of the appropriate eigenbundles of the "fermionic mass operator" that is induced by spontaneous symmetry breaking. If the spectrum is non-degenerated (like in the case of the Standard Model) the eigenbundles are Hermitian line bundles which one may consider to geometrically model "asymptotically free fermions".

Let ξ_{F} be a chiral fermion bundle with respect to some Dirac triple $(\mathrm{G}, \rho_{\mathrm{F}}, D)$.

Definition 3.1 *A Dirac type operator $D' \in \mathcal{D}(\xi_{\mathrm{F}})$ is called of "simple type" if the Bochner-Laplacian of D' is defined by a Clifford connection, i.e. $\hat{A}'_{\mathrm{D}} \in \mathcal{A}_{\mathrm{Cl}}(\xi_{\mathrm{F}}) \subset \mathcal{A}(\xi_{\mathrm{F}})$.*

We denote the corresponding covariant derivative again by $\partial_{\!A}$. Then, the covariant derivative of the Dirac connection $A'_{\mathrm{D}} \in \mathcal{A}(\xi_{\mathrm{F}})$ reads

$$\partial_{D'} = \partial_{\!A} + \varpi'_{\mathrm{D}} \tag{27}$$

with a unique one-form $\varpi'_{\mathrm{D}} \in \Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{E}))$. The next Proposition permits us to characterize the Dirac forms of simple type Dirac operators of arbitrary signature.

Proposition 3.1 *A Dirac type operator $D' \in \mathcal{D}(\xi_{\mathrm{F}})$ is of simple type if and only if it reads*

$$D' = \partial\!\!\!/_{\!A} + \gamma_{\mathrm{M}} \otimes \phi, \tag{28}$$

with $\phi \in \Gamma(\mathrm{End}^-_{\mathrm{Cl}}(\xi_{\mathrm{F}}))$.

Proof: The proof of the statement is lengthy and somewhat technical though elementary. It is similar to the proof already presented in [AT '96] for the special case $s = n$. A detailed proof for arbitrary signature s can be found in the Appendix. □

Note that a simple type Dirac operator is fully determined by a Clifford connection in the case where ξ_{F} is not chiral and thus has a vanishing Dirac form. In general, however, the Dirac connection of a simple type Dirac operator is given by a unique Clifford connection $A \in \mathcal{A}_{\rm Cl}(\xi_{\rm F})$ together with the specific Dirac form

$$\varpi'_{\rm D} = \Theta \wedge (\gamma_{\rm M} \otimes \phi). \tag{29}$$

With respect to a local trivialization (U_α, χ_α) of $\xi_{\rm F}$ the Dirac form is determined by

$$\theta_\alpha = \pm\tfrac{1}{n}\, g_{ij}\, X^i \otimes \gamma^j_\alpha \gamma_{\rm M} \otimes \phi_\alpha, \tag{30}$$

with $1 \otimes \phi_\alpha := \chi_\alpha \circ (1 \otimes \phi) \circ \chi^{-1}_\alpha \in \mathcal{C}^\infty(U_\alpha, {\rm End}^-(\mathbb{C}^{\rm N_F}))$ and $\theta_\alpha \equiv \chi^{-1*}_\alpha \varpi'_{\rm D}$.

Dirac operators of simple type define the largest class of Dirac type operators with the corresponding Bochner-Laplace operators defined by Clifford connections. Of course, the most important sub-class of Dirac type operators is given by $D' = \not{\partial}_A$. They correspond to "twisted Spin-Dirac operators" in the case where $\mathcal{M}$ denotes a spin manifold. Notice that in the elliptic case, Dirac operators of simple type turn out to be of importance in the discussion of the family index theorem (c.f. [Bis '86], [Qui '85]). They are also known to play a fundamental role in the description of the minimal Standard Model within the realm of non-commutative geometry (please see, for example, the corresponding references already cited in the introduction). This kind of first order differential operator is thus well-known in physics (please, see below), as well as in mathematics. However, in this paper we discuss them from a purely geometrical perspective of gauge theories.

We turn now to the discussion of spontaneous symmetry breaking within the realm of the presented geometrical frame. For this let again $\xi_{\rm F}$ be a chiral fermion bundle with respect to $({\rm G}, \rho_{\rm F}, D)$ where D is of simple type.

Proposition 3.2 *Let D be a global solution of the Euler-Lagrange equation*

$$\mathcal{EL}_{\rm D}(D) = 0 \tag{31}$$

such that $\mathcal{G}_{\rm YM}$ acts transitively on the image of $D - \not{\partial}_A$. Then there exists a constant (skew-Hermitian) section $\mathcal{D} \in \Gamma({\rm End}^-_{\rm Cl}(\xi_{\rm F}))$ such that $(\mathcal{M}, g_{\rm M})$ is an Einstein manifold with the scalar curvature given by

$$r_{\rm M} = \lambda\, \|{\rm M_F}\|^2. \tag{32}$$

Here, $\|\mathrm{M_F}\|^2 \equiv \mathrm{tr}(\mathrm{M_F^\dagger M_F})$ *with* $i\mathrm{M_F} := \gamma_{\mathrm{M}} \otimes \mathcal{D}$ *representing the "total fermionic mass operator";* $\lambda \in \mathbb{R}$ *is an appropriate non-zero constant which may also depend on a suitable normalization of* $\mathcal{L}_{\mathrm{D}}(D)$*.*

Proof: The Dirac-Lagrangian of a simple type Dirac operator reads

$$\mathcal{L}_{\mathrm{D}}(D) = 2^k(\mathrm{N_F}\, r_{\mathrm{M}} + \mathrm{tr}\phi^2)\mu_{\mathrm{M}}. \tag{33}$$

We remark that this Lagrangian depends on the connection that is defined only with respect to (g_{M}, ϕ). Moreover, the Euler-Lagrange equation concerning $\phi \in \Gamma(\mathrm{End}^-_{\mathrm{Cl}}(\xi_{\mathrm{F}}))$ is trivial. Whence, one may conclude that a global solution of (31) yields: $D = \not{\partial}_A$ with $A \in \mathcal{A}_{\mathrm{Cl}}(\xi_{\mathrm{F}})$ arbitrary and $(\mathcal{M}, g_{\mathrm{M}})$ Ricci flat. However, there is actually a bigger class of solutions of (31). Since the latter does not provide any dynamical condition on the sections ϕ one may treat the latter as "background fields", similar to the metric in the case of pure Yang-Mills gauge theory. The Euler-Lagrange equations with respect to the corresponding Dirac-Lagrangian then reduces to the Einstein equation

$$\mathrm{Ric}(g_{\mathrm{M}}) = \lambda_{\mathrm{gr}}\, \mathrm{tr}\phi^2\, \mathrm{id}_{\mathrm{TM}} \tag{34}$$

with $\lambda_{\mathrm{gr}} \in \mathbb{R}$ being some non-zero constant which also depends on the chosen normalization of $\mathcal{L}_{\mathrm{D}}(D)$. It also takes into account the appropriate physical (length) dimension, where ϕ is accordingly re-scaled. The section $\mathrm{Ric} \in \Gamma(\mathrm{End}(\tau_{\mathrm{M}}))$ denotes the Ricci tensor with respect to g_{M}. From the Einstein equation it follows that $\mathrm{d}(\mathrm{tr}\phi^2) = 0$. Hence, the Dirac-Lagrangian (33) of a simple type Dirac operator reduces, in general, to the Einstein-Hilbert Lagrangian with "cosmological constant" included. This constant is generated by a section $\phi \in \Gamma(\mathrm{End}^-_{\mathrm{Cl}}(\xi_{\mathrm{F}}))$ subject to the condition that $\|\phi\|^2 := <\phi, \phi> \equiv \mathrm{tr}(\phi^\dagger\phi)$ must be constant. Note that $\phi^\dagger = \pm\phi$, depending on whether D is supposed to be Hermitian or skew-Hermitian. The basic idea then is to make a polar decomposition $\phi \overset{loc.}{=} \rho_{\mathrm{F}}(g)\circ\mathcal{D}\circ\rho_{\mathrm{F}}(g)^{-1}$ with $\mathcal{D}$ being a fixed vector of the same length as ϕ. To make this more precise let $\mathcal{W} := \mathrm{End}^-_{\mathrm{Cl}}(\xi_{\mathrm{F}})$ be the Hermitian vector bundle of (complex) rank $\mathrm{N_F^2}$ with total space $W := \mathrm{End}^-_{\mathrm{Cl}}(\mathcal{E})$. Accordingly, let $\mathfrak{P} := (P, \mathcal{M}, \pi_{\mathrm{P}})$ be the frame bundle associated with $\mathcal{W}$. Also, let $\mathfrak{E} := (E, \mathcal{M}, \pi_{\mathrm{E}})$ be the associated Hermitian vector bundle with total space defined by $E := P \times_{\mathrm{G}} \mathrm{End}(\mathbb{C}^{\mathrm{N_F}})$. Then, by construction $\mathfrak{E} \simeq \mathcal{W}$ and we do not distinguish between these two vector bundles. In particular, we may write $W \ni \mathfrak{Z} = [(p, \mathfrak{z})]$. Equivalently, if $\phi \neq 0$ we may consider the normalized section $\varphi := \phi/\|\phi\| \in \Gamma(\mathcal{S})$ with $\mathcal{S} \subset \mathcal{W}$ being the sphere sub-bundle. According to the identification $\mathfrak{E} \simeq \mathcal{W}$ any section φ corresponds to a G-equivariant mapping $\tilde{\varphi} : P \to \mathrm{S}^{\mathrm{N}'-1}$ ($\mathrm{N}' = 2\mathrm{N}_{\mathrm{F}}^2$), such that $\varphi(x) = [(p, \tilde{\varphi}(p))]|_{p\in\pi_{\mathrm{P}}^{-1}(x)}$. By assumption, G acts transitively on $\mathrm{im}(\tilde{\varphi}) \subset \mathrm{S}^{\mathrm{N}'-1}$. Hence, for arbitrarily chosen $\mathfrak{z}_0 \in \mathrm{im}(\tilde{\varphi})$ we may identify the orbit of $\mathfrak{z}_0$, $\mathrm{orbit}(\mathfrak{z}_0)$, with $\mathrm{im}(\tilde{\varphi})$. Let $\mathrm{I}(\mathfrak{z}_0) \subset \rho_{\mathrm{F}}(\mathrm{G})$ be the isotropy group of $\mathfrak{z}_0$. The mapping

$$\begin{array}{rcl} \nu_\phi : P & \longrightarrow & \mathrm{orbit}(\mathfrak{z}_0) \\ p & \mapsto & \rho_{\mathrm{F}}(g)\mathfrak{z}_0\rho_{\mathrm{F}}(g^{-1}), \end{array} \tag{35}$$

defines an "H-reduction" $(\mathcal{Q}_\phi, \iota_\phi)$ of $\mathfrak{P}$ with $g \in \mathrm{G}$ being determined (modulo $\mathrm{I}(\mathfrak{z}_0)$) by the relation $\tilde{\varphi}(pg) = \mathfrak{z}_0$. Indeed, the corresponding section

$$\begin{array}{rcl} \mathcal{V}_\phi : \mathcal{M} & \longrightarrow & P \times_{\mathrm{G}} \mathrm{G}/\mathrm{H} \\ x & \mapsto & [(p, \nu_\phi(p))]|_{p\in\pi_{\mathrm{P}}^{-1}(x)} \end{array} \tag{36}$$

is known to be equivalent to a specific principal H-bundle $\mathcal{Q}_\phi \equiv (Q, \mathcal{M}, \pi_{\mathrm{Q}}, \mathrm{H})$ together with an equivariant embedding $\iota_\phi : \mathcal{Q}_\phi \hookrightarrow \mathcal{P}$ of principal bundles (c.f., for example, in [KN '96]). For "bundle reduction" in the context of Yang-Mills-Higgs gauge theories see also, for example, in [CW '89], [Ster '95] and [Trau '80]. Here, $\mathrm{H} \subset \mathrm{G}$ is the unique sub-group equivalent to $\mathrm{I}(\mathfrak{z}_0)$, and thus $\mathrm{orbit}(\mathfrak{z}_0) \simeq \mathrm{G}/\mathrm{H}$. Finally, we may define $\mathcal{D} \in \Gamma(\mathrm{End}^-_{\mathrm{Cl}}(\xi_{\mathrm{F}}))$ by the section

$$\begin{array}{rcl} \mathcal{D} : \mathcal{M} & \longrightarrow & E \\ x & \mapsto & [(\iota_\phi(q), \mathfrak{z}_0)]|_{q\in\pi_{\mathrm{Q}}^{-1}(x)}. \end{array} \tag{37}$$

Of course, the section $\mathcal{D}$ also gives rise to an (equivalent) H-reduction $(\mathcal{Q}, \iota)$ of $\mathfrak{P}$ which may be identified with $(\mathcal{Q}_\phi, \iota_\phi)$ by $\mathrm{H} \simeq \mathrm{I}(\tilde{g}\mathfrak{z}_0\tilde{g}^{-1})$. Here, $\tilde{g} \in \mathrm{G}$ is determined (up to $\mathrm{I}(\mathfrak{z}_0)$) by a choice of $q_0 \in Q_\phi$ and the corresponding relation $\varphi(\iota_\phi(q_0)) \equiv \tilde{\mathfrak{z}}_0 =: \tilde{g}\mathfrak{z}_0\tilde{g}^{-1}$. The rest of the statement is a direct consequence of the Einstein equation. $\square$

A simple type Dirac operator D is said to be in the "unitary gauge" provided it reads

$$D = \not{\partial}_{\!A} + \gamma_{\mathrm{M}} \otimes \mathcal{D}. \tag{38}$$

A necessary condition for the existence of the unitary gauge is that $D - \not{\partial}_{\!A} \neq 0$. If $\mathcal{G}_{\mathrm{YM}}$ acts transitively on the image of the latter operator, this condition is also sufficient. A simple type Dirac operator in the unitary gauge spontaneously breaks the Yang-Mills gauge symmetry since in general

$$\mathcal{H}_{\mathrm{YM}} := \{g \in \mathcal{G}_{\mathrm{YM}} \,|\, [\mathcal{D}, g] = 0\} \tag{39}$$

is a proper sub-group of the Yang-Mills gauge group $\mathcal{G}_{\rm YM} \subset \mathcal{G}_{\rm F}$. In this case, the Lagrangian $\mathcal{L}_{\rm D}(D)$ is said to define a "spontaneously broken fermionic gauge theory". Note that in the case where $\mathcal{G}_{\rm YM}$ acts transitively on the sphere sub-bundle $\mathcal{S} \subset {\rm End}_{\rm Cl}(\xi_{\rm F})$ any global solution of (31) satisfying $D - \partial\!\!\!/_{\rm A} \neq 0$ defines a spontaneously broken fermionic gauge theory.

