Complex vector bundles with a given top Chern class

by

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Preprint no.: 28 2005
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April 14, 2005

Abstract
We give a detailed proof of Thom’s theorem on the existence of a complex vector bundle over a compact oriented manifold $M^n$ whose top Chern class lies in a given positive ray in $H^{even}(M^n, \mathbb{Z})$.

MSC: 55F10

1 Introduction.

The problem of representing a homology class $A \in H_*(V, \mathbb{Z})$ by a smooth submanifold $W \subset V$ was investigated by R. Thom in his famous paper [Thom1954]. The main idea of Thom is to study the relations between 3 objects: smooth submanifolds, (co)homology classes and smooth vector bundles. Let us now recall some important notions and results in [Thom1954].
• $B_G$: - the classifying space of a group $G \subset O(k)$ and $E_G \to B_G$
the corresponding universal fibration,

• $A_G$: - the associated fibration $E_G \times_G D^k \to B_G$, where $D^k$ is
the unit ball in $\mathbb{R}^k$,

• $M_G$: - the Thom space of the fibration $D^k \to A_G \to B_G$.

Recall that the algebra $H^*(M(G))$ is obtained from that one of
$H^*(B_G)$ by shifting all dimension by $k$. Namely we have the following Thom’s
isomorphism

$$\phi_G: H^{r-k}(B_G) \to H^r_k(A_G = A_G \setminus \partial A_G) \cong H^r(M(G)),$$

here $H^*_k$ denotes the cohomologies with compact support. The class
$H^k(M(G)) \ni U = \phi_G(u^0)$ is called the fundamental (Thom’s)
class of $M(G)$, where $u_G \in H^0(B_G)$ is the unit of the ring $H^*(B_G)$.

We observe that there is sequence of natural mappings

$$B_G \xrightarrow{i} A_G \xrightarrow{e} M_G$$

where $i$ is the embedding via the zero section, and $e$ is the map which
is identity in $A_G \setminus \partial A_G$ and which maps the boundary $\partial A_G$ to a point.

It is easy to see that

$$i^* \circ e^*(U) = e_k$$

where $e_k$ is the Euler class of the universal bundle $V_G = E_G \times_G V^k \to
B_G$.

A cohomology class $u \in H^k(A)$ of a space $A$ is called realizable
w.r.t. a group $G \subset O(k)$, if there exists a map $f: A \to M(G)$ such
that $f^*(U) = u$.

1.1. Thom Criterion. [Thom1954 Theorem II.1] Suppose that
$V^n$ is a compact orientable smooth manifold. A class $z \in H_{n-k}(V^n)$
can be realized by a submanifold $W^{n-k}$ whose normal bundle has a $G$
structure, if and only if the Poincaré dual class $PD(z) \in H^k(V^n)$ is
realizable w.r.t. $G$.

We shall say that a cohomology class $u \in H^k(A)$ of a space $A$ is
strongly realizable w.r.t. a group $G \subset O(k)$, if there exists a
map $g: A \to B_G$ such that $g^*(e_k) = u$. Because of (1.1) the strong
realizability of $u$ implies the realizability of $u$ (w.r.t. $G$). Clearly the
strong realizability of \( u \) is equivalent to the condition that \( u \) is the Euler class of a \( G \)-vector bundle over \( M^m \).

In [Thom1954] Thom also proved the following existence theorem.

1.2. **Theorem** [Thom1954 Theorem II.25]. For each cohomology class \( z \in H^k(M^m, \mathbb{Z}) \) there exists a number \( N(k, n) \) such that the class \( N(k, n) \cdot z \) is strongly realizable w.r.t. \( G = SO(k) \). If \( k = 2l \), then there exists a number \( N_1(k, n) \geq N(k, n) \) such that the class \( N_1(k, n) \cdot z \) is strongly realizable w.r.t. \( G = U(k) \).

Thom gave a detailed proof of Theorem 1.2 for \( G = SO(k) \). He noticed that his proof also works for \( G = U(k) \) or \( Sp(k) \). Since we use Thom’s theorem for \( G = U(k) \) in [Le2004] and [Le2005] we feel a need for a detailed proof of Thom’s theorem 1.2 in this case.

I thank Dietmar Salamon for stimulating discussions on Thom’s theorems in [Thom1954].

2 Proof of Theorem 1.2 for \( G = U(k) \).

2.1. **The strategy to find a map** \( g : M^m \to B_{U(k)} \). Suppose that \( u \in H^{2k}(M^m, \mathbb{Z}) \). Then there is a map

\[
f : M^m \to K(\mathbb{Z}, 2k)
\]

such that \( f^*(\tau) = u \), where \( \tau \) is the fundamental class of \( H^k(K(\mathbb{Z}, 2k), \mathbb{Z}) \). Moreover we can assume that \( f(M^m) \subset K^m(\mathbb{Z}, 2k) \), where \( K^q(\mathbb{Z}, 2k) \) is the \( q \)-skeleton of the Eilenberg-McLane space \( K(\mathbb{Z}, 2k) \). To prove Theorem 2.1 it suffices to find a map

\[
h : K^m(\mathbb{Z}, 2k) \to B_{U(k)}
\]

such that for some positive number \( N_1(k, m) \) we have

\[
(2.2) \quad h^*(e_{2k}) = N_1(k, m)j^*(\tau),
\]

where \( e_{2k} = e_k \) is the top Chern class of the universal bundle \( V_{U(k)} \) over \( B_{U(k)} \) and \( j \) is the embedding \( K^m(\mathbb{Z}, 2k) \to K(\mathbb{Z}, 2k) \).

