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by

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Abstract

We consider the Cauchy problem for the scalar wave equation in the Kerr geometry for smooth initial data supported outside the event horizon. We prove that the solutions decay in time in L^∞_{loc}. The proof is based on a representation of the solution as an infinite sum over the angular momentum modes, each of which is an integral of the energy variable ω on the real line. This integral representation involves solutions of the radial and angular ODEs which arise in the separation of variables.

Contents

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1 Introduction

In this paper we study the long-time dynamics of scalar waves in the Kerr geometry. We prove that solutions of the Cauchy problem with smooth initial data which is compactly supported outside the event horizon, decay in L^∞_{loc}. Our starting point is the integral representation for the propagator [4], which involves an integral over a complex contour in the energy variable ω. In order to study the long-time dynamics, we must deform the contour to the real line. To this end, we carefully analyze the solutions of the associated radial and angular ODEs which arise in the separation of variables. In particular, we show that the integrand in our representation has no poles on the real axis. We call such poles *radiant modes*, because in a dynamical situation they would lead to continuous radiation coming out of the ergosphere.

We now set up some notation and state our main result. As in [4], we choose Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$, in which the Kerr metric takes the form

$$ds^2 \;=\; \frac{\Delta}{U}\,(dt \;-\; a\,\sin^2\vartheta\,d\varphi)^2 \;-\; U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) \;-\; \frac{\sin^2\vartheta}{U}\,(a\,dt \;-\; (r^2+a^2)\,d\varphi)^2 \quad (1.1)$$

with

$$U(r,\vartheta) \;=\; r^2 + a^2\,\cos^2\vartheta\;, \qquad \Delta(r) \;=\; r^2 - 2Mr + a^2\;,$$

where M and aM denote the mass and the angular momentum of the black hole, respectively. We restrict attention to the *non-extreme case* $M^2 > a^2$, where the function Δ has two distinct zeros,

$$r_0 \;=\; M \;-\; \sqrt{M^2-a^2} \qquad \text{and} \qquad r_1 \;=\; M \;+\; \sqrt{M^2-a^2}\;,$$

corresponding to the Cauchy and the event horizon, respectively. We consider only the region $r > r_1$ outside the event horizon, and thus $\Delta > 0$. The ergosphere is the region where the Killing vector $\frac{\partial}{\partial t}$ is space-like, that is where

$$r^2 - 2Mr + a^2\,\cos^2\vartheta < 0\,. \qquad (1.2)$$

The ergosphere lies outside the event horizon $r = r_1$, and its boundary intersects the event horizon at the poles $\vartheta = 0, \pi$.

Theorem 1.1 *Consider the Cauchy problem for the wave equation in the Kerr geometry for smooth initial data which is compactly supported outside the event horizon and has fixed angular momentum in the direction of the rotation axis of the black hole, i.e. for some* $k \in \mathbb{Z}$*,*

$$(\Phi_0, \partial_t \Phi_0) \;=\; e^{-ik\varphi}\,(\Phi_0, \partial_t\Phi_0)(r,\vartheta) \;\in\; C^\infty_0((r_1,\infty)\times S^2)^2\,.$$

Then the solution decays in $L^\infty_{loc}((r_1,\infty)\times S^2)^2$ *as* $t \to \infty$*.*

The study of linear hyperbolic equations in a black hole geometry has a long history. Regge and Wheeler [10] considered the radial equation for metric perturbations of the Schwarzschild metric. In the late 1960s and early 1970s, Carter, Teukolsky and Chandrasekhar discovered that the equations describing scalar, Dirac, Maxwell and linearized gravitational fields in the Kerr geometry are separable into ordinary differential equations (see [1]). Much research has been done concerning the long-time behavior of the solutions of these equations, through both numerical and analytical methods. Price [8] gave arguments which indicated decay of solutions of the scalar wave equation in the Schwarzschild geometry. Press and Teukolsky [7] did a numerical study which strongly suggested the absence of unstable modes, and Whiting [9] later proved that for ω in the complex plane, such unstable modes cannot exist. This "mode stability" does not rule out that there might be unstable modes for real ω (what we call radiant modes). Furthermore, mode stability does not lead to any statement on the Cauchy problem. Kay and Wald [6] used energy estimates to prove a boundedness result for solutions of the scalar wave equation in the Schwarzschild geometry.

Unfortunately, these energy methods cannot be used in a rotating black hole geometry, because the energy density is indefinite inside the ergosphere, making it impossible to introduce a positive definite conserved scalar product. This difficulty was dealt with in [4, 5], where Whiting's mode stability result was combined with estimates for the resolvent and for the radial and angular ODEs. In [4] we established an integral representation which expresses the solution as a contour integral of an integrand involving the separated radial and angular eigenfunctions over a contour staying within a neighborhood located arbitrarily close to the real axis. This integral representation is the starting point of the present paper. After deforming the contours onto the real axis, we can prove decay using the Riemann-Lebesgue Lemma, similar to the case of the Dirac equation [3].

We remark that the problem considered here is closely related to one of the major open questions in general relativity; namely the problem of linearized stability of the Kerr metric. For the stability under metric perturbations one considers the equation for linearized gravitational waves, which can be identified with the general wave equation for spin $s = 2$ (see [1]). Thus replacing scalar waves ($s = 0$) by gravitational waves ($s = 2$), the above theorem would prove linearized stability of the Kerr metric. However, the analysis for $s = 2$ would be considerably more difficult due to the complexity of the linearized Einstein equations. Nevertheless, we regard this paper as a first step towards proving linearized stability of the Kerr metric.

2 Preliminaries

We recall a few constructions and results from [4, 5] which will be needed later on. As radial variable we usually work with the Regge-Wheeler variable $u \in \mathbb{R}$ defined by

$$\frac{du}{dr} = \frac{r^2 + a^2}{\Delta} \; ; \tag{2.1}$$

then $u = -\infty$ corresponds to the event horizon. It is most convenient to write the wave equation in the Hamiltonian form

$$i\,\partial_t \Psi = H\,\Psi, \tag{2.2}$$

where $\Psi = (\Phi, i\partial_t \Phi)$. The Hamiltonian can be written as

$$H = \begin{pmatrix} 0 & 1 \\ A & \beta \end{pmatrix}, \tag{2.3}$$

where

$$A = \frac{1}{\rho}\left[-\frac{\partial}{\partial u}(r^2 + a^2)\frac{\partial}{\partial u} - \frac{\Delta}{r^2 + a^2}\,\Delta_{S^2} - \frac{a^2 k^2}{r^2 + a^2} \right] \tag{2.4}$$

$$\beta = -\frac{2ak}{\rho}\left(1 - \frac{\Delta}{r^2+a^2}\right) \tag{2.5}$$

$$\rho = r^2 + a^2 - a^2 \sin^2\vartheta \, \frac{\Delta}{r^2+a^2} \, . \tag{2.6}$$

The operators A and β are symmetric on the Hilbert space $L^2(\mathbb{R} \times S^2, d\mu)^2$ with the measure

$$d\mu := \rho \, du \, d\cos\vartheta \, . \tag{2.7}$$

It is immediately verified that the Hamiltonian is symmetric with respect to the bilinear form

$$<\Psi_1, \Psi_2> = \int_{\mathbb{R}\times S^2} \langle \Psi_1, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi_2 \rangle_{\mathbb{C}^2} \, d\mu \, . \tag{2.8}$$

As is worked out in detail in [4], the inner product $<\Psi, \Psi>$ is the physical energy of Ψ. Therefore, we refer to $<.,.>$ as the *energy scalar product.* The fact that the energy scalar product is not positive definite can be understood from the fact that the operator A is not positive on $L^2(\mathbb{R} \times S^2, d\mu)$.

Using the ansatz

$$\Phi(t, r, \vartheta, \varphi) = e^{-i\omega t - ik\varphi} \, R(r) \, \Theta(\vartheta), \tag{2.9}$$

the wave equation can be separated into an angular and a radial ODE,

$$\mathcal{R}_{\omega,k} \, R_\lambda = -\lambda \, R_\lambda, \qquad \mathcal{A}_{\omega,k} \, \Theta_\lambda = \lambda \, \Theta_\lambda \, . \tag{2.10}$$

Here the angular operator $\mathcal{A}_{\omega,k}$ is also called the *spheroidal wave operator.* The separation constant λ is an eigenvalue of $\mathcal{A}_{\omega,k}$ and can thus be regarded as an angular quantum number. In [5] it was shown that if ω is in a small neighborhood of the real line, more precisely if

$$\omega \in U_\varepsilon := \left\{ \omega \in \mathbb{C} \mid |\mathrm{Im}\, \omega| < \frac{\varepsilon}{1 + |\mathrm{Re}\, \omega|} \right\},$$

then for sufficiently small $\varepsilon > 0$ the angular operator $\mathcal{A}_{\omega,k}$ has a purely discrete spectrum $(\lambda_n)_{n\in\mathbb{N}}$ with corresponding one-dimensional eigenspaces which span the Hilbert space $L^2(S^2)$. We denote the projections onto the eigenspaces by $Q_n(k, \omega)$. These projections as well as the corresponding eigenvalues λ_n are holomorphic in $\omega \in U_\varepsilon$. In analogy to the eigenvalues $l(l+1)$ of the Laplacian on the sphere, the angular eigenvalues λ_n grow quadratically for large n in the sense that there is a constant $C(k, \omega) > 0$ such that

$$|\lambda_n(k, \omega)| \geq \frac{n^2}{C(k, \omega)} \qquad \text{for all } n \in \mathbb{N} \, . \tag{2.11}$$

We set

$$\omega_0 = -\frac{ak}{r_1^2 + a^2} \tag{2.12}$$

with r_1 the event horizon and use the notation

$$\Omega(\omega) = \omega - \omega_0 \, . \tag{2.13}$$

In order to bring the radial equation into a convenient form, we introduce a new radial function $\phi(r)$ by

$$\phi(r) = \sqrt{r^2 + a^2} \, R(r) \, .$$

Then in the Regge-Wheeler variable, the radial equation can be written as the "Schrödinger equation"

$$\left(-\frac{d^2}{du^2} + V(u)\right) \phi(u) \;=\; 0 \tag{2.14}$$

with the potential

$$V(u) \;=\; -\left(\omega + \frac{ak}{r^2+a^2}\right)^2 \;+\; \frac{\lambda_n(\omega)\,\Delta}{(r^2+a^2)^2} \;+\; \frac{1}{\sqrt{r^2+a^2}}\,\partial_u^2 \sqrt{r^2+a^2}\,. \tag{2.15}$$

In [4] we derived an integral representation for the solution of the Cauchy problem of the following form,

$$\begin{aligned} &\Psi(t,r,\vartheta,\varphi) \\ &= -\frac{1}{2\pi i} \sum_{k\in\mathbf{Z}} e^{-ik\varphi} \sum_{n\in\mathbb{N}} \lim_{\varepsilon\searrow 0} \left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}}\right) d\omega\; e^{-i\omega t}\; (Q_{k,n}(\omega)\; S_\infty(\omega)\; \Psi_0^k)(r,\vartheta)\,. \end{aligned} \tag{2.16}$$

Here the integration contour C_ε must lie inside the set U_ε.

3 Asymptotic Estimates for the Radial Equation

3.1 Holomorphic Families of Radial Solutions

In this section we fix the angular quantum numbers k, n and consider solutions $\acute{\phi}$ and $\grave{\phi}$ of the Schrödinger equation (2.14) which satisfy the following asymptotic boundary conditions on the event horizon and at infinity, respectively,

$$\lim_{u\to-\infty} e^{-i\Omega u}\,\acute{\phi}(u) \;=\; 1\,, \qquad \lim_{u\to-\infty}\left(e^{-i\Omega u}\,\acute{\phi}(u)\right)' \;=\; 0 \tag{3.1}$$

$$\lim_{u\to\infty} e^{i\omega u}\,\grave{\phi}(u) \;=\; 1\,, \qquad \lim_{u\to\infty}\left(e^{i\omega u}\,\grave{\phi}(u)\right)' \;=\; 0\,. \tag{3.2}$$

These solutions were introduced in [4] for ω in the lower complex half plane intersected with U_ε. Here we will show that they are holomorphic in ω, and we will extend their definition to a larger ω-domain. More precisely, we prove the following two theorems.

Theorem 3.1 *The solutions $\acute{\phi}$ are well-defined on the domain*

$$D \;=\; U_\varepsilon \cap \left\{\omega\in\mathbb{C} \,|\, \mathrm{Im}\,\omega \;\leq\; \frac{r_1-r_0}{2(r_1^2+a^2)}\right\}.$$

They form a holomorphic family of solutions in the sense that for every fixed $u \in \mathbb{R}$ and $n \in \mathbb{N}$, the function $\acute{\phi}(u)$ is holomorphic in $\omega \in D$.

Theorem 3.2 *For every angular momentum number n there is an open set E containing the real line except for the origin,*

$$E \;\supset\; E_0 \;:=\; U_\varepsilon \cap \{\omega\in\mathbb{C} \,|\, \mathrm{Im}\,\omega \leq 0 \textit{ and } \omega\neq 0\}\,, \tag{3.3}$$

such that the solutions $\grave{\phi}$ are well-defined for all $\omega \in E$ and form a holomorphic family on E.

For the proofs we will rewrite the Schrödinger equation with boundary conditions (3.1, 3.2) as an integral equation (which in different contexts is called Lipman-Schwinger or Jost equation). Then we will perform a perturbation expansion and get estimates for all the terms of the expansion. To introduce the method, we begin with the solutions $\acute{\phi}$; the solutions $\grave{\phi}$ will be treated later with a similar technique. First we write the Schrödinger equation (2.14) in the form

$$\left(-\frac{d^2}{du^2} - \Omega^2\right) \acute{\phi}(u) \;=\; -W(u)\, \acute{\phi}(u) \tag{3.4}$$

with a potential $W = \Omega^2 + V(u)$ which vanishes at $u = -\infty$. We define the *Green's function* of the differential operator $-\partial_u^2 - \Omega^2$ by the distributional equation

$$(-\partial_v^2 - \Omega^2)\, S(u,v) \;=\; \delta(u-v)\,. \tag{3.5}$$

The Green's function is not unique; we choose it such that its support is contained in the region $v \leq u$; i.e.

$$S(u,v) \;=\; \Theta(u-v) \;\times\; \left\{ \begin{array}{cl} \dfrac{1}{2i\Omega} \left(e^{-i\Omega(u-v)} - e^{i\Omega(u-v)}\right) & \text{if } \Omega \neq 0 \\ v-u & \text{if } \Omega = 0\,. \end{array} \right. \tag{3.6}$$

(Here Θ denotes the Heaviside function defined by $\Theta(x) = 1$ if $x \geq 0$ and $\Theta(x) = 0$ otherwise.) We multiply (3.4) by the Green's function and integrate,

$$\int_{-\infty}^{\infty} S(u,v) \left((-\partial_v^2 - \Omega^2)\, (\acute{\phi}(v) - e^{i\Omega u})\right) dv \;=\; -\int_{-\infty}^{\infty} S(u,v)\, W(v)\, \acute{\phi}(v)\, dv\,.$$

If we assume for the moment that $\acute{\phi}$ satisfies the desired boundary conditions (3.1), we can integrate by parts on the left and use (3.5). This gives the Lipman-Schwinger equation

$$\acute{\phi}(u) \;=\; e^{i\Omega u} - \int_{-\infty}^{u} S(u,v)\, W(v)\, \acute{\phi}(v)\, dv\,,$$

which in the context of potential scattering is also called *Jost equation* (see e.g. [2]). Its significance lies in the fact that we can now easily perform a perturbation expansion in the potential W. Namely, taking for $\acute{\phi}$ the ansatz as the perturbation series

$$\acute{\phi} \;=\; \sum_{l=0}^{\infty} \phi^{(l)}\,, \tag{3.7}$$

we are led to the iteration scheme

$$\left. \begin{array}{rcl} \phi^{(0)}(u) &=& e^{i\Omega u} \\ \phi^{(l+1)}(u) &=& -\displaystyle\int_{-\infty}^{u} S(u,v)\, W(v)\, \phi^{(l)}(v)\, dv\,. \end{array} \right\} \tag{3.8}$$

This iteration scheme can be used for constructing solutions of the Jost equation, and this will give us the functions $\acute{\phi}$ with the desired properties.