Remark:

The notion of unitary gauge and its existence is similar to that presented in [Tol '03(a)] (Prop. 3.2) in the case of rotationally symmetric Higgs potentials. However, the "mass term" $\|\phi\|^2$ in the Lagrangian of a simple type Dirac operator itself does not break the symmetry, of course. The symmetry breaking is caused by assuming that the fermionic mass generates a non-trivial geometry. Indeed, the geometry is fully determined by the spectrum of the (square of the) fermionic mass operator ${\rm M}_{\rm F}^2 \in \Gamma({\rm End}(\xi_{\rm F}))$. *Also, since the spectrum* ${\rm spec}({\rm M}_{\rm F}^2)$ *is constant throughout $\mathcal{M}$, one may decompose the fermion bundle into the Whitney sum of the corresponding eigenbundles of* ${\rm M}_{\rm F}^2$, *i.e.*

$$\begin{aligned} \xi_{\rm F} &= \bigoplus_{{\rm m}^2 \,\in\, {\rm spec}({\rm M}_{\rm F}^2)} \xi_{\rm F,m^2} \\ &= \ker({\rm M}_{\rm F}^2) \oplus \left[\bigoplus_{{\rm m}^2 \,\in\, {\rm spec}({\rm M}_{\rm F}^2)\backslash\{0\}} \xi_{\rm F,m^2} \right]. \end{aligned} \tag{40}$$

The total curvature on $\xi_{\rm F}$ with respect to a simple type Dirac operator satisfying (31) is given by

$$\mathcal{F}_D = R\!\!\!/ + F_A + {\rm M}_{\rm F}^2\, \Theta \wedge \Theta - \,\partial_A^{{\rm End}(\mathcal{E})} {\rm M}_{\rm F} \wedge \Theta. \tag{41}$$

Here, respectively, $R\!\!\!/ \in \Omega^2(\mathcal{M}, {\rm End}(\mathcal{E}))$ is the lifted (semi-)Riemannian curvature with respect to $g_{\rm M}$, and $F_A \equiv \mathcal{F}_{\partial\!\!\!/_{\rm A}} - R\!\!\!/ \in \Omega^2(\mathcal{M}, {\rm End}(\mathcal{E}))$ is the "twisting curvature" with respect to the Clifford connection $A \in \mathcal{A}_{\rm Cl}(\xi_{\rm F})$ that is determined by D. In contrast to $R\!\!\!/$, which is determined by the spectrum of ${\rm M}_{\rm F}^2$, *the twisting curvature F_A is completely arbitrary. For this reason it is natural to assume that A is purely topological, i.e. flat. In this case, the curvature of $\xi_{\rm F}$ is fully determined by the spectrum of the fermionic mass operator. As a consequence, for $n = 4$ the chiral fermion bundle must indeed be real. If in addition $\mathcal{M}$ is a spin-manifold, then $\xi_{\rm F,m^2} \simeq \tau_{\rm spin} \otimes_{\mathcal{M}} \zeta_{\rm F,m^2}$, where the latter is an Hermitian line bundles if and only if* ${\rm spec}({\rm M}_{\rm F}^2)\backslash\{0\}$ *is non-degenerated. Consequently, when restricted to the residual group H, the fermionic representation $\rho_{\rm F}$ decomposes into the sum of the trivial representation and irreducible U(1)-representations[3]. The latter are either trivial, and hence*

[3]To date, electromagnetism is the only Abelian gauge theory that is physically well-established.

$\xi_{\mathrm{F,m^2}}$ corresponds to electrically uncharged but massive fermion or, for non-trivial representations, $\xi_{\mathrm{F,m^2}}$ corresponds to a massive electrically charged particles. Apparently, together with spin, the assumption that the Clifford connection A is flat imposes crucial restrictions on the fermion bundle. In fact, in this case (up to algebraic torsion) $\xi_{\mathrm{F}} \simeq \bigoplus_{k=1}^{\mathrm{N_F}} \tau_{\mathrm{spin}}$. Note that, if $n = 4$ and $\mathrm{spec}(\mathrm{M}_{\mathrm{F}}^2)$ is non-degenerate, the existence of a flat Clifford connection on ξ_{F} (again, up to torsion) becomes equivalent to the reality of the latter.

Definition 3.2 *A fermion bundle ξ_{F} is said to be in the "unitary gauge" provided it is defined with respect to a Dirac triple $(\mathrm{G}, \rho_{\mathrm{F}}, D)$ such that D is in the unitary gauge. More generally, a fermion bundle is called "massive" if it is gauge equivalent to a fermion bundle in the unitary gauge. The corresponding element of $\mathcal{G}_{\mathrm{YM}} \subset \mathcal{G}_{\mathrm{F}}$ is referred to as a "unitary gauge transformation".*

On a massive fermion bundle there exists a distinguished class of connections.

Definition 3.3 *A connection $A \in \mathcal{A}(\xi_{\mathrm{F}})$ on a massive fermion bundle ξ_{F} is called compatible with D provided the corresponding covariant derivative $\nabla^{\mathcal{E}}$ commutes with the appropriate total fermionic mass operator. That is,*

$$\nabla_X^{\mathrm{End}(\mathcal{E})} \mathrm{M}_{\mathrm{F}} = 0 \tag{42}$$

for all smooth tangent vector fields $X \in \Gamma(\tau_{\mathrm{M}})$.

This definition expresses the H-reducibility of a connection on ξ_{F} in terms of Dirac type operators which spontaneously break the gauge symmetry. The Definition (3.3) is in fact analogous to the Definition 2.1 in [Tol '03(a)] for a spontaneously broken Yang-Mills-Higgs gauge theory. Note that (42) is equivalent to the condition

$$D' \circ \mathrm{M}_{\mathrm{F}} = -\mathrm{M}_{\mathrm{F}} \circ D', \tag{43}$$

with $D' \in \mathcal{D}(\xi_{\mathrm{F}})$ being identified with $\gamma \circ \nabla^{\mathcal{E}}$. In particular, one may assume that the Clifford connection which defines the Bochner-Laplacian of $D = \not{\partial}_{\mathrm{A}} + i\mathrm{M}_{\mathrm{F}}$ is compatible with the latter. This holds true if and only if

$$D^2 = \not{\partial}_{\mathrm{A}}^{\,2} - \mathrm{M}_{\mathrm{F}}^2. \tag{44}$$

Hence, the Clifford connection of the Bochner-Laplacian $\triangle_{\mathrm{D}}$ is compatible with spontaneous symmetry breaking if and only if "the square of the sum equals the sum

Moreover, as a matter of fact massless but electrically charged particles are unknown in nature.

of the squares". We note that, from a geometrical point of view, it is the condition $\nabla^{\mathrm{End}(\mathcal{E})}\mathrm{M}_{\mathrm{F}} \neq 0$ that yields "massive vector bosons" (please see below). In other words, the existence of a non-trivial "Yang-Mills mass operator" can be expressed by the violation of the compatibility condition (44).

Definition 3.4 *We call a simple type Dirac operator D to define a "(semi-classical) fermionic vacuum" if D is gauge equivalent to $\not{\partial}_{\mathrm{A}} + i\mathrm{M}_{\mathrm{F}}$ where the corresponding Clifford connection $A \in \mathcal{A}(\xi_{\mathrm{F}})$ is purely topological. In this case, D in the unitary gauge is denoted by*

$$\not{\partial}_{\mathcal{D}} \equiv \not{\partial} + i\mathrm{M}_{\mathrm{F}}. \tag{45}$$

Clearly, when restricted to the appropriate eigenbundles this operator corresponds to Dirac's well-known first order differential operator $i\not{\partial} - \mathrm{m}$ and thus provides us with the appropriate physical interpretation of $\mathrm{spec}(\mathrm{M}_{\mathrm{F}}^2)$ (and hence also with $\mathcal{D}$). For example, in the case of $(n, s) = (4, \mp 2)$ there is always a local frame such that the total symbol $\sigma(i\not{\partial}_{\mathcal{D}})$ coincides with the principal symbol of (45). Every time-like $\xi \in T^*\mathcal{M} \subset \mathrm{End}(\mathcal{E})$ and eigenvector $\mathfrak{z} \in \mathcal{E}$ of $\mathrm{M}_{\mathrm{F}}^2$ (with eigenvalue m^2) yields $\sigma(i\not{\partial})(\xi)\mathfrak{z} \overset{loc}{=} \gamma(\xi)\mathfrak{z} = \pm\mathrm{m}\mathfrak{z}$. Hence, one obtains the usual relation between momentum and mass: $g_{\mathrm{M}}(\xi, \xi) = \pm\mathrm{m}^2$ of a point-like particle.

From a geometrical point of view a "fermionic vacuum" may be regarded as a fermion bundle $\xi_{\mathrm{F,red}} := (\mathcal{E}_{\mathrm{red}}, \mathcal{M}, \pi_{\mathcal{E},\mathrm{red}})$ with respect to the Dirac triple $(\mathrm{H}, \rho_{\mathrm{F,red}}, \not{\partial}_{\mathcal{D}})$. Here, respectively, $\mathcal{E}_{\mathrm{red}} := Q \times_{\mathrm{H}} \mathbb{C}^{2^{\mathrm{k}}} \otimes \mathbb{C}^{\mathrm{N_F}}$ and $\rho_{\mathrm{F,red}} := \rho_{\mathrm{F}}|_{\mathrm{H}}$. Notice that $\xi_{\mathrm{F}} \simeq \xi_{\mathrm{F,red}}$ via the bundle mapping $[(q, z)] \mapsto [(\iota(q), z)]$. Accordingly, we shall not distinguish between these two bundles and proceed to say that a fermion bundle ξ_{F} can be generated from a fermionic vacuum if it is determined by a Dirac triple of the form $(\mathrm{H}, \rho_{\mathrm{F,red}}, \not{\partial}_{\mathcal{D}})$. In other words, ξ_{F} is generated from a fermionic vacuum provided the corresponding frame bundle $\mathcal{P}$ can be considered as a prolongation of the frame bundle $\mathcal{Q}$ that corresponds to some fermion bundle $\xi_{\mathrm{F,red}}$. Finally, the Dirac potential of a fermionic vacuum has the particular simple form

$$V_{\mathrm{D}}(\not{\partial}_{\mathcal{D}}) = \frac{\lambda}{2} < \mathrm{M}_{\mathrm{F}}^2 >, \tag{46}$$

where $<\mathrm{M}_{\mathrm{F}}^2> := \frac{1}{\mathrm{N_F}} \sum_{a=1}^{\mathrm{N_F}} \mathrm{m}_a^2$ and $\lambda \in \mathbb{R}$ is a suitable non-zero constant.

The idea of a fermionic vacuum is mainly motivated by a geometrical description of perturbation theory used in quantum field theory. As already mentioned above the fermion bundle $\xi_{\rm F}$ is considered as a "perturbation" of a fermionic vacuum $\xi_{\rm F,red}$. Such a perturbation cannot change the topology of $\xi_{\rm F}$ but its geometry. The notion of a fermionic vacuum itself puts severe topological restrictions on a fermion bundle[4]. Before we explain this in more detail, however, we shall discuss in the next section a more specific class of simple type Dirac operators which takes into account that, within the Standard Model of Particle Physics, the Higgs boson is described by a sub-representation of $\rho_{\rm F}$ instead of the fundamental representation. Moreover, we shall discuss the need of "fermionic doubling" and the fermionic Lagrangian within the presented setup.

4 Dirac-Yukawa Type Operators and the Fermionic Lagrangian

In the last section we discussed a distinguished class of Dirac type operators on a fermion bundle. Their basic feature is to give rise to a reduction of the underlying gauge symmetry. Moreover, these Dirac type operators also determine a distinguished class of connections on the fermion bundle. In the next two sections we specialize the presented frame in order to geometrically describe the action of the Standard Model of Particle Physics in terms of a specific Dirac-Lagrangian. For this, we first discuss a certain "refinement" of simple type Dirac operators which will then be called "Dirac-Yukawa operators". In what follows, we also discuss an important consequence of the occurrence of the grading involution $\gamma_{\rm M}$ in the definition of simple type Dirac operators. This turns out to parallel the occurrence of this grading involution in A. Connes' non-commutative geometry (c.f., for example, in [Con '94], [GIS '98], [GV '93], [KS '96], [LMMS '96], [LMMS '97] and [SZ '95]).

4.1 Yukawa Bundles and Dirac Operators of Yukawa Type

To start with, let again $\xi_{\rm F}$ be a chiral fermion bundle with respect to $({\rm G}, \rho_{\rm F}, D)$, where D is of simple type. Also let $\xi_{\rm H} \subset \xi_{\rm F}$ be a sub-vector bundle of rank ${\rm N_H} < {\rm N_F}$ on which $\tau_{\rm Cl}$ acts trivially. We denote its dual by $\xi^*_{\rm H}$. The structure group of $\xi_{\rm H}$ is a specific sub-group of $\rho_{\rm F}({\rm G})$. It will be denoted by $\rho_{\rm H}({\rm G})$. The gauge group of $\xi_{\rm H}$ is accordingly

[4] One might speculate that "quantum fluctuations" will lead to a change of the topology of the fermionic vacuum for it basically adds "quantum corrections" to the fermionic mass spectrum.

denoted by $\mathcal{G}_{\rm H} \subset \mathcal{G}_{\rm YM} \subset {\rm Aut}(\xi_{\rm H}) \subset {\rm Aut}(\xi_{\rm F})$ (the bundle automorphisms of $\xi_{\rm H}$ over the identity on $\mathcal{M}$.)

Definition 4.1 *Let $E_{\rm H} \subset \mathcal{E}$ be the total space of $\xi_{\rm H}$, and let $\pi_{\rm H}$ be the appropriate projection mapping onto the base manifold $\mathcal{M}$. Also, let again $W := {\rm End}^-_{\rm Cl}(\mathcal{E})$. We call the sub-vector bundle $\xi_{\rm Y} \subset \xi^*_H \otimes_{\mathcal{M}} \xi_{\rm W}$ the "Yukawa bundle" (with respect to the above data) if its structure group acts as follows: For each $h \in {\rm Aut}(E_{\rm H})$ there is a unique $g \in {\rm Aut}^+_{\rm Cl}(\mathcal{E})$ such that $\mathcal{Y}(h^{-1}\mathfrak{z}) = {\rm Ad}_{g^{-1}}(\mathcal{Y}(\mathfrak{z}))$ for all $\mathfrak{z} \in E_{\rm H}$ and $\mathcal{Y} \in E^*_{\rm H} \otimes W$. In this case we call $\xi_{\rm H}$ the "Higgs bundle" (again, with respect to the above given data). A section $\mathcal{Y} \in \Gamma(\xi_{\rm Y})$ of the Yukawa bundle is called a "Yukawa mapping" provided that it fulfills the following conditions: Considered as a bundle mapping the Yukawa mapping $\mathcal{Y}$ is injective and anti-Hermitian, i.e. $\mathcal{Y}(\mathfrak{z})^\dagger = -\mathcal{Y}(\mathfrak{z})$ for all $\mathfrak{z} \in E_{\rm H}$. Moreover, we assume that it satisfies the requirement $\mathcal{Y}(\partial_{A,X}\varphi) = [\partial_{A,X}, \mathcal{Y}(\varphi)]$ for all Clifford connections on $\xi_{\rm F}$ (and thus for all induced connections on $\xi_{\rm H}$), sections $\varphi \in \Gamma(\xi_{\rm H})$ and tangent vector fields $X \in \Gamma(\tau_{\rm M})$.*

Note that for each connection on $\xi_{\rm F}$ with covariant derivative $\nabla^{\mathcal{E}}$, the operator $[\nabla^{\mathcal{E}}_X, \mathcal{Y}(\varphi)] - \mathcal{Y}(\nabla^{E_{\rm H}}_X\varphi)$ on the fermion bundle $\xi_{\rm F}$ defines a connection on $\xi^*_{\rm H} \otimes \xi_{\rm W}$ with the covariant derivative $\nabla^{E^*_{\rm H}\otimes W} \equiv \nabla^{E^*_{\rm H}} \otimes 1 + 1 \otimes \nabla^W$. Hence, a Yukawa mapping is assumed to be covariantly constant with respect to any Clifford connection. By the definition of the Yukawa bundle it then follows that a Yukawa mapping has to be a constant section. For instance, in the case of the Standard Model the Yukawa mapping (4.1) is parameterized by the "Yukawa coupling constants". The representations $\rho_{\rm H}$ and $\rho_{\rm F}$ are known to be related by the "hyper-charges" of the fermions and the Higgs boson.

