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**Almost-holomorphic and totally real solenoids in complex surfaces**

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by


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# ALMOST-HOLOMORPHIC AND TOTALLY REAL LAMINATIONS IN COMPLEX SURFACES

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ABSTRACT. We show that there exists a Lipschitz almost-complex structure $J$ on $\mathbf{C}P^2$, arbitrarily close to the standard one, for which there exists a compact lamination by $J$-holomorphic curves satisfying the following properties: it is minimal, it has hyperbolic holonomy and it is transversely lipschitz. Its transverse Hausdorff dimension can be any number $\delta$ in an interval $(0, \delta_{max})$ where $\delta_{max} = 1.6309...$. We also show that there is a compact lamination by totally real surfaces in $\mathbf{C}^2$ with the same properties, unless the transverse dimension can be any number $0 < \delta < 1$. Our laminations are transversally totally disconnected.

## 1. Introduction

In [16], Sullivan introduced a family of laminations by surfaces of a compact space, for the study of $C^1$-conjugacy classes of expanding endomorphisms of the circle. These laminations are usually called the *Sullivan's solenoids.* For every prime integer $k$, the $k$-th Sullivan's solenoid $\mathcal{S}^k$ is the quotient of the trivial lamination $\mathbf{H} \times \mathbf{Q}_k$ by the diagonal action of the group of affine transformations $x \mapsto ax + b$, where $a$ is a power of $k$, and $b$ belongs to the ring $\mathbf{Z}[1/k]$. When $k$ is not prime, one can construct a similar lamination. These solenoids carry a very rich dynamics. Among their properties, one can note that: (i) all the leaves are dense, (ii) $\mathcal{S}^k$ has hyperbolic holonomy, (iii) $\mathcal{S}^k$ is transversally lipschitz and carries a transverse conformal structure of arbitrary Hausdorff dimension. In this work we construct various examples of embeddings of the Sullivan's solenoids in complex surfaces.

1.1. **Almost-holomorphic laminations in $\mathbf{C}P^2$.** In a complex surface, a lamination by holomorphic curves is a closed subset which decomposes as a disjoint union of non singular holomorphic curves, called the leaves.

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*Does there exist a compact lamination by holomorphic curves in* $\mathbf{C}P^2$ *whose leaves are not algebraic?*

Our main motivation for this problem is the *Exceptional Minimal Set Conjecture*, asserting that a leaf of a singular holomorphic singular foliation of $\mathbf{C}P^2$ accumulates on a singularity [3]. If this conjecture were not true, there would exist a minimal compact lamination by holomorphic curves in $\mathbf{C}P^2$ tangent to a singular holomorphic foliation. It has been proved that such a lamination would have hyperbolic holonomy [2].

Recent progress has been made on this question when the total space of the lamination is a sufficiently smooth real hypersurface. A real hypersurface of $\mathbf{C}P^2$ is called Levi-flat if it is foliated by holomorphic curves. In the works [15, 13, 4, 7] it is proved that there is no compact Levi-flat hypersurface in $\mathbf{C}P^2$ with various type of regularity.

However we conjecture that there is a compact lamination by holomorphic curves in $\mathbf{C}P^2$ without algebraic leaf, whose transverse space is totally disconnected. To comfort our conjecture, we prove (see corollary 5.3) that *for every* $k \geq 2$*, there exist a lipschitz almost-complex structure* $J$ *on* $\mathbf{C}P^2$*, arbitrarily close to the standard structure in the uniform topology, and a compact lamination by* $J$*-holomorphic curves which is diffeomorphic to the* $k$*-th Sullivan's solenoid* $\mathcal{S}^k$. In particular, the leaves of these laminations are symplectic surfaces. Moreover, the transverse space is transversally discontinous and its Hausdorff dimension can be any number in an interval $(0, \delta_k)$; the maximum of the $\delta_k$ is $\delta_{max} = 1.6309...$ attained for $k = 6$. Unfortunately, our laminations tend to the union of three lines when the almost-complex structure approaches the standard one, so that the construction does not give an affirmative answer to our conjecture.

1.2. **Totally real laminations in $\mathbf{C}^2$.** In the context of totally real geometry, we also have examples of laminations. A real surface $\Sigma$ in a complex surface $S$ is called *totally real* if $TS|_\Sigma = T\Sigma \oplus iT\Sigma$. In $\mathbf{C}^2$, the torus $\mathbf{S}^1 \times \mathbf{S}^1$ is totally real, and its normal bundle is trivial. Thus there exists a 3-dimensional torus $\mathbf{T}^3$ in its neighborhood equipped with a foliation $\mathcal{F}$ by oriented totally real surfaces which is diffeomorphic to a minimal linear foliation of $\mathbf{R}^3/\mathbf{Z}^3$. However, these foliations have transversely invariant metric, and thus do not carry hyperbolic holonomy. In this work (see corollary 5.4) we prove that: *there is a lamination by smooth totally real surfaces in* $\mathbf{C}^2$*, which is diffeomorphic to the second Sullivan's solenoid, and whose Hausdorff dimension is any number* $0 < \delta < 1$. Moreover, there are also examples of non orientable solenoids verifying these properties.

1.3. **Structure of the proof.** The Sullivan's solenoids can be smoothly approximated by *branched surfaces* $\overline{\Sigma}^k$, obtained from an annulus by identifying its boundary components with a $k:1$ map, where $k$ is an integer greater than 2. The branched surface $\overline{\Sigma}^k$ is equipped with a smooth structure, and the approximation works in the smooth topology. The part 4 recalls this construction. The 1-dimensional analog of this process is the theory of *train-tracks*, invented by Thurston to study geodesic laminations of hyperbolic surfaces [1]. In higher dimension, the approximation of a transversely disconnected lamination by branched manifolds has been studied by Williams [17] and Gambaudo [9].

In a complex hermitian surface, an immersed real 2-dimensional surface $\Sigma$ is called $\varepsilon$-holomorphic if the angle between $T\Sigma$ and $iT\Sigma$ is uniformly bounded by $\varepsilon$. If there is no holomorphic branched surface because of analytic continuation, there does exist $\varepsilon$-holomorphic surfaces when $\varepsilon$ is positive, at least locally. We prove that for every $\varepsilon > 0$, and every integer $k \geq 1$ there exists a smooth embedding of $\overline{\Sigma}^k$ in $\mathbf{C}P^2$ whose image is an $\varepsilon$-holomorphic branched surface. We also prove that there exists a smooth embedding of $\overline{\Sigma}^2$ in $\mathbf{C}^2$, whose image is a totally real branched surface. These results are proved in part 2.

In part 3, we prove a neighborhood theorem for a smooth embedding of the branched surfaces $\overline{\Sigma}^k$ in an oriented 4-manifold. These neighborhoods are topologically characterized by an odd integer, that we call *transverse braid class*: it is a conjugacy class of the braid group on $k$ strings, and measures how the sheets are turning around the branched locus, using a natural trivialisation of the normal bundle of $\overline{\Sigma}^k$.

The last step consists to construct in a smooth oriented 4-manifold a smoothly embedded Sullivan's solenoid whose leaves are arbitrarily close to the sheets of an embedded branched surface diffeomorphic to $\overline{\Sigma}^k$. This lamination is obtained by peeling the branched locus off, and is transversely Lipschitz. This construction is the goal of part 5.1.

1.4. **Acknowledgments.** Sidney Frankel made remarks on a preliminary version of this paper [5] that permitted me to avoid the case of higher genus laminations. Étienne Ghys communicated to me some examples of laminated *automorphic* functions which inspired me to do the construction of the complex surface containing Sullivan's solenoid as a lamination by holomorphic curves, and Bruno Sevennec lead numerical experiments showing that these functions do not raise to holomorphic embbedings of Sullivan's solenoid in the plane (see 5.1). Alexei Glutsyuk observed that Sullivan's solenoid does not embed symplectically in $\mathbf{R}^4$ with its standard symplectic structure, thus answering a question initially asked to me by Jean-Claude Sikorav (see 5.4). It was also a pleasure to discuss this work with Emmanuel Giroux, André Haefliger, Misha Lyubich, Martin Pinsonnault and Jean-Yves Welschinger. I thank all these persons for their very useful comments and suggestions. This work has been possible with the hospitality of the Université de Genève, the University of Toronto, and the Max Planck Institut für Mathematik in Leipzig.