Proof of Theorem 3.1. Fix $\omega \in D$. As the potential W is smooth in r and vanishes on the event horizon, we know that W has near r_1 the asymptotics $W = \mathcal{O}(r - r_1)$. This means in the Regge-Wheeler variable (2.1) that W decays exponentially as $u \to -\infty$. More precisely, there is a constant $c > 0$ such that

$$|W(u)| \;\leq\; c\, e^{\gamma u} \qquad \text{with} \qquad \gamma \;:=\; \frac{r_1 - r_0}{r_1^2 + a^2}\,. \tag{3.9}$$

Let us show inductively that

$$|\phi^{(l)}(u)| \;\leq\; \mu^l\, e^{-\mathrm{Im}\,\Omega\, u} \qquad \text{with} \qquad \mu \;:=\; \frac{c\, e^{\gamma u}}{(\gamma - \mathrm{Im}\,\Omega - |\mathrm{Im}\,\Omega|)^2}\,. \tag{3.10}$$

In the case $l = 0$, the claim is obvious from (3.8). Thus assume that (3.10) holds for a given l. Then, estimating the integral equation in (3.8) using (3.9), we obtain

$$|\phi^{(l+1)}(u)| \;\leq\; c\, \mu^l \int_{-\infty}^u |S(u,v)|\, e^{(\gamma - \mathrm{Im}\,\Omega)\, v}\, dv\,. \tag{3.11}$$

The Green's function (3.6) can be estimated in the case $v \leq u$ by

$$|S(u,v)| \;=\; \frac{u-v}{2} \left| \int_0^1 e^{-i\Omega(u-v)\,\tau}\, d\tau \right| \;\leq\; (u-v)\, e^{|\mathrm{Im}\,\Omega|\,(u-v)}\,.$$

Substituting this inequality in (3.11) gives

$$|\phi^{(l+1)}(u)| \;\leq\; c\, \mu^l\, e^{|\mathrm{Im}\,\Omega|\, u} \int_{-\infty}^u (u-v)\, e^{(\gamma - \mathrm{Im}\,\Omega - |\mathrm{Im}\,\Omega|)\, v}\, dv\,.$$

Since the parameter $\alpha := \gamma - \mathrm{Im}\,\Omega - |\mathrm{Im}\,\Omega|$ is positive according to the definition of D, we can carry out the integral as follows,

$$\int_{-\infty}^u (u-v)\, e^{\alpha v}\, dv \;=\; \left(u - \frac{d}{d\alpha}\right) \int_{-\infty}^u e^{\alpha v}\, dv \;=\; \left(u - \frac{d}{d\alpha}\right) \frac{e^{\alpha u}}{\alpha} \;=\; \frac{e^{\alpha u}}{\alpha^2}\,.$$

This gives (3.10) with l replaced by $l+1$.

Since for u on a compact interval, the analytic dependence of the solutions in ω from the coefficients and the initial conditions follows immediately from the Picard-Lindelöf Theorem, it suffices to consider the region $u < u_0$ for any $u_0 \in \mathbb{R}$. By choosing u_0 sufficiently small, we can arrange that $\mu < 1/2$ for all $u < u_0$. Then the estimate (3.10) shows that the perturbation series (3.7) converges absolutely, uniformly in $u \in (-\infty, u_0)$. Using similar estimates for the u-derivatives of $\phi^{(l)}$, one sees furthermore that the perturbation series (3.10) can be differentiated term by term, and using (3.5) we find that $\acute{\phi}$ is indeed a solution of (3.4). Furthermore,

$$\acute{\phi}(u) - e^{i\Omega u} \;=\; \sum_{l=1}^\infty \phi^{(l)}(u)\,,$$

and taking the limit $u \to \infty$ and using (3.10) we find that the right side goes to zero. Using the same argument for the first derivatives, we obtain (3.1).

In order to prove that $\acute{\phi}$ is analytic in ω, we first note that if $\Omega \neq 0$, we can differentiate the perturbation series (3.7) term by term and verify that the Cauchy-Riemann equations are satisfied (note that λ_n is holomorphic in ω according to [5]). Since $\acute{\phi}$ is bounded near $\Omega = 0$, it is also analytic at $\Omega = 0$. ∎

We come to the solutions $\grave{\phi}$. In analogy to (3.4), we now write the Schrödinger equation as

$$\left(-\frac{d^2}{du^2} - \omega^2 \right) \grave{\phi}(u) \;=\; -W(u)\, \grave{\phi}(u) \tag{3.12}$$

with

$$W(u) \;=\; -\omega \frac{2ak}{r^2+a^2} \;-\; \frac{(ak)^2}{(r^2+a^2)^2} \;+\; \frac{\lambda_n\, \Delta}{(r^2+a^2)^2} \;+\; \frac{1}{\sqrt{r^2+a^2}}\, \partial_u^2 \sqrt{r^2+a^2}\,. \tag{3.13}$$

Assuming that $\omega \neq 0$, we choose the Green's function as

$$S(u,v) \;=\; \frac{1}{2i\omega} \left(e^{-i\omega(v-u)} - e^{i\omega(v-u)} \right) \Theta(v-u)\,. \tag{3.14}$$

The corresponding Jost equation is

$$\grave{\phi}(u) \;=\; e^{-i\omega u} - \int_u^\infty S(u,v)\, W(v)\, \grave{\phi}(v)\, dv\,.$$

The perturbation series ansatz

$$\grave{\phi} \;=\; \sum_{l=0}^\infty \phi^{(l)} \tag{3.15}$$

leads to the iteration scheme

$$\left. \begin{aligned} \phi^{(0)}(u) \;&=\; e^{-i\omega u} \\ \phi^{(l+1)}(u) \;&=\; -\int_u^\infty S(u,v)\, W(v)\, \phi^{(l)}(v)\, dv\,. \end{aligned} \right\} \tag{3.16}$$

Note that, in contrast to the exponential decay (3.9), now the potential W, (3.13), has only polynomial decay. As a consequence, the iteration scheme allows us to construct $\grave{\phi}$ only inside the set E_0 as defined in (3.3).

Lemma 3.3 *The solutions $\grave{\phi}$ are well-defined for every $\omega \in E_0$. They form a holomorphic family in the interior of E_0.*

Proof. Fix $\omega \in E_0$. Then $\omega \neq 0$ and $\mathrm{Im}\,\omega \leq 0$, and this allows us to estimate the potential (3.13) and the Green's function (3.14) for $u, v > u_0$ and some $u_0 > 0$ by

$$|W(v)| \;\leq\; \frac{c}{v^2}\,, \qquad |S(u,v)| \;\leq\; \frac{1}{|\omega|}\, e^{\mathrm{Im}\,\omega\,(u-v)}\,. \tag{3.17}$$

Let us show by induction that

$$|\phi^{(l)}(u)| \;\leq\; \frac{1}{l!} \left(\frac{c}{|\omega|\, u} \right)^l e^{\mathrm{Im}\,\omega\, u}\,.$$

For $l = 0$ this is obvious from (3.16), whereas the induction step follows by estimating the integral equation in (3.16) with (3.17),

$$\begin{aligned} |\phi^{(l+1)}(u)| \;&\leq\; \frac{1}{l!} \left(\frac{c}{|\omega|} \right)^n \int_u^\infty \frac{1}{|\omega|}\, e^{\mathrm{Im}\,\omega\,(u-v)}\, \frac{c}{v^{2+l}}\, e^{\mathrm{Im}\,\omega\, v} \\ &=\; \frac{1}{(l+1)!} \left(\frac{c}{|\omega|\, u} \right)^{l+1} e^{\mathrm{Im}\,\omega\, u}\,. \end{aligned}$$

Hence the perturbation series (3.15) converges absolutely, locally uniformly in u. It is straightforward to check that $\acute{\phi}$ satisfies the Schrödinger equation (3.12) with the correct boundary values (3.2). If $\mathrm{Im}\,\omega < 0$, one can differentiate the series (3.15) term by term with respect to ω and verify that the Cauchy-Riemann equations are satisfied. ∎

It remains to analytically extend the solutions $\grave{\phi}$ for fixed n to a neighborhood of any point $\omega_0 \in \mathbb{R} \setminus \{0\}$. To this end, we need good estimates of the derivatives of $\grave{\phi}$ with respect to ω and u. It is most convenient to work with the functions

$$\psi^{(l)}(u) \;:=\; (2i\omega)^l \, e^{i\omega u} \, \phi^{(l)}(u)\,, \tag{3.18}$$

for which the iteration scheme (3.16) can be written as $\psi^{(0)} = 1$ and

$$\psi^{(l+1)}(u) \;=\; \int_u^\infty (e^{-2i\omega(v-u)} - 1)\, W(v)\, \psi^{(l)}(v)\, dv\,. \tag{3.19}$$

Lemma 3.4 *For every $\omega_0 \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$, there are positive constants c, K, δ, such that for all $\omega \in E_0 \cap B_\delta(\omega_0)$ and all $p, q, n \in \mathbb{N}$ the following inequality holds,*

$$\left| \left(\frac{\partial}{\partial \omega} \right)^p \left(\frac{\partial}{\partial u} \right)^q \psi^{(l)}(u) \right| \;\leq\; c^{1+l+p}\, K^q\, \frac{p!\, q!}{l!}\, \frac{1}{u^{l+q}}\,. \tag{3.20}$$

Proof. According to [5], λ_n is holomorphic in a neighborhood of ω_0, and thus (for example using the Cauchy integral formula) its derivatives can be bounded in $B_\delta(\omega_0)$ by

$$|\partial^p_\omega \lambda_n(\omega)| \;\leq\; \left(\frac{K}{2} \right)^{1+p} p!$$

for suitable $K > 0$. Since the potential W, (3.13), is also holomorphic in r (in a suitable neighborhood of the positive real axis) and has quadratic decay, its derivatives can be estimated by

$$|\partial^p_\omega \partial^q_u W(u)| \;\leq\; \left(\frac{K}{2} \right)^{1+p+q} \frac{p!\, q!}{u^{2+q}}\,. \tag{3.21}$$

We choose c so large that the following conditions hold,

$$c \;>\; 16\,K\,, \qquad\qquad \frac{K\, e^{\frac{1}{K}}}{(\omega_0 - \delta)\, c} \;\leq\; \frac{1}{2}\,. \tag{3.22}$$

We proceed to prove (3.20) by induction in l. For $l = 0$ there is nothing to prove. Thus assume that (3.20) holds for a given l. Using the induction hypothesis together with (3.21), we can then estimate the derivatives of the product $W\psi^{(l)}$ as follows,

$$\begin{aligned} |\partial^p_\omega \partial^q_u (W\psi^{(l)})| \;&\leq\; \sum_{a=0}^p \binom{p}{a} \sum_{b=0}^q \binom{q}{b} \left(\frac{K}{2} \right)^{1+a+b} \frac{a!\, b!}{u^{2+b}}\, \frac{c^{1+l+p-a}\, K^{q-b}}{u^{l+q-b}}\, \frac{(p-a)!\,(q-b)!}{l!} \\ &=\; \frac{c^{1+l+p}\, K^{1+q}}{u^{2+l+q}}\, \frac{p!\, q!}{l!} \sum_{a=0}^p \left(\frac{K}{2c} \right)^a \sum_{b=0}^q \left(\frac{1}{2} \right)^b . \end{aligned}$$

According to (3.22), the two remaining sums can be bounded by the geometric series $\sum_{m=0}^\infty 2^{-m} = 2$, and thus

$$|\partial^p_\omega \partial^q_u (W\psi^{(l)})| \;\leq\; 4\, \frac{c^{1+l+p}\, K^{1+q}}{u^{2+l+q}}\, \frac{p!\, q!}{l!}\,. \tag{3.23}$$

Next we differentiate the integral equation (3.19),

$$\partial_\omega^p \partial_u^q \psi^{(l+1)}(u) \;=\; \sum_{r=0}^p \binom{p}{r} \int_{-\infty}^\infty \partial_\omega^r \partial_u^q \left[\Theta(v-u)\,(e^{-2i\omega(v-u)}-1)\right]\,\partial_\omega^{p-r}\left(W\psi^{(l)}(v)\right)\,dv\,.$$

After manipulating the partial derivatives as follows,

$$\partial_\omega^r \partial_u^q \left[\Theta(v-u)\,(e^{-2i\omega(v-u)}-1)\right] \;=\; (-\partial_v)^q \left[\Theta(v-u)\left(\frac{v-u}{\omega}\,\partial_v\right)^p (e^{-2i\omega(v-u)}-1)\right],$$

the resulting v-derivatives can all be integrated by parts. The boundary terms drop out, and we obtain

$$\partial_\omega^p \partial_u^q \psi^{(l+1)}(u) \;=\; \sum_{r=0}^p \binom{p}{r} \int_u^\infty (e^{-2i\omega(v-u)}-1)\left(\partial_v\,\frac{u-v}{\omega}\right)^r \partial_v^q \partial_\omega^{p-r}\left(W\psi^{(l)}(v)\right)dv\,.$$

Since ω is in the lower half plane, we have the inequality $|e^{-2i\omega(v-u)}| \leq 1$. We conclude that

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 2\sum_{r=0}^p \binom{p}{r} \int_u^\infty \left|\left\{\partial_v\,\frac{u-v}{\omega}\right\}^r \partial_v^q \partial_\omega^{p-r}\left(W\psi^{(l)}(v)\right)\right| dv\,. \tag{3.24}$$

The v-derivatives in the curly brackets can act either on one of the factors $(u-v)$ or on the function $W\psi^{(l)}$. Taking into account the combinatorics, we obtain