Definition 4.2 *We call a Dirac type operator D on a fermion bundle $\xi_{\rm F}$ a "Dirac-Yukawa operator" if there is a section of the Higgs bundle, $\varphi \in \Gamma(\xi_{\rm H})$, such that*

$$D = \not{\partial}_{\rm A} + \gamma_{\rm M} \otimes \mathcal{Y}(\varphi). \tag{47}$$

According to its physical interpretation we call the section $\mathcal{Y}(\varphi) \in \Gamma({\rm End}^-_{\rm Cl}(\xi_{\rm F}))$ the "Yukawa coupling term" with respect to $(\mathcal{Y}, \varphi) \in \Gamma(\xi_{\rm Y} \times_{\mathcal{M}} \xi_{\rm H})$.

A Yukawa mapping defines an additional data on a fermion bundle which in some sense is not natural within the frame of Dirac type gauge theories. For this reason we shall refer to the data $({\rm G}, \rho_{\rm F}, D)$, with D being a Dirac-Yukawa operator, as a "Dirac-Yukawa model". A necessary condition for a Dirac-Yukawa operator to spontaneously break the underlying gauge symmetry is that $\varphi \in \Gamma(\xi_{\rm H})$ does not vanish. Again, this condition is also sufficient provided G acts transitively on the image of the section $\mathcal{Y}(\varphi)$. Assuming this is the case it follows from the definition of the Higgs bundle and the Yukawa mapping that there must exist a constant section $\mathcal{V} \in \Gamma(\xi_{\rm H})\backslash\{\mathcal{O}\}$ (with $\mathcal{O}$ being the zero-section) such that in the unitary gauge

$$D = \not{\partial}_{\rm A} + \gamma_{\rm M} \otimes \mathcal{Y}(\mathcal{V}). \tag{48}$$

Analogous to our previous definition we consider a Dirac-Yukawa operator to define a (semi-classical) fermionic vacuum if it is gauge equivalent to $\not{\partial}_{\mathcal{V}} \equiv \not{\partial} + i\mathrm{M}_{\rm F}$ with the total fermionic mass operator $i\mathrm{M}_{\rm F} := \gamma_{\rm M} \otimes \mathcal{Y}(\mathcal{V})$. Notice that the spectrum of the total fermionic mass operator is independent of the choice of $\mathfrak{z}_0 \in \mathrm{End}(\mathbb{C}^{\rm N_F})$. This reduces to $\mathfrak{z}_0 = \mathrm{G_Y}(\mathbf{z}_0)$ in the case where the gauge symmetry is spontaneously broken by a Dirac-Yukawa operator. Here, $\mathrm{G_Y} \in \mathrm{Hom}(\mathbb{C}^{\rm N_H}, \mathrm{End}(\mathbb{C}^{\rm N_F}))$ is the matrix of the "Yukawa coupling constants" and $\mathbf{z}_0 \in \mathbb{C}^{\rm N_H}$. In particular, we obtain $\mathrm{orbit}(\mathfrak{z}_0) = \mathrm{G_Y}(\mathrm{orbit}(\mathbf{z}_0))$. Hence, from the properties of the Yukawa mapping it can be inferred that the "little group" $\mathrm{H} \subset \mathrm{G}$ crucially depends on $\rho_{\rm H} \subset \rho_{\rm F}$.

4.2 The Fermionic Lagrangian

Next, we discuss the fermionic Lagrangian within the presented frame. By definition, the grading involution of a chiral fermion bundle $\xi_{\rm F} = \xi_{\rm F}^+ \oplus \xi_{\rm F}^-$ reads $\Gamma = \gamma_{\rm M} \otimes \chi$. Consequently, the total space $\mathcal{E}$ of the fermion bundle decomposes as

$$\begin{aligned} \mathcal{E} &= \mathcal{E}^+ \oplus \mathcal{E}^- \\ &= (\mathcal{E}_{\rm LL} \oplus \mathcal{E}_{\rm RR}) \oplus (\mathcal{E}_{\rm RL} \oplus \mathcal{E}_{\rm LR}) \end{aligned} \tag{49}$$

where, respectively,

$$\begin{aligned} \mathcal{E}_{\rm LL} &:= \{\mathfrak{z} \in \mathcal{E} \,|\, (\gamma_{\rm M} \otimes 1)\mathfrak{z} = -\mathfrak{z},\ (1 \otimes \chi)\mathfrak{z} = -\mathfrak{z}\}, \\ \mathcal{E}_{\rm RR} &:= \{\mathfrak{z} \in \mathcal{E} \,|\, (\gamma_{\rm M} \otimes 1)\mathfrak{z} = \mathfrak{z},\ (1 \otimes \chi)\mathfrak{z} = \mathfrak{z}\}, \\ \mathcal{E}_{\rm RL} &:= \{\mathfrak{z} \in \mathcal{E} \,|\, (\gamma_{\rm M} \otimes 1)\mathfrak{z} = \mathfrak{z},\ (1 \otimes \chi)\mathfrak{z} = -\mathfrak{z}\}, \\ \mathcal{E}_{\rm LR} &:= \{\mathfrak{z} \in \mathcal{E} \,|\, (\gamma_{\rm M} \otimes 1)\mathfrak{z} = -\mathfrak{z},\ (1 \otimes \chi)\mathfrak{z} = \mathfrak{z}\}. \end{aligned} \tag{50}$$

Let $\pi_{\rm R/L} := \frac{1}{2}(1 \pm (\gamma_{\rm M} \otimes 1))$ and $\rho_{\rm R/L} := \frac{1}{2}(1 \pm (1 \otimes \chi))$. The appropriate projection mappings of the respective subspaces (50) of $\mathcal{E}$ are denoted by $\pi_{\rm LL} \equiv \pi_{\rm L} \circ \rho_{\rm L} = \rho_{\rm L} \circ \pi_{\rm L}$, $\pi_{\rm RR} \equiv \pi_{\rm R} \circ \rho_{\rm R} = \rho_{\rm R} \circ \pi_{\rm R}$, $\pi_{\rm RL} \equiv \pi_{\rm R} \circ \rho_{\rm L} = \rho_{\rm L} \circ \pi_{\rm R}$ and $\pi_{\rm LR} \equiv \pi_{\rm L} \circ \rho_{\rm R} = \rho_{\rm R} \circ \pi_{\rm L}$. Consequently, $\pi_+ = \pi_{\rm RR} + \pi_{\rm LL}$ and $\pi_- = \pi_{\rm RL} + \pi_{\rm LR}$. For $\phi \in \Gamma(\mathrm{End_{Cl}}(\xi_{\rm F}))$ we also define $1 \otimes \phi_{\rm LL} := \rho_{\rm L} \circ (1 \otimes \phi) \circ \rho_{\rm L} \in \Gamma(\mathrm{End_{Cl}}(\xi_{\rm F,LL} \oplus \xi_{\rm F,RL})) \simeq \Gamma(\mathrm{End_{Cl}}(\xi_{\rm F,LL})) \oplus$

$\Gamma(\mathrm{End}_{\mathrm{Cl}}(\xi_{\mathrm{F,RL}}))$, $1\otimes\phi_{\mathrm{RL}} := \rho_{\mathrm{R}}\circ(1\otimes\phi)\circ\rho_{\mathrm{L}} \in \Gamma(\mathrm{Hom}_{\mathrm{Cl}}(\xi_{\mathrm{F,LL}}\oplus\xi_{\mathrm{F,RL}},\xi_{\mathrm{F,LR}}\oplus\xi_{\mathrm{F,RR}})) \simeq \Gamma(\mathrm{Hom}_{\mathrm{Cl}}(\xi_{\mathrm{F,LL}},\xi_{\mathrm{F,LR}}))\oplus\Gamma(\mathrm{Hom}_{\mathrm{Cl}}(\xi_{\mathrm{F,RL}},\xi_{\mathrm{F,RR}}))$, etc.

If $\mathcal{M}$ denotes a spin manifold, then $\mathcal{E}\simeq S\otimes E_{\mathrm{F}}$, where S is the total space of the spinor bundle τ_{spin} (with respect to some chosen spin structure) and E_{F} is the total space of some Hermitian vector bundle ζ_{F}. In this case, the fermion bundle $\xi_{\mathrm{F}}\simeq\tau_{\mathrm{spin}}\otimes\zeta_{\mathrm{F}}$ is chiral if and only if ζ_{F} is $\mathbb{Z}_2-$graded, i.e. $E_{\mathrm{F}} = E_{\mathrm{F,R}}\oplus E_{\mathrm{F,L}}$. Here, $E_{\mathrm{F,R/L}}$ are considered as the eigenspaces of χ with respect to the eigenvalues ±1. Then, for instance, $\mathcal{E}_{\mathrm{LL}}\simeq S_{\mathrm{L}}\otimes E_{\mathrm{F,L}}$, etc. Consequently, like in non-commutative geometry, the fermionic degrees of freedoms are doubled in the geometrical description presented here (c.f. again the corresponding discussion in [LMMS '96], [LMMS '97]). Indeed, as far as the Standard Model is concerned only

$$\mathcal{E}_{\mathrm{phy}}\equiv\mathcal{E}^{+} = (\mathcal{E}_{\mathrm{LL}}\oplus\mathcal{E}_{\mathrm{RR}}) \tag{51}$$

represents the "true" physical degrees of freedom.

With this in mind the "fermionic Lagrangian" of D may be defined as the following specific quadratic form on $\Gamma(\xi_{\mathrm{F}})$ (taking its value in the top forms of $\mathcal{M}$):

$$\begin{array}{rcl}\mathcal{L}_{\mathrm{F}} : \mathcal{D}(\xi_{\mathrm{F}}) & \longrightarrow & \Gamma(\xi_{\mathrm{F}}\otimes_{\mathcal{M}}\Lambda^{\mathrm{n}}\tau^*_{\mathrm{M}}) \\ D & \mapsto & \left\{\begin{array}{ccc}\Gamma(\xi_{\mathrm{F}}) & \longrightarrow & \Omega^{\mathrm{n}}(\mathcal{M}) \\ \psi & \mapsto & <\psi, D_+\psi>_{\mathcal{E}}\,\mu_{\mathrm{M}}.\end{array}\right.\end{array} \tag{52}$$

Here, $<\cdot,\cdot>_{\mathcal{E}}$ is the Hermitian product on $\mathcal{E}$ and $D_\pm\equiv\pi_\mp\circ D\circ\pi_\pm : \Gamma(\xi^\pm_{\mathrm{F}})\to\Gamma(\xi^\mp_{\mathrm{F}})$ such that $D\in\mathcal{D}(\xi_{\mathrm{F}})$ reads

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0\end{pmatrix} : \begin{array}{c}\Gamma(\xi^+_{\mathrm{F}})\\ \oplus \\ \Gamma(\xi^-_{\mathrm{F}})\end{array} \longrightarrow \begin{array}{c}\Gamma(\xi^+_{\mathrm{F}})\\ \oplus \\ \Gamma(\xi^-_{\mathrm{F}})\end{array}. \tag{53}$$

It is common use to also refer to the operators $D_\pm$ themselves as Dirac type operators although the square of these operators is usually not defined[5]. The Hermitian product on $\mathcal{E}$ depends on the signature of D. For instance, in the respective cases of

[5]Equivalently, if, for instance, the operator $D_+ : \Gamma(\xi^+_{\mathrm{F}})\to\Gamma(\xi^-_{\mathrm{F}})$ is identified by the operator $\begin{pmatrix} 0 & 0 \\ D_+ & 0\end{pmatrix} : \Gamma(\xi_{\mathrm{F}})\to\Gamma(\xi_{\mathrm{F}})$ it follows that $D_+^2\equiv0$. Hence, it is not a Dirac type operator in the sense presented here. However, every (anti-) symmetric Dirac type operator D is fully determined by D_+.

Lorentzian and Euclidean signature the following is obtained for all $\mathfrak{z}, \mathfrak{z}' \in \mathcal{E}$:

$$<\mathfrak{z}, \mathfrak{z}'>_{\mathcal{E}} := \begin{cases} \overline{\mathfrak{z}_-}\, \mathfrak{z}'_+ \; + \; \overline{\mathfrak{z}_+}\, \mathfrak{z}'_- & \text{(Lorentzian sign.)}, \\ \overline{\mathfrak{z}_+}\, \mathfrak{z}'_+ \; + \; \overline{\mathfrak{z}_-}\, \mathfrak{z}'_- & \text{(Euclidean sign.)}, \end{cases}$$

where $\bar{\mathfrak{z}}$ means either the Dirac or Hermitian conjugate of the "spinor degrees" of freedom of $\mathfrak{z}$. More precisely, let $\pi : \mathrm{F}\mathcal{E} \to \mathcal{M}$ be the frame bundle of ξ_{F}, such that $\mathcal{E} \ni \mathfrak{z} \simeq [(p, \mathbf{z} = \sum_{i=1}^{2^{\mathrm{k}}} \mathbf{s}_i \otimes \mathbf{z}_i)] \in \mathrm{F}\mathcal{E} \times_{\mathrm{spin(n)} \times \rho_{\mathrm{F}}(\mathrm{G})} \mathbb{C}^{2^{\mathrm{k}}} \otimes \mathbb{C}^{\mathrm{N_F}}$. Then, the notation $\overline{\mathfrak{z}_1}\, \mathfrak{z}_2$ means: $\overline{\mathfrak{z}_1}\, \mathfrak{z}_2 \equiv \overline{\mathbf{z}_1}\, \mathbf{z}_2 := \sum_{i=1}^{2^{\mathrm{k}}} (\overline{\mathbf{s}_{1,i}}\, \mathbf{s}_{2,i})(\mathbf{z}_{1,i}^{\dagger}\, \mathbf{z}_{2,i})$. By the definition of the fermion bundle, this value is clearly independent of the choice of $p \in \mathrm{F}\mathcal{E}$ and thus independent of the representative $\mathbf{z}$ of $\mathfrak{z}$. Hence, in the cases considered, the fermionic Lagrangian (52) reads

$$\mathcal{L}_{\mathrm{F}}(D)(\psi) \; := \; \begin{cases} \left(\overline{\psi_+}\; D_+ \psi_+\right) \mu_{\mathrm{M}} & \text{(Lorentzian sign.)}, \\ \left(\overline{\psi_-}\; D_+ \psi_+\right) \mu_{\mathrm{M}} & \text{(Euclidean sign.)}. \end{cases}$$