## 2. Real branched surfaces in complex surfaces

2.1. **The branched surfaces $\overline{\Sigma}^k$.** Let $k \geq 2$ be an integer. Consider an oriented compact smooth annulus $A$, a smooth embedded curve $b$ that separates $A$ in the disjoint union of two annulus $A_l$ and $A_r$, and a connected non ramified $k:1$ covering $R : \widetilde{A} \to A$. We identify two points of $\widetilde{A}$ if they belongs to the same fiber $R^{-1}(x)$, where $x$ is a point of the closure of $A_r$. The resulting quotient space $P : \widetilde{A} \to \overline{A}^k$ is a *branched surface* (see [17]). The branched locus is a circle $b$ separating $\overline{A}^k$ in two annulus $\overline{A}_l$ and $\overline{A}_r$, where $\overline{A}_r$ is collapsed to $b$ by a diffeomorphism, and $\overline{A}_l$ is collapsed to $b$ by a $k:1$ map. Thus, the neighborhood of the branched locus is locally an open book with $k+1$ sheets.

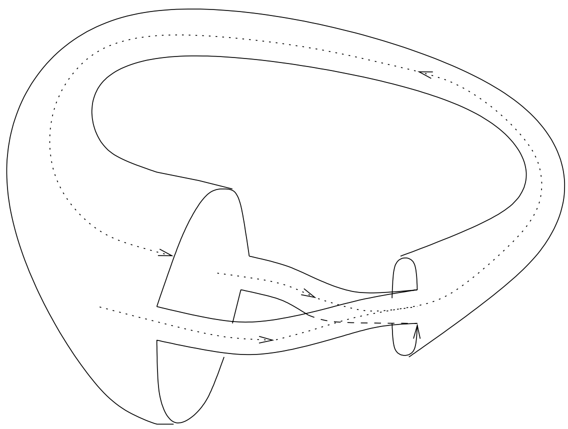


Figure 1. $\overline{\Sigma}$

The branched surface $\overline{A}^k$ has a smooth structure: for $l = 0, 1, \dots, \infty$, a function $f : \overline{A}^k \to V$ with values in a smooth manifold $V$ is of class $C^l$ if $f \circ P$ is a function of class $C^l$ in the usual sense. Define the tangent space to a point $x$ as usual: it is the quotient $\mathcal{I}_x/\mathcal{I}_x^2$, where $\mathcal{I}_x$ is the ideal of smooth functions vanishing at $x$. The reader can easily verify that the tangent space is a 2-dimensional real plane, even if the point belongs to the branched locus. It is important to observe that $\overline{A}^k$ is oriented. This orientation determines an orientation on the branched locus $b$, defined by the fact that $\overline{A}_l$ lies on the left of $b$ and $\overline{A}_r$ on its right.

Collapse the two boundaries of $\overline{A}^k$ by a diffeomorphism reversing the orientation, in order to obtain a compact oriented smooth branched surface without boundary $\overline{\Sigma}^k$. Smooth maps $\overline{A}^k, \overline{\Sigma}^k \to V$ in a smooth manifold are called *embeddings* if their derivative $T\overline{A}^k, T\overline{\Sigma}^k \to TV$ is injective, and *immersions* if their derivative restricted to the tangent space at every point is injective.

## 2.2. Almost-holomorphic embeddings in $\mathbf{C}P^2$.

In a complex hermitian surface, an immersed surface $S \subset \mathbf{C}^2$ is called *$\varepsilon$-holomorphic* if the angle between $T\Sigma$ and $iT\Sigma$ is uniformly bounded by $\varepsilon$. Smooth immersions of $\overline{A}^k$ or $\overline{\Sigma}^k$ are called $\varepsilon$-holomorphic if their image is an $\varepsilon$-holomorphic branched surface. Of course there does not exist holomorphic immersion of $\overline{\Sigma}^k$ in a complex surface, because of analytic continuation. However, there does exist $\varepsilon$-immersion for $\varepsilon > 0$ arbitrarily small, as we shall see.

We construct almost-holomorphic properly embedded annulus and branched annulus in the bidisc $B$ in $\mathbf{C}^2$, defined in some affine coordinates $x, y$ by $B = \{|x| \leq 1,\ |y| \leq 1\}$. The following result does not depend on the hermitian structure of the bidisc, because it is compact.

**Lemma 2.1.** *For every $\varepsilon > 0$ and every $k \geq 1$, there exist smooth proper $\varepsilon$-holomorphic embeddings of the branched annulus $\overline{A}^k$ such that their images coincide with the union of the axis $\{xy = 0\}$ on a neighborhood of $\partial B$.*

*Proof.* Consider the holomorphic non-singular curve $C_\varepsilon = \{xy^k = \varepsilon^k\}$. Its intersection with the bidisc is an annulus which is a non ramified cover over an annulus in the $x$-axis: it is defined by the graph of the multi-valued function $\varepsilon/x^{1/k}$ of $x$. The determinations of this function are $\varepsilon$-close to the $x$-axis when $|x|$ is close to 1, so that we can glue them together in a neighborhood of the circle $\{|x| = 1\}$ to obtain an $\varepsilon$-holomorphic branched annulus diffeomorphic to $\overline{A}^k$.

Now some details. Decompose the bidisc $B$ as the union of $B_1 = \{\varepsilon^{1/k} \leq |x| \leq 1, |y| \leq 1\}$ and $B_2 = \{0 \leq |x| \leq \varepsilon^{1/k}, |y| \leq 1\}$. The curve $C_\varepsilon$ intersects $B_1$ and $B_2$ in two annulus $A_1$ and $A_2$ that intersect in their common boundary $\{xy^k = \varepsilon^k, |x| = \varepsilon^{1/k}\}$.

Let $A_x = \{x \in \mathbf{C} \mid \varepsilon^{1/k} \leq |x| \leq 1\}$, $\widetilde{A}_x = \{x \in \mathbf{C} \mid \varepsilon^{1/k^2} \leq |x| \leq 1\}$ and $R = x^k : \widetilde{A}_x \to A_x$. Consider a smooth function $\varphi : [0,1] \to [0,1]$ which is identically 0 on $[1/2, 1]$, does not vanish on $[0, 1/2)$ and is identically 1 on $[0, 1/4]$. The map $\pi_1 : x \in \widetilde{A}_x \mapsto (x^k, \varepsilon\varphi(|x|)/x) \in B_1$ is a smooth immersion. It is holomorphic if $|x| \leq 1/2$ or if for $|x| \geq 1/4$. If $1/4 \leq |x| \leq 1/2$, its derivative $d\pi_1$ is $c\varepsilon$-close to the $k$-bilipschitz $\mathbf{C}$-linear embedding $x \in \mathbf{C} \mapsto (kx, 0) \in \mathbf{C}^2$, where $c > 0$ is a constant depending only on $\varphi$. Thus the map $\pi_1$ is a $c\varepsilon$-holomorphic immersion. Note $A_{x,l} = \{x|\ \varepsilon^{1/k} \leq |x| < 1/2\}$, $A_{x,r} = \{x|\ 1/2 < |x| \leq 1\}$ and $\overline{A}_x^k$ the smooth branched surface constructed as in 2.1. The map $\pi_1$ induces a $c\varepsilon$-holomorphic smooth embedding $\overline{\pi_1} : \overline{A}_x^k \to B_1$, whose image coincides with the $x$-axis on $\{(x,y)|\ 1/4 \leq |x| \leq 1\}$ and with $C_\varepsilon$ on $\{(x,y)|\ \varepsilon^{1/k} \leq |x| \leq 1/4^k\}$.

Let $A_y = \{y|\ \varepsilon^{1-1/k^2} \leq |y| \leq 1\}$. The image of the map $\pi_2 : y \in A_y \mapsto (\varepsilon^k\varphi(|y|)/y^k, y) \in B_2$ is a $c\varepsilon$-holomorphic annulus properly embedded in $B_2$ which coincides with the $y$-axis on $\{|x| \leq 1,\ |y| \geq 1/2\}$ and with $C_\varepsilon$ on $\{|x| \leq 1,\ |y| \leq 1/4\}$. The images of $\pi_1$ and $\pi_2$ in $B$ have a common boundary and their union is a $c\varepsilon$-holomorphic smooth branched surface which is a smooth embedding of $\overline{A}^k$. By construction, this branched surface coincides with the union of the $x$-axis and the $y$-axis on a neighborhood of $\partial B$. The lemma is proved.

We are now able to construct almost-holomorphic embeddings of $\overline{\Sigma}$ in $\mathbf{C}P^2$. Again, the result does not depend on the hermitian structure on the complex projective plane $\mathbf{C}P^2$.