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 2\sum_{r=0}^p \binom{p}{r} \frac{1}{\omega^r} \sum_{s=0}^r \binom{r}{s} r^{r-s} \int_u^\infty (v-u)^s \left|\partial_\omega^{p-r}\partial_v^{q+s}\left(W\psi^{(l)}\right)\right| dv\,.$$

Using (3.23), we get

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 8\sum_{r=0}^p \sum_{s=0}^r \frac{p!\,(q+s)!}{s!\,(r-s)!\,l!}\,\omega^{-r}\,r^{r-s}\,c^{1+l+p-r}\,K^{1+q+s} \int_u^\infty \frac{(v-u)^s}{v^{2+l+q+s}}\,dv\,.$$

Introducing the new variable $\tau = \frac{u}{v}$, the integral can be computed with iterative integrations by parts,

$$\begin{aligned}
\int_u^\infty \frac{(v-u)^s}{v^{2+l+q+s}}\,dv \;&=\; \frac{1}{u^{1+l+q}} \int_0^1 (1-\tau)^s\,\tau^{l+q}\,d\tau \\
&=\; \frac{1}{u^{1+l+q}}\,\frac{(l+q)!}{(l+q+s)!} \int_0^1 (1-\tau)^s\,\frac{d^s}{d\tau^s}\tau^{l+q+s}\,d\tau \\
&=\; \frac{1}{u^{1+l+q}}\,\frac{(l+q)!\,s!}{(l+q+s)!} \int_0^1 \tau^{l+q+s}\,d\tau \;=\; \frac{1}{u^{1+l+q}}\,\frac{(l+q)!\,s!}{(1+l+q+s)!}\,.
\end{aligned}$$

We thus obtain

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 8\,\frac{c^{1+l+p}\,K^{1+q}}{u^{1+l+q}} \sum_{r=0}^p \sum_{s=0}^r \frac{p!\,(q+s)!}{(r-s)!\,l!}\,\frac{K^s\,r^{r-s}}{(\omega c)^r}\,\frac{(l+q)!}{(1+l+q+s)!}\,.$$

Using the elementary estimate

$$\frac{(q+s)!\,(l+q)!}{(1+l+q+s)!} \;=\; \frac{q!}{q+l+1}\cdot\frac{q+1}{q+l+2}\cdots\frac{q+s}{q+l+s+1} \;\leq\; \frac{q!}{l+1}\,,$$

we obtain

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 8\, \frac{c^{1+l+p}\, K^{1+q}}{u^{1+l+q}}\, \frac{p!\, q!}{(l+1)!} \sum_{r=0}^p \left(\frac{K}{\omega c}\right)^r \sum_{s=0}^r \frac{1}{(r-s)!} \left(\frac{r}{K}\right)^{r-s} .$$

The last sum can be estimated by an exponential,

$$\sum_{s=0}^r \frac{1}{(r-s)!} \left(\frac{r}{K}\right)^{r-s} \;\leq\; \sum_{a=0}^r \frac{1}{a!} \left(\frac{r}{K}\right)^a \;\leq\; \sum_{a=0}^\infty \frac{1}{a!} \left(\frac{r}{K}\right)^a \;=\; \exp\left(\frac{r}{K}\right).$$

According to (3.22), we can now estimate the remaining sum over r by a geometric series,

$$\sum_{r=0}^p \left(\frac{K}{\omega c}\right)^r \exp\left(\frac{r}{K}\right) \;\leq\; \sum_{r=0}^\infty \left(\frac{K\, e^{\frac{1}{K}}}{\omega c}\right)^r \;\leq\; 2\, .$$

We thus obtain

$$\left|\partial_\omega^p \partial_u^q \psi^{(l+1)}(u)\right| \;\leq\; 16\, \frac{c^{1+l+p}\, K^{1+q}}{u^{1+l+q}}\, \frac{p!\, q!}{(l+1)!} \;\leq\; \frac{c^{2+l+p}\, K^q}{u^{1+l+q}}\, \frac{p!\, q!}{(l+1)!}\, ,$$

where in the last step we again used (3.22). ■

Proof of Theorem 3.1. According to (3.15, 3.18),

$$\grave{\phi}(\omega, u) \;=\; e^{-i\omega u} \sum_{l=0}^\infty \frac{1}{(2i\omega)^l}\, \psi^{(l)}(\omega, u)\, .$$

Expanding $\psi^{(l)}$ in a Taylor series in ω, we obtain the formal expansion

$$\grave{\phi}(\omega + \zeta, u) \;=\; e^{-i\omega u} \sum_{l=0}^\infty \frac{1}{(2i(\omega+\zeta))^l} \sum_{p=0}^\infty \frac{\zeta^p}{p!}\, \partial_\omega^p \psi^{(l)}(\omega, u)\, .$$

Lemma 3.4 allows us to estimate this expansion as follows,

$$|\grave{\phi}(\omega + \zeta, u)| \;\leq\; c \sum_{l=0}^\infty \frac{1}{l!} \left(\frac{c}{|\omega+\zeta|\, u}\right)^l \sum_{p=0}^\infty (c|\zeta|)^p\, .$$

This expansion converges uniformly for $|\zeta| < \frac{c}{2}$. It is straightforward to check that the ω-derivatives also converge uniformly, and that the Cauchy-Riemann equations are satisfied. Thus the above expansion allows us to extend $\grave{\phi}$ analytically to the ball $|\zeta| < \frac{c}{2}$. ■

3.2 A Continuous Family of Solutions near $\omega = 0$

In Theorem 3.2 we made no statement about the behavior of the fundamental solutions $\grave{\phi}$ at $\omega = 0$. Indeed, we cannot expect the solutions to have a holomorphic extension in a neighborhood of $\omega = 0$. But at least, after suitable rescaling, these solutions have a well-defined limit at $\omega = 0$:

Theorem 3.5 *For every angular momentum number n, there is a real solution ϕ_0 of the Schrödinger equation (2.14) for $\omega = 0$ with the asymptotics*

$$\lim_{u\to\infty} u^{\mu-\frac{1}{2}}\, \phi_0(u) \;=\; \frac{\Gamma(\mu)}{\sqrt{\pi}} \qquad \textit{with} \qquad \mu \;:=\; \sqrt{\lambda_n(0)+\frac{1}{4}}\,. \tag{3.25}$$

This solution can be obtained as a limit of the solutions from Theorem 3.2, in the sense that for all $u \in \mathbb{R}$,

$$\phi_0(u) \;=\; \lim_{E_0 \ni \omega \to 0} \omega^\mu\, \grave{\phi}(u) \qquad \textit{and} \qquad \phi_0'(u) \;=\; \lim_{E_0\ni\omega\to 0} \omega^\mu\, \grave{\phi}'(u)\,.$$

Note that the λ_n are the eigenvalues of the Laplacian on the sphere. They are clearly non-negative, and thus the parameter μ in (3.25) is positive.

Unfortunately, the function ϕ_0 cannot be constructed with the iteration scheme (3.16) because if we put in the Green's function for $\omega = 0$ (which is obtained from (3.14) by taking the limit $\omega \to 0$), we get the for $\phi^{(1)}$ the equation

$$\phi^{(1)}(u) \;=\; \int_u^\infty (v-u)\, W(v)\, dv\,,$$

and since W decays at infinity only quadratically, the integral diverges. To overcome this problem, we combine the quadratically decaying part of the potential with the unperturbed operator. More precisely, for any ω in the set

$$F \;:=\; \left\{\omega \in \mathbb{C} \mid \operatorname{Im}\omega \le 0 \text{ and } |\omega| \le (16ak)^{-1}\right\}\,,$$

we write the Schrödinger equation as

$$\left(-\frac{d^2}{du^2} + \frac{\mu^2 - \frac{1}{4}}{u^2} - \omega^2\right)\phi(u) \;=\; -W(u)\,\phi(u)\,,$$

where $\mu(\omega) = (\lambda_n(\omega) - 2ak\omega + \frac{1}{4})^{\frac{1}{2}}$. The potential W is continuous in ω and bounded by

$$|W(u)| \;\le\; \frac{c}{u^3} \qquad \text{for all } \omega \in F\,. \tag{3.26}$$

The solutions of the unperturbed Schrödinger equation can be expressed with Bessel functions,

$$h_1(u) \;=\; \sqrt{\frac{\pi u}{2}}\, J_\mu(\omega u)\,, \qquad h_2(u) \;=\; \sqrt{\frac{\pi u}{2}}\, Y_\mu(\omega u)\,.$$

They have the following asymptotics,

$$\begin{cases} h_1(u) \;\sim\; \cos(\omega u) \;, & h_2(u) \;\sim\; \sin(\omega u) \qquad \text{if } \omega u \gg 1 \\ h_1(u) \;\sim\; \dfrac{\sqrt{\pi}\,\omega^\mu}{\Gamma(\mu+1)\, 2^{\mu+\frac{1}{2}}}\, x^{\mu+\frac{1}{2}}\;, & h_2(u) \;\sim\; \dfrac{\Gamma(\mu)\, 2^{\mu-\frac{1}{2}}}{\sqrt{\pi}\,\omega^\mu}\, x^{-\mu+\frac{1}{2}} \qquad \text{if } \omega u \ll 1. \end{cases}$$

The Green's function can be expressed in terms of the two fundamental solutions by the standard formula

$$S(u,v) \;=\; \Theta(v-u)\, \frac{h_1(u)\, h_2(v) - h_1(v)\, h_2(u)}{w(h_1,h_2)}\,,$$

where $w(h_1, h_2) = h_1' h_2 - h_1 h_2' = -\omega$ is the Wronskian. The perturbation series ansatz

$$\phi \;=\; \sum_{l=1}^{\infty} \phi^{(l)} \tag{3.27}$$

now leads to the integral equation

$$\phi^{(l+1)}(u) \;=\; \int_u^\infty S(u,v)\, W(v)\, \phi^{(l)}(v)\, dv\,. \tag{3.28}$$

We choose the function $\phi^{(0)}$ such that its asymptotics at infinity is a multiple times the plane wave $e^{-i\omega u}$, whereas for $\omega = 0$, it has the asymptotics (3.25),

$$\phi^{(0)}(u) \;=\; \omega^\mu\, (h_1 - i h_2)(u)\,. \tag{3.29}$$

Lemma 3.6 *For any fixed n there is $u_0 \in \mathbb{R}$ such that the iteration scheme (3.29, 3.28) converges uniformly for all $u > u_0$ and $\omega \in F$. The functions ϕ defined by (3.27) are solutions of the Schrödinger equation (2.14) with the asymptotics*

$$\left| \frac{\phi(u)}{\phi^{(0)}(u)} - 1 \right| \;\leq\; \frac{c}{u}$$

and a constant $c = c(n)$.

Proof. Using the asymptotic formulas for the Bessel functions, one sees (similar to the estimate [2, eqn. (4.4)] for $\mu = l + \frac{1}{2}$ and integer l) that for all $v \geq u$ and $\omega \in F$, the Green's function is bounded by

$$|S(u,v)| \;\leq\; C\, e^{\mathrm{Im}\,\omega\,(v-u)} \left(\frac{u}{1+|\omega|\,u} \right)^{-\mu+\frac{1}{2}} \left(\frac{v}{1+|\omega|\,v} \right)^{\mu+\frac{1}{2}}\,. \tag{3.30}$$

Similarly, we can bound the Bessel functions in (3.29) to get

$$\frac{1}{C} \;\leq\; |\phi^{(0)}|\, e^{-\mathrm{Im}\,\omega u} \left(\frac{u}{1+|\omega|\,u} \right)^{\mu-\frac{1}{2}} \;\leq\; C\,. \tag{3.31}$$

Let us show inductively that

$$|\phi^{(l)}| \;\leq\; C\, e^{\mathrm{Im}\,\omega\,u} \left(\frac{u}{1+|\omega|\,u} \right)^{-\mu+\frac{1}{2}} \left(\frac{Cc}{u} \right)^l\,. \tag{3.32}$$

For $l = 0$ there is nothing to prove. The induction step follows from (3.28, 3.26, 3.30)

$$\begin{aligned} |\phi^{(l+1)}| \;&\leq\; C\, e^{-\mathrm{Im}\,\omega u} \left(\frac{u}{1+|\omega|\,u} \right)^{-\mu+\frac{1}{2}} \int_u^\infty e^{2\,\mathrm{Im}\,\omega v} \left(\frac{v}{1+|\omega|\,v} \right) \frac{cC}{v^3} \left(\frac{Cc}{v} \right)^l dv \\ &\leq\; C\, e^{\mathrm{Im}\,\omega u} \left(\frac{u}{1+|\omega|\,u} \right)^{-\mu+\frac{1}{2}} \left(\frac{Cc}{u} \right)^l \int_u^\infty \frac{cC}{v^2}\, dv \end{aligned}$$

The lemma now follows immediately from (3.32, 3.31) and by differentiating the series (3.27) with respect to u. ∎

Proof of Theorem 3.5. From the asymptotics at infinity, it is clear that

$$\phi \;=\; \begin{cases} \omega^\mu \,\grave{\phi} & \text{if } \omega \neq 0 \\ \phi_0 & \text{if } \omega = 0\,. \end{cases}$$

Denoting the ω-dependence of ϕ by a subscript, we thus need to prove that for all $u \in \mathbb{R}$,

$$\lim_{F \ni \omega \to 0} \phi_\omega(u) \;=\; \phi_0(u)\,, \qquad \lim_{F \ni \omega \to 0} \phi'_\omega(u) \;=\; \phi'_0(u)\,. \tag{3.33}$$

To simplify the problem, we first note that for u on a compact intervals, the continuous dependence on ω follows immediately from the Picard-Lindelöf Theorem (i.e. the continuous dependence of solutions of ODEs on the coefficients and initial values). Thus it suffices to prove (3.33) for large u. Furthermore, writing the Schrödinger equation as

$$(\partial_u - i\omega)(\partial_u + i\omega)\,\phi_\omega \;=\; -U\,\phi\,,$$

the potential U has quadratic decay at infinity. Thus, after the substitution $(\partial_u - i\omega) = e^{i\omega u}\partial_u e^{-i\omega u}$, we can multiply the above equation by $e^{-i\omega u}$ and integrate to obtain

$$e^{-i\omega u}\,(\partial_u + i\omega)\,\phi_\omega(u) \;=\; \int_u^\infty e^{-i\omega v}\,U(v)\,\phi_\omega(v)\,dv\,.$$

Here we emphasized the ω-dependence by a subscript; note also that the integral is well-defined in view of the asymptotics of ϕ_ω at infinity. This equation shows that ϕ'_ω converges pointwise once we know that $\phi_\omega(u)$ converges uniformly in u. Hence it remains to show that for every $\epsilon > 0$ there is u_0 and $\delta > 0$ such that for all $\omega \in F$ with $|\omega| < \delta$,

$$|\phi_\omega(u) - \phi_0(u)| \;<\; \varepsilon \qquad \text{for all } u > u_0. \tag{3.34}$$

To prove (3.34) we use the uniform convergence of the functions $\phi^{(0)}_\omega$, (3.29), to choose δ such that for all $\omega \in F$ with $|\omega| < \delta$,

$$|\phi^{(0)}_\omega(u) - \phi^{(0)}_0(u)| \;<\; \frac{\varepsilon}{3} \qquad \text{for all } u > u_0.$$

According to Lemma 3.6, we can by choosing u_0 sufficiently large arrange that

$$|\phi^{(0)}_\omega(u) - \phi_\omega(u)| \;<\; \frac{\varepsilon}{3} \qquad \text{for all } u > u_0 \text{ and } \omega \in F.$$

Now (3.34) follows immediately from the estimate

$$|\phi_\omega - \phi_0| \;\leq\; |\phi_\omega - \phi^{(0)}_\omega| + |\phi^{(0)}_\omega - \phi^{(0)}_0| + |\phi^{(0)}_0 - \phi_0|\,.$$

∎

4 Global Estimates for the Radial Equation

Let Y_1 and Y_2 be two real fundamental solutions of the Schrödinger equation (2.14) for a general real and smooth potential V. Then their Wronskian

$$w \;:=\; Y_1(u)\,Y_2'(u) - Y_1'(u)\,Y_2(u) \tag{4.1}$$

is a constant. By flipping the sign of Y_2, we can always arrange that $w > 0$. We combine the two real solutions into the complex function

$$z = Y_1 + iY_2 ,$$

and denote its polar decomposition by

$$z = \rho\, e^{i\varphi} \tag{4.2}$$

with real functions $\rho(u) \geq 0$ and $\varphi(u)$. By linearity, z is a solution of the complex Schrödinger equation

$$z'' = V z . \tag{4.3}$$

Note that z has no zeros because at every u at least one of the fundamental solutions does not vanish.