The D_+ part of simple type Dirac operators has the form

$$\begin{aligned} D_+ \;&=\; \begin{pmatrix} \partial\!\!\!/_A & \Phi_{\mathrm{LR}} \\ -\Phi_{\mathrm{RL}} & \partial\!\!\!/_A \end{pmatrix} \\ &\equiv\; \partial\!\!\!/_A + \gamma_{\mathrm{M}} \otimes \phi_+ \end{aligned} \tag{54}$$

where, respectively, $\Phi_{\mathrm{LR}} := \gamma_{\mathrm{M}} \otimes \tilde{\phi}_{\mathrm{LR}} \in \Gamma(\mathrm{Hom}(\xi_{\mathrm{F,RR}}, \xi_{\mathrm{F,RL}}))$ and $\Phi_{\mathrm{RL}} := -\gamma_{\mathrm{M}} \otimes \tilde{\phi}_{\mathrm{RL}} \in \Gamma(\mathrm{Hom}(\xi_{\mathrm{F,LL}}, \xi_{\mathrm{F,LR}}))$. The mapping $\tilde{\phi}_{\mathrm{LR}}$ equals ϕ_{LR} restricted to $\Gamma(\mathrm{Hom}_{\mathrm{Cl}}(\xi_{\mathrm{F,LL}}, \xi_{\mathrm{F,LR}}))$ and $\tilde{\phi}_{\mathrm{RL}}$ equals $-\phi_{\mathrm{RL}}$, restricted to the sub-space $\Gamma(\mathrm{Hom}_{\mathrm{Cl}}(\xi_{\mathrm{F,RR}}, \xi_{\mathrm{F,RL}}))$. Since (54) formally looks like a simple type Dirac operator, we also refer to it as a Dirac operator of simple type. For Lorentzian or Euclidean signature the corresponding fermionic Lagrangian reads:

$$\begin{aligned} \mathcal{L}_{\mathrm{F}}(\partial\!\!\!/_A + \gamma_{\mathrm{M}} \otimes \phi)(\psi) \;&=\; \begin{cases} \left(\overline{\psi_+}(\partial\!\!\!/_A + \gamma_{\mathrm{M}} \otimes \phi_+)\psi_+\right) \mu_{\mathrm{M}} & \text{(Lorentzian sign.)}, \\ \left(\overline{\psi_-}(\partial\!\!\!/_A + \gamma_{\mathrm{M}} \otimes \phi_+)\psi_+\right) \mu_{\mathrm{M}} & \text{(Euclidean sign.)}. \end{cases} \\ &=\; \begin{cases} \left(\overline{\psi_{\mathrm{LL}}}\; \partial\!\!\!/_A \psi_{\mathrm{LL}} \;+\; \overline{\psi_{\mathrm{RR}}}\; \partial\!\!\!/_A \psi_{\mathrm{RR}}\right) \mu_{\mathrm{M}} \;+ \\ \left(\overline{\psi_{\mathrm{LL}}}(1 \otimes \tilde{\phi}_{\mathrm{LR}})\psi_{\mathrm{RR}} \;+\; \overline{\psi_{\mathrm{RR}}}(1 \otimes \tilde{\phi}_{\mathrm{RL}})\psi_{\mathrm{LL}}\right) \mu_{\mathrm{M}}, \\ \left(\overline{\psi_{\mathrm{RL}}}\; \partial\!\!\!/_A \psi_{\mathrm{LL}} \;+\; \overline{\psi_{\mathrm{LR}}}\; \partial\!\!\!/_A \psi_{\mathrm{RR}}\right) \mu_{\mathrm{M}} \;+ \\ \left(\overline{\psi_{\mathrm{RL}}}(1 \otimes \tilde{\phi}_{\mathrm{LR}})\psi_{\mathrm{RR}} \;+\; \overline{\psi_{\mathrm{LR}}}(1 \otimes \tilde{\phi}_{\mathrm{RL}})\psi_{\mathrm{LL}}\right) \mu_{\mathrm{M}}. \end{cases} \end{aligned}$$

Note that D is formally self-adjoint if and only if $D_- = D_+^\dagger$. Also note that $\phi^\dagger = -\phi$ if and only if $\phi_+^\dagger = -\phi_+$, which in turn is equivalent to $(1 \otimes \tilde{\phi}_{\rm RL}) = (1 \otimes \tilde{\phi}_{\rm LR})^\dagger$. Here, all mappings are considered to be defined on the total space $\Gamma(\xi_{\rm F})$. In case of D being (anti-) Hermitian we may set, respectively, $(1 \otimes \tilde{\phi}) := (1 \otimes \tilde{\phi}_{\rm LR})$ and $\tilde{\Phi} \equiv \gamma_{\rm M} \otimes \tilde{\phi}$.

Finally, for a Dirac-Yukawa operator one obtains

$$\begin{aligned} D_+ &= \begin{pmatrix} \not{\partial}_{\!A} & \mathcal{G}_{\rm Y}(\varphi) \\ -\mathcal{G}_{\rm Y}(\varphi)^\dagger & \not{\partial}_{\!A} \end{pmatrix} \\ &\equiv \not{\partial}_{\!A} + \gamma_{\rm M} \otimes \tilde{\mathcal{Y}}(\varphi), \end{aligned} \tag{55}$$

with a smooth mapping

$$\begin{aligned} \mathcal{G}_{\rm Y} : \Gamma(\xi_{\rm H}) &\longrightarrow \Gamma({\rm Hom}(\xi_{\rm F,RR}, \xi_{\rm F,RL})) \\ \varphi &\mapsto \gamma_{\rm M} \otimes \tilde{\phi} := \mathcal{G}_{\rm Y}(\varphi) \end{aligned} \tag{56}$$

that is induced by an appropriate Yukawa mapping (4.1) and where $\varphi \in \Gamma(\xi_{\rm H})$ is a section of the Higgs bundle. We may therefore formally refer to the operator (55) also as a Dirac-Yukawa operator.

As an example, we consider the fermionic Lagrangian of a Dirac-Yukawa type operator of Lorentzian signature which spontaneously breaks the gauge symmetry. In the case of $\mathrm{N}_{\rm F,L} := 2, \mathrm{N}_{\rm F,R} := 1$ the fermionic Lagrangian (52) reads

$$\mathcal{L}_{\rm F}(iD)(\psi) \;= <\nu_{\rm L}, i\not{\partial}\nu_{\rm L}>_{\mathcal{E}_\nu} \mu_{\rm M} + \; <e, (i\not{\partial} - m)e>_{\mathcal{E}_e} \mu_{\rm M}, \tag{57}$$

with the suggestively physical notation $\psi_{\rm LL} \equiv (\nu_{\rm L}, e_{\rm L})$ and $\psi_{\rm RR} \equiv e_{\rm R}$ for the "state" of the left-handed and right-handed leptons, respectively. Here, $\nu_{\rm L} \equiv \nu_{\rm LL} \oplus \nu_{\rm RL}$ and $e \equiv e_{\rm L} \oplus e_{\rm R}$ are considered as eigen sections of the total fermionic mass matrix which correspond to the eigenvalues zero and $m \in \mathbb{R}_+^\times$. Physically, one may interpret the corresponding (isomorphism class of) eigenbundles $\xi_{\rm F}^\nu$ and $\xi_{\rm F}^e$ (with $\xi_{\rm F} \simeq \xi_{\rm F}^\nu \oplus \xi_{\rm F}^e$) as "asymptotically free particles".

Remark:

To "lowest order" (c.f. our discussion in the next section) the energy-momentum current $\mathcal{L}_{\rm tot}^* \vartheta_{\rm M} \in \Gamma({\rm End}(\tau_{\rm M}))$ *of the "total Lagrangian"*

$$\mathcal{L}_{\rm tot}(i\not{\partial} - \mathrm{M}_{\rm F})(\psi) \equiv \mathcal{L}_{\rm F}(i\not{\partial} - \mathrm{M}_{\rm F})(\psi) + \mathcal{L}_{\rm D}(i\not{\partial} - \mathrm{M}_{\rm F}) \tag{58}$$

reads

$$\mathcal{L}^*_{\rm tot}\vartheta_{\rm M} \ \sim_{\epsilon\to 0}\ \lambda_{\rm gr}\,{\rm tr}{\rm M}^2_{\rm F}\,{\rm id}_{\rm TM} + \mathcal{O}(\epsilon). \tag{59}$$

This holds true for every gauge theory that is based on a Dirac-Yukawa type operator.

In this section we introduced the Higgs bundle as a specific Hermitian sub-vector bundle of a chiral fermion bundle and discussed a specific sub-class of simple type Dirac operators, called Dirac-Yukawa operators. We also introduced the fermionic Lagrangian within our geometrical setup. In particular, in the case of the Lorentzian signature the definition of the fermionic Lagrangian simply looks like the restriction to the physical sub-bundle $\xi_{\rm phy}$ of the fermion bundle. However, this is not the case. In order to obtain the "correct" fermionic couplings one also needs $\xi^-_{\rm F} \subset \xi_{\rm F}$. Indeed this doubling of the fermionic degrees of freedom is necessary in order to consider a Dirac type operator as an endomorphism on the vector space of sections of a fermion bundle. It is only in this case that one can make use of the general Lichnerowicz decomposition of (the square of) a Dirac type operator which in turn permits to consider the universal Lagrangian (20) as a canonical mapping between the affine set of all Dirac type operators on a fermion bundle and the top forms of the underlying base manifold $\mathcal{M}$.

In the next section we will consider a natural generalization of Dirac-Yukawa type operators which encodes the dynamics of the sections of the Higgs bundle $\xi_{\rm H}$ and the "Yang-Mills bundle" $\xi_{\rm YM}$. It also yields the appropriate mass matrices in such a way that spontaneous symmetry breaking induced by a minimum of the Higgs potential is in accordance with spontaneous symmetry breaking induced by the Yukawa coupling and gravity.

5 The Lagrangian of the Standard Model as the "Square" of Pauli-Dirac-Yukawa Type Operators

From our discussion of the preceding section it follows that the total Lagrangian of a simple type Dirac operator to lowest order only yields the "free field" equations of the eigen sections of the fermionic mass matrix[6]. Moreover, space-time should be an Einstein manifold that is physically determined by the (sum of the) fermionic masses.

[6]This is because the energy momentum current is at least homogeneous of degree two with respect to the appropriate sections.

As a consequence, one has to appropriately generalize simple type Dirac operators in order to obtain non-trivial Euler-Lagrange equations also for the Yang-Mills gauge fields and the sections of the Higgs bundle. Of course, such a generalization of a simple type Dirac operator on a fermion bundle must be done in such a way that it is consistent with spontaneous symmetry breaking induced by the Yukawa coupling and gravity. For this we first introduce a new class of Dirac type operators which we call "Dirac operators of Pauli type" (PD). These operators act on sections of a specific sub-bundle of the doubled fermion bundle, where the latter is defined by the data of a simple type Dirac operator that underlies the corresponding PD. The doubling of the fermion bundle has the physical meaning to simultaneously deal with "particles and anti-particles". The above mentioned sub-bundle turns out to be equivalent to the fermion bundle one starts with and the corresponding fermionic Lagrangian reduces to the one which is defined only by the underlying Dirac operator of simple type. To make this precise, we have to consider real fermion bundles.

5.1 Real Fermion Bundles and Operators of Pauli Type

Let $\zeta_{2\mathrm{F}}$ be a real vector bundle of rank $2\,\mathrm{N}$ and total space $\mathcal{W}_{2\mathrm{F}}$. Also let $\mathcal{I}_{2\mathrm{F}} \in \mathrm{End}_{\mathbb{R}}(\zeta_{2\mathrm{F}})$ be a complex structure. We denote by ξ_{F} the complex vector bundle of rank N which is defined by the $\mathbb{C}-$action: $z\mathfrak{z} := x\mathfrak{z} + y\mathcal{I}_{2\mathrm{F}}(\mathfrak{z})$, for all $z \equiv x + iy \in \mathbb{C}$ and $\mathfrak{z} \in \mathcal{W}_{2\mathrm{F}}$. The corresponding total space is denoted again by $\mathcal{E}$. Also, let $\xi_{2\mathrm{F}} := \mathbb{C} \otimes \zeta_{2\mathrm{F}}$ with total space $\mathcal{E}_{2\mathrm{F}} := \mathbb{C} \otimes \mathcal{W}_{2\mathrm{F}}$. The complex vector bundle $\xi_{2\mathrm{F}}$ of rank $2\,\mathrm{N}$ is naturally $\mathbb{Z}_2-$graded since

$$\xi_{2\mathrm{F}} \simeq \xi_{\mathrm{F}} \oplus \overline{\xi_{\mathrm{F}}}. \tag{60}$$

Here, $\overline{\xi_{\mathrm{F}}}$ is the conjugate complex vector bundle of ξ_{F}. The elements of its total space $\overline{\mathcal{E}}$ are denoted by $\overline{\mathfrak{z}}$. They may be identified either with elements $\mathfrak{z} \in \mathcal{W}_{2\mathrm{F}}$, such that $z\mathfrak{z} := x\mathfrak{z} - y\mathcal{I}_{2\mathrm{F}}(\mathfrak{z})$, or considered as anti-linear functionals on $\mathcal{E}^*$ (dual of $\mathcal{E}$). Of course, the subspaces of the decomposition (60) are but the eigen spaces of $\mathcal{I}_{2\mathrm{F}}$ (considered as a complex linear mapping) with respect to the eigenvalues $\pm i$.

The canonical real structure on $\xi_{2\mathrm{F}}$ is denoted by $\mathcal{J}_{2\mathrm{F}}$. It is given by the action $\mathcal{J}_{2\mathrm{F}}(\mathfrak{z}_1, \overline{\mathfrak{z}_2}) := (\mathfrak{z}_2, \overline{\mathfrak{z}_1})$. The corresponding real sub-space

$$\{(\mathfrak{z}, \overline{\mathfrak{z}}) \in \mathcal{E}_{2\mathrm{F}} \,|\, \mathfrak{z} \in \mathcal{E}\} \simeq \mathcal{W}_{2\mathrm{F}} \tag{61}$$

can be identified with $\mathcal{E}$ via the canonical complex structure: $i(\mathfrak{z}, \overline{\mathfrak{z}}) := (i\mathfrak{z}, -i\overline{\mathfrak{z}})$. Note that, likewise, $\mathcal{E}_{2\mathrm{F}}$ may be viewed as the complex space $\mathcal{W}_{4\mathrm{F}} \equiv \mathcal{W}_{2\mathrm{F}} \oplus \mathcal{W}_{2\mathrm{F}}$ with the complex structure given by the action $\mathcal{I}_{4\mathrm{F}}(\mathfrak{w}_1,\mathfrak{w}_2) := (-\mathfrak{w}_2,\mathfrak{w}_1)$. Clearly, this complex structure in turn can be identified with $\mathcal{I}_{2\mathrm{F}}$ under the identification of $\mathcal{W}_{2\mathrm{F}}$ with $\mathcal{E}$.