**Theorem 2.2.** *For every integer $k \geq 2$ and every real $\varepsilon > 0$, there is a smooth $\varepsilon$-holomorphic embedding $\pi : \overline{\Sigma}^k \to \mathbf{C}P^2$.*

*Proof.* Consider a non degenerate triangle $a, b, c$ in $\mathbf{C}P^2$. Let $B_a, B_b$ and $B_c$ be bidiscs around $a, b$ and $c$. The lemma 2.1 shows that there is a smooth $\varepsilon$-holomorphic proper embedding of $\overline{A}^k$ in $B_a$ whose image $\overline{A}_a^k$ coincides with the union of the lines $(ab)$ and $(ac)$ on a neighborhood of $\partial B_a$. There is also a smooth $\varepsilon$-holomorphic properly embedded annulus $A_b$ (resp. $A_c$) in $B_b$ (resp. $B_c$) which coincides with the union of the lines $(ba)$ and $(bc)$ (resp. $(ca)$ and $(cb)$) in a neighborhood of $\partial B_b$ (resp. $\partial B_c$). The union

$$\overline{A}_a^k \cup A_b \cup A_c \cup \big(((ab) \cup (bc) \cup (ca)) - (B_a \cup B_b \cup B_c)\big)$$

is the image of a smooth $\varepsilon$-holomorphic embedding $\pi : \overline{\Sigma}^k \to \mathbf{C}P^2$.

2.3. **Totally real embeddings in $\mathbf{C}^2$.** A smooth immersion $\pi : \overline{\Sigma}^k \to \mathbf{C}^2$ is *totally real* if:

$$T\mathbf{C}^2|_{\overline{\Sigma}^k} = T\overline{\Sigma}^k \oplus iT\overline{\Sigma}^k.$$

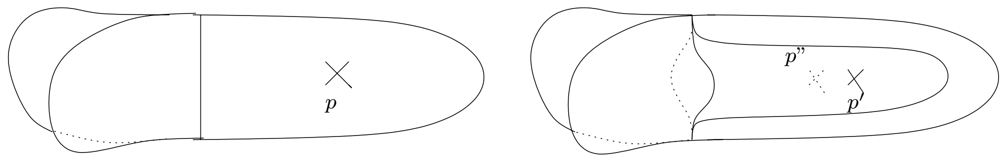
Observe that one can construct a smooth embedding $\pi : \overline{\Sigma}^2 \to \mathbf{C}^2$. Take the branched annulus $\overline{A}$ in the bidisc $B$ constructed in lemma 2.1. It is trivial to see that there exists a smooth annulus $A$ properly embedded in $\mathbf{C}^2 - B$, which coincides with the axis $x$ and $y$ on a neighborhood of $\partial B$. The union of $A$ and $\overline{A}^2$ is an embedding of $\overline{\Sigma}^2$ in $\mathbf{C}^2$.

**Theorem 2.3.** *Every smooth embedding $\overline{\Sigma}^2 \to \mathbf{C}^2$ is isotopic to a totally real embedding.*

*Proof.* The proof of the theorem uses the work of Forstneric [8]. By Thom's transversality theorem, a generic embedding $\pi : \overline{\Sigma}^2 \to \mathbf{C}^2$ has only a finite number of complex tangents (a *complex tangent* is a point $p$ of the image of $\overline{\Sigma}^2$ whose tangent plane is invariant by $i$), and one can suppose that there are not located on the branched locus. To every of these complex tangents, it is associated an index $I(p, \pi)$ defined in the following way [8]: let $U$ be a small disc neighborhood of $p$ in $\pi(\overline{\Sigma}^2)$. In suitable holomorphic coordinates $(z, w)$ centered at $p$, the surface $\pi(\overline{\Sigma}^2)$ is a graph of a smooth complex valued function $f(z)$ defined on a neighborhood of 0. Since the graph of $f$ is totally real at a point $(z, f(z))$ if and only if $f_{\overline{z}}(z) \neq 0$, the origin $z = 0$ is an isolated zero of the function $f_{\overline{z}}$. The winding number of this function around the origin is by definition the indice $I(p, \pi)$. One defines also the indices $I_+(\pi)$ (resp. $I_-(\pi)$) by the sum of the indices of the positive (resp. negative) complex tangent.

For *non-branched* compact oriented surface, the indices $I_+$ and $I_-$ are invariant under isotopy. However, this is not true in the case of branched surfaces, because they can jump when the complex tangents cut the branched locus. We will take advantage of this fact. In [8], it is proved that there exists a smooth isotopy $\pi_t : \overline{\Sigma} \to \mathbf{C}^2$, $t \in [0, 1]$, such that: if $I_+(\pi) \neq 0$, the image of $\pi_1$ has two positive complex tangents $p$ and $q$ of index $I(p, \pi) = -I_+(\pi)$ and $I(q, \pi) = 2I_+(\pi)$. If the index $I_+(\pi, r)$ vanishes at a point $r$, then $r$ is not a positive complex tangent. Of course, the result is also valid for negative tangents by changing the orientation.

Let $\gamma_+$ be a smooth embedded curve in $\overline{\Sigma}^2$ from $p$ to a point of the branched locus, which arrives to the branched locus by the right hand side and cuts the branched locus only at its extremity, and whose




$\pi_1(\overline{\Sigma}^2)$ $\pi_2(\overline{\Sigma}^2)$

FIGURE 2. The isotopy $\pi_t$, $1 \leq t \leq 2$

image does not contain a complex tangent. A neighborhood of $\gamma^+$ is the union of the image $D_i$ of two smooth embeddings $f_i : \mathbf{D} \to \overline{\Sigma}^2$, $i = 1, 2$ which coincide on the right part of the unit disc $\mathbf{D}$ in $\mathbf{R}^2$. We can suppose that the images of the disc by the $f_i$, $i = 1, 2$ do not contain complex tangent, and that the curve $f_i^{-1} \circ \gamma^+$ is the segment $[0, 1/2]$. Now consider an isotopy $F_{i,t}$, $i = 1, 2$ and $1 \leq t \leq 2$ of the maps $F_{i,1} = \pi_1 \circ f_i : \mathbf{D} \to \mathbf{C}^2$ in such a way that:

- they take different values on an increasing semi-disc $S_t \subset D$,
- they take the same value outside $S_t$,
- they coincide with $F_{i,1}$ on a neighborhood of the unit circle,
- and $1/2$ belongs to $S_2$.

If the $F_{i,t}$ are close enough to $F_{i,1}$ in the uniform topology, one define in this way an isotopy $\pi_t : \overline{\Sigma}^2 \to \mathbf{C}^2$, $1 \leq t \leq 2$, whose image is the union of the image of the $F_{i,t}$ and the image of $\overline{\Sigma}^2 - (D_1 \cup D_2)$. Because for non branched surface, the index $I_+$ is invariant by isotopy, we have $I_+(\pi_2) = 0$. Applying Forstneric's theorem, one can eliminate the positive complex tangents after an isotopy.

In the same way, one can eliminate the negative complex tangents by an isotopy. The theorem is proved.

*Remark* 2.4. For the branched surfaces $\overline{\Sigma}^k$, $k \geq 3$, the class modulo $k - 1$ of the indices $I_+(\pi)$ and $I_-(\pi)$ are invariant under isotopy. One has the same result: if this class vanishes, there is an isotopy that makes the branched surface totally real.

One can also consider smooth non-oriented branched surfaces $\overline{\Sigma}^k_{nonor}$, obtained from an annulus by identifying one of its boundary to the other by a $k : 1$ map reversing the orientation. The same theorem is true for these embeddings, if and only if $k$ is even. In particular, there exists a smooth totally real embedding $\overline{\Sigma}^2_{nonor}$ in $\mathbf{C}^2$.

## 3. A neighborhood theorem for branched surfaces

This section is devoted to the study of the neighborhood of some *smooth oriented branched surfaces* in oriented 4-manifold. Let $k \geq 2$ be an integer and $\pi : \overline{\Sigma}^k \to V$ be a smooth embedding of $\overline{\Sigma}^k$ in an oriented smooth 4-manifold $V$. Our technique also works for higher genus branched surfaces (see [5]).

### 3.1. The transverse braid class.

The normal bundle of $\pi$ is the 2-dimensional oriented bundle $\pi^*TV/T\overline{\Sigma}^k$. By the tubular neighborhood theorem, the neighborhood of a *non-branched* submanifold is characterized by its normal bundle. However, this is not true for the embeddings $\pi$; their neighborhoods are characterized by another invariant called the *transverse braid class*. It is a conjugacy class of the braid group on $k$ strings. Its definition is the goal of this paragraph.