4.1 The Complex Riccati Equation

We introduce the function y by

$$y = \frac{z'}{z} . \tag{4.4}$$

Since z has no zeros, the function y is smooth. Moreover, it satisfies the complex Riccati equation

$$y' + y^2 = V . \tag{4.5}$$

The fact that the solutions of the complex Riccati equation are smooth will be helpful for getting estimates. Conversely, from a solution of the Riccati equation one obtains the corresponding solution of the Schrödinger equation by integration,

$$\log z|_u^v = \int_u^v y . \tag{4.6}$$

Using (4.2) in (4.4) gives separate equations for the amplitude and phase of z,

$$\rho' = \rho\, \mathrm{Re}\, y , \qquad \varphi' = \mathrm{Im}\, y ,$$

and integration gives

$$\log \rho|_u^v = \int_u^v \mathrm{Re}\; y \tag{4.7}$$

$$\varphi|_u^v = \int_u^v \mathrm{Im}\; y . \tag{4.8}$$

Furthermore, the Wronskian (4.1) gives a simple algebraic relation between ρ and y. Namely, w can be expressed by $w = \mathrm{Im}\,(\overline{z}\, z') = \rho^2\, \mathrm{Im}\, y$ and thus

$$\rho^2 = \frac{w}{\mathrm{Im}\, y} . \tag{4.9}$$

Since ρ^2 and w are non-negative, we see that

$$\mathrm{Im}\, y(u) > 0 \qquad \text{for all } u. \tag{4.10}$$

4.2 Invariant Disk Estimates

We now explain a method for getting estimates for the complex Riccati equation. This method was first used in [5] for estimates in the case where the potential is negative (Lemma 4.1). Here we extend the method to the situation when the potential is positive (Lemma 4.2). For sake of clarity, we develop the method again from the beginning, but we point out that the proof of Lemma 4.1 is taken from [5]. Let $y(u)$ be a solution of the complex Riccati equation (4.5). We want to estimate the Euclidean distance of y to a given curve $m(u) = \alpha + i\beta$ in the complex plane. A direct calculation using (4.5) gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{du}|y-m|^2 \;&=\; (\mathrm{Re}\, y - \alpha)\,(\mathrm{Re}\, y-\alpha)' + (\mathrm{Im}\, y - \beta)\,(\mathrm{Im}\, y - \beta)' \\
&= (\mathrm{Re}\, y - \alpha)\left[V - (\mathrm{Re}\, y)^2 + (\mathrm{Im}\, y)^2 - \alpha'\right] - (\mathrm{Im}\, y - \beta)\left[2\,\mathrm{Re}\, y\,\mathrm{Im}\, y + \beta'\right] \\
&= (\mathrm{Re}\, y - \alpha)\left[V - (\mathrm{Re}\, y)^2 - (\mathrm{Im}\, y)^2 + 2\beta\,\mathrm{Im}\, y - \alpha'\right] \;+\; (\mathrm{Re}\, y - \alpha)\, 2(\mathrm{Im}\, y - \beta)\,\mathrm{Im}\, y \\
&\quad -(\mathrm{Im}\, y - \beta)\left[\beta' + 2\alpha\,\mathrm{Im}\, y\right] \;-\; (\mathrm{Im}\, y - \beta)\, 2(\mathrm{Re}\, y - \alpha)\,\mathrm{Im}\, y \\
&= (\mathrm{Re}\, y - \alpha)\left[V - (\mathrm{Re}\, y - \alpha)^2 - (\mathrm{Im}\, y - \beta)^2 - \alpha^2 + \beta^2 - \alpha'\right] \\
&\quad -(\mathrm{Im}\, y - \beta)\left[\beta' + 2\alpha\beta\right] - 2\alpha\left((\mathrm{Re}\, y - \alpha)^2 + (\mathrm{Im}\, y - \beta)^2\right).
\end{aligned}$$

Choosing polar coordinates centered at m,

$$y \;=\; m + Re^{i\varphi}\,, \qquad R \;:=\; |y - m|\,,$$

we obtain the following differential equation for R,

$$R' + 2\alpha R \;=\; \cos\varphi\left[V - R^2 - \alpha^2 + \beta^2 - \alpha'\right] - \sin\varphi\left[\beta' + 2\alpha\beta\right]. \tag{4.11}$$

In order to use this equation for estimates, we assume that α is a given function (to be determined later). With the abbreviations

$$U \;=\; V - \alpha^2 - \alpha' \qquad \text{and} \qquad \sigma(u) \;=\; \exp\left(2\int_0^u \alpha\right), \tag{4.12}$$

the ODE (4.11) can then be written as

$$(\sigma R)' \;=\; \sigma\left[U - R^2 + \beta^2\right]\cos\varphi - (\sigma\beta)'\,\sin\varphi\,.$$

To further simplify the equation, we want to arrange that the square bracket vanishes. If U is negative, this can be achieved by the ansatz

$$\beta \;=\; \frac{\sqrt{|U|}}{2}\left(T + \frac{1}{T}\right), \qquad R \;=\; \frac{\sqrt{|U|}}{2}\left(T - \frac{1}{T}\right) \qquad (U<0), \tag{4.13}$$

with $T > 1$ a free function. In the case $U > 0$, we make similarly the ansatz

$$\beta \;=\; \frac{\sqrt{U}}{2}\left(T - \frac{1}{T}\right), \qquad R \;=\; \frac{\sqrt{U}}{2}\left(T + \frac{1}{T}\right) \qquad (U>0) \tag{4.14}$$

with a function $T > 0$. Using (4.13, 4.14), the ODE (4.11) reduces to the simple equation $(\sigma R)' = -(\sigma\beta)'\sin\varphi$. If we now replace this equation by a strict inequality,

$$(\sigma R)' \;>\; -(\sigma\beta)'\sin\varphi\,, \tag{4.15}$$

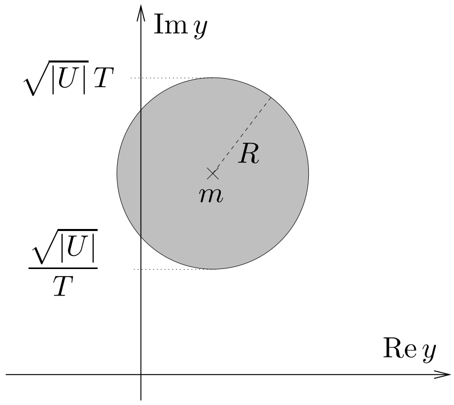

Figure 1: Invariant disk estimate for $U < 0$.

with R a general positive function, the inequality $|y - m| \leq R$ will be preserved as u increases. In other words, the disk $\overline{B_R}(m)$ will be an invariant region for the flow of y. In the next two lemmas we specify the function T in the cases $U < 0$ and $U > 0$, respectively. To avoid confusion, we note that it is only a matter of convenience to state the lemmas on the interval $[0, u_{\max}]$; by translation we can later immediately apply the lemmas on any closed interval.

Lemma 4.1 *Let α be a real function on $[0, u_{max}]$ which is continuous and piecewise C^1, such that the corresponding function U, (4.12), is negative,*

$$U \leq 0 \quad on \quad [0, u_{max}] \, .$$

For a constant $T_0 \geq 1$ we introduce the function T by

$$T(u) \;=\; T_0 \, \exp\left(\frac{1}{2}\, TV_{[0,u)} \log|\sigma^2 U|\right) , \tag{4.16}$$

define the functions β and R by (4.13) and set $m = \alpha + i\beta$. If a solution y of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$|y - m| \;\leq\; R \, , \tag{4.17}$$

then this condition holds for all $u \in [0, u_{max}]$ (for illustration see Figure 1).

Proof. For $\varepsilon > 0$ we set

$$T_\varepsilon(u) \;=\; T_0 \, \exp\left(\frac{1}{2}\int_0^u \left|\frac{|\sigma^2 U|'}{|\sigma^2 U|}\right| + \varepsilon(1 - e^{-u})\right) \tag{4.18}$$

and denote corresponding functions α, R, m, and σ by an additional subscript ε. Since $T_\varepsilon(0) = T(0)$ and $\lim_{\varepsilon \searrow 0} T_\varepsilon = T$, it suffices to show that for all $\varepsilon > 0$ the following statement holds,

$$|y - m_\varepsilon|(0) \leq R_\varepsilon(0) \quad \Longrightarrow \quad |y - m_\varepsilon|(u) \leq R_\varepsilon(u) \;\text{ for all } u \in [0, u_{\max}].$$

In differential form, we get the sufficient condition

$$|y - m_\varepsilon|(u) = R_\varepsilon(u) \quad \Longrightarrow \quad |y - m_\varepsilon|'(u) < R'_\varepsilon(u) \, .$$

According to (4.15), this last condition will be satisfied if

$$(\sigma_\varepsilon R_\varepsilon)' \;>\; |(\sigma_\varepsilon \beta_\varepsilon)'|\,. \tag{4.19}$$

From now on we omit the subscripts ε.

In order to prove (4.19), we first use (4.13, 4.12) to rewrite the functions $\sigma\beta$ and σR as

$$\left.\begin{aligned} \sigma\beta &= \frac{1}{2}\left(\sqrt{|\sigma^2 U|}\,T + \sqrt{|\sigma^2 U|}\,T^{-1}\right) \\ \sigma R &= \frac{1}{2}\left(\sqrt{|\sigma^2 U|}\,T - \sqrt{|\sigma^2 U|}\,T^{-1}\right). \end{aligned}\right\} \tag{4.20}$$

By definition of T_ε (4.18),

$$\frac{T'}{T} \;=\; \frac{1}{2}\left|\frac{|\sigma^2 U|'}{|\sigma^2 U|}\right| + \varepsilon e^{-u}\,.$$

It follows that

$$\begin{cases} (\sqrt{|\sigma^2 U|}\,T^{-1})' \;=\; -\varepsilon e^{-u}\,(\sqrt{|\sigma^2 U|}\,T^{-1}) & \text{if } |\sigma^2 U|' \geq 0 \\ (\sqrt{|\sigma^2 U|}\,T)' \;=\; \varepsilon e^{-u}\,(\sqrt{|\sigma^2 U|}\,T) & \text{if } |\sigma^2 U|' < 0\,. \end{cases}$$

Hence when we differentiate through (4.20) and set $\varepsilon = 0$, either the first or the second summand drops out in each equation, and we obtain $(\sigma R)' = |\sigma\beta|'$. If $\varepsilon > 0$, an inspection of the signs of the additional terms gives (4.19). ■

Lemma 4.2 *Let α be a real function on $[0, u_{max}]$ which is continuous and piecewise C^1, such that the corresponding function U, (4.12), satisfies on $[0, u_{max}]$ the conditions*

$$U \geq 0 \qquad \text{and} \qquad U' + 4U\alpha \;\geq\; 0\,. \tag{4.21}$$

For a constant $T_0 \geq 0$ we introduce the function T by

$$T(u) \;=\; T_0\,\sqrt{\frac{U(0)}{\sigma^2 U}}\,, \tag{4.22}$$

define the functions β and R by (4.14) and set $m = \alpha + i\beta$. If a solution y of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$|y - m| \;\leq\; R\,,$$

then this condition holds for all $u \in [0, u_{max}]$ (see Figure 2). Furthermore,

$$\text{Re}\, y \;\geq\; \alpha - \sqrt{U} - T_0\,\frac{\sqrt{U(0)}}{2\sigma}\,. \tag{4.23}$$

Proof. For $\varepsilon > 0$ we set

$$T_\varepsilon \;=\; T_0\,(\sigma^2 U)^{-\frac{1}{2}}\,(1 - \varepsilon e^{-u})\,.$$

Using (4.14, 4.12) we can write the functions $\sigma\beta$ and σR as

$$\left.\begin{aligned} \sigma\beta &= -\frac{1}{2}\left(T_0^{-1}\,\sigma^2 U\,(1 - \varepsilon e^{-u})^{-1} - T_0\,(1 - \varepsilon e^{-u})\right) \\ \sigma R &= \frac{1}{2}\left(T_0^{-1}\,\sigma^2 U\,(1 - \varepsilon e^{-u})^{-1} + T_0\,(1 - \varepsilon e^{-u})\right), \end{aligned}\right\}$$

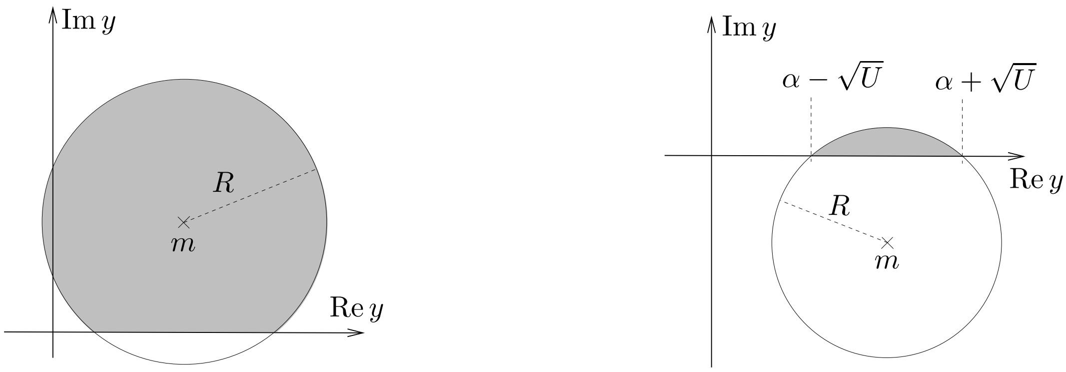

Figure 2: Invariant disk estimate for $U > 0$, in the cases $T > 1$ (left) and $T < 1$ (right).

where we again omitted the subscript ε. Differentiation gives

$$(\sigma R)' \;>\; -(\sigma\beta)' \;=\; \frac{1}{2}\, T_0^{-1} \left(\sigma^2 U \,(1-\varepsilon e^{-u})^{-1}\right)' - \frac{1}{2}\, T_0 \left(1-\varepsilon e^{-u}\right)' \,. \tag{4.24}$$

According to the second inequality in (4.21), the function $\sigma^2 U$ is strictly increasing and thus the expression on the right of (4.24) is positive for sufficiently small ε. Hence (4.19) is satisfied. Letting $\varepsilon \to 0$, we obtain that the circle $\overline{B}_R(m)$ is invariant.