In what follows, it is assumed that the complex vector bundle ξ_{F} is a fermion bundle with respect to $(\mathrm{G},\rho_{\mathrm{F}},D)$. Both, the signature $s \in \mathbb{Z}$ of D and the dimension $n = 2k \in \mathbb{N}$ of the orientable base manifold $\mathcal{M}$ are again arbitrary, although we are mainly interested in the physically distinguished case of $(n,s) = (4,\mp 2)$. Likewise, the complex vector bundle $\overline{\xi_{\mathrm{F}}}$ is treated as the conjugate complex ("charge conjugate") fermion bundle with respect to $(\mathrm{G},\overline{\rho}_{\mathrm{F}},\bar{D})$. Here, $\overline{\rho}_{\mathrm{F}}$ is the conjugate representation of G and the ("charge conjugate") Dirac type operator $\bar{D}$ is defined by $\bar{D}\overline{\psi} := \overline{D\psi}$, for all $\overline{\psi} \in \Gamma(\overline{\xi_{\mathrm{F}}})$. If $<\cdot,\cdot>_{\mathcal{E}}$ denotes again the Hermitian product[7] on $\mathcal{E}$, then $<\bar{\mathfrak{z}}_1,\bar{\mathfrak{z}}_2>_{\overline{\mathcal{E}}} := <\mathfrak{z}_2,\mathfrak{z}_1>_{\mathcal{E}}$. Hence, the sum $<\psi, D\psi>_{\mathcal{E}} + <\overline{\psi},\bar{D}\overline{\psi}>_{\overline{\mathcal{E}}}$ vanishes if D is anti-symmetric.

Although they are anti-isomorphic to each other, there is no natural way to identify the fermion bundle ξ_{F} with its charge conjugate $\overline{\xi_{\mathrm{F}}}$. In order to do so we still have to give additional input. For this let $\mathcal{J}$ be a real structure on ξ_{F} such that

$$\begin{array}{rcl} \mathcal{C}:\mathcal{E} & \longrightarrow & \overline{\mathcal{E}} \\ \mathfrak{z} & \mapsto & \overline{\mathcal{J}(\mathfrak{z})} \end{array} \tag{62}$$

defines a linear bundle isomorphism over the identity on $\mathcal{M}$, usually referred to as "charge conjugation" (see, for instance, in [BT '88] in the context of Clifford algebras and in [Con '95] in the context of non-commutative geometry). Notice that $\mathcal{C}^{-1}(\bar{\mathfrak{z}}) = \mathcal{J}(\mathfrak{z})$. Then, the charge conjugate Dirac operator may be written as

$$\bar{D} = \mathcal{C}_{\mathrm{J}} \circ D \circ \mathcal{C}_{\mathrm{J}}^{-1}, \tag{63}$$

where $\mathcal{C}_{\mathrm{J}}(\mathfrak{z}) := \mathcal{C}(\mathcal{J}(\mathfrak{z})) = \bar{\mathfrak{z}}$.

The existence of $\mathcal{J}$ depends on the topology of ξ_{F}. Indeed, it can be shown that a complex vector bundle possesses a real structure if and only if all of its odd Chern classes vanish (see, for instance, in [GS '99]).

Definition 5.1 *Let ξ_{F} be a real fermion bundle over $\mathcal{M}$ with respect to the Dirac triple $(\mathrm{G},\rho_{\mathrm{F}},D)$. Also, let $F_D \in \Omega^2(\mathcal{M},\mathrm{End}^+_{\mathrm{Cl}}(\mathcal{E}))$ be the twisting curvature of ∂_D. We call*

[7] The Hermitian product on $\mathcal{E}$ is assumed to be anti-linear in the first, and linear in the second argument. Also, the "bar" notation, as for instance $\bar{\mathfrak{z}}$, should not be confounded with the Dirac conjugation in the case of the Lorentz signature.

the associated first order differential operator

$$D_{\mathrm{P}} := \begin{pmatrix} D + i\gamma(F_D) & 0 \\ 0 & \mathcal{C}_{\mathrm{J}}^{-1} \circ \overline{(D - i\gamma(F_D))} \circ \mathcal{C}_{\mathrm{J}} \end{pmatrix} : \begin{array}{c} \Gamma(\xi_{\mathrm{F}}) \\ \oplus \\ \Gamma(\xi_{\mathrm{F}}) \end{array} \longrightarrow \begin{array}{c} \Gamma(\xi_{\mathrm{F}}) \\ \oplus \\ \Gamma(\xi_{\mathrm{F}}) \end{array} \tag{64}$$

a Dirac operator of "Pauli type" (or "Pauli-Dirac operator") with respect to the grading involution $\Gamma_{2\mathrm{F}}$ that is defined by the action $\Gamma_{2\mathrm{F}}(\mathfrak{z}_1, \mathfrak{z}_2) := (\Gamma(\mathfrak{z}_2), \Gamma(\mathfrak{z}_1))$ and the real structure $\mathcal{J}$.

Equivalently, one may also express a Pauli-Dirac operator with respect to the diagonal representation of the grading involution $\Gamma_{2\mathrm{F}}$ (i.e., where $\Gamma_{2\mathrm{F}} = \mathrm{diag}(\Gamma, -\Gamma)$), in which case

$$\begin{aligned} D_{\mathrm{P}} &= \begin{pmatrix} D & -\gamma(F_D) \\ \gamma(F_D) & D \end{pmatrix} \\ &\equiv D + \mathcal{I} \otimes \gamma(F_D). \end{aligned} \tag{65}$$

The bundle mapping $\mathcal{I} \in \mathrm{End}_{\mathbb{C}}(\mathcal{E} \oplus \mathcal{E})$, which is defined by $\mathcal{I}(\mathfrak{z}_1, \mathfrak{z}_2) := (-\mathfrak{z}_2, \mathfrak{z}_1)$, corresponds to the complex structure $\mathcal{I}_{4\mathrm{F}}$ with help of the identification of $\mathcal{W}_{2\mathrm{F}} \subset \mathcal{E} \oplus \overline{\mathcal{E}}$ with $\mathcal{E}$.

If $D \equiv \not{\partial}_{\mathrm{A}}$, then the zero order term $D_{\mathrm{P}} - \not{\partial}_{\mathrm{A}}$ formally looks like the well-known "Pauli-term" $i\gamma(F_{\mathrm{A}})$ which has been introduced by physicists in order to correctly describe the anomalous magnetic moment of the proton. However, the first order operator $\not{\partial}_{\mathrm{A}} + i\gamma(F_{\mathrm{A}})$ is not a Dirac type operator in our sense for the Pauli term is an even operator. To remedy this flaw we again have to "double the fermionic degrees of freedom", in this case, however, by adding the corresponding "anti-fermions". As a consequence, for diagonal sections, which one may physically interpret as representing the state of a "particle-anti-particle"[8], $\Psi \equiv (\psi, \psi) \in \Gamma(\xi_{\mathrm{F}}) \oplus \Gamma(\xi_{\mathrm{F}}) = \Gamma(\xi_{\mathrm{F}} \oplus \xi_{\mathrm{F}}) \simeq \Gamma(\xi_{2\mathrm{F}})$, we obtain the identity

$$< \Psi, D_{\mathrm{P}}\Psi >_{\mathcal{E}_{2\mathrm{F}}} = 2 < \psi, D\psi >_{\mathcal{E}} . \tag{66}$$

Hence, the Pauli term does not contribute to the fermionic Lagrangian as far as "particle-anti-particle states" are simultaneously taken into account. This is certainly desirable for it is well-known that the coupling of the fermions to the curvature actually spoils the theory of their renormalizability. Hence, to lift the first order differential

[8]With help of the identification $\overline{\xi_{\mathrm{F}}} \simeq \xi_{\mathrm{F}}$

operator $\not{\partial}_{\mathrm{A}} + i\gamma(F_{\mathrm{A}})$ to a "true" Dirac type operator restores a basic feature of (perturbative) quantum field theory. Again, by formal similarity we also refer to the operator $\not{\partial}_{\mathrm{A}} + i\gamma(F_{\mathrm{A}})$ itself as a Dirac operator of Pauli type, analogous to the operator (55) is formally referred to as Dirac operator of Yukawa type.

Let ξ_{F} be the real chiral fermion bundle with respect to $(\mathrm{G}, \rho_{\mathrm{F}}, D)$, with D being of simply type.

Proposition 5.1 *The top form of D_{P} decomposes into the sum*

$$\mathcal{L}_{\mathrm{D}}(D_{\mathrm{P}}) = \mathcal{L}_{EH} \pm \mathcal{L}_{YM} \pm \mathcal{L}_{H} \tag{67}$$

where, respectively, $\mathcal{L}_{EH}$ is the Einstein-Hilbert Lagrangian, $\mathcal{L}_{YM}$ the Yang-Mills Lagrangian and $\mathcal{L}_{H}$ the "Higgs" Lagrangian of the Standard Model of Particle Physics.

Proof: The proof is basically a copy of the proof of the corresponding statement that has been presented already in [Tol '98] in the case of $s = n$ (c.f. Theorem 1). We note that the top form $\mathcal{L}_{\mathrm{D}}(D') \in \Omega^{\mathrm{n}}(\mathcal{M})$ is independent of the connection representing $D' \in \mathcal{D}(\xi_{\mathrm{F}})$. Hence, one may choose any representative of the connection class that corresponds to D_{P} to define the Pauli term $i\gamma(F_A)$. The relative signs of (67) depend on the signature of D and of the definition of the Clifford multiplication. In particular, the relative sign in front of the kinetic term $< \partial_A^W \phi, \partial_A^W \phi >$ of the Higgs Lagrangian depends on whether τ_{Cl} or $\tau_{\mathrm{Cl}}^{\mathrm{op}}$ is considered to act on ξ_{F}. Finally, we stress that the decomposition (67) is actually independent of the existence of a real structure on ξ_{F}. In particular, it does not depend on the choice of $\mathcal{J}$. $\square$

The top form (67) clearly reduces to the combined Einstein-Hilbert-Yang-Mills Lagrangian in the case where ξ_{F} is not chiral. However, if D denotes a Dirac-Yukawa type operator, then

$$\begin{aligned} \mathcal{L}_{\mathrm{tot}}(D_{\mathrm{P}})(\Psi) &\equiv \mathcal{L}_{\mathrm{F}}(D_{\mathrm{P}})(\Psi) + \mathcal{L}_{\mathrm{D}}(D_{\mathrm{P}}) \\ &= \mathcal{L}_{\mathrm{F}}(D)(\psi) + \mathcal{L}_{\mathrm{D}}(D_{\mathrm{P}}) \end{aligned} \tag{68}$$

equals the total Lagrangian of the Standard Model, including Einstein's theory of gravity. Here, we used the homogeneity property of the fermionic density: $\mathcal{L}_{\mathrm{F}}(D_{\mathrm{P}})(\lambda\Psi) = \lambda^2 \mathcal{L}_{\mathrm{F}}(D_{\mathrm{P}})(\Psi)$ and put $\Psi \equiv (\psi, \psi)/\sqrt{2}$. Note that the corresponding Euler-Lagrange equations form a dynamically closed system. For this reason, we refer to D_{P} also as a Dirac operator of "Pauli-Yukawa" type (or "Pauli-Dirac-Yukawa" operator, PDY) if the operator (64) is defined in terms of a Dirac-Yukawa type operator (47). Therefore,

$$(\xi_{\rm F}, D_{\rm P}) \tag{69}$$

may be regarded as a "square root" of (the Lagrangian of) the Standard Model[9].

5.2 "Fluctuation" of a Fermionic Vacuum and the YM-Mass Matrix

Before we proceed let us come back to the notion of a "(semi-classical) fermionic vacuum" and how this is related to the reality of a fermion bundle. Essentially, a chiral fermion bundle $\xi_{\rm F} = \xi_{\rm F}^+ \oplus \xi_{\rm F}^-$ is related to a Dirac triple $({\rm G}, \rho_{\rm F}, D)$, with D being of simple type. The existence of a fermionic vacuum crucially depends on the existence of a non-vanishing section $\phi \in \Gamma({\rm End}^-_{\rm Cl}(\xi_{\rm F}))$ and a purely topological Clifford connection $A \in \mathcal{A}_{\rm Cl}(\xi_{\rm F})$. This in fact reduces the above Dirac triple to $({\rm H}, \rho_{\rm F,red}, \partial\!\!\!/)$ and $\xi_{\rm F}$ may be regarded, accordingly, as a perturbation of the corresponding $\xi_{\rm F,red}$. Clearly, such a reduction gives sever topological restrictions on a fermion bundle. Of course, this holds true also for the existence of a Dirac-Yukawa type operator. For example, in the case of the electroweak interaction a fermionic vacuum exists if and only if the corresponding Yang-Mills gauge bundle of the electroweak interaction is trivial. This in turn holds true if and only if the (charged) electroweak vector bosons are charge conjugate to each other (c.f. [Tol '05]). In the (algebraic) torsion free case this is equivalent to the existence of a flat Yang-Mills connection. This example may motivate the following

Definition 5.2 *A fermion bundle $\xi_{\rm F}$ is called "perturbative" provided there is a Dirac type operator $D \in \mathcal{D}(\xi_{\rm F})$ such that $\mathcal{F}_D = R\!\!\!/$.*

A fermionic vacuum is thus geometrically described by a perturbative massive fermion bundle. Next, we introduce a specific sub-vector bundle of $\xi_{\rm F}^* \otimes_{\mathcal{M}} \xi_{\rm F}$ and discuss the "bosonic mass matrix" within the presented fermionic frame.

[9]Of course, the data $(\xi_{\rm F}, D_{\rm P})$ covers the geometrical properties of the Standard Model only up to the "semi-classical approximation" of the latter. It also seems worth noting that because the decomposition (67) is independent of the existence of the reality of the fermion bundle, it is possible to also take into account magnetic monopoles within the Standard Model as topologically non-trivial ground states of the Higgs boson. Moreover, it is well-known that the weak interaction actually spoils the symmetry under charge conjugation.

Definition 5.3 *Let again* $\xi_{\rm F}$ *be a massive fermion bundle with respect to a Dirac-Yukawa model* $({\rm G}, \rho_{\rm F}, D)$*. The real sub-bundle*

$$\xi_{\rm YM} := \tau^*_{\rm M} \otimes_{\mathcal{M}} {\rm End}^+_{\rm Cl}(\xi_{\rm F}) \subset {\rm End}(\xi_{\rm F}) \tag{70}$$

is called the "Yang-Mills bundle" with respect to the appropriate fermionic vacuum $\xi_{\rm F,red}$*.*

With respect to a fermionic vacuum the (real form of the) Higgs bundle decomposes into the Whitney sum of two real vector bundles

$$\xi_{\rm H} \simeq \xi_{\rm G} \oplus \xi_{\rm H,phys} \tag{71}$$

with $\xi_{\rm G} \subset \xi_{\rm H} \subset \xi_{\rm F}$ being the "Goldstone bundle" and $\xi_{\rm H,phys} \subset \xi_{\rm F}$ being the "physical Higgs bundle" (c.f. Lemma 3.1 in [Tol '03(a)] for Yang-Mills-Higgs gauge theories). Therefore, any Dirac-Yukawa type operator on a massive fermion bundle $\xi_{\rm F}$ is parameterized by $(A, \varphi_{\rm H}) \in \Gamma(\xi_{\rm YM} \times_{\mathcal{M}} \xi_{\rm H,phys})$. In particular, for $t \in [0,1]$ one may consider the one-parameter family $(A_t, \varphi_t) \in \mathcal{A}(\xi_{\rm H}) \times \Gamma(\xi_{\rm H})$ which is defined by $\partial_{A,t} := \partial + tA$, $\varphi_t := \mathcal{V} + t\varphi_{\rm H}$. Hence, the "Yang-Mills-Higgs pair" $(A, \varphi_{\rm H}) \in \Gamma(\xi_{\rm YM} \times_{\mathcal{M}} \xi_{\rm H,phys})$ may be physically regarded as a "fluctuation" of the corresponding fermionic vacuum $\xi_{\rm F,red}$.