**Lemma 3.1.** *The normal bundle of $\pi$ is trivial. Moreover, two trivialisations of it are homotopically equivalent on the branched locus.*

*Proof.* Because the normal bundle is oriented, it suffices to see that $H^2(\overline{\Sigma}^k, \mathbf{Z}) = 0$ to prove the first part of the lemma. Let $\overline{B}$ be a neighborhood of the branched locus of $\overline{\Sigma}^k$, and $\Sigma$ the exterior of the branched locus in $\overline{\Sigma}^k$: the branched locus separates $\overline{B}$ in right and left annulus $\overline{B}_r$ and $\overline{B}_l$. The Mayer-Vietoris exact sequence associated to the decomposition $\overline{\Sigma}^k = \Sigma \cup \overline{B}$ is:

$$H^1(\Sigma) \oplus H^1(\overline{B}) \xrightarrow{i^*_\Sigma \oplus i^*_{\overline{B}}} H^1(\overline{B}_r \cup \overline{B}_l) \to H^2(\overline{\Sigma}^k) \to H^2(\Sigma) \oplus H^2(\overline{B}),$$

the last term being 0 because the top cohomology of a non compact surface vanishes, and $\overline{B}$ is homotopically equivalent to a circle. Thus it suffices to prove that $i^*_\Sigma \oplus i^*_{\overline{B}}$ is surjective. The group $H^1(\overline{B}_r \cup \overline{B}_l)$ is the free abelian group generated by $b_r^*$, $b_l^*$ (the dual basis), where $b_r$, $b_l$ are the generators of $H_1(\overline{B}_r)$ and $H_1(\overline{B}_l)$ giving the same orientation to the branched locus. Because $b_r = b_l$ in $H_1(\Sigma)$, there exists a linear form $\lambda_\Sigma \in H^1(\Sigma)$ such that $\lambda_\Sigma(b_r) = \lambda_\Sigma(b_l) = 1$. On the other hand, we have $b_l = kb_r$ in $H_1(\overline{B})$, so that there exists a linear form $\lambda_{\overline{B}} \in H^1(\overline{B})$ such that $\lambda_{\overline{B}}(b_r) = 1$ and $\lambda_{\overline{B}}(b_l) = k$. Thus the forms $i^*_\Sigma\lambda_\Sigma$ and $i^*_{\overline{B}}\lambda_{\overline{B}}$ generate $H^1(B_l \cup B_r)$, and the first part of the lemma is proved.

The second part of the lemma follows essentially from the fact that $(k-1)[b] = 0$ in $H_1(\overline{\Sigma}^k, \mathbf{Z})$. If $s_i : \overline{\Sigma}^k \to N$, $i = 1, 2$ are non vanishing continuous sections of the normal bundle of $\overline{\Sigma}^k$, there exists a non vanishing function $f : \overline{\Sigma}^k \to \mathbf{C}^*$ such that $s_2 = fs_1$. Because $(k-1)[b] = 0$ in $H_1(\overline{\Sigma}^k, \mathbf{Z})$ the indice of $f$ on $(k-1)b$ vanishes. Thus, because $k-1\neq 0$ the indice of $f$ on $b$ is 0, and $f|_b : b \to \mathbf{C}^*$ is homotopic to a constant.

We analyse the neighborhood of the branched locus. We begin by some notations. Let $B\subset A$ be a small annular neighborhood of $b$. We note $\widetilde{B} = R^{-1}(B)$, $B_l$ and $B_r$ the left and right components of $b$, and $P:\widetilde{B}\to\overline{B}$ the quotient of $\widetilde{B}$ by the relation identifying the fibers $R^{-1}(x)$ in $\widetilde{B}$ if $x$ belongs to the closure of $B_r$. Note that we have a natural smooth immersion $Q:\overline{B}\to B$.

**Lemma 3.2** (Neighborhood of the branched locus)**.** *Let* $s:\overline{\Sigma}^k\to N$ *be a non-vanishing section of the normal bundle of* $\overline{\Sigma}^k$*. There exists a neighborhood* $\mathcal{W}$ *of the branched locus* $\pi(b)$ *in* $V$ *and a diffeomorphism* $\psi=(q,\zeta):\mathcal{W}\to B\times\mathbf{D}$ *where* $B$ *is an annular neighborhood of* $b$ *in* $A$*, such that*

- $q\circ\pi = Q$.
- *On the branched locus* $b$: $\frac{\partial}{\partial\zeta} = s$.

*Moreover, the isotopy type of these coordinates is uniquely determined by the second condition.*

*Proof.* First, we construct the map $q$ *locally* at the neighborhood of the branched locus. On the neighborhood $D$ of a point $x$ of $b\subset B$, we have $k$ determinations $S_i : D\to\overline{B}$, $i=1,\dots,k$ such that $Q\circ S_i = id$. The maps $\pi\circ S_i : D\to V$ are smooth embeddings that coincide on the right part of $D$. By the tubular neighborhood theorem, there exists neighborhoods $\mathcal{W}_i'$ of $\pi\circ S_i(D)$ in $V$ and submersions $q_i' : \mathcal{W}_i'\to D$ such that $q_i'\circ\pi\circ S_i = id$. Think of $B$ as being an embedded annulus in the complex line $\mathbf{C}$. Let $q' = (q_1'+\ldots+q_k')/k$ be the submersion from the neighborhood $\mathcal{W}' = \mathcal{W}_1'\cap\ldots\cap\mathcal{W}_k'$ of $\pi(x)$ to a neighborhood of $x$. The maps $q'\circ\pi\circ S_i$, $i=1,\dots,k$ are equal to a diffeomorphism $F$ from a neighborhood of $x$ into its image. On the right hand part of $D$ this diffeomorphism is the identity. The submersion $q = F^{-1}\circ q'$ is defined on a neighborhood of $\pi(x)$, and verify $q\circ\pi = Q$. To get a global map $q$, it suffices to use a partition of the unity, and to consider a sufficiently small annular neighborhood $B$ of $b$.

Because $V$ is oriented the branched locus $\pi(b)$ is transversely oriented in the 3-manifold $\widetilde{R}^{-1}(b)$. Thus there exists a smooth submersion $\zeta$ from a neighborhood of $\pi(b)$ in $\widetilde{R}^{-1}(b)$ to $\mathbf{D}$ such that the map $(q,\zeta)$ is a diffeomorphism into the product $b\times\mathbf{D}$. Of course, one can choose $\zeta$ so that $\frac{\partial}{\partial\zeta} = s$. Then, $\zeta$ can be smoothly extended to a submersion $\zeta:\mathcal{W}\to\mathbf{D}$ from a neighborhood $\mathcal{W}$ of $b$ in $V$ to $\mathbf{D}$ in such a way that $\psi=(q,\zeta):\mathcal{W}\to B\times\mathbf{D}$ is a diffeomorphism. The isotopy type of the coordinates $\psi$ is determined by the homotopy type of the restriction of $s$ to the branched locus, which is uniquely determined by lemma 3.1. The lemma is proved.

Recall that a conjugacy class of the braid group on $k$ strings is represented by a smooth embedded curve in the solid torus $\mathbf{S}^1 \times \mathbf{D}$, transverse to the vertical fibration and intersecting the fibers in $k$ points. In fact the conjugacy classes of the braid group on $k$ strings are in correspondance with the isotopy classes of these transverse curves, modulo isotopy tangent to the vertical fibration.

**Definition 3.3.** Let $\psi : \mathcal{W} \to A \times \mathbf{D}$ be the coordinates constructed in lemma 3.2, which are well-defined up to isotopy. Let $b_l \subset B$ be a smooth oriented circle obtained by pushing $b$ in the left part of $B$. The curve $\psi(\overline{\Sigma}^k) \cap (b_l \times \mathbf{D})$ in the solid torus $b_l \times \mathbf{D}$ defines a conjugacy class of a braid on $k$ strings: it is called the *normal braid class.* Remark that this braid induces a cyclic permutation on the set with $k$ elements.

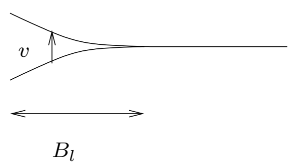



FIGURE 3. The section $v$ of the normal bundle

*Remark* 3.4. If $k = 2$, the braid group is $\mathbf{Z}$, and the normal braid class is naturally identified to an odd integer. In that case, there is a more intuitive way to think of the transverse braid class. On a left annular neighborhood $B_l$ of the branched locus, let $v$ be the normal vector field that looks to the opposite sheet of $\overline{\Sigma}$, as shown in figure 3.1. Let $n : \overline{\Sigma} \to N$ be a global trivialisation of the normal bundle of $\overline{\Sigma}$. The normal braid class is the winding number of the function $v/n : B_l \to \mathbf{C}^*$.