In order to prove (4.23) we note that in the case $T < 1$ the inequality is obvious because even $\mathrm{Re}\, y \leq \alpha - \sqrt{U}$ (see Figure 2). Thus we can assume $T \geq 1$, and the estimate

$$\mathrm{Re}\, y \;\geq\; \alpha - R \;\geq\; \alpha - \frac{\sqrt{U}}{2}\,(T+2)$$

together with (4.22) gives the claim. ■

If the potential V is monotone increasing, by choosing $\alpha \equiv 0$ we obtain the following simple estimate.

Corollary 4.3 *Assume that the potential V is monotone increasing on $[0, u_{max}]$. For a constant $T_0 > 0$ with $T_0^2 \geq -V(0)$ we introduce the functions*

$$\beta \;=\; \frac{1}{2}\left(T_0 - \frac{V}{T_0}\right), \qquad R \;=\; \frac{1}{2}\left(T_0 + \frac{V}{T_0}\right). \tag{4.25}$$

If a solution of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$y \;\in\; \Big\{z \;|\; |z - i\beta| \leq R,\ \mathrm{Re}\, z, \mathrm{Im}\, z \geq 0\Big\} \cup \left\{z \;\Big|\; \left|z - \frac{iT_0}{2}\right| \leq \frac{T_0}{2},\ \mathrm{Re}\, z \leq 0\right\},$$

then this condition holds for all $u \in [0, u_{max}]$ (see Figure 3).

Proof. Choosing $\alpha \equiv 0$ and β, T according to (4.25), we know from Lemma 4.1 and Lemma 4.2 that the circles $|y - m| \leq R$ are invariant. Furthermore, we note that the arc Λ in Figure 3 is the flow line of the equation $y' + y^2 = 0$, and thus it cannot be crossed from the right to the left when V is positive. This gives the result in the case that V has no zeros. If V has a zero, the invariant disks in the regions $V \leq 0$ and $V \geq 0$ coincide at

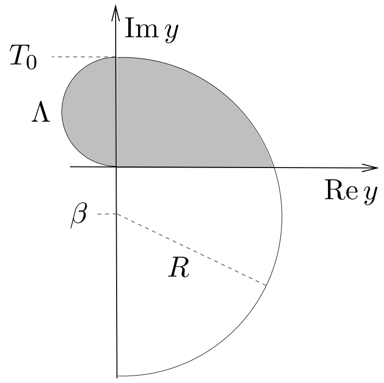

Figure 3: Invariant region estimate for monotone V.

the zero of V. ■

The invariant disk estimates of Lemma 4.1 and Lemma 4.2 can also be used if the functions α and U have a discontinuity at some $v \in [0, u_{\max}]$, i.e.

$$\alpha_l \;:=\; \lim_{u \nearrow v} \alpha(u) \;\neq\; \lim_{u \searrow v} \alpha(u) \;=:\; \alpha_r\;, \qquad U_l \;:=\; \lim_{u \nearrow v} U(u) \;\neq\; \lim_{u \searrow v} U(u) \;=:\; U_r\;.$$

In this case we choose the function T also to be discontinuous at v,

$$T_l \;:=\; \lim_{u \nearrow v} T(u) \;\neq\; \lim_{u \searrow v} T(u) \;=:\; T_r\;,$$

in such a way that the circle corresponding to (α_r, U_r, T_r) contains that corresponding to (α_l, U_l, T_l) (see Figure 4). In the next lemma we give sufficient "jump conditions" for this "matching."

Lemma 4.4 (matching of invariant disks) *Suppose that $U_l < 0$. Depending on the sign of U_r, we set*

$$T_r \;=\; T_l\, \frac{(\alpha_r - \alpha_l)^2 + |U_l + U_r|}{\sqrt{|U_l|\,|U_r|}} \qquad \textit{if } U_r < 0 \tag{4.26}$$

$$T_r \;=\; T_l\, \frac{(\alpha_r - \alpha_l)^2 + |U_l + U_r| + \sqrt{|U_l|\,U_r}}{\sqrt{|U_l|\,U_r}} \qquad \textit{if } U_r > 0 \tag{4.27}$$

Let $B_{l/r}$ be the disks with centers $m_{l/r} = \alpha_{l/r} + i\beta_{l/r}$ and radii $R_{l/r}$ as given by (4.13) or (4.14). Then $B_l \subset B_r$.

Proof. We must satisfy the condition $R_r \geq |m_r - m_l| + R_l$. Taking squares, we obtain the equivalent conditions $R_r \geq R_l$ and

$$(R_r - R_l)^2 \;\geq\; (\alpha_r - \alpha_l)^2 + (\beta_r - \beta_l)^2\,.$$

This last condition can also be written as

$$(\alpha_r - \alpha_l)^2 + (\beta_l^2 - R_l^2) + (\beta_r^2 - R_r^2) \;\leq\; 2\,(\beta_l \beta_r - R_l R_r)\,. \tag{4.28}$$

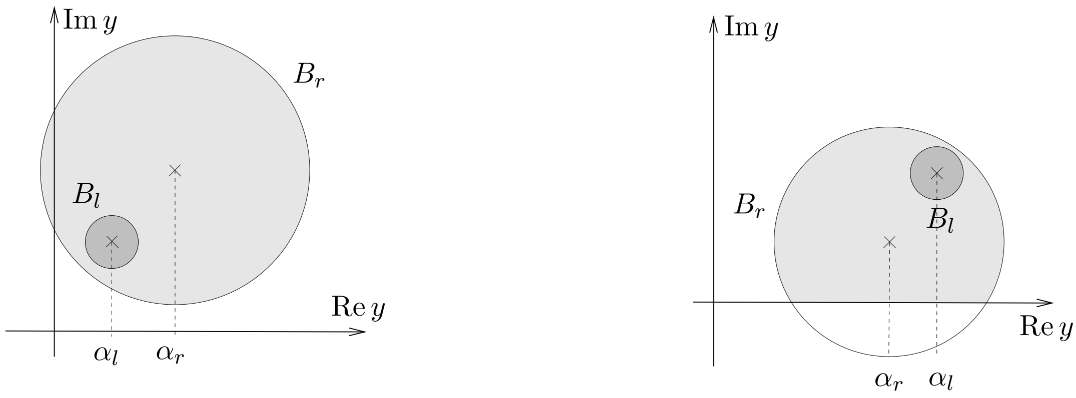

Figure 4: Matching of invariant disks in the cases $U_r < 0$ (left) and $U_r > 0$ (right).

In the case $U_r < 0$, we can substitute the ansatz (4.13) into (4.28) to obtain the equivalent inequality

$$(\alpha_r - \alpha_l)^2 + |U_l| + |U_r| \;\leq\; \sqrt{|U_l|\,|U_r|}\left(\frac{T_r}{T_l} + \frac{T_l}{T_r}\right).$$

Dropping the last summand on the right and solving for T_r, we obtain (4.26), which is thus a sufficient condition.

In the case $U_r > 0$, we substitute (4.13, 4.14) into (4.28) to obtain the equivalent condition

$$(\alpha_r - \alpha_l)^2 + |U_l| - U_r \;\leq\; \sqrt{|U_l|\,U_r}\left(\frac{T_r}{T_l} - \frac{T_l}{T_r}\right).$$

Using the inequality $|U_l| - U_r \leq |U_l + U_r|$, replacing the factor T_l/T_r on the right by one and solving for T_r, we obtain the sufficient condition (4.27). ∎

4.3 Bounds for the Wronskian and the Fundamental Solutions

We now consider the solutions $\acute{\phi}$ and $\grave{\phi}$ as defined in Section 3.1 for ω on the real axis and set

$$\acute{y} \;=\; \frac{\acute{\phi}'}{\acute{\phi}}\,, \qquad \grave{y} \;=\; \frac{\grave{\phi}'}{\grave{\phi}}\,.$$

We keep k fixed. Since taking the complex conjugate of the separated wave equation flips the sign of k, we may assume that $k \geq 0$. Then ω_0 as defined by (2.12) is negative.

Proposition 4.5 *If $\omega \not\in [\omega_0, 0]$, the Wronskian $w(\acute{\phi}, \grave{\phi})$ is non-zero.*

Proof. According to (2.13), ω and Ω have the same sign. From (4.10) we know that the functions $\acute{y}$ and $\grave{y}$ both stay either in the upper or lower half plane. In view of the asymptotics (3.1, 3.2), we know that they must be in be in opposite half planes. Thus

$$w(\acute{\phi}, \grave{\phi}) \;=\; \acute{\phi}\,\grave{\phi}\,(\acute{y} - \grave{y}) \;\neq\; 0\,.$$ ∎

In the case $\omega \in (\omega_0, 0)$, we need the following global estimate for large λ.

Proposition 4.6 *For any $u_1 \in \mathbb{R}$ there are constants $c, \lambda_0 > 0$ such that*

$$\left| \frac{\acute{\phi}(u)}{w(\acute{\phi}, \grave{\phi})} \right| \;\leq\; \frac{c}{\lambda} \qquad \text{for all } \lambda > \lambda_0,\; \omega \in (\omega_0, 0),\; u < u_1 \,.$$

The remainder of this section is devoted to the proof of this proposition. Let $u_1 \in \mathbb{R}$ and $\omega \in (\omega_0, 0)$. Possibly by increasing u_1 and λ_0 we can clearly arrange that V is monotone decreasing on $[u_1, \infty)$. Then we have the following estimate.

Lemma 4.7 *The functions $\grave{\phi}$ and $\grave{y}$ satisfy the inequalities*

$$|\grave{\phi}(u)| \;\geq\; 1\,, \quad \mathrm{Re}\, \grave{y}(u) \;\leq\; |\omega| \qquad \text{on } [u_1, \infty)\,.$$

Proof. From the asymptotics (3.2) we know that $\lim_{u \to \infty} \grave{y}(u) = -i\omega$. Thus for v sufficiently large, $|\grave{y}(v) - i|\omega|| < \varepsilon$, and we can apply Corollary 4.3 on the interval $[u_1, v]$ backwards in u with $T_0 = |\omega| + 2\varepsilon$. Since ε can be chosen arbitrarily small, we conclude that Corollary 4.3 applies even on $[u_1, \infty)$ with $T_0 = |\omega|$. This means that

$$0 \;\leq\; \mathrm{Im}\, \grave{y} \;\leq\; |\omega|\,, \quad \mathrm{Re}\, \grave{y} \;\leq\; |\omega| \qquad \text{on } [u_1, \infty)\,.$$

Finally, we use (4.9) with $w = i|\omega|$. ∎

We now come to the estimates for $\acute{\phi}$, which are more difficult because we need a stronger result. The next lemma specifies the behavior of the potential on $(-\infty, 2u_1]$.

Lemma 4.8 *For any $u_1 \in \mathbb{R}$ there are constants c, λ_0 such that the potential V has for all $\omega \in (\omega_0, 0)$ and all $\lambda > \lambda_0$ the following properties. There are unique points $u_- < u_0 < u_+ < u_1$ such that*

$$V(u_-) \;=\; -\frac{\Omega^2}{2}\,, \qquad V(u_0) \;=\; 0\,, \qquad V(u_+) \;=\; \Omega^2\,.$$

V is monotone increasing on $(-\infty, u_+]$. Furthermore,

$$u_+ - u_- \;\leq\; c \tag{4.29}$$

$$\gamma\, u_+ \;\geq\; \log \Omega^2 - \log \lambda - c \tag{4.30}$$

$$|V'|^{\frac{2}{3}} + |V''|^{\frac{1}{2}} \;\leq\; \frac{1}{4}\, |V| \qquad \text{on } [u_+, 2u_1], \tag{4.31}$$

with γ as in (3.9).

Proof. We expand V in a Taylor series around the event horizon,

$$V \;=\; -\Omega^2 + (\lambda + c_0)(r - r_1) + \lambda\, \mathcal{O}((r - r_1)^2)\,.$$

Hence for sufficiently large λ_0 there are near the event horizon unique points u_-, u_0, u_+ where the potential has the required value. Integrating (2.1) we get near the event horizon the asymptotic formula

$$u \;\sim\; \frac{1}{\gamma}\, \log(r - r_1)\,.$$

Getting asymptotic expansions for $u_\pm$ we immediately obtain (4.29, 4.30). Furthermore, using (2.1) to transform r-derivatives into u-derivatives, we obtain in the region $(r_1, r_1 + \varepsilon) \cap (u_+, \infty)$ the estimate

$$\begin{aligned} \frac{\lambda}{c}\, e^{\gamma u} \;\leq\; V(u) \;&\leq\; \lambda\, c\, e^{\gamma u} \\ |V'(u)| + |V''(u)| \;&\leq\; \lambda\, c\, e^{\gamma u}\,, \end{aligned}$$

uniformly in λ and ω. Hence for sufficiently large λ_0, (4.31) will be satisified near the event horizon.

In the region $r > r_1 + \varepsilon$ away from the event horizon, V is strictly positive, $V > \lambda/c$, and since the derivatives of V can clearly be bounded by $|V'| + |V''| < c\lambda$, it follows that (4.31) is again satisfied. ∎

First we apply Corollary 4.3 on the interval $(-\infty, u_-)$ to obtain the following result.

Corollary 4.9 *There is a constant $c > 0$ such that for all $\omega \in (\omega_0, 0)$ and $\lambda > \lambda_0$,*

$$\frac{\Omega}{2} \;\leq\; \mathrm{Im}\, y \;\leq\; \Omega\,, \quad |\mathrm{Re}\, y| \;\leq\; \frac{\Omega}{2} \qquad \text{on } (-\infty, u_-]\,.$$

Also, at $u = u_-$ we have an invariant disk with

$$\alpha_l \;=\; 0\,, \qquad U_l \;=\; -\frac{\Omega^2}{2}\,, \qquad T_l \;=\; \sqrt{2}\,. \tag{4.32}$$

On the interval $[u_-, u_+]$ we use the method described in the next lemma.