Like in Yang-Mills-Higgs gauge theories a fluctuation $(A, \varphi_{\rm H})$ of a fermionic vacuum yields a self-adjoint section ${\rm M_H} \in \Gamma({\rm End}(\xi_{\rm H})) \subset \Gamma({\rm End}_{\rm Cl}(\xi_{\rm F}))$ such that the rank of the Goldstone bundle equals the dimension of the kernel of the "Higgs mass operator" ${\rm M_H}$. Moreover, $\xi_{\rm H,phys}$ decompose into the Whitney sum of eigenbundles of the Higgs mass matrix. Likewise, since in general $A \in \Gamma(\xi_{\rm YM})$ gives rise to a connection on $\xi_{\rm F}$ that is not compatible with the fermionic vacuum (i.e. the corresponding covariant derivative does not commute with the total fermionic mass operator), a fluctuation of the fermionic vacuum also yields a non-trivial "Yang-Mills mass operator" ${\rm M_{YM}} \in \Gamma({\rm End}(\xi_{\rm YM}))$. (see [Tol '03(a)]). As a consequence, the Yang-Mills bundle decomposes into the eigenbundles of ${\rm M_{YM}}$ for again ${\rm spec}({\rm M_{YM}})$ is constant throughout $\mathcal{M}$. In particular, one obtains the decomposition (see, again, [Tol '03(a)])

$$\xi_{\rm YM} \simeq \tau^*_{\rm M} \otimes_{\mathcal{M}} (\mathfrak{ad}(\mathcal{Q}) \oplus \xi_{\rm G}) \tag{72}$$

with $\mathfrak{ad}(\mathcal{Q}) \equiv {\rm Lie}(\mathcal{H}_{\rm YM})$ being the "adjoint bundle" of the reduced frame bundle $\mathcal{Q} \overset{\iota}{\hookrightarrow} \mathcal{P}$, which is associated with the fermionic vacuum $\xi_{\rm F,red}$. Since ${\rm rk}({\rm M_{YM}}) = {\rm rk}(\xi_{\rm G})$ the equivalence (72) is a geometrical variant of the famous "Higgs-Dinner". It follows that

$A \in \Gamma(\xi_{\rm YM})$ decomposes into $A = A_{\rm YM} + A_{\rm G}$. Hence, the deviation from A being compatible with the fermionic vacuum can be expressed by

$$\partial_A^{\mathrm{End}(\mathcal{E})}\mathrm{M}_{\rm F} = \mathrm{ad}(A_{\rm G})\mathrm{M}_{\rm F}. \tag{73}$$

As already mentioned, the non-vanishing of the right hand side (i.e. of $A_{\rm G} \in \tau^*_{\rm M} \otimes_{\mathcal{M}} \xi_{\rm G}$) yields a non-trivial Yang-Mills mass operator $\mathrm{M}_{\rm YM}$. In fact, one has

$$\mathrm{M}_{\rm YM}(A) = \mathrm{ad}(\mathrm{M}_{\rm F})(A) \tag{74}$$

with $\|\mathrm{M}_{\rm YM}(A)\|^2 = \mathrm{M}^2_{\rm YM}(A, A)$ and the symmetric bilinear form

$$\begin{array}{rccl} \mathrm{M}^2_{\rm YM} : & \Gamma(\xi_{\rm YM} \times_{\mathcal{M}} \xi_{\rm YM}) & \longrightarrow & \mathcal{C}^\infty(\mathcal{M}) \\ & (A, A') & \mapsto & \frac{1}{2}\mathrm{M}^2_{\rm YM}(T_a, T_b)\, g_{\rm M}(A^a, A'^b). \end{array} \tag{75}$$

Here, respectively, $A \stackrel{\rm loc}{=} A^a \otimes T_a$, $A' \stackrel{\rm loc}{=} A'^a \otimes T_a$ and

$$\mathrm{M}^2_{\rm YM}(T_a, T_b)|_x := 2\|\mathcal{G}_{\rm Y}\|^2 < \mathcal{V}(x), [T_a, T_b]_+ \mathcal{V}(x) >_{\mathcal{E}} \tag{76}$$

is the (squared) "Yang-Mills mass matrix", with $[\cdot,\cdot]_+$ being the anti-commutator. Note that we used $\xi_{\rm H} \subset \xi_{\rm F}$, such that a vacuum section $\mathcal{V}$ can also be regarded as a section of the fermion bundle. We also extensively used the properties of the Yukawa mapping (4.1). In particular, we used that $\mathrm{ad}(\mathcal{D})A = \mathcal{Y}(A\mathcal{V})$ where, by abuse of notation, A refers to two different representations. Also note that the eigenvalues of (76) are actually independent of $x \in \mathcal{M}$. Of course, the rank of (76) equals the rank of the Goldstone bundle $\xi_{\rm G} \subset \xi_{\rm F}$. Accordingly, one may re-write (73) for a Clifford connection to be non-compatible with the fermionic vacuum as

$$\|\partial_A^{\mathrm{End}(\mathcal{E})}\mathrm{M}_{\rm F}\|^2 = 2^n \mathrm{M}^2_{\rm YM}(A, A). \tag{77}$$

That is, the fermionic mass matrix is covariantly constant with respect to a Clifford connection on a massive fermion bundle iff this Clifford connection is in the kernel of the Yang-Mills mass matrix. The latter, of course, is in one-to-one correspondence with the residual gauge fields.

Let $D \in \mathcal{D}(\xi_{\rm F})$ be a Dirac operator of simple type such that $D - \partial\!\!\!/_A \neq 0$ and $\mathcal{G}_{\rm YM}$ acts transitively on the image of $D - \partial\!\!\!/_A$. Then, there is a non-vanishing smooth function $\chi \in \mathcal{C}^\infty(\mathcal{M})$ such that $D = \partial\!\!\!/_A + i\chi\,\mathrm{M}_{\rm F}$. Let $\xi_{\rm F,red} \simeq \xi_{\rm F}$ be a fermionic vacuum with respect to $(\mathrm{H}, \rho_{\rm F,red}, \partial\!\!\!/_{\mathcal{D}})$. Then, D defines a fluctuation of $\xi_{\rm F,red}$ iff

$$D = \partial\!\!\!/_A + \chi(\partial\!\!\!/_{\mathcal{D}} - \partial\!\!\!/). \tag{78}$$

Note that this condition is in full accordance with the usual definition of the Higgs boson to be in the "unitary gauge". Here, however, this condition is expressed purely in terms of fermions.

Proposition 5.2 *Let $\xi_{\mathrm{F,red}} \simeq \xi_{\mathrm{F}}$ be a fermionic vacuum with respect to a Dirac-Yukawa model $(\mathrm{H}, \rho_{\mathrm{F,red}}, \not{\partial}_{\mathcal{V}})$. Also, let $(A, \varphi_{\mathrm{H}}) \in \Gamma(\xi_{\mathrm{YM}} \times_{\mathcal{M}} \xi_{\mathrm{H,phys}})$ be a fluctuation of the fermionic vacuum. Then, the total curvature on ξ_{F} of the connection determined by the Dirac-Yukawa operator*

$$\begin{aligned} D &= \not{\partial}_A + \gamma_{\mathrm{M}} \otimes \phi \\ &= \not{\partial} + \gamma_{\mathrm{M}} \otimes \mathcal{Y}(\mathcal{V}) + \gamma(A) + \gamma_{\mathrm{M}} \otimes \mathcal{Y}(\varphi_{\mathrm{H}}) \\ &\equiv \not{\partial}_D + \gamma(A_{\mathrm{fl}}) \end{aligned} \tag{79}$$

reads

$$\begin{aligned} \mathcal{F}_D &= \not{R} + F_A + F_{\mathrm{H}} + F_{\mathrm{mass}} \\ &= \not{R} + F_{\mathrm{YM}} + F_{\mathrm{G}} + F_{\mathrm{H}} + F_{\mathrm{mass}}. \end{aligned} \tag{80}$$

Here, respectively,

$$\begin{aligned} F_{\mathrm{YM}} &:= \partial A_{\mathrm{YM}} + A_{\mathrm{YM}} \wedge A_{\mathrm{YM}}, \\ F_{\mathrm{G}} &:= \partial A_{\mathrm{G}} + A_{\mathrm{G}} \wedge A_{\mathrm{G}}, \\ F_{\mathrm{H}} &:= \partial A_{\mathrm{H}} + A_{\mathrm{H}} \wedge A_{\mathrm{H}} \end{aligned} \tag{81}$$

are the Yang-Mills curvature with respect to the reduced Yang-Mills gauge group $\mathcal{H}_{\mathrm{YM}} \subset \mathcal{G}_{\mathrm{YM}} \subset \mathcal{G}_{\mathrm{F}}$, the curvature on ξ_{F} of the (massive) vector boson that corresponds to the Goldstone boson and the curvature induced by the (physical part of the) Higgs boson according to the decomposition

$$\begin{aligned} A_{\mathrm{fl}} &= A + A_{\mathrm{H}} \\ &= A_{\mathrm{YM}} + A_{\mathrm{G}} + A_{\mathrm{H}}, \end{aligned} \tag{82}$$

with $A_{\mathrm{H}} := \mathrm{ext}_{\Theta}(\gamma_{\mathrm{M}} \otimes \mathcal{Y}(\varphi_{\mathrm{H}}))$.

Finally, the "mass-curvature" $F_{\mathrm{mass}} \in \Omega^2(\mathcal{M}, \mathrm{End}(\mathcal{E}))$ is given by

$$\begin{aligned} F_{\mathrm{mass}} &:= (1 - 2\|\varphi_{\mathrm{H}}\|)\, \mathrm{M}_{\mathrm{F}}^2 \Theta \wedge \Theta + (1 + \|\varphi_{\mathrm{H}}\|)\, \mathrm{M}_{\mathrm{YM}}(A_{\mathrm{G}}) \wedge \Theta \qquad (83) \\ &= \mathrm{ext}_{\Theta}[(1 - 2\|\varphi_{\mathrm{H}}\|)\, \mu_{\mathrm{F}} + (1 + \|\varphi_{\mathrm{H}}\|)\, \mu_{\mathrm{YM}}]. \end{aligned}$$

We call, respectively, $\mu_{\mathrm{F}} := \mathrm{ext}_{\Theta}(\mathrm{M}_{\mathrm{F}}^2) \in \Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{E}))$ and $\mu_{\mathrm{YM}} := \mathrm{M}_{\mathrm{YM}}(A_{\mathrm{G}}) \equiv \gamma_{\mathrm{M}} \otimes \mathrm{M}_{\mathrm{YM}}(A) \in \Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{E}))$ the "fermionic mass form" and the "Yang-Mills mass form".

Proof: First, note that the Yang-Mills mass form $\mu_{\rm YM}$ contributes to the total curvature even if $F_{\rm A} = \varphi_{\rm H} = 0$ is supposed to hold true. Hence, it also gives rise to a "fluctuation" of $g_{\rm M}$. In contrast to what one may infer from $F_{\rm mass}$, however, the contribution of the bosonic mass is of "higher order" in comparison to the curvature that is induced by the fermionic mass. In other words, $F_{\rm mass} = {\rm ext}_\Theta(\mu_{\rm F}) + \mathcal{O}(t)$ in accordance with (41). We stress that (80) indeed reduces to (41) if $\varphi_{\rm H} = 0$. Hence, it gives a physical interpretation, in particular, of the last term of the decomposition (41) of the curvature of a simple type Dirac operator which spontaneously breaks the gauge symmetry. One may express this also in more physical terms by saying that it is the interaction of the gauge field with the fermionic vacuum that yields massive vector bosons.

To prove the decomposition (80) one uses the decomposition (71) and the Higgs Dinner (72), as well as $[{\rm M}_{\rm F}^2, \Theta] = [{\rm M}_{\rm YM}, \Theta] = 0$. Moreover, due to our remark above concerning fluctuations one may take into account that $\varphi_{\rm H} = \|\varphi_{\rm H}\|\mathcal{V}$ (where $\|\mathcal{V}\| = 1$ is assumed without loss of generality). Note also that both the (lifted) soldering form Θ and the Yukawa-mapping $\mathcal{Y}$ are covariantly constant with respect to any Clifford connection. Finally, taking also into account that ∂ acts on $A_{\rm fl}$ like the usual exterior derivative, the prove actually becomes a straightforward calculation. $\square$

We emphasize that spontaneous symmetry breaking induced by a fermionic vacuum is compatible with spontaneous symmetry breaking induced by the Higgs potential arising from a fluctuation of the fermionic vacuum (i.e. $\not{\partial}_{\mathcal{D}} \mapsto D_{\rm P}$). Clearly, G acts transitively on ${\rm im}(\phi) \subset {\rm Orbit}(\mathfrak{z}_0) \simeq {\rm P}\times_{\rm G}{\rm G/H}$ for any chosen minimum $\mathfrak{z}_0 \in {\rm End}(\mathbb{C}^{\rm N_H})$ of the Higgs potential induced by $D_{\rm P}$. Therefore, the condition $\phi \in \Gamma({\rm End}(\xi_{\rm H}))\backslash\{\mathcal{O}\}$ is necessary and sufficient for the unitary gauge to exist. In particular, if $\xi_{\rm F}$ is defined with respect to a Dirac-Yukawa model, then for each $\varphi \in \Gamma(\xi_{\rm H})\backslash\{\mathcal{O}\}$ there exists a "vacuum section" $\mathcal{V}_\varphi \in \Gamma(\mathcal{O}rbit(\mathbf{z}_0)) \subset \Gamma(\xi_{\rm H})$ such that $\varphi \in \Gamma(\xi_{\rm H,phys})$. This holds true for any rotationally symmetric Higgs potential (like the Higgs potential generated by a Pauli-Dirac type operator). By the very definition of the Yukawa mapping the structure group G then acts transitively also on ${\rm im}(\mathcal{Y}(\varphi)/\|\mathcal{Y}(\varphi)\|) \subset \mathcal{S} \subset {\rm End}_{\rm Cl}(\xi_{\rm F})$.