In order to find the map \( h \) satisfying (2.2) we first notice that \( \pi_{2k}(B_{U(k)}) \otimes \mathbb{Q} = 1 \) (see Proposition 2.6), so we can define a map \( G_{2k} : K^{2k}(\mathbb{Z}, 2k) = S^{2k} \to B_{U(k)} \) by sending \( S^{2k} \) to the generator of
the free part of $\pi_{2k}(B_{U(k)})$. Next we shall extend map $G_q$ inductively on $q$ by applying the following

**2.3. Lemma.** ([Thom1954, Lemma 2.24]) Suppose that $Y$ is a space such that the free component of $\pi_k(Y)$ is isomorphic to $\mathbb{Z}$ with a generator $t$. If for all $q \geq k$ the group $\pi_q(Y)$ is of finite type, and $H^{q+1}(K(Z,k), \pi_q(Y))$ is finite, then for each $q$ there exists a map $G_q : K(Z,k) \to Y$ such that $(G_q)_*(\pi_k(K(Z,k)) = < N(q,k)t >_{\mathbb{Z}}$.

In order to apply Lemma 2.3 to $Y = B_{U(k)}$ we must verify that

1. $\pi_{2k}(B_{U(k)}) \otimes \mathbb{Q} = \mathbb{Q}$,
2. $\pi_q(B_{U(k)})$ is of finite type, $\forall q \geq k$,
3. $H^{q+1}(K(Z,2k), \pi_q(B_{U(k)})) \otimes \mathbb{Q} = 0$, $\forall q \geq k$.

Once the conditions (2.3.1), (2.3.2) and (2.3.3) hold, Lemma 2.3 gives us a map

$h : K^m(Z,2k) \to B_{U(k)}$

such that $h_*(w) = N(m,k)t$, where $w$ is a generator of $\pi_{2k}(K(Z,2k)) = \mathbb{Z}$. In Lemma 2.8 we shall show that the pairing of the Chern class $c_k$ with $t$ is positive. Hence we get

$h^*(c_k) = N'(m,k)t$,

where $N'(m,k) > 0$. So the map $h$ satisfies (2.2).

**2.5. Homotopy type of $B_{U(k)}$ and verifying conditions (2.3.1), (2.3.2).**

Clearly the conditions (2.3.1) and (2.3.2) are consequences of the following

**2.6. Proposition.** We have

1. $\pi_{2i}(B_{U(k)}) \otimes \mathbb{Q} = \mathbb{Q}$, if $i \leq k$
2. $\pi_j(B_{U(k)}) \otimes \mathbb{Q} = 0$, for all other $j$.

*Proof of Proposition 2.6.* There are two proofs of Proposition 2.6. In the first proof we use Thom’s argument based on applying the $\mathcal{C}$-version of the Whitehead theorem to a map $f : B_{U(k)} \to K(Z,2) \times \cdots \times$
and we use a computation of \( H^*(B_{U(k)}, \mathbb{Z}) \), see the proof of Lemma 2.8 below. In the second proof we use an induction argument. The Proposition 2.6 is clearly true for \( B_{U(1)} = \mathbb{CP}^\infty \). Suppose that the Proposition is valid till \( k \). To show its validity for \( k+1 \) we consider the following exact homotopy sequence

\[
\rightarrow \pi_q(S^{2k+1}) \rightarrow \pi_q(B_{U(k)}) \rightarrow \pi_q(B_{U(k+1)}) \rightarrow \pi_{q-1}(S^{2k+1}) \rightarrow ...
\]

related to the fibration \( S^{2k+1} \rightarrow B_{U(k)} \rightarrow B_{U(k+1)} \). Next we use the well-known fact that \( \pi_q(S^{2k+1}) \) is zero, if \( q \leq 2k \), it is finite if \( q \geq 2k+2 \), and \( \pi_{2k+1}(S^{2k+1}) = \mathbb{Z} \), (see e.g. [Spanier1966, 9.7]). Applying this fact and our induction assumption to the above exact sequence after tensoring with \( \mathbb{Q} \) we get the induction statement for \( k+1 \).