Lemma 4.10 *Assume that the potential V is monotone increasing on $[0, u_{max}]$. We set*

$$\alpha \;=\; \sqrt{\max(2\, V(u_{max}), 0)}$$

and introduce for a given constant $T_0 > 1$ the functions U, σ, β, R, and T by (4.12, 4.14) and

$$T(u) \;=\; T_0\, e^{2\alpha u}\, \frac{\sqrt{|U(0)|}}{\sqrt{|U(u)|}}\,. \tag{4.33}$$

If a solution y of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$|y - m| \;\leq\; R\,,$$

then this condition holds for all $u \in [0, u_{max}]$.

Proof. By definition of α, the function $U = V - \alpha^2$ is negative and monotone increasing. Using furthermore that $\sigma = e^{2\alpha u}$, we can estimate the total variation in (4.16) as follows,

$$\mathrm{TV}_{[0,u)} \log |\sigma^2 U| \;=\; \int_0^u \left(4\alpha - \frac{|U|'}{|U|} \right) \;=\; 4\alpha u + \log |U(0)| - \log |U(u)|.$$

This gives (4.33). ∎

Thus we match the invariant disk (4.32) to a disk with $U_r = V(u_-) - \alpha_r^2$ and $\alpha_r = \sqrt{2}\,\Omega$. From (4.29) we see that $(u_+ - u_-)\,\alpha$ is uniformly bounded, and thus we obtain the following estimate.

Corollary 4.11 *There is a constant $c > 0$ such that for all $\omega \in (\omega_0, 0)$ and $\lambda > \lambda_0$,*

$$\frac{\Omega}{c} \;\leq\; \mathrm{Im}\, y \;\leq\; c\,\Omega\,, \qquad |\mathrm{Re}\, y| \;\leq\; c\,\Omega \qquad\qquad \text{on } [u_-, u_+]\,.$$

At $u = u_+$ we get an invariant disk with

$$0 \;\leq\; \alpha_l \;\leq\; \Omega\,, \quad -c\,U_l \;=\; -\Omega^2\,, \qquad T_l \;\leq\; c\,. \tag{4.34}$$

In the remaining interval $[u_+, 2u_1]$ an approximate solution of the Schrödinger equation (2.14) is available from semi-classical analysis: the WKB wave function

$$\phi(u) \;=\; V^{-\frac{1}{4}}\, \exp\left(\int^u \sqrt{V} \right).$$

The corresponding function y is given by

$$y(u) \;=\; \sqrt{V} - \frac{V'}{4V}\,.$$

In order to get an invariant disk estimate which quantifies the exponential increase of φ, we choose α such that it also becomes large as $V \gg 0$. For technical simplicity, we choose

$$\alpha(u) \;=\; \frac{7}{8}\, \sqrt{V(u)}\,, \tag{4.35}$$

giving rise to the following general estimate.

Lemma 4.12 *Assume that the potential V is positive on $[0, u_{max}]$ and that*

$$|V'(u)| \;\leq\; \frac{1}{2}\, V(u)^{\frac{3}{2}}\,, \qquad\qquad |V''(u)| \;\leq\; \frac{1}{4}\, V(u)^2\,. \tag{4.36}$$

We introduce for a given constant $T_0 > 0$ the functions α, U, σ, β, R, and T by (4.35, 4.12, 4.14, 4.22). If a solution y of the complex Riccati equation (4.5) satisfies at $u = 0$ the condition

$$|y - m| \;\leq\; R\,,$$

then this condition holds for all $u \in [0, u_{max}]$. Furthermore,

$$\mathrm{Re}\, y \;\geq\; \frac{\sqrt{V}}{8} \;-\; \frac{T_0}{2}\,. \tag{4.37}$$

Proof. A short calculation yields

$$\begin{aligned} U \;&=\; V - \alpha^2 - \alpha' \;=\; \frac{15}{64}\, V - \frac{7}{16}\, \frac{V'}{\sqrt{V}} \\ U' + 4\alpha U \;&=\; \frac{105}{128}\, V^{\frac{3}{2}} - \frac{83}{64}\, V' + \frac{7}{32}\, \frac{V'^2}{V^{\frac{3}{2}}} - \frac{7}{16}\, \frac{V''}{\sqrt{V}}\,. \end{aligned}$$

Using (4.36) we obtain the estimates

$$\frac{V}{64} \;\leq\; U \;\leq\; \frac{V}{2} \qquad\quad \text{and} \qquad\quad U' + 4\alpha U \;\geq\; \frac{V^{\frac{3}{2}}}{16}\,.$$

Hence the conditions (4.21) are satisified, and Lemma 4.2 applies. The inequality (4.37) follows from (4.23), the just-derived upper bound for U and the fact that $\sigma \geq 1$. ■

Matching the invariant disk (4.34) to the invariant disk with $\alpha_r = \alpha(u_r)$ and $U_r = V(u_r) - \alpha^2(u_r) - \alpha'(u_r)$ with α according to (4.35), we obtain

$$U_r \;\leq\; \Omega^2 \,, \qquad\quad T_r \;\leq\; c\,. \tag{4.38}$$

We can then apply the last lemma on the interval $[u_+, 2u_1]$.

Proof of Proposition 4.6. Suppose that $u < u_1$. Using the definition of $\acute{y}$ and $\grave{y}$, we can rewrite the Wronskian as

$$w(\acute{\phi}, \grave{\phi}) \;=\; \acute{\phi}\,\grave{\phi}\,(\acute{y} - \grave{y})\,.$$

Applying Lemma 4.7 at $u = 2u_1$ gives

$$\left|\frac{\acute{\phi}(u)}{w(\acute{\phi}, \grave{\phi})}\right| \;\leq\; \left|\frac{\acute{\phi}(u)}{\acute{\phi}(2u_1)}\right| \, \frac{1}{\mathrm{Re}\,\acute{y}(2u_1) - |\omega|}\,. \tag{4.39}$$

We combine (4.37) with $T_0 = T_r$ satisfying (4.38) to get

$$\mathrm{Re}\,\acute{y} \;\geq\; \frac{\sqrt{V}}{8} \;-\; c\Omega\,. \tag{4.40}$$

Since the potential V is strictly positive on the interval $[u_1, 2u_1]$, we can, possibly by increasing λ_0 and c, arrange that

$$\sqrt{V} \;\geq\; \frac{\sqrt{\lambda}}{c} \qquad\quad \text{on } [u_1, 2u_1] \tag{4.41}$$

and thus also that

$$\mathrm{Re}\,\acute{y} \;\geq\; \frac{\sqrt{\lambda}}{16\,c} \qquad\quad \text{on } [u_1, 2u_1]\,. \tag{4.42}$$

This inequality allows us to bound the fraction in (4.39),

$$\left|\frac{\acute{\phi}(u)}{w(\acute{\phi}, \grave{\phi})}\right| \;\leq\; \left|\frac{\acute{\phi}(u)}{\acute{\phi}(2u_1)}\right|\,. \tag{4.43}$$

Thus it remains to control the last quotient. We omit the accent and use the notation $\rho = |\phi|$. In the case $u < u_+$, we can use (4.9),

$$\frac{\rho(u)^2}{\rho(u_+)^2} \;=\; \frac{\mathrm{Im}\, y(u_+)}{\mathrm{Im}\, y(u)}\,,$$

and the last quotient is controlled from above and below by Corollary 4.9 and Corollary 4.11. Hence, rewriting the quotient on the right of (4.43) as

$$\frac{\rho(u)}{\rho(2u_1)} \;=\; \frac{\rho(u)}{\rho(u_+)}\,\frac{\rho(u_+)}{\rho(2u_1)}\,,$$

it remains to consider the case $u \geq u_+$. Applying (4.7) and (4.40), we obtain

$$A \;:=\; \log\left|\frac{\phi(u)}{\phi(2u_1)}\right| \;=\; -\int_u^{2u_1} \mathrm{Re}\, y(u) \;\leq\; c\,\Omega\,(2u_1 - u_+) \;-\; \frac{1}{8}\int_u^{2u_1} \sqrt{V}\,.$$

Now we use (4.30) and the fact that the function $\Omega \log \Omega$ is bounded,

$$A \;\leq\; c \,\log \lambda \;-\; \frac{1}{8} \int_u^{2u_1} \sqrt{V}\,.$$

Estimating the last summand with (4.41),

$$\int_u^{2u_1} \sqrt{V} \;\geq\; \int_{u_1}^{2u_1} \sqrt{V} \;\geq\; \frac{\sqrt{\lambda}}{c}\, u_1\,,$$

we conclude that for large λ this summand dominates the term $c \log \lambda$, and thus (4.43) decays in λ even like $\exp(-\sqrt{\lambda}/c)$. ■

5 Contour Deformations to the Real Axis

In this section we fix the angular momentum number k throughout and omit the angular variable φ. We can again assume without loss of generality that $k \geq 0$. Also, since here we are interested in the situation only locally in u, we evaluate weakly. Thus we write the integral representation (2.16) for compactly supported initial data Ψ_0 and a test function $\eta \in C^\infty_0(\mathbb{R} \times S^2)^2$ as

$$<\eta,\, \Psi(t)> \;=\; -\frac{1}{2\pi i} \sum_{n \in \mathbb{N}} \lim_{\varepsilon \searrow 0} \left(\int_{C_\varepsilon} - \int_{\overline{C_\varepsilon}} \right) d\omega\, e^{-i\omega t} <\eta, Q_n(\omega)\, S_\infty(\omega)\, \Psi_0> . \qquad (5.1)$$

The integration contour in (5.1) can be moved to the real axis provided that the integrand is continuous. In the next lemma we specify when this is the case and simplify the integrand. For ω real, the complex conjugates of $\acute{\phi}$ and $\grave{\phi}$ are again solutions of the ODE. Thus, apart from the exceptional cases $\omega \in \{0, \omega_0\}$, we can express $\grave{\phi}$ as a linear combination of $\acute{\phi}$ and $\overline{\acute{\phi}}$,

$$\grave{\phi} \;=\; \alpha\, \acute{\phi} + \beta\, \overline{\acute{\phi}} \qquad\qquad (\omega \in \mathbb{R} \setminus \{0, \omega_0\})\,. \qquad (5.2)$$

The complex coefficients α and β are called *transmission coefficients*. The Wronskian of $\acute{\phi}$ and $\grave{\phi}$ can then be expressed by

$$w(\acute{\phi}, \grave{\phi}) \;=\; \beta\, w(\acute{\phi}, \overline{\acute{\phi}}) \;=\; 2i\Omega\, \beta\,, \qquad (5.3)$$

where in the last step we used the asymptotics (3.1). Furthermore, it is convenient to introduce the real fundamental solutions

$$\phi_1 \;=\; \mathrm{Re}\, \acute{\phi}\,, \qquad\qquad \phi_2 \;=\; \mathrm{Im}\, \acute{\phi}\,,$$

and to denote the corresponding solutions of the wave equation in Hamiltonian form by $\Psi^{\omega n}_{1\!/\!2}$.

Lemma 5.1 *If the Wronskian $w(\acute{\phi}, \grave{\phi})$ is non-zero at $\omega \in \mathbb{R} \setminus \{0, \omega_0\}$, then the integrand in (5.1) is continuous at ω and*

$$(\lim_{\varepsilon \nearrow 0} - \lim_{\varepsilon \searrow 0})(Q_n(\omega + i\varepsilon)\, S_\infty(\omega + i\varepsilon)\, \Psi)(r, \vartheta) \;=\; -\frac{i}{\omega \Omega} \sum_{a,b=1}^2 t^{\omega n}_{ab}\; \Psi^{\omega n}_a <\Psi^{\omega n}_b, \Psi>\,, \qquad (5.4)$$

where the coefficients t_{ab} are given by

$$t_{11} \;=\; 1 + \mathrm{Re}\,\frac{\alpha}{\beta}\,, \qquad t_{12} \;=\; t_{21} \;=\; -\mathrm{Im}\,\frac{\alpha}{\beta}\,, \qquad t_{22} \;=\; 1 - \mathrm{Re}\,\frac{\alpha}{\beta}\,. \tag{5.5}$$

Proof. We start from the explicit formula for the operator product $Q_n S_\infty$ given in [4, Proposition 5.4]. Since the angular operator $Q_n(\omega{+}i\varepsilon)$ can be diagonalized for ε sufficiently small, the kernel $g(u,u')$ is simply the Green's function of the radial ODE, i.e. for ω in the lower half plane,

$$g(u,u') \;:=\; \frac{1}{w(\acute{\phi},\grave{\phi})} \times \left\{ \begin{array}{ll} \acute{\phi}(u)\,\grave{\phi}(u') & \text{if } u \leq u' \\ \grave{\phi}(u)\,\acute{\phi}(u') & \text{if } u > u'\,. \end{array} \right. \,, \tag{5.6}$$

whereas the formula in the upper half plane is obtained by complex conjugation. Using that $\lim_{\varepsilon \nearrow 0} \acute{\phi} = \lim_{\varepsilon \searrow 0} \overline{\acute{\phi}}$ and $\lim_{\varepsilon \nearrow 0} \grave{\phi} = \lim_{\varepsilon \searrow 0} \overline{\grave{\phi}}$, we find that

$$(\lim_{\varepsilon \nearrow 0} - \lim_{\varepsilon \searrow 0})\, g(u,u') \;=\; 2i\,\mathrm{Im}\, g(u,u')\,,$$

and a short calculation using (5.2, 5.3) gives

$$(\lim_{\varepsilon \nearrow 0} - \lim_{\varepsilon \searrow 0})\, g(u,u') \;=\; -\frac{i}{\Omega} \sum_{a,b=1}^{2} t_{ab}\, \phi^a(u)\, \phi^b(u')$$

with t_{ab} according to (5.5).