6 Outlook

We discussed a certain class of gauge theories with the basic property of having a "square root" in the sense of the data of Dirac type operators. These Dirac type gauge theories have in common that they are derived by a universal Lagrangian which is shown to be equivariant with respect to bundle automorphisms. Moreover, these gauge theories naturally include Einstein's theory of gravity, and the fermionic gauge group of the universal (Dirac-) Lagrangian contains both Yang-Mills and Einstein-Hilbert type symmetry groups. In particular, the action of the diffeomorphism group of the base manifold is naturally represented by pull-back. We also considered a distinguished class of Dirac type operators whose associated top form gives rise to spontaneous symmetry breaking without using Higgs like potentials. Indeed, the latter naturally arises when a fluctuation of the fermionic vacuum is taken into account. The geometrical meaning of the induced bosonic mass operators can be shown to consists of defining the extrinsic curvature of the "physical space-time" $\mathcal{M}_{\rm phys}$. The intrinsic curvature of the latter, however, was shown to be defined by the fermionic vacuum. In the case where the fermionic vacuum is defined with respect to a Dirac-Yukawa model, the appropriate Higgs and Yang-Mills bundle can be naturally regarded as specific sub-vector bundles of $\xi_{\rm F}$, resp. of $\xi^*_{\rm F} \otimes_{\mathcal{M}} \xi_{\rm F}$. For this we discussed the Yukawa couplings from a geometrical point of view in terms of specific sections of the "Yukawa bundle" which is shown to yield the connection between the fermion and the Higgs bundle. To consider the Yukawa bundle $\xi_{\rm Y}$ as a specific sub-vector bundle of $\xi^*_{\rm H} \otimes_{\mathcal{M}} \xi^*_{\rm F} \otimes_{\mathcal{M}} \xi_{\rm F}$ permits a geometrical understanding of the well-known "hypercharge relations" between the physical Higgs boson $\xi_{\rm H,phy}$ and the asymptotically free fermions $\xi_{\rm F,m^2} \subset \xi_{\rm F}$ in the case of the minimal Standard Model. In this sense, the presented frame makes it possible to treat the geometrical properties of spontaneously broken Yang-Mills-Higgs gauge theories, as discussed in [Tol '03(a)], in terms of fermions. In particular, it is shown that this kind of gauge theories can be expressed in the geometrical setup needed to describe fermions without use of spin structures. Note that the latter actually has no obvious physical meaning. Indeed, all experiments carried out to date demonstrating the physical significance of the two-fold cover of SO(3) are local. The assumption of orientability, however, is necessary to derive the Einstein equation from a globally defined density which seems to also have some significance in our understanding of mass.

The "fermion doubling" within the presented geometrical setup is shown to be tied to the Lichnerowicz decomposition of a Dirac type operator. Since the latter gives rise to the universal Lagrangian and, moreover, to a specific class of Dirac type operators, which yield spontaneous symmetry breaking, the projection onto the physical sub-space $\xi_{\rm phy} \subset \xi_{\rm F}$ clearly indicates a non-trivial relation between the fermionic Lagrangian $\mathcal{L}_{\rm F}$ and the Dirac Lagrangian $\mathcal{L}_{\rm D}$.

Since the Dirac Lagrangian is a canonical element within the presented geometrical frame, it will be useful to discuss it also in terms of the geometry of variational bi-complexes. This may offer a more profound mathematical understanding of operators of Pauli-Dirac type as has been introduced here as a "fluctuation of a fermionic vacuum". These kinds of Dirac type operators obviously play a fundamental role in the Standard Model of Particle Physics. In a forthcoming paper we shall thus discuss the Dirac triple of the Standard Model in more detail. In particular, we shall show how this triple permits specification of $\mathrm{spec}(\mathrm{M_H})$. In the case of the "minimal" Standard Model $\mathrm{rk}(\xi_{\rm H,phys}) = 1$ which allows a prediction of the mass of the Higgs boson. For this, however, one still has to carefully take into account possible "coupling constants" within the frame of Dirac type gauge theories. In general, one may modify the total Lagrangian $\mathcal{L}_{\rm tot}$ as

$$\mathcal{L}_{\rm tot}(D)(\psi) \rightsquigarrow \mathcal{L}_{\rm phys}(D)(\psi) := \mathcal{L}_{\rm F}(D)(\psi) + \lambda\, \mathcal{L}_{\rm D}(D), \tag{84}$$

with the Dirac-Lagrangian being refined by

$$\mathcal{L}_{\rm D}(D) := *\mathrm{tr}(\zeta[D^2 - \triangle_{\rm D}]). \tag{85}$$

Here, respectively, $\lambda \in \mathbb{R}$ is a "relative weight" between the fermionic and bosonic Lagrangian and ζ is the most general element of the commutant with respect to the fermionic representation $\rho_{\rm F}$ of the structure group G. More precisely, $\zeta \in \Gamma(\mathrm{End}^+_{\rm Cl}(\xi_{\rm F}))$ is a positive Hermitian operator satisfying: $[D,\zeta] = 0 = [\zeta, g]$, for all $g \in \mathcal{G}_{\rm YM}$. It therefore may be considered as generalizing the Yang-Mills coupling constant of a "pure" Yang-Mills gauge theory. Actually, the constant λ may be fixed by an appropriate normalization of the Einstein-Hilbert Lagrangian.

Due to formula (76) the Yang-Mills mass matrix is proportional to the (squared) norm of the Yukawa-coupling constants $\mathcal{G}_{\rm Y}$. However, the "physical" Yang-Mills mass matrix is known to be proportional to the Yang-Mills coupling constants $g_{\rm YM} > 0$ which parameterize the most general Killing form on Lie(G). Hence, we have to re-scale $A_{\rm G}$ by a positive constant $g_{\rm G}$ for each simple factor of G, i.e. $A^a_{\rm G} \rightsquigarrow A^a_{\rm G}/g^{(a)}_{\rm G}$ (no summation involved), such that

$$g_{\rm YM}^{(a)} = g_{\rm G}^{(a)} g_{\rm Y} \tag{86}$$

with the abbreviation $g_{\rm Y} \equiv \|\mathcal{G}_{\rm Y}\|$.

Finally, one also has to take into account that in general $\|\mathcal{V}\| \neq 1$, and that the various differential forms defining the Dirac type operator in question have different dimensions. Besides the "Planck scale" (which comes in because of the generic Einstein-Hilbert part of the total Lagrangian) this will bring in an additional length scale within Dirac type gauge theories. However, in the case of the Lagrangian of a PDY this additional length scale turns out to be proportional to the (inverse of the) Higgs mass. Hence, in the case of the Standard Model the two length scales decouple within GTDT and gravity effects can be neglected as it is commonly expected. For this to be consistent, however, we stress again the necessity of the compatibility of the two different symmetry reductions obtained by the fermionic vacua (i.e. simple type Dirac operators) and the ground states of the Higgs boson (i.e. Pauli type Dirac operators).

We finish with some (rather) speculative remarks on how "quantum corrections" might be incorporated in Dirac type gauge theories. For this let again $\xi_{\rm F,red}$ be a fermionic vacuum with respect to a given Dirac-Yukawa model $({\rm H}, \rho_{\rm F,red}, \partial\!\!\!/_{\mathcal{V}})$. Accordingly, let $\mathcal{M}_{\rm phys} \equiv {\rm im}(\mathcal{V}) \subset {\rm E_H} \subset \mathcal{E}$ be the "physical space-time" with respect to the fermionic vacuum. As mentioned above, the geometry of $\mathcal{M}_{\rm phys}$ is determined by ${\rm M_F}$, ${\rm M_{YM}}$ and ${\rm M_H}$, respectively, in the sense that $g_{\rm M}$ is determined by the spectrum of the fermionic mass operator and the Higgs potential evaluated with respect to $\mathcal{V}$. The normal sections of $\mathcal{M}_{\rm phys}$ are determined by the eigenbundles of the bosonic mass operators that correspond to the massive bosons. Hence, a change of the fermionic vacuum leads to a change of the geometry of $\mathcal{M}_{\rm phys}$, provided the respective spectra of the corresponding mass operators are changed. Naively, this will be caused by "quantum corrections" to the propagators of the "asymptotically free particles" like, for instance, of $\partial\!\!\!/_{\mathcal{V}}^{-1}$ in the case of asymptotically free fermions (40). In this respect, the geometrical frame presented so far mimics perturbation theory to lowest order in the Planck-constant $\hbar$. Of course, the task then consists in expressing "quantum corrections" in terms of an appropriate "quantum stochastic" on the moduli space of simple type Dirac operators of the form (which, however, is known to be not well-defined for arbitrary signature[10]):

$$\mathfrak{W}(g_{\rm M}, A, \varphi_{\rm H}) \equiv \log\frac{\det_\Lambda(\not\partial_{\mathcal{V}} + \not A_{\rm fl})}{\det_\Lambda \not\partial_{\mathcal{V}}}. \tag{87}$$

Here, respectively, Λ is some "regularizing cut-off" and $\not A_{\rm fl} := \gamma(A) + \gamma_{\rm M} \otimes \mathcal{Y}(\varphi_{\rm H}) \in \Gamma({\rm End}(\xi_{\rm F}))$, with $(A, \varphi_{\rm H}) \in \Gamma(\xi_{\rm YM} \times_{\mathcal{M}} \xi_{\rm H,phys})$ being a fluctuation of $\xi_{\rm F,red}$. Notice again, that a quantum fluctuation of the fermionic vacuum would yield a fluctuation of both the inner as well as the exterior geometry of $\mathcal{M}_{\rm phys}$ and hence a fluctuation of all bosons. This again emphasizes the geometrical role of fermions.

7 Appendix

Because of its relevance within Dirac type gauge theories we present here in some detail the proof of Proposition (3.1). In particular, it is shown that it holds true for arbitrary signature of D. In [AT '96] a similar proof was presented for the special case of elliptic Dirac type operators.

7.1 Tensor Decomposition

In this sub-section we collect some useful formulae which will be needed to prove the explicit form of the Dirac forms $\varpi_{\rm D} \in \Omega^1(\mathcal{M}, {\rm End}(\mathcal{E}))$ of simple type Dirac operators. Though interesting in its own we will not prove these formulae here (since the proof would be technical but straightforward).

To get started let $\omega \in \Gamma(\tau^*_{\rm M} \otimes_{\mathcal{M}} \Lambda^{\rm n-1}\tau^*_{\rm M} \otimes_{\mathcal{M}} {\rm End}(\xi_{\rm F}))$. Throughout this Appendix, let $(X_1, \ldots, X_n)$ be a locally defined orthonormal frame on $\mathcal{M}$ and $(X^1, \ldots, X^n)$ its dual frame. Then, locally one has

$$\begin{aligned} \omega(X_{i_1}, \ldots, X_{i_n}) &=: \omega_{i_1\cdots i_n} \equiv \omega_{i_1[i_2\cdots i_n]}, \\ \gamma(\omega) &=: \not\omega \equiv \gamma^{i_1} \ldots \gamma^{i_n} \circ \omega_{i_1\cdots i_n}. \end{aligned} \tag{88}$$

Here, respectively, the brackets $[\cdots]$ indicate skew-symmetrization with the convention: $n!\,\omega_{[i_1\cdots i_n]} = \sum_{\sigma\in S_n} {\rm sgn}\sigma\, \omega_{\sigma({\rm i}_1)\cdots\sigma({\rm i}_n)}$ and, again, $\gamma^k \equiv \gamma(X^k)$. In what follows, we restrict ourselves to the Clifford relation $\alpha\beta + \beta\alpha = +2g_{\rm M}(\alpha, \beta)$ for all $\alpha, \beta \in T^*M \hookrightarrow Cl(M)$

[10] In the case where $\mathcal{M}$ is compact and $\not\partial_{\mathcal{V}}$ is elliptic, the propagator $\not\partial_{\mathcal{V}}^{-1}$ is well-defined in terms of Fourier integral operators and one may choose, for instance, the "zeta-function" to regularize formal expressions like $\log\det(1 + \not\partial_{\mathcal{V}}^{-1} \circ \not A_{\rm fl})$.

(the total space of τ_{Cl}).

First, we have the following decomposition[11]

$$\omega_{i_1\cdots i_n} = \omega_{[i_1\cdots i_n]} + \frac{1}{n}\sum_{j=2}^{n}\left(\omega_{i_1\cdots i_n} + \omega_{\, i_j \; i_2 \; \cdots \; \underset{\underset{j}{\uparrow}}{i_1} \; \cdots \; i_n}\right). \tag{89}$$

As a consequence, it follows that $\gamma(\omega)$ may locally be written as

$$\begin{aligned}\gamma^{i_1}\gamma^{i_3}\cdots\gamma^{i_n}\circ\omega_{i_1\mu i_3\cdots i_n} \;&=\; -\frac{n}{n-1}\,\gamma^{i_1}\gamma^{i_3}\cdots\gamma^{i_n}\circ\omega_{[\mu i_1 i_3\cdots i_n]} \\ &\quad + \frac{1}{n-1}\,\gamma^{i_1}\gamma^{i_3}\cdots\gamma^{i_n}\circ\omega_{\mu i_1 i_3\cdots i_n} \\ &\quad - (n-2)\, g^{\alpha\beta}\,\gamma^{i_4}\cdots\gamma^{i_n}\circ\omega_{\alpha\beta\mu i_4\cdots i_n}\end{aligned} \tag{90}$$

where, again, $g^{ij} \equiv g_{\mathrm{M}}(X^i, X^j)$.

Using these two formulae one finally proves the following local decomposition which turns out to be particularly useful in what follows:

$$\gamma(\omega) \;\overset{loc.}{=}\; \gamma^{i_1}\cdots\gamma^{i_n}\circ\omega_{[i_1\cdots i_n]} + \,(n-1)\,g^{\alpha\beta}\,\gamma^{i_3}\cdots\gamma^{i_n}\circ\omega_{\alpha\beta i_3\cdots i_n}. \tag{91}$$

7.2 Proof of Proposition 3.1

Let $\xi = (\mathcal{E}, \mathcal{M}, \pi_{\mathcal{E}})$ be an arbitrary $\mathbb{Z}_2-$graded Clifford module bundle over any smooth (semi-)Riemannian manifold $(\mathcal{M}, g_{\mathrm{M}})$ with $\dim\mathcal{M} = n$ and n even. Every Dirac type operator D may be globally decomposed as $D = \not{\partial}_A + \not{\omega}$ with A being a Clifford connection and $\omega \in \Omega^1(\mathcal{M}, \mathrm{End}^+(\mathcal{E}))$ being given by $\omega := \Theta \wedge (D - \not{\partial}_A)$. Notice again that ω may also depend on the choice of A unless D is of simple type. Locally, ω reads

$$\begin{aligned}\omega \;&\overset{loc.}{=}\; X^i \otimes \omega_i^a \otimes \mathfrak{e}_a \\ &\equiv\; X^i \otimes \left(\sum_{k=0}^{n}\gamma^{i_1}\cdots\gamma^{i_k}\omega^a_{ii_1\cdots i_k}\right)\otimes\mathfrak{e}_a,\end{aligned} \tag{92}$$

with $\omega^a_{ii_1\cdots i_k} = \omega^a_{i[i_1\cdots i_k]}$ and $(\mathfrak{e}_1, \ldots, \mathfrak{e}_{\mathrm{N}})$ being a local frame in $\mathrm{End}_{\mathrm{Cl}}(\xi)$ such that ω is odd with respect to the total grading.

[11] The "$\underset{j}{\uparrow}$" means that i_1 is at the jth position.