**Example 3.5.** Let $\mathcal{B}$ be a conjugacy class of the braid group on $k$ strings, inducing a cyclic permutation. We construct a smooth oriented 4-manifold $V_{\mathcal{B}}$ with boundary and a smooth embedding $\pi : \overline{\Sigma}^k \to V_{\mathcal{B}}$ whose transverse braid class is $\mathcal{B}$. This shows that the transverse braid class is a non-trivial invariant.

Let $f : \widetilde{A} \to \mathbf{D}$ be a smooth function, vanishing on $A_r$, such that $(R, f) : \widetilde{A} \to A \times \mathbf{D}$ induces a smooth embedding $\pi : \overline{A}^k \to A \times \mathbf{D}$, and such that the intersection of the image of $\pi$ with $\partial_l A \times \mathbf{D}$ represent the class $\mathcal{B}$. There is a canonical trivialisation of the normal bundle of $\partial_{l,r}\pi(\overline{A})$ in $\partial A \times \mathbf{D}$ given by the product structure. Thus there exists a canonical choice, up to isotopy, of a diffeomorphism $\Phi$ from a neighborhood of $\partial_l \pi(\overline{A})$ to the solid torus $\partial_r \pi(\overline{A}) \times \mathbf{D}$. We glue these tori by $\Phi$, in order to build a smooth oriented 4-manifold $V_{\mathcal{B}}$ and a smooth embedding $\pi : \overline{\Sigma}^k \to V_{\mathcal{B}}$. The transverse braid class of this embedding is $\mathcal{B}$.

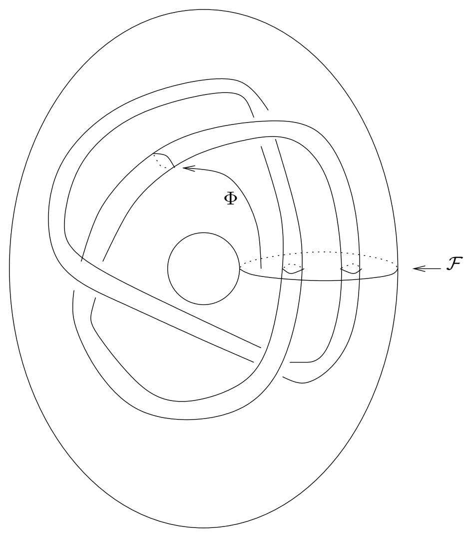



Figure 4. Hirsch foliation

The boundary of $V_{\mathcal{B}}$ has corners but can be smoothed. It is homeomorphic to the Hirsch manifold, obtained by identifying without twist the boundary components of the domain between the two solid tori showed in figure 3.5, where the solid torus inside represents the conjugacy class $\mathcal{B}$.

It is interesting to note that, given a non-ramified $k:1$ map $r$ from the circle to itself, there are two geometric constructions that code the dynamics of $r$:

- the vector field on the branched surface $\overline{\Sigma}^k$ represented in figure 2.1.
- the foliation $\mathcal{F}$ by surfaces on the Hirsch manifold represented in figure 3.5.

In fact, the leaves of Hirsch foliations can be thought of as the boundaries of neighborhoods of the orbits of the vector field on $\overline{\Sigma}^k$, as is shown in figure 3.5. This construction gives an interesting duality between the Hirsch foliations and the branched surfaces $\overline{\Sigma}^k$ equipped with a vector field both realizing the dynamics of an endomorphism of the circle.

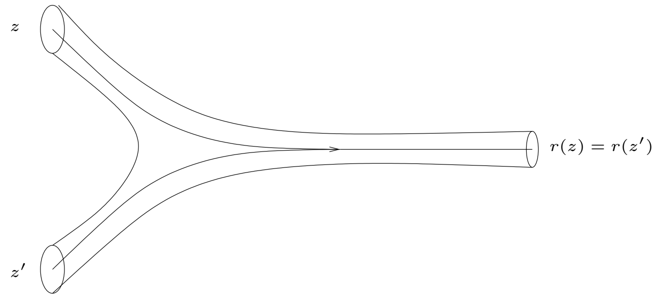



FIGURE 5. Duality between Hirsch foliation and vector fields on branched surfaces

**Theorem 3.6** (Neighborhood theorem)**.** *Let $k \geq 2$ be an integer, and $\pi_i : \overline{\Sigma}^k \to V_i$, $i = 1, 2$ be smooth embeddings of $\overline{\Sigma}^k$ in oriented smooth 4-manifolds $V_i$, having the same transverse braid class. For $i = 1, 2$, there exist neighborhoods $\mathcal{V}_i$ of $\pi_i(\overline{\Sigma}^k)$ in $V_i$ and a diffeomorphism $\Psi : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\Psi \circ \pi_1$ is arbitrarily close to $\pi_2$ in the smooth topology.*

*Remark* 3.7. Even on the neighborhood of a point of the branched locus, it is not true that one can find $\Psi$ such that $\Psi \circ \pi_1 = \pi_2$, like in the tubular neighborhood theorem.

*Proof.* Take the neighborhood $W_i$ of the branched locus and the coordinates $\psi_i$ of $\pi_i(\overline{\Sigma})$ constructed in 3.2, in such a way that the curves $\psi_i \circ \pi_i(\overline{\Sigma}^k) \cap (b_l \times \mathbf{D})$ are isotopic, by an isotopy preserving the vertical fibration of $b_l \times \mathbf{D}$. In such coordinates, it is easy to construct a diffeomorphism $\Phi : W_1 \to W_2$ isotopic to the identity such that $\Phi \circ \pi_1$ is arbitrary close to $\pi_2$ in the smooth topology, and coincide with $\pi_2$ on a neighborhood of $\partial \overline{B}$. In coordinates $(X, \zeta)$ of $B \times \mathbf{D}$, we have the following facts:

- the section $\frac{\partial}{\partial \zeta}$ of the normal bundle of $\pi_1(\overline{\Sigma})$ can be extended to a global non vanishing section.
- the section $\phi_* \frac{\partial}{\partial \zeta}$ of the normal bundle of $\pi_2(\overline{\Sigma})$ can be extended to a global non vanishing section.

This is because the transverse braid classes of the $\pi_i$'s are the same. These non vanishing sections induce isomorphisms between the normal bundles of the $\pi_i$ outside the $W_i$. By the classical techniques of the tubular neighborhood theorem using riemannian metrics, we find an extension of $\phi$ to a neighborhood of $\pi_1(\overline{\Sigma})$ such that outside $W_1$ one has $\pi_2 = \phi \circ \pi_1$. The theorem is proved.

## 4. Solenoids and branched surfaces

A *Riemann surface lamination* of a topological space $X$ is an atlas $\mathcal{L}$ made by homeomorphisms $\varphi : U \to \mathbf{D} \times T$ from open sets of $X$ to the product of a disc by a transverse space $T$, such that:

(i) the open sets $U$ cover $X$.

(ii) The change of coordinates preserve the local fibrations by discs.

(iii) They are holomorphic along the fibers. A Riemann surface lamination inherits a smooth structure [10]. For $l = 0, 1, \ldots, \infty$, a function $f : X \to V$ with values in a smooth manifold is of class $C^l$ if in the charts $(z, t) : U \to \mathbf{D} \times T$ of $\mathcal{L}$, the functions

$$f_t = f(., t) : \mathbf{D} \to V$$

are of class $C^l$ and depend continuously of $t$ in the $C^l$ topology. This definition is independant of the coordinate chart, because the change of coordinates are holomorphic along the fibers and continuous transversely.

4.1. **Sullivan's solenoids** [16]. Let $\mathbf{D}^* = \{q \in \mathbf{C} \mid 0 < |q| < 1\}$ be the punctured disc, $k \geq 2$ an integer and $R$ the endomorphism

$$q \in \mathbf{D}^* \mapsto R(q) = q^k \in \mathbf{D}^*.$$

The inverse limit of the tower of holomorphic non ramified coverings $\ldots \xrightarrow{R} \mathbf{D}^* \to \ldots \xrightarrow{R} \mathbf{D}^*$ is the space $\mathbf{D}^*_\infty$ of bi-infinite sequences $\hat{q} = (q_n)_{n \in \mathbf{Z}}$ of points of $\mathbf{D}^*$ such that $q_{n+1} = q_n^k$. It is equipped with the product topology.