Now we shall see that condition (2.3.3) is a consequence of a computation of the cohomology ring \( H^*(K(\mathbb{Z}, k), \mathbb{Q}) \), obtained by Serre and Cartan. Namely we have

2.7. Lemma. (see e.g.[F-F, 3.25]) a) If \( n \) is odd, then
\[
H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x), \quad \dim x = n.
\]
b) If \( n \) is even, then
\[
H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \mathbb{Q}[x], \quad \dim x = n.
\]

Lemma 2.7 can be easily proved by using induction and the cohomology spectral sequence associated with the fibration \( K(\mathbb{Z}, n-1) \cong \Omega K(\mathbb{Z}, n) \rightarrow K(\mathbb{Z}, n) \), whose fiber is contractible.

2.8. Lemma. The pairing of \( c_k \in H^k(B_{U(k)}, \mathbb{Z}) \) with the generator \( t \) of \( \pi_{2k}(B_{U(k)} \otimes \mathbb{Q}) \) is positive.

Proof. First we notice that the cohomology ring \( H^*(B_{U(k)}, \mathbb{Z}) \) is a polynomial ring generated by the Chern classes \( c_1(V_{U(k)}), \cdots, c_k(V_{U(k)}) \), moreover there are no polynomial relations between these generators (see e.g. [M-S1974, 14.5]). Now we use the existence of a map

\[
f : B_{U(k)} \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times \cdots \times K(\mathbb{Z}, 2k)
\]
such that

\[
f^*(\tau_{2l}) = c_l,
\]
where $\tau_2l$ is the fundamental class of $H^{2l}(K(\mathbb{Z}, 2l), \mathbb{Z})$. By Lemma 2.7.b the induced map $f^*: H_*(BU(k), \mathbb{Z}) \to H_*(K(\mathbb{Z}, 2) \times \cdots \times K(\mathbb{Z}, 2k), \mathbb{Z})$ is a $C$-isomorphism, where $C$ is the class of finite groups. Hence the generalized Whitehead theorem of Serre [Serre1953] gives us an isomorphism

$$f^*: \pi_{2k}(BU(k)) \otimes \mathbb{Q} \to \pi_{2k}(K(\mathbb{Z}, 2k)) \otimes \mathbb{Q} = Q.$$  

Now using (2.9) we get immediately that

$$<c_k, t > = <\tau_{2k}, f^*(t) > > 0.$$  

To make a good feeling of completeness we shall outline the Thom proof of Lemma 2.3 here. Lemma 2.3 shall be proved by induction on the dimension $q$. Clearly Lemma 2.3 for $q = k$ is trivial. To construct a map $G_q : K^{q+1}(\mathbb{Z}, 2k) \to Y$ from a map $G_q : K^q(\mathbb{Z}, 2k) \to Y$ we use the existence of a “covering map” $F_N : K(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$ with the following property

$$F_N^*(\tau) = N\tau.$$  

By theorem of simplicial approximation we can assume that $F_N$ sends $K^q(\mathbb{Z}, n)$ to $K^q(\mathbb{Z}, n)$ for each $q \geq n$.

The idea of Thom is that by using this covering map we can modify map $G_q$ to a map

$$G'_q : K^q(\mathbb{Z}, 2k) \xrightarrow{F_N} K^q(\mathbb{Z}, 2k) \to Y$$

such that $G'_q$ can be extended to $K^{q+1}$, since the obstruction to this extension lies in the group $F_N^*(H^{q+1}(K(\mathbb{Z}, 2k), \pi_q(Y))$ which is trivial under the condition of Lemma 2.3 and taking into account the following Lemma.

### 2.10. Lemma. [Thom1954, Lemma 2.23].

Let $G$ be a finitely generated abelian group and suppose that all the elements of $H^r(K(\mathbb{Z}, n), G)$ is of finite order $N$. Then there exists a number $m \neq 0$ such that then endomorphism $(F_m)^*: H^r(K(\mathbb{Z}, n), G)$ is trivial.

In closing this note we remark that the proof of main technical Lemma 2.3 is somewhat similar to the proof of Serre of the following

### 2.11. Proposition. [Serre 1953, V.2.2].

Let $K$ be a finite polyhedral, $n$ be an odd number and $x \in H^n(K, \mathbb{Z})$. Then there are a
number \( N \neq 0 \) and a map \( K \to S^n \) such that \( f^*(u) = Nx \), where \( u \) is the fundamental class of \( H^n(S^n, \mathbb{Z}) \).

Since \( n \) is odd we get that for \( i > n \) the group \( \pi_i(S^n) \) is finite. Hence any map \( K^q \to S^n \) after composing with a certain covering map \( F_N : S^n \to S^n \) can be extended to \( K^{q+1} \to S^n \). So in the Serre proof we compose the old map with a covering map of the target space \( S^n \), while in the Thom proof we compose the old map with a covering map on the domain space \( K(\mathbb{Z}, 2k) \).

Since \( S^{2k+1} \) is the Thom space for \( G = e \in SO(2k+1) \) we remark that the proof of Theorem 1.2 for \( G = U(k) \) together with Proposition 2.11 and Proposition 1.1 are sufficient to get the following Thom’s stability statement. For each \( x \in H^*(M^m, \mathbb{Z}) \) there is a positive number \( N \) such that \( Nx \) is the Poincare dual to the fundamental class of a closed submanifold in \( M^m \).

References