Except for the function $g(u,u')$, all the functions appearing in the formula for $Q_n S_\infty$ in [4, Proposition 5.4] are continuous on the real axis. A direct calculation shows that

$$\begin{aligned} &(\lim_{\varepsilon \nearrow 0} - \lim_{\varepsilon \searrow 0})(Q_{k,n}(\omega+i\varepsilon)\, S_\infty(\omega+i\varepsilon)\, \Psi) \\ &= \; -\frac{i}{\omega\Omega} \sum_{a,b=1}^{2} t_{ab}\, \Psi_a <\Psi_b, \begin{pmatrix} (\omega-\beta)\omega & 0 \\ 0 & 1 \end{pmatrix} \Psi>_{L^2(d\mu)} \end{aligned}$$

with $d\mu$ given by (2.7). Since the Ψ_b are eigenfunctions of the Hamiltonian, we know according to (2.3) that $A\Psi_b = (\omega-\beta)\omega\Psi_b$. Using furthermore that the operator A is symmetric on $L^2(d\mu)$, we conclude that

$$<\Psi_b, \begin{pmatrix} (\omega-\beta)\omega & 0 \\ 0 & 1 \end{pmatrix} \Psi>_{L^2(d\mu)} \;=\; <\Psi_b, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi>_{L^2(d\mu)} \;=\; <\Psi_b,\, \Psi>\,,$$

where in the last step we used (2.8). ■

Let us now consider for which values of ω and n the contour can be moved to the real axis. According to Proposition 4.5, the Wronskian $w(\acute{\phi},\grave{\phi})$ is non-zero unless $\omega \in [\omega_0, 0]$. We now analyze carefully the exceptional cases $\omega = 0, \omega_0$. From Theorem 3.1, Theorem 3.2 and Theorem 3.5 we know that the functions $\acute{\phi}$ and $\phi_\omega = \omega^\mu \grave{\phi}$ are continuous for all $\omega \in \mathbb{R}$. If $ak = 0$ and $\omega = 0$, the functions $\acute{\phi}$ and ϕ_ω degenerate to real solutions with the asymptotics

$$\lim_{u \to -\infty} \acute{\phi}(u) \;=\; 1\,, \qquad \lim_{u \to \infty} u^{\mu - \frac{1}{2}}\, \phi_0(u) \;=\; \frac{\Gamma(\mu)}{\sqrt{\pi}}\,.$$

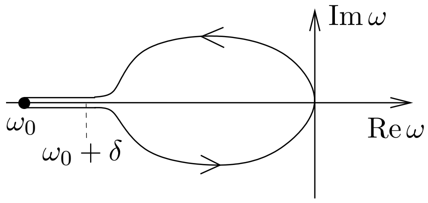

Figure 5: The integration contour D_δ.

Noting that the function

$$\partial_u \sqrt{r^2+a^2} \;=\; \frac{r\,\Delta}{(r^2+a^2)^{\frac{3}{2}}} \;=\; \frac{r}{\sqrt{r^2+a^2}} \left(1 - \frac{2Mr}{r^2+a^2}\right)$$

is monotone increasing, the potential V, (2.15), is everywhere positive. Hence solutions of the Schrödinger equation (2.14) are convex. This implies that the functions $\acute{\phi}$ and $\grave{\phi}$ do not coincide, and thus their Wronskian is non-zero. As a consequence, the Green's function (5.6), and thus the whole integrand in (5.1), is bounded and continuous near $\omega = 0$ (note that (5.6) is invariant under rescalings of $\grave{\phi}$, and thus we can in this formula replace $\grave{\phi}$ by ϕ_ω). In the case $ak \neq 0$ and $\omega = 0$, the function ϕ_0 is real, whereas $\acute{\phi}$ is complex, and thus $w(\acute{\phi}, \phi_0) \neq 0$. If on the other hand $ak \neq 0$ and $\omega = \omega_0$, $\acute{\phi}$ is real and $\grave{\phi}$ is complex, and again $w(\acute{\phi}, \phi_0) \neq 0$. Hence the integrand in (5.1) is continuous and bounded at the points $\omega = 0, \omega_0$. We conclude that for every $n \in \mathbb{N}$, the integrand in (5.1) is continuous on an open neighborhood of on $\omega \in \mathbb{R} \setminus (\omega_0, 0)$. Furthermore, according to Proposition 4.6, $w(\acute{\phi}, \grave{\phi}) \neq 0$ if $\omega \in (\omega_0, 0)$ and λ is sufficiently large. We have thus proved the following result.

Proposition 5.2 *There is $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for every $\Psi_0 \in C^\infty_0(\mathbb{R} \times S^2)^2$, the completeness relation*

$$\begin{aligned} \Psi_0 \;=\; & \frac{1}{2\pi} \left(\sum_{n=0}^{\infty} \int_{\mathbb{R}\setminus[\omega_0,0]} + \sum_{n>n_0} \int_{\omega_0}^{0} \right) \frac{d\omega}{\omega\Omega} \sum_{a,b=1}^{2} t^{\omega n}_{ab}\, \Psi^{\omega n}_a <\Psi^{\omega n}_b, \Psi_0> \\ & + \sum_{n \leq n_0} \oint_{D_\delta} (Q_n\, S_\infty\, \Psi_0)\, d\omega \end{aligned}$$

holds, with the contour D_δ as in Figure 5.

We point out that the contour D_δ passes along the line segment $[\omega_0, \omega_0 + \delta)$ twice, once as the limit of the contour in the lower half plane, and once as limit of the contour in the upper half plane. These two integrals can be combined to one integral over $[\omega_0, \omega_0 + \delta)$ with the integrand given by (5.4).

Let us now consider how the remaining contour integrals over C_ε can be moved to the real line. According to Theorems 3.1 and 3.2, the functions $\acute{\phi}$ and $\grave{\phi}$ have for every $n \leq n_0$ and for every $\omega \in (\omega_0, 0)$ a holomorphic extension to a neighborhood of ω. Thus their Wronskian is also holomorphic in this neighborhood, and consequently they can have only isolated zeros of finite order. Since $w(\acute{\phi}, \grave{\phi}) \neq 0$ for ω near 0 and ω_0, we conclude that the numbers of zeros must be finite. Since we only need to consider a finite number of angular momentum modes, there is only a finite number of points $\omega_1, \ldots, \omega_K \in (\omega_0, 0)$, $K \geq 0$, where any of the Wronskians $w(\acute{\phi}_n, \grave{\phi}_n)$ has a zero. We denote the maximum of the orders of these zeros at ω_i by $l_i \in \mathbb{N}$.

The above zeros of the Wronskian lead to poles in the integrand of (5.1) and correspond to radiant modes. We will prove in Section 7 by contradiction that these radiant modes are actually absent. Therefore, we now make the assumption that there are radiant modes, i.e. that the Wronskians $w(\acute{\phi}_n, \grave{\phi}_n)$ have at least one zero on the real axis. As a preparation for the analysis of Section 7, we now choose a special configuration where radiant modes appear, but in the simplest possible way. We choose new initial data

$$\Phi_0 \;=\; \mathcal{P}(H)\, \Psi_0 \,, \tag{5.7}$$

where $\mathcal{P}$ is the polynomial

$$\mathcal{P}(x) \;=\; \omega\, (\omega - \omega_0)\; (x - \omega_1)^{l_1 - 1} \prod_{i=2}^{K} (x - \omega_i)^{l_i} \,.$$

Then Φ_0 again has compact support, and using the spectral calculus, the corresponding solution $\Phi(t)$ of the Cauchy problem is obtained from (5.1) by multiplying the integrand by $\mathcal{P}(\omega)$. Then the poles of the integrand at $\omega_2, \ldots, \omega_K$ disappear, and at ω_1 a simple pole remains. Subtracting this pole, the integrand becomes analytic, whereas for the pole itself we get a contour integral which can be computed with residues.

Let us summarize the result of the above construction with a compact notation. For a test function $\eta \in C^\infty_0(\mathbb{R} \times S^2)^2$ we introduce the vectors $\eta^{\omega n}$ by

$$\eta^{\omega n} \;=\; (\eta^{\omega n}_1, \eta^{\omega n}_2) \qquad \text{where} \qquad \eta^{\omega n}_a \;=\; <\Psi^{\omega n}_a, \eta> .$$

Proposition 5.3 *Assume that there are radiant modes, $K > 0$. Then the Cauchy development $\Phi(t)$ of the initial data (5.7) satisfies the relation*

$$<\eta, \Phi(t)> \;=\; \frac{1}{2\pi} \sum_{n \in \mathbb{N}} \int_{-\infty}^{\infty} e^{-i\omega t}\, \langle \eta^{\omega n}, T^{\omega n} \Phi^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega \;+\; e^{-i\omega_1 t} \sum_{n \leq n_0} \langle \eta^{\omega_1 n}, \sigma^n \rangle_{\mathbb{C}^2} \,.$$

Here $\omega_1 \in (\omega_0, 0)$. The $(\sigma^n)_{n=1,\ldots,n_0}$ are vectors in $\mathbb{C}^2$, at least one of which is non-zero. The matrices $T^{\omega n}$ have the following properties,

(1) *If $\omega \notin [\omega_0, 0]$ or $n > n_0$,*

$$(T^{\omega n})_{ab} \;=\; t^{\omega n}_{ab} \tag{5.8}$$

with t_{ab} according to (5.5).

(2) *For each n, the function $T^{\omega n}$ is continuous in $\omega \in \mathbb{R}$ and analytic in $(\omega_0, 0)$.*

6 Energy Splitting Estimates

In this section, we consider the family of test functions

$$\eta_L(u) \;=\; \eta(u + L)$$

for a fixed $\eta \in C^\infty_0(\mathbb{R} \times S^2)^2$. Our goal is to control the inner product $<\eta_L, \Phi(t)>$ in the limit $L \to \infty$ when the support of η_L moves towards the event horizon. Our method is to split up the inner product into a positive and an indefinite part. Once the indefinite part is bounded using the ODE estimates of Section 4, we can use the Schwarz inequality and energy conservation to also control the positive part.

We choose $u_1 \in \mathbb{R}$ and general test functions $\eta, \zeta \in C_0^\infty((-\infty, u_1) \times S^2)^2$ which are supported to the left of u_1 (for later use we often work more generally with ζ instead of Φ_0). Since for each fixed n, the $T^{\omega n}$ are continuous and the eigensolutions $\Psi_a^{\omega n}(u)$ are, according to Theorem 3.1, also continuous in ω, uniformly for $u \in (-\infty, u_1)$, we have no difficulty controlling the expressions $\langle \eta^{\omega n}, T^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2}$ for $n \leq n_0$ and $\omega \in [\omega_0, 0]$. Hence we only need to consider the case when the matrix $T^{\omega n}$ is given by (5.8). Using (5.5), the eigenvalues $\lambda_\pm$ of this matrix are

$$\lambda_\pm \;=\; 1 \pm \left| \frac{\alpha}{\beta} \right| . \tag{6.1}$$

In order to determine the sign of these eigenvalues, we first use the asymptotics (3.1, 3.2) to compute the Wronskians $w(\grave{\phi}, \overline{\grave{\phi}}) = -2i\omega$ and $w(\acute{\phi}, \overline{\acute{\phi}}) = 2i\Omega$. Furthermore, we obtain from (5.2) and its complex conjugate that

$$w(\grave{\phi}, \overline{\grave{\phi}}) \;=\; (|\alpha|^2 - |\beta|^2)\, w(\acute{\phi}, \overline{\acute{\phi}}) \,.$$

Combining these identities, we find that

$$|\alpha|^2 - |\beta|^2 \;=\; -\frac{\omega}{\Omega} \tag{6.2}$$

From (6.1, 6.2) we see that in the case $\omega \not\in [\omega_0, 0]$, where ω and Ω have the same sign, the eigenvalues $\lambda_{1/2}$ are both positive. However, if $\omega \in (\omega_0, 0)$, one of the eigenvalues is negative. This result is not surprising, because the lack of positivity corresponds to the fact that for $\omega \in [\omega_0, 0]$ the energy density can be negative inside the ergosphere. In the case when $T^{\omega n}$ is not positive, we decompose it into the difference of two positive matrices,

$$T^{\omega n} \;=\; T_+^{\omega n} - T_-^{\omega n} \qquad \text{for } \omega \in (\omega_0, 0),\ n > n_0 \,,$$

where

$$T_-^{\omega n} \;=\; -\lambda_- \, \mathbb{1} \,.$$

In the next lemma we bound the integral over $T_-^{\omega n}$ using ODE techniques.

Lemma 6.1 *For any $\varepsilon > 0$ we can, possibly by increasing n_0, arrange that for all $L \leq 0$,*

$$\sum_{n > n_0} \int_{\omega_0}^0 \left| \langle \eta_L^{\omega n}, T_-^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2} \right| d\omega \;\leq\; \varepsilon \,.$$

Proof. Using (6.2, 5.3) we can estimate the norm of T_- by

$$\|T_-\| \;=\; |\lambda_-| \;=\; \frac{|\alpha|^2 - |\beta|^2}{|\beta|\,(|\alpha| + |\beta|)} \;\leq\; \left| \frac{\omega}{\Omega} \right| \frac{1}{2\,|\beta|^2} \;=\; \frac{2|\omega\Omega|}{|w(\acute{\phi}, \grave{\phi})|^2} \,.$$

Hence

$$\sum_{n > n_0} \int_{\omega_0}^0 \left| \langle \eta_L^{\omega n}, T_-^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2} \right| d\omega \;\leq\; 2 \sum_{n > n_0} \int_{\omega_0}^0 \frac{|\eta_L^{\omega n}|}{|w(\acute{\phi}, \grave{\phi})|} \, \frac{|\zeta^{\omega n}|}{|w(\acute{\phi}, \grave{\phi})|} \, |\omega\Omega| \, d\omega \,.$$

Writing out the energy scalar product using [4, eq. (2.14)] and expressing the fundamental solutions $\Psi_a^{\omega n}$ in terms of the radial solution $\acute{\phi}$, one sees that

$$|\eta_L^{\omega n}| \;\leq\; c \sup_{\mathbb{R}} |\eta_L\, \acute{\phi}|\;, \qquad\qquad |\zeta^{\omega n}| \;\leq\; c \sup_{\mathbb{R}} |\zeta\, \acute{\phi}|\;,$$

where the constant $c = c(\omega)$ is independent of λ. Now we apply Proposition 4.6 and use that the eigenvalues λ_n grow quadratically in n, (2.11). ■

Lemma 6.2 *There is a constant $C > 0$ such that for all $L \geq 0$,*

$$\sum_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus[\omega_0,0]} |\langle \eta_L^{\omega n}, T^{\omega n} \zeta^{\omega n}\rangle_{\mathbb{C}^2}|\; d\omega \;\leq\; C\;.$$

Proof. First of all, using the positivity of the matrix T,

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus[\omega_0,0]} |\langle \eta_L^{\omega n}, T^{\omega n} \zeta^{\omega n}\rangle_{\mathbb{C}^2}|\; d\omega \\
&\quad\leq\; \frac{1}{2} \sum_{n\in\mathbb{N}} \int_{\mathbb{R}\setminus[\omega_0,0]} \left(\langle \eta_L^{\omega n}, T^{\omega n} \eta_L^{\omega n}\rangle_{\mathbb{C}^2} \;+\; \langle \zeta^{\omega n}, T^{\omega n} \zeta^{\omega n}\rangle_{\mathbb{C}^2}\right) d\omega.
\end{aligned}$$

The two summands can be treated in exactly the same way; we treat the summand involving $\eta_L^{\omega n}$ because of the additional L-dependence. Applying Proposition 5.2 and dropping all negative terms, we get

$$\begin{aligned}
\int_{\mathbb{R}\setminus[\omega_0,0]} \langle \eta_L^{\omega n}, T_+^{\omega n} \eta_L^{\omega n}\rangle_{\mathbb{C}^2}\, d\omega \;&\leq\; <\!\eta_L, H\,(H-\omega_0)\,\eta_L\!> \\
&+ \sum_{n>n_0} \int_{\omega_0}^0 \langle \eta_L^{\omega n}, T_-^{\omega n} \eta_L^{\omega n}\rangle_{\mathbb{C}^2}\, d\omega \;+\; \sum_{n\leq n_0} \oint_{D_\delta} |<\!\eta_L,\, (Q_n\, S_\infty\, H\,(H-\omega_0)\,\eta_L)\!>|\; d\omega\;.
\end{aligned}$$