**Lemma 4.1.** *The inverse limit $\mathbf{D}^*_\infty$ has a natural structure of a Riemann surface lamination, given by the fact that the map $q_0 : \mathbf{D}^* \to \mathbf{D}^*$ is a fibration by the Cantor set $\mathbf{Z}_k$ of $k$-adic integers.*

*Proof.* To see that, think of $\mathbf{D}^*_\infty$ as being a quotient of the trivial product Riemann surface lamination $\mathbf{H} \times \mathbf{Z}_k$, where $\mathbf{H}$ is the upper half plane in the complex line. Indeed, if $s = s_0 + ks_1 + \ldots + k^n s_n + \ldots$ is a $k$-adic integer, and $\tau$ an element of the upper half plane, one can construct the element $\hat{q}$ of $\mathbf{D}^*_\infty$ defined by the bi-infinite sequence

$$\forall n \geq 0, \ q_n = e^{k^n i\tau} \ \text{ and } \ q_{-n} = e^{(i\pi S_{n-1} + i\tau)/k^n},$$

where $S_n = s_0 + ks_1 + \ldots + k^n s_n$. The map $(\tau, s) \in \mathbf{H} \times \mathbf{Z}_k \mapsto \hat{q}(\tau, s) \in \mathbf{D}^*_\infty$ induces an homeomorphism between the quotient of $\mathbf{H} \times \mathbf{Z}_k$ by the translation $(\tau, s) \mapsto (\tau + 1, s + 1)$ and $\mathbf{D}^*_\infty$. Because this translation is a Riemann surface lamination isomorphism of $\mathbf{H} \times \mathbf{Z}_k$ to itself, it induces a Riemann surface lamination structure on $\mathbf{D}^*_\infty$. This ends the proof of the lemma.

Observe that the double cover $R$ lifts to a homeomorphism $R_\infty$ of Riemann surface lamination from $\mathbf{D}^*_\infty$ to itself, defined by $R_\infty(\hat{q}) = (q_n^k)_{n\in\mathbf{Z}}$. In fact $R_\infty$ lifts to the map $(\tau, s) \mapsto (k\tau, ks)$ from $\mathbf{H} \times \mathbf{Z}_k$ to itself, and thus $R_\infty$ is an isomorphism of Riemann surface lamination. It acts properly, cocompactly and without fixed points. Thus the quotient $\mathcal{S}^k$ of $\mathbf{D}^*_\infty$ by $R_\infty$ is a compact Riemann surface lamination. This lamination has been introduced by Sullivan in [16] for the study of $C^1$-conjugacy classes of expanding endomorphisms of the circle. We call $\mathcal{S}^k$ the $k$-th *Sullivan's solenoid.*

**Proposition 4.2.** *The solenoid $\mathcal{S}^k$ has hyperbolic holonomy and its leaves are dense. Moreover, there are transversely invariant conformal structures on $\mathcal{S}^k$ given by the ultra-metrics $d_\alpha$ on $\mathbf{Z}_k$ defined by*

$$d_\alpha(s, s') = \alpha^{v_k(s-s')},$$

*for every real number $\alpha$ verifying $0 < \alpha < 1$. Observe that the transverse space is of Hausdorff dimension $\delta = -\frac{\log k}{\log \alpha}$ for the conformal structure $d_\alpha$.*

*Proof.* This is a consequence from the fact that the holonomy pseudo-group is conjugated to the pseudo-group of affine tranformations acting on $\mathbf{Z}_k$, of the form $x \mapsto ax + b$ where $a$ is a power of $k$ and $b$ belong to the ring $\mathbf{Z}[1/k]$.

4.2. **Approximation of $\mathcal{S}^k$ by $\overline{\Sigma}^k$.** A point of $\mathcal{S}^k$ is a part $\{q_n\}_{n\in\mathbf{Z}}$ of points of $\mathbf{D}^*$ for which $q_{n+1} = q_n^k$. Such a part is called an *orbit* of $R$. The topology on $\mathcal{S}^k$ is induced by the Hausdorff topology on compact subsets of the punctured disc.

For every $0 < c < 1$, identify two orbits $O, O' \in \mathcal{S}^k$ if they coincide on the open punctured disc of radius $c$, and let $P_c : \mathcal{S} \to \overline{\Sigma}^k_c$ be the quotient map. For $l = 0, 1, \ldots, , \infty$, a function $f : \overline{\Sigma}^k_c \to V$ with value in a smooth manifold is of class $C^l$ if $f \circ P_c$ is a function of class $C^l$ on $\mathcal{S}^k$.

**Lemma 4.3.** *The space $\overline{\Sigma}^k_c$ is diffeomorphic to $\overline{\Sigma}^k$, and the map $P_c$ : $\mathcal{S}^k \to \overline{\Sigma}^k_c$ is a smooth immersion.*

*Proof.* Let $c < c' < \sqrt{c}$ be a small real number. Consider the annulus $A' = \{c^2 \leq |q| \leq c'\}$, and the space $P' : A' \to \overline{\Sigma}'$ obtained from $A'$ by identifying the points $q$, $-q$ and $q^k$ if $|q| \geq c$. This space is clearly diffeomorphic to the branched surface $\overline{\Sigma}^k$, if it is equipped with the following smooth structure: for $l = 0, 1, \ldots, \infty$, a function $f : \overline{\Sigma}' \to V$ is of class $C^l$ if $f \circ P'$ is of class $C^l$. If $O$ is an orbit of $R$, its intersection with $A'$ is either one point, or three points of the form $\{q, -q, q^k\}$ where

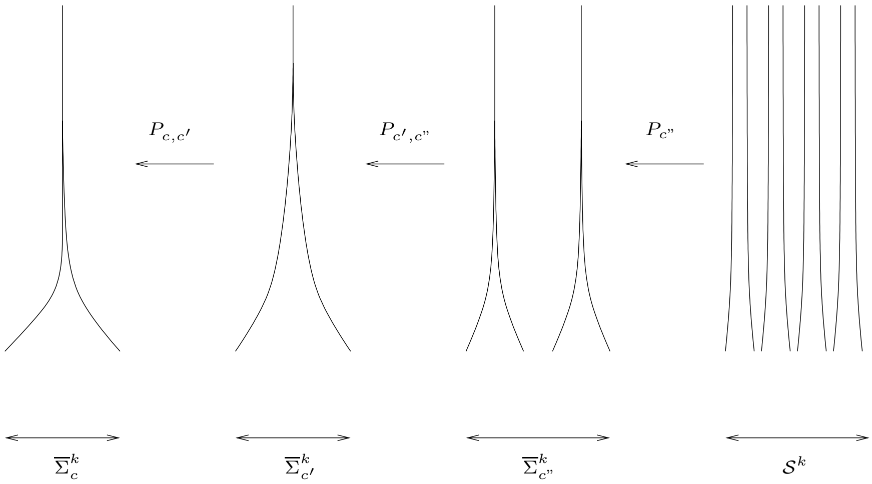



FIGURE 6. Peeling the branched locus off (with the values $c < c' < c" = \sqrt{c}$)

$|q| \geq c$. Thus we have a canonical map $P : \mathcal{S} \to \overline{\Sigma}'$. It is obvious that this map is a smooth immersion. The lemma is proved.

Because the fibers of $P_{c'}$ are contained in the fibers of $P_c$ if $0 < c < c' < 1$, there exists a continuous map

$$P_{c,c'} : \overline{\Sigma}_{c'}^k \to \overline{\Sigma}_c^k$$

for which $P_c = P_{c,c'} \circ P_c$. Observe that the diameter of the fibers of $P_c$ tends uniformly to 0 as $c$ tends to 1, so that the solenoid $\mathcal{S}^k$ is the inverse limit of the system of maps $\{P_{c,c'}\}$. The maps $P_{c,c'} : \overline{\Sigma}_{c'}^k \to \overline{\Sigma}_c^k$ are smooth immersions. In [17, 5], it is proved that $\mathcal{S}^k$ is the inverse limit of the system $\{P_{c,c'}\}$ in the *smooth topology*, but we will not need this fact here. Thus $\mathcal{S}^k$ is obtained from $\overline{\Sigma}_c$ by "peeling the branched locus off".