By definition, D is of simple type if the Clifford connection A also defines the Bochner-Laplacian of D. Using the general Bochner-Lichnerowicz-Weizenböck decomposition of D^2 it can be shown that, independently of the signature of g_{M}, this holds true if and only if[12]

$$2g^{ij}\omega_j^a + \gamma^j[\omega_j^a, \gamma^i] = 0. \tag{93}$$

Since this relation is linear with respect to the frame $(\mathfrak{e}_1, \ldots, \mathfrak{e}_{\mathrm{N}})$ we may suppress the index a in what follows.

Lemma 7.1 *Let $\omega \in \Gamma(\tau_{\mathrm{M}}^* \otimes_{\mathcal{M}} \tau_{\mathrm{Cl}})$ be a Clifford algebra valued one-form where the coefficients ω_ν fulfill the relation (93). Then, the most general form of ω_ν reads*

$$\omega_\nu = \sum_{k=0}^{n} \gamma^{i_1} \cdots \gamma^{i_k} \omega^{(k)}_{\nu[i_1\cdots i_k]} \tag{94}$$

where the coefficients satisfy the relations:

$$\begin{aligned}
&\omega^{(n)}_{[\nu i_1\cdots i_n]} = 0,\\
&\omega^{(n-1)}_{[\nu i_1\cdots i_{n-1}]} = \varepsilon_{\nu i_1\cdots i_{n-1}} f,\\
&k g^{\alpha\beta}\omega^{(k)}_{\alpha\beta i_1\cdots i_{k-1}} + \omega^{(k-2)}_{[i_1\cdots i_{k-1}]} = 0, \quad k = n-1, \ldots, 2,\\
&g^{\alpha\beta}\omega^{(1)}_{\alpha\beta} = 0.
\end{aligned} \tag{95}$$

Here, respectively, $\omega^{(k)}_{\nu i_1\cdots i_k} \equiv \omega^{(k)}_{\nu[i_1\cdots i_k]} := \omega^{(k)}_\nu(X_{i_1}, \ldots X_{i_k})$ are the local coefficients of appropriate k-forms $\omega^{(k)}_\nu \in \Omega^1(U)$ ($U \subset \mathcal{M}$ open, $\nu = 1, \ldots, n$), $f \in \mathcal{C}^\infty(U)$ and $\epsilon_{i_1\ldots i_n} \equiv \mu_{\mathrm{M}}(X_{i_1}, \ldots, X_{i_n})$ the Levi-Civita symbol.

Proof: To get started we re-write condition (93) as $\gamma^\mu\gamma^\nu\omega_\nu + \gamma^\nu\omega_\nu\gamma^\mu = 0$ and then appropriately re-arrange both terms on the left hand side.

$$\gamma^\nu\omega_\nu\gamma^\mu = \sum_{k=0}^{n}(-1)^k(\gamma^\nu\gamma^\nu\gamma^{i_1}\cdots\gamma^{i_k}\omega^{(k)}_{\nu i_1\cdots i_k} - 2k\, g^{\mu i_1}\gamma^\nu\gamma^{i_2}\cdots\gamma^{i_k}\omega^{(k)}_{\nu i_1 i_2\cdots i_k}). \tag{96}$$

Using this re-arrangement and formula (90) one obtains:

$$\begin{aligned}
0 \;=\; \sum_{k=0}^{n}\Big(&(1-(-1)^k)(\gamma^\mu\gamma^\nu\gamma^{i_1}\cdots\gamma^{i_k}\omega^{(k)}_{[\nu i_1\cdots i_k]} + k\gamma^\mu\gamma^{i_2}\cdots\gamma^{i_k}g^{\alpha\beta}\omega^{(k)}_{\alpha\beta i_2\cdots i_k})\\
&+(-1)^k\, 2(k+1)\, g^{\mu\nu}\gamma^{i_1}\cdots\gamma^{i_k}\omega^{(k)}_{[\nu i_1\cdots i_k]}\\
&+(-1)^k\, 2k(k-1)\, g^{\mu\nu}\gamma^{i_3}\cdots\gamma^{i_k}g^{\alpha\beta}\omega^{(k)}_{\alpha\beta\nu i_3\cdots i_k}\Big).
\end{aligned} \tag{97}$$

[12] In the case $s = n$ this has been proved in [AT '96]. The more general case of arbitrary signature has been proved in [Thum '02].

This sum may be further split into two sums of an even and odd number of Clifford elements. Since these terms are linearly independent one may evaluate each sum separately. For example, the sum of an odd number of Clifford elements gives rise to the condition:

$$\begin{aligned} 0 \;=\; & \sum_{\substack{k=1 \\ \text{(k odd)}}}^{n} \Big(\gamma^\mu\gamma^\nu\gamma^{i_1}\cdots\gamma^{i_k}\omega^{(k)}_{[\nu i_1\cdots i_k]} + k\,\gamma^\mu\gamma^{i_2}\cdots\gamma^{i_k}g^{\alpha\beta}\omega^{(k)}_{\alpha\beta i_2\cdots i_k} \\ & -(k+1)\,g^{\mu\nu}\gamma^{i_1}\cdots\gamma^{i_k}\omega^{(k)}_{[\nu i_2\cdots i_k]} - k(k-1)\,g^{\mu\nu}\gamma^{i_3}\cdots\gamma^{i_k}g^{\alpha\beta}\omega^{(k)}_{\alpha\beta\nu i_3\cdots i_k}\Big)\,. \end{aligned} \tag{98}$$

Since[13] $\gamma^\mu\gamma^\nu\gamma^{i_1}\cdots\gamma^{i_{n-1}}\omega^{(n-1)}_{[\nu i_1\cdots i_{n-1}]} = n\,g^{\mu\nu}\gamma^{i_1}\cdots\gamma^{\hat{i}_\mu}\cdots\gamma^{i_n}\omega^{(n-1)}_{[\nu i_1\cdots \hat{i_\mu}\cdots i_n]}$, the condition (98) becomes equivalent to

$$\begin{aligned} 0 \;=\; & \gamma^\mu\gamma^{i_2}\cdots\gamma^{i_{n-1}}\Big((n-1)g^{\alpha\beta}\omega^{(n-1)}_{\alpha\beta i_2\cdots i_{n-1}} + \omega^{(n-3)}_{[i_2\cdots i_{n-1}]}\Big) \\ & + \sum_{\substack{k=3 \\ \text{(k odd)}}}^{n-3} \Big(-(k+1)\,g^{\mu\nu}\gamma^{i_1}\cdots\gamma^{i_k}((k+2)g^{\alpha\beta}\omega^{(k+2)}_{\alpha\beta\nu i_1\cdots i_k} + \omega^{(k)}_{[\nu i_1\cdots i_k]}) \\ & + \gamma^\mu\gamma^{i_2}\cdots\gamma^{i_k}(kg^{\alpha\beta}\omega^{(k)}_{\alpha\beta i_2\cdots i_k} + \omega^{(k-2)}_{[i_2\cdots i_k]})\Big) + \gamma^\mu g^{\alpha\beta}\omega^{(1)}_{\alpha\beta}. \end{aligned} \tag{99}$$

The term with the highest degree in the γ^{i_j} vanishes. By an induction argument one ends up with the recursion relation:

$$k\,g^{\alpha\beta}\omega^{(k)}_{\alpha\beta i_2\cdots i_k} + \omega^{(k-2)}_{[i_2\cdots i_k]} = 0, \quad k = 3,\ldots n-1. \tag{100}$$

As a consequence, it follows that $g^{\alpha\beta}\omega^{(1)}_{\alpha\beta} = 0$. Moreover, the term $\omega^{(n-1)}_{[\nu i_1\ldots i_{n-1}]}$ drops out and thus is undetermined. Its most general form is given by

$$\omega^{(n-1)}_{[\nu i_1\ldots i_{n-1}]} = \varepsilon_{\nu i_1\ldots i_{n-1}} f, \tag{101}$$

with f being an arbitrary locally defined smooth function on $\mathcal{M}$.

Next, we consider the sum of an even number of Clifford elements. This yields the relation

$$\sum_{\substack{k=0 \\ \text{(k even)}}}^{n} \Big((k+1)\,g^{\mu\nu}\gamma^{i_1}\cdots\gamma^{i_k}\omega^{(k)}_{[\nu i_1\cdots i_k]} + k(k-1)\,g^{\mu\nu}\gamma^{i_3}\cdots\gamma^{i_k}g^{\alpha\beta}\omega^{(k)}_{\alpha\beta\nu i_3\cdots i_k}\Big) = 0, \tag{102}$$

[13] The $\hat{\cdot}$ denotes the omission of the "hated" object.

which in turn gives rise to the following constraint equations:

$$\begin{aligned}
0 &= (n+1)\, g^{\mu\nu}\gamma^{i_1}\cdots\gamma^{i_n}\omega^{(n-1)}_{[\nu i_1\cdots i_n]},\\
0 &= (n-1)\, g^{\mu\nu}\gamma^{i_1}\cdots\gamma^{i_{n-2}}(n g^{\alpha\beta}\omega^{(n)}_{\alpha\beta\nu i_1\cdots i_{n-2}} + \omega^{(n-2)}_{[\nu i_1\cdots i_{n-2}]}),\\
&\vdots\\
0 &= (k+1)\, g^{\mu\nu}\gamma^{i_1}\cdots\gamma^{i_k}((k+2) g^{\alpha\beta}\omega^{(k+2)}_{\alpha\beta\nu i_1\cdots i_k} + \omega^{(k)}_{[\nu i_1\cdots i_k]}),\\
&\vdots\\
0 &= g^{\mu\nu}\gamma^{i}\gamma^{j}(2 g^{\alpha\beta}\omega^{(2)}_{\alpha\beta\nu} + \omega^{(0)}_{\nu}).
\end{aligned} \tag{103}$$

These are satisfied provided that

$$\begin{aligned}
0 &= \omega^{(n)}_{[\nu i_1\cdots i_n]}\\
0 &= (k+2) g^{\alpha\beta}\omega^{(k+2)}_{\alpha\beta\nu i_1\cdots i_k} + \omega^{(k)}_{[\nu i_1\cdots i_k]}, \quad k = 0,\ldots,n
\end{aligned} \tag{104}$$

which, when combined with our previous result with respect to the sum of an odd number of Clifford elements, finally proves the statement. □

Corollary 7.1 *Let ξ_{F} be the chiral fermion bundle with respect to the Dirac triple $(\mathrm{G}, \rho_{\mathrm{F}}, D)$, with D being of simple type and of arbitrary signature. The Dirac form of D reads $\varpi_{\mathrm{D}} = \Theta \wedge (\gamma_{\mathrm{M}} \otimes \phi)$, with $\phi \in \Gamma(\mathrm{End}^{-}_{\mathrm{Cl}}(\xi_{\mathrm{F}}))$ uniquely determined by D.*

Proof: Again, in the sequel we shall suppose that the induced Clifford relations, defining τ_{Cl}, are given by $\alpha\beta + \beta\alpha = +2g_{\mathrm{M}}(\alpha,\beta)$. Locally, we may write $\varpi_{\mathrm{D}}(X_\mu) = \omega^a_\mu \otimes \mathfrak{e}_a$ and, again, decompose the coefficients into the sum of odd and even terms with respect to the canonical involution $\alpha \mapsto -\alpha$ for all $\alpha \in T^*M \hookrightarrow Cl(M)$:

$$\begin{aligned}
\omega^a_\mu &= \sum_{k=1}^{n} \gamma^{i_1}\cdots\gamma^{i_k}\omega^a_{\mu[i_1\cdots i_k]}\\
&= \sum_{\substack{k=1\\ \text{(k odd)}}}^{n-1} \gamma^{i_1}\cdots\gamma^{i_k}\omega^a_{\mu[i_1\cdots i_k]} + \sum_{\substack{k=0\\ \text{(k even)}}}^{n} \gamma^{i_1}\cdots\gamma^{i_k}\omega^a_{\mu[i_1\cdots i_k]}\\
&\equiv \alpha^a_\mu + \beta^a_\mu.
\end{aligned} \tag{105}$$

We then compute $\gamma^\mu\omega^a_\mu \equiv \gamma^\mu\alpha^a_\mu + \gamma^\mu\beta^a_\mu$ to show that $\gamma^\mu\omega^a_\mu \otimes \mathfrak{e}_a = \gamma_{\mathrm{M}} \otimes \phi$.

With help of formula (91) one obtains[14]

$$\begin{aligned}\gamma^\mu\alpha_\mu &= \gamma^\mu\gamma^{i_1}\cdots\gamma^{i_{n-1}}\omega^{(n-1)}_{[\mu i_1\cdots i_{n-1}]} + 2\, g^{ij}\omega^{(1)}_{ij} \\ &+ \sum_{\substack{k=1\\ \text{(k odd)}}}^{n-3}\Big((k+2)\,\gamma^{i_2}\cdots\gamma^{i_{k+2}}g^{ij}\omega^{(k+2)}_{ij[i_2\cdots i_{k+2}]} + \gamma^\mu\gamma^{i_1}\cdots\gamma^{i_k}\omega^{(k)}_{[\mu i_1\cdots i_k]}\Big). \qquad (106)\end{aligned}$$

Hence, using Lemma 7.1, one concludes that

$$\gamma^\mu\alpha_\mu = \gamma^\mu\gamma^{i_1}\cdots\gamma^{i_{n-1}}\omega^{(n-1)}_{[\mu i_1\cdots i_{n-1}]}. \qquad (107)$$

Next, we consider $\gamma^\mu\beta_\mu$ and find, using similar arguments like above, that

$$\begin{aligned}\gamma^\mu\beta_\mu &= \gamma^\mu\gamma^{i_1}\cdots\gamma^{i_n}\omega^{(n)}_{[\mu i_1\cdots i_n]} \\ &+ \sum_{\substack{k=0\\ \text{(k even)}}}^{n-2}\Big((k+2)\,\gamma^{i_2}\cdots\gamma^{i_{k+2}}\, g^{ij}\omega^{(k+2)}_{ij[i_2\cdots i_{k+2}]} + \gamma^\mu\gamma^{i_1}\cdots\gamma^{i_k}\omega^{(k)}_{[\mu i_1\cdots i_k]}\Big) \\ &= 0. \qquad (108)\end{aligned}$$

Finally, using Lemma 7.1 again, we end up with

$$\begin{aligned}\gamma^\mu\omega_\mu &= \gamma^\mu\gamma^{i_1}\cdots\gamma^{i_{n-1}}\omega^{(n-1)}_{[\mu i_1\cdots i_{n-1}]} \\ &= f\,\gamma^1\cdots\gamma^n \\ &= \tilde{f}\,\gamma_{\mathrm{M}}. \qquad (109)\end{aligned}$$

If we set $\phi \equiv \tilde{f}^a\mathfrak{e}_a$, where $(\mathfrak{e}_1,\ldots\mathfrak{e}_{\mathrm{N}^-})$ is a local frame in $\mathrm{End}^-_{\mathrm{Cl}}(\xi_{\mathrm{F}})$, we obtain the desired result and thus have also proved Proposition 3.1 in the case of τ_{M}. Of course, for $\tau^{\mathrm{op}}_{\mathrm{Cl}}$ the proof is similar. □

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[14]For notational convenience the index a is again suppressed.

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