## 5. Embeddings of Sullivan's solenoid in 4-manifolds

5.1. **Ghys's automorphic functions.** Ghys has discovered some remarcable *automorphic* functions on the Sullivan's solenoids $\mathcal{S}^k$. Consider a complex number $\alpha$ of modulus $0 < |\alpha| < 1$, an integer $p$, and the following series

$$G_p(\hat{q}) = \sum_{n \in \mathbf{Z}} \alpha^n q^{kp+1/k^n},$$

the notation $q^{k^n} = q_n$ being suggested by Ghys. For $n$ positive it is equivalent to a geometric series because the $k^n$ roots of $q$ converge to the unit circle. For negative $n$ the geometric series $\alpha^n$ is killed by the term $q^{k^n}$ which converges faster than any exponential to 0. The series $G_p$ is then convergent, and it verifies the relation

$$G_p(R_\infty \hat{q}) = \alpha G_p(\hat{q}).$$

Thus $G_p$ behaves as an automorphic function. Indeed, if $p_0, \ldots, p_N$ are distinct integers, the map

$$G = [G_{p_0} : \ldots : G_{p_N}]$$

defines a "birational" map from $\mathcal{S}^k$ to $\mathbf{C}P^N$, which is holomorphic at any point $\hat{q}$ for which the $G_{p_i}$'s do not vanish simultaneously. In [6], it is proved that one can obtain this way:

- an holomorphic embedding of $\mathcal{S}^k$ in $\mathbf{C}P^N$ if $N$ is large enough.
- an holomorphic map in $\mathbf{C}P^2$, which is an immersion along the leaves.

Numerical experiments lead by Sevennec have suggested that these automorphic functions do not raise to a holomorphic embedding in $\mathbf{C}P^2$. To comfort these experiments, we showed in [6] that these immersions are not $d_\alpha$-bilipschitz embeddings if $|\alpha| \neq k^{(1-\sqrt{13})/2}$. It is interesting to note that for these values of $\alpha$, the Hausdorff dimension of $d_\alpha$ is $\delta = (1+\sqrt{13})/2$, independant of $k$.

However, the Sullivan's solenoids can be equipped with different Riemann surface structures. It would be interesting to study similar automorphic functions, and the possible obstructions to obtain holomorphic $d_\alpha$-bilipschitz holomorphic embeddings in the complex projective plane.

5.2. **Peeling the branched locus off in the 4-manifold.** This paragraph is devoted to the construction of embeddings of Sullivan's solenoid in a smooth oriented 4-manifold, arbitrarily close in the smooth topology to a smooth embedding of the branched surface $\overline{\Sigma}^k$.

**Theorem 5.1.** *Let $k \geq 2$ be an integer, $0 < c < 1$ and $\mathcal{B}$ a conjugacy class of the braid group on $k$ strings, inducing a cyclic permutation. There exists a number $0 < \delta_{\mathcal{B}} < 2$ with the following property. Let $\pi_c : \overline{\Sigma}^k_c \to V$ be a smooth embedding of $\overline{\Sigma}^k_c$ in an oriented smooth 4-manifold, whose transverse braid class is $\mathcal{B}$. There exists a smooth embedding $\pi : \mathcal{S}^k \to V$ which is arbitrarily close to $\pi_c \circ P_c$ in the smooth topology. Moreover, $\pi$ can be chosen to be $d_\alpha$-bilipschitz for every $\alpha$ such that $-\log k / \log \alpha < \delta_{\mathcal{B}}$.*

*Proof.* According to theorem 3.6, it suffices to show the statement in a particular example of a smooth embedding $\pi_c : \overline{\Sigma}^k_c \to V$ with arbitrary transverse braid class (inducing a cyclic permutation on the set with $k$ elements). We do that in the $C^l$ topology for every integer $l \geq 1$.

Let $\mathcal{B}$ be a conjugacy class of the braid group on $k$ strings, inducing a cyclic permutation. The class $\mathcal{B}$ can be represented by a smooth 1-dimensional manifold of $\mathbf{S}^1 \times \mathbf{D}$ transverse to the vertical fibration by discs. Let

$$z \in \mathbf{S}^1 \mapsto (z^k, f(z)) \in \mathbf{S}^1 \times \mathbf{D}$$

be a parametrization of it. By approximating $f$ by an analytic function, one can suppose that $f$ is analytic. Thus it can be extended to a holomorphic function on a neighborhood of the unit circle.

Let $0 < d < 1$ be a real number close to 1, $\alpha > 0$ a small real number, and $D = D(0, M/(1-\alpha))$ the closed disc around 0 in $\mathbf{C}$ of radius $M/(1-\alpha)$, where $M = \sup |f|$. The map $\Phi : (q, \zeta) \in \{|q| = d\} \times D \mapsto (q^k, f(q) + \alpha\zeta) \in \{|q| = d^k\} \times D$ is a well-defined embedding if $\alpha$ is small enough. Let $V$ be the oriented 4-manifold obtained from $\{d^k \leq |q| \leq d\} \times D$ by attaching the point $(q, \zeta)$ to the point $\Phi(q, \zeta) = (q^k, f(q) + \alpha\zeta)$ if $|q| = d$ and $\zeta \in D$. The topology of this smooth 4-manifold with boundary and corner has already been studied in 3.5.

Let $c \geq 1/4$, and $g : \mathbf{D}^* \to [0, 1]$ be a smooth function which is identically 1 on a neighborhood of $\{|q| \leq d^{1/4}\}$, strictly positive on $\{|q| < c\}$, and vanishing on $\{|q| \geq c\}$. We also suppose that $g$ depends only on $|q|$ and is a decreasing function of it. Consider the finite series

$$F(\hat{q}) = g(q^{1/k})f(q^{1/k}) + \alpha g(q^{1/k^2})f(q^{1/k^2}) + \ldots + \alpha^{n-1} g(q^{1/k^n})f(q^{1/k^n}) + ..$$

The function $F$ is smooth on $\mathbf{D}^*_\infty$, strictly bounded by $\frac{M}{1-\alpha}$.

We have the relation

$$F(R_\infty \hat{q}) = f(q) + \alpha F(\hat{q})$$

if $|q| = d$. The set $\hat{A} = q^{-1}(A)$ contained in $\mathbf{D}^*_\infty$ is a fundamental domain for the action of $R_\infty$. The smooth immersion $P \circ (q, F) : \hat{A} \to V$ is constant along the orbits of $R_\infty$. Thus it induces a smooth immersion $\mathcal{S} \to V$. Because this map depends only on the intersection of an orbit with the punctured disc of radius $c$, it induces a smooth immersion $\pi_c = P \circ (q, F) : \overline{\Sigma}^k_c \to V$.

**Lemma 5.2.** *If $\alpha > 0$ is small enough, the map $\pi_c$ is a smooth embedding of $\overline{\Sigma}^k_c$ in $V$ whose transverse braid class is $\mathcal{B}$.*

*Proof.* It suffices to prove that for any $z$ in $A$, the map $F_c$ takes the same value on two points of $q^{-1}(z)$ if and only if they coincide on $\{0 < |q| < c\}$. Let $\hat{q}_1 \neq \hat{q}_2 \in q^{-1}(z)$, and $n \geq 1$ be the smallest integer such that $q_1^{1/k^n} \neq q_2^{1/k^n}$. Observe that there exists a $k$-th root of the unity $\zeta \neq 1$ such that $q_1^{1/k^n} = \zeta q_2^{1/k^n}$. Because $g$ is a decreasing function of $|q|$, we get the inequality

$$(5.1) \qquad |F(\hat{q}_2) - F(\hat{q}_1)| \geq \alpha^{n-1} g(|q_i|^{1/k^n}) C(\alpha, f),$$

where $C(\alpha, f) = m - \frac{2\alpha M}{1-\alpha}$, $M = \sup |f|$ and $m = \inf(|f(q) - f(\zeta q)| \mid |q| \geq d,\ \zeta^k = 1,\ \zeta \neq 1)$. Thus, if $\alpha > 0$ is small enough, $C(\alpha, f) > 0$ and $\pi_c$ is a smooth embedding.

To see that the transverse braid class of $\pi_c$ is $\mathcal{B}$, it suffices to observe that there is a non-vanishing section of the normal bundle of $\overline{\Sigma}_c$ of the form

$$\beta(q) \frac{\partial}{\partial \zeta},$$

where $\beta$ is a non-vanishing function which takes the value $\alpha$ if $|q| = d^2$ and 1 if $|q| = d$. The lemma is proved.

We are going to peel the branched locus of $\pi_c(\overline{\Sigma}_c^k)$ off in $V$. Let $\varepsilon > 0$ be a small number, and $l \geq 1$ an integer. Consider a smooth non vanishing function $g_\varepsilon : \mathbf{D} \to (0, 1]$ such that

- $g_\varepsilon$ is 1 on a neighborhood of $\{|q| \leq d^{1/4}\}$,
- $g_\varepsilon$ is a decreasing function of $|q|$,
- $g_\varepsilon$ is $\varepsilon$-close to $g$ in the $C^l$-topology (the derivatives are computed with respect to the hyperbolic metric of $\mathbf{D}^*$),
- $g_\varepsilon$ is identically $\varepsilon$ on $\{|q| \geq c\}$.

Let $f_\varepsilon = g_\varepsilon f$ and consider the infinite series

$$F_\varepsilon(\hat{q}) = f_\varepsilon(q^{1/2}) + \alpha f_\varepsilon(q^{1/4}) + \ldots + \alpha^{n-1} f_\varepsilon(q^{1/2^n}) + \ldots.$$

This series converges to a smooth function strictly bounded by $\frac{M}{1-\alpha}$, so that as before, we have a smooth immersion $\pi = P \circ (q, F_\varepsilon) : \mathcal{S}^k \to V$. The estimates (5.1) shows that if $z \in A$ and $\hat{q}_1 \neq \hat{q}_2 \in q^{-1}(z)$ then

$$|F_\varepsilon(\hat{q}_2) - F_\varepsilon(\hat{q}_1)| \geq C\alpha^{n-1},$$

where $C > 0$ is a constant depending on $f$, $\alpha$ and $\epsilon$. Thus $\pi$ is a smooth $d_\alpha$-bilipschitz embedding of $\mathcal{S}^k$ in $V$. Moreover, by construction the function $g - g_\varepsilon$ is uniformly bounded by $\varepsilon$ in the $C^k$ topology, so that $\pi$ is $\varepsilon$-close to $\pi_c$ in the $C^l$ topology. The theorem is proved.

**Corollary 5.3** (Almost-holomorphic laminations in $\mathbf{C}P^2$)**.** *For every $k \geq 2$, there exists a number $\delta_k > 0$ such that for every $0 < \delta < \delta_k$, there exist a lipschitz almost-complex structure $J$ on $\mathbf{C}P^2$, arbitrarily close to the standard one in the uniform topology, and a compact lamination by $J$-holomorphic curves, which is transversely Lipschitz of transverse dimension $\delta$. This lamination is a smooth embedding of $\mathcal{S}^k$. The maximum of the $\delta_k$ is $\delta_6 = 1.6309\ldots$.*

*Proof.* For every $k \geq 2$, let $\mathcal{B}_k$ be the transverse braid class of the almost-holomorphic embeddings of $\overline{\Sigma}^k$ in $\mathbf{C}P^2$ constructed in 2.2. By our construction, the braid $\mathcal{B}_k$ is represented by some curves in the solid torus of the form

$$f : q \in \mathbf{S}^1 \mapsto (q^k, q^{s+rk}) \in \mathbf{S}^1 \times \mathbf{D},$$

where $s, r$ are integers and $s$ has no common divisors with $k$. Let $d_k$ be the euclidian distance between two consecutive $k$-th roots of the unity. Observe that if

$$0 < \alpha < \alpha_k = \frac{d_k}{2 + d_k},$$

the conditions on $\alpha$ in the proof of theorem 5.1 are satisfied. Thus, for every $0 < \alpha < \alpha_k$, and for every $\varepsilon > 0$, there exists a smooth $\varepsilon$-holomorphic embedding of $\mathcal{S}^k$ in $\mathbf{C}P^2$, which is $d_\alpha$-bilipschitz. Remark that the Hausdorff dimension $\delta_k$ of the distance $d_{\alpha_k}$ is

$$\delta_k = -\log k / \log \alpha_k.$$

The maximum $\delta_{max}$ of these Hausdorff dimension is attained for $k = 6$: one has $\delta_{max} = \delta_6 = 1.6309...$

Given an $\varepsilon$-holomorphic and $d_\alpha$-bilispchitz embedding of $\mathcal{S}^6$ in $\mathbf{C}P^2$, it suffices to construct the almost-complex structure $J$. We define it on the tangent bundle $T\mathcal{S}$, in such a way that it is lipschitz and $\varepsilon$-close to the standard one. This almost complex structure defined on $T\mathcal{S}$ can be extended to a lipschitz almost complex structure on $T\mathbf{C}P^2$ which is $\varepsilon$-close to the standard one. The corollary is proved.

**Corollary 5.4** (Totally real laminations in $\mathbf{C}^2$)**.** *For every $0 < \delta < 1$, there exists a compact lamination by totally real smooth surfaces, which is transversely Lipschitz of transverse dimension $\delta$. This lamination is a smooth embedding of $\mathcal{S}^2$.*

*Proof.* Because $\delta_2 = 1$ (see theorem 5.1 and the first paragraph of the proof of corollary 5.3), for every $0 < \delta < 1$ one can perturb a smooth embedding of $\overline{\Sigma}_2$ in an oriented 4-manifold by a smooth embedding of $\overline{\Sigma}^2$ which is $d_\alpha$-bilipschitz, where $\delta = -\log 2 / \log \alpha$. Thus the corollary is a consequence of 2.3.

*Remark* 5.5. One also can peel the branched locus of the non orientable branched surface $\overline{\Sigma}^2_{nonor}$ off (see remark 2.4), and obtain a lamination by non orientable totally real surfaces in $\mathbf{C}^2$.

*Question* 5.6. In the introduction, we have seen an example of a 3-dimensional compact torus $T^3 \subset \mathbf{C}^2$, equipped with a minimal linear foliation by totally real surfaces. If $V^3 \subset S$ is a 3-dimensional manifold in a complex surface, equipped with a foliation by totally real surfaces, then $i$ times the transverse direction defines a tangent non vanishing vector field along the leaves. Most of the foliations by surfaces on 3-manifolds do not carry such flows. There are however examples with hyperbolic holonomy; for instance the stable foliation of the geodesic flow on the tangent bundle of a surface a negative curvature. In which compact complex surfaces is it possible to realize these examples as a foliation by totally real surfaces of an embedded (or immersed) 3-manifold $V$?

5.3. **Holomorphic embeddings of $\mathcal{S}^k$ in complex surfaces.** For every conjugacy class of braid $\mathcal{B}$ in the braid group on $k$ strings, the $k$-th Sullivan's solenoid embeds *holomorphically* in the complex surface $V$, if $d$ is sufficiently close to 1. Indeed, consider the following series defined on $\mathbf{D}^*_\infty$:

$$F(\hat{q}) = f(q^{1/k}) + \alpha f(q^{1/k^2}) + \ldots + \alpha^{n-1} f(q^{1/k^n}) + \ldots$$

It verifies the relation

$$F(R_\infty \hat{q}) = f(q) + \alpha F(\hat{q}),$$

and is strictly bounded by $\frac{1}{1-\alpha}$. The set $\hat{A} = q^{-1}(A)$ contained in $\mathbf{D}^*_\infty$ is a fundamental domain for the action of $R_\infty$. The map $P \circ (q, F)$ : $\hat{A} \to V$ is holomorphic and constant along the orbits of $R_\infty$. Thus it induces a *holomorphic map* $\pi : \mathcal{S} \to V$. This map is a $d_{|\alpha|}$-bilipschitz embedding if $d$ is close to 1, as it has been shown in lemma 5.2.

5.4. **Symplectic embeddings of $\mathcal{S}^k$ in $\mathbf{R}^4$.** The leaves of the solenoids that we have constructed in 5.3 are in particular symplectic surfaces. Because the top homology of $\mathcal{S}$ is vanishing, Sikorav raised the question wether it is possible to embed symplectically $\mathcal{S}$ into $\mathbf{R}^4$, with its standard symplectic structure. Glutsyuk beautifully answered this question by the negative. If it was possible, one would be able to construct a lipschitz almost-complex structure $J$ on $\mathbf{C}P^2$, calibrated on the standard symplectic form, for which there would exist a lamination $\mathcal{S}$ by $J$-holomorphic curves, and a $J$-holomorphic line disjoint from $\mathcal{S}$. But by a theorem of Gromov [11], by two distinct points passes a unique $J$-holomorphic line. Thus, by connectedness of the space of lines, no lines intersect $\mathcal{S}$, because the intersection between lines and leaves have to be positive. This contradicts the fact that some line cuts the lamination.

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