Using the asymptotic form of the energy scalar product and the Hamiltonian near the event horizon, it is obvious that the first term stays bounded as $L \to \infty$. The second term is bounded according to Lemma 6.1. For the contour integrals we can use the formula (5.4) on the real interval $[\omega_0, \omega_0+\delta)$. Since Theorem 3.1 gives us control of the asymptotics of fundamental solution $\acute{\phi}$ uniformly as $u \to -\infty$, it is clear that the integral over $[\omega_0, \omega_0+\delta)$ is bounded uniformly in L. For the contour in the complex plane, we cannot work with (5.4), but we must instead consider the formula for the operator product $Q_n\, S_\infty$ give in [4, Proposition 5.4] together with the estimate for the Green's function given in Lemma 6.3 below. ■

Lemma 6.3 *For every $\tilde{\omega} \in D_\delta$ with $\omega \neq \omega_0$, there are constants $C, \epsilon > 0$ and $u_0 \in \mathbb{R}$ such the Green's function satisfies for all $\omega \in D_\delta \cap B_\epsilon(\tilde{\omega})$ the inequality*

$$|g(u,v)| \;\leq\; C \qquad\quad \textit{for all } u, v \leq u_0\;.$$

Proof. It suffices to consider the case $\mathrm{Im}\,\omega \leq 0$, because the Green's function in the upper half plane is obtained simply by complex conjugation. By symmetry, we can furthermore assume that $u \leq v$. Thus, according to (5.6), we must prove the inequality

$$\left| \frac{\acute{\phi}(u)\,\grave{\phi}(v)}{w(\acute{\phi}, \grave{\phi})} \right| \;\leq\; C \qquad \text{for all } u \leq v \leq u_0\,.$$

According to Whiting's mode stability [9], the Wronskian $w(\acute{\phi}, \grave{\phi})$ has no zeros away from the real line, and thus by choosing δ so small that $B_\delta(\tilde{\omega})$ lies entirely in the lower half plane, we can arrange that $|w(\acute{\phi}, \grave{\phi})|$ is bounded away from zero on $B_\delta(\tilde{\omega})$. Hence our task is to bound the factor $|\acute{\phi}(u)\,\grave{\phi}(v)|$. Solving the defining equation for $w(\acute{\phi}, \grave{\phi})$ for $\grave{\phi}$ and integrating, we obtain

$$\frac{\grave{\phi}}{\acute{\phi}}\Big|_v^{u_0} \;=\; -w(\acute{\phi}, \grave{\phi}) \int_v^{u_0} \frac{du}{\acute{\phi}(u)^2}\,.$$

Substituting the identity

$$\frac{1}{\acute{\phi}(u)^2} \;=\; \frac{e^{-2i\Omega u}}{2i\Omega}\, \frac{d}{du} \left(\frac{e^{2i\Omega u}}{\acute{\phi}(u)^2} \right) \;-\; \frac{1}{2i\Omega}\, \frac{d}{du} \left(\frac{1}{\acute{\phi}(u)^2} \right),$$

the integral over the last term gives a boundary term,

$$\int_v^{u_0} \frac{1}{2i\Omega}\, \frac{d}{du} \left(\frac{1}{\acute{\phi}(u)^2} \right) \;=\; \frac{1}{2i\Omega}\, \frac{1}{\acute{\phi}(u)^2} \Big|_v^{u_0}\,.$$

The integral over the other term can be estimated by

$$\frac{1}{2\Omega} \int_v^{u_0} \left| e^{-2i\Omega u} \left(\frac{e^{2i\Omega u}}{\acute{\phi}(u)^2} \right)' \right| dv \;\leq\; \frac{e^{-2\mathrm{Im}\,\Omega\, v}}{\Omega} \int_v^{u_0} \left| \frac{(e^{-i\Omega u}\, \acute{\phi}(u))'}{(e^{-i\Omega u}\, \acute{\phi}(u))^3} \right| dv\,.$$

Using the asymptotics (3.1) one sees that the last integrand vanishes at the event horizon. From the series expansion for $\acute{\phi}$, (3.7, 3.8), we see that this integrand decays even exponentially fast. Therefore, the last integral is finite, uniformly in v and locally uniformly in ω. Collecting all the obtained terms and using the known asymptotics (3.1) of $\acute{\phi}$, the result follows. ∎

Lemma 6.4 *For any $\varepsilon > 0$ we can, possibly by increasing n_0, arrange that for all $L \geq 0$,*

$$\sum_{n > n_0} \int_{\omega_0}^0 \left| \langle \eta_L^{\omega n}, T_+^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2} \right| d\omega \;\leq\; \varepsilon\,.$$

Proof. Again using positivity, it suffices to bound the terms

$$\sum_{n > n_0} \int_{\omega_0}^0 \langle \eta_L^{\omega n}, T_+^{\omega n} \eta_L^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega \quad \text{and} \quad \sum_{n > n_0} \int_{\omega_0}^0 \langle \zeta^{\omega n}, T_+^{\omega n} \zeta^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega\,.$$

They can be treated similarly, consider for example the first term. For any $n_1 > n_0$,

$$\inf_{(\omega_0,0)} \lambda_{n_1}^2 \sum_{n \geq n_1} \int_{\omega_0}^0 \langle \eta_L^{\omega n}, T_+^{\omega n} \eta_L^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega \;\leq\; \sum_{n > n_0} \int_{\omega_0}^0 \langle (\mathcal{A}\eta_L)^{\omega n}, T_+^{\omega n} (\mathcal{A}\eta_L)^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega$$

$$\leq\; <\mathcal{A}\eta_L, \, H\,(H-\omega_0)\,\mathcal{A}\eta_L> + \sum_{n > n_0} \int_{\omega_0}^0 \langle (\mathcal{A}\eta_L)^{\omega n}, T_-^{\omega n} (\mathcal{A}\eta_L)^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega$$

$$+ \sum_{n \leq n_0} \oint_{D_\delta} |<\mathcal{A}\eta_L, \, (Q_n \, S_\infty \, H\,(H-\omega_0)\,\mathcal{A}\eta_L)>| \; d\omega \,.$$

Here $\mathcal{A}$ is the angular operator. When it acts on a test function, we always get rid of the time-derives with the replacement $i\partial_t \to H$. We now argue as in the proof of Lemma 6.2 (with η_L replaced by $\mathcal{A}\eta_L$) and choose n_1 sufficiently large. ■

7 Absence of Radiant Modes

We now use a causality argument together with the estimates of the previous section to show that the radiant mode in Proposition 5.3 must be absent. This will be a contradiction to the assumption that there are radiant modes, ruling out the possibility that there are radiant modes at all.

Let us return to the setting of Proposition 5.3. Choosing the ϑ-dependence of η such that it is orthogonal to the angular wave functions $(\Psi_a^{\omega_1 n})_{n \leq n_0}$ except for one n, and choosing the u-dependence of η such that it is orthogonal only to one of the plane waves $e^{\pm i(\omega_1 - \omega_0)u}$, we can clearly arrange that

$$\limsup_{L \to \infty} |\sigma(L)| \;=:\; \kappa \;>\; 0 \qquad \text{where} \qquad \sigma(L) \;:=\; \sum_{n \leq n_0} <\eta_L^{\omega_1 n}, \sigma^n>_{\mathbb{C}^2} \,. \tag{7.1}$$

Furthermore, we choose η such that its support lies to the left of Φ_0, i.e.

$$\mathrm{dist}(\mathrm{supp}\, \eta_L, \, \mathrm{supp}\, \Phi_0) \;>\; L \qquad \text{for all } L \geq 0.$$

Due to the finite propagation speed (which in the (t, u)-coordinates is equal to one),

$$\mathrm{supp}\, \eta_L \,\cap\, \mathrm{supp}\, \Phi(t) \;=\; \emptyset \qquad \text{if } |t| \leq L.$$

Hence for all $L > 0$,

$$\begin{aligned} 0 \;&=\; \frac{1}{2L} \int_{-L}^{L} e^{i\omega_1 t} <\eta_L, \, \Phi(t)> \\ &=\; \frac{1}{2\pi} \sum_{n \in \mathbb{N}} \int_{-\infty}^{\infty} \frac{\sin((\omega - \omega_1)L)}{(\omega - \omega_1)L} \, \langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2} \, d\omega \;+\; \sigma(L) \,. \end{aligned}$$

We apply Lemma 6.1 and Lemma 6.4 with $\varepsilon = \kappa/(8\pi)$ to obtain

$$\frac{1}{2\pi} \sum_{n > n_0} \int_{\omega_0}^0 |\langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2}| \; d\omega \;\leq\; \frac{\kappa}{2} \,.$$

Furthermore, Lemma 6.2 gives rise to the estimate

$$\left| \int_{\mathbb{R}\setminus[\omega_0,0]} \frac{\sin((\omega-\omega_1)L)}{(\omega-\omega_1)L} \,\langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2} \,d\omega \right| \;\leq\; C \sup_{\mathbb{R}\setminus[\omega_0,0]} \left| \frac{\sin((\omega-\omega_1)L)}{(\omega-\omega_1)L} \right| ,$$

and since this supremum tends to zero as $L \to \infty$, we conclude that the expression on the left vanishes in the limit $L \to \infty$. Combining these estimates with (7.1), we obtain

$$\limsup_{L\to\infty} \left| \sum_{n \leq n_0} \int_{\omega_0}^0 \frac{\sin((\omega-\omega_1)L)}{(\omega-\omega_1)L} \,\langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2} \,d\omega \right| \;\geq\; \pi\kappa \,. \tag{7.2}$$

Since the matrices $T^{\omega n}$ are continuous in ω and the fundamental solutions $\Psi_a^{\omega n}(u)$ are according to Theorem 3.1 uniformly bounded as $u \to -\infty$, there is a constant C such that

$$|\langle \eta_L^{\omega n}, T^{\omega n} \Phi_0^{\omega n} \rangle_{\mathbb{C}^2}| \;\leq\; C \qquad \text{for all } L \geq 0 \text{ and } \omega \in (\omega_0, 0),\; n \leq n_0 \,.$$

Hence we can apply Lebesgue's dominated convergence theorem on the left of (7.2) and take the limit $L \to \infty$ inside the integral, giving zero. This is a contradiction.

8 An Integral Spectral Representation on the Real Axis

Since radiant modes have been ruled out, we know that the Wronskian $w(\acute{\phi}, \grave{\phi})$ has no zeros on the real axis. Thus we can move all contours up to the real axis. This gives the following integral representation for the propagator.

Theorem 8.1 *For any initial data $\Psi_0 \in C_0^\infty(\mathbb{R} \times S^2)^2$, the solution of the Cauchy problem has the integral representation*

$$\Psi(t,r,\vartheta,\varphi) \;=\; \frac{1}{2\pi} \sum_{k\in\mathbf{Z}} e^{-ik\varphi} \sum_{n\in\mathrm{I\!N}} \int_{-\infty}^{\infty} \frac{d\omega}{\omega\Omega}\, e^{-i\omega t} \sum_{a,b=1}^{2} t_{ab}^{k\omega n}\, \Psi^a_{k\omega n}(r,\vartheta) <\Psi^b_{k\omega n}, \Psi_0>$$

with the coefficients t_{ab} as given by (5.5, 5.2). Here the sums and the integrals converge in L^2_{loc}.

Proof of Theorem 1.1. We choose an interval $[r_L, r_R] \subset (r_1, \infty)$ and let K be the compact set $K = [r_L, r_R] \times S^2$. As a consequence of Theorem 8.1, we have for any $\eta \in C_0^\infty(\mathring{K})^2$ the integral representation

$$<\eta, H\,(H-\omega_0)\,\Psi(t)> \;=\; \frac{1}{2\pi} \sum_{n\in\mathrm{I\!N}} \int_{-\infty}^{\infty} e^{-i\omega t}\, \langle \eta^{\omega n},\, T^{\omega n}\, \Psi_0^{\omega n} \rangle_{\mathbb{C}^2}\, d\omega \,. \tag{8.1}$$

It is useful to introduce the short notation

$$(\eta, \zeta) \;=\; <\eta, H\,(H-\omega_0)\,\zeta> . \tag{8.2}$$

We consider on K the Hilbert space $\mathcal{H} = L^2(K, d\mu)^2$ and denote its scalar product by $<.,.>_{\mathcal{H}}$. We can represent the inner product (8.2) as

$$(\eta, \zeta) \;=\; <\eta, B\,\zeta>_{\mathcal{H}}$$

with the operator B given by

$$B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} H\,(H-\omega_0) = \begin{pmatrix} A^2 & A(\beta-\omega_0) \\ (\beta-\omega_0)A & A+\beta\,(\beta-\omega_0) \end{pmatrix}.$$

This operator, densely defined on $C_0^\infty(K)^2 \subset \mathcal{H}$, is obviously symmetric. We now construct a self-adjoint extension. We decompose B in the form

$$B = B_0 + E \qquad \text{with} \qquad B_0 := \begin{pmatrix} A^2 & 0 \\ 0 & A \end{pmatrix}.$$

The elliptic operator A on the compact domain K is essentially self-adjoint and has compact resolvent (see [4, Section 3]). Thus we can choose a domain $\mathcal{D}(B_0)$ which makes B_0 self-adjoint. Denoting the resolvents by $R^0_\lambda = (B_0-\lambda)^{-1}$ and $R_\lambda = (B-\lambda)^{-1}$, the resolvent identity reads

$$R^0_\lambda = (\mathbb{1} + R^0_\lambda\, E)\, R_\lambda = \sqrt{R^0_\lambda}\left(\mathbb{1} + \sqrt{R^0_\lambda}\, E \sqrt{R^0_\lambda}\right)\sqrt{B_0-\lambda}\, R_\lambda\,.$$

Writing out the operator inside the brackets,

$$\begin{aligned} F &:= \sqrt{R^0_\lambda}\, E \sqrt{R^0_\lambda} \\ &= \begin{pmatrix} 0 & (A^2-\lambda)^{-\frac{1}{2}}\, A(\beta-\omega_0)\,(A-\lambda)^{-\frac{1}{2}} \\ (A-\lambda)^{-\frac{1}{2}}\,(\beta-\omega_0)A(A^2-\lambda)^{-\frac{1}{2}} & (A-\lambda)^{-\frac{1}{2}}\,\beta\,(\beta-\omega_0)\,(A-\lambda)^{-\frac{1}{2}} \end{pmatrix}, \end{aligned}$$

and using that $\|(A^2-\lambda)^{-\frac{1}{2}}\, A\|_{\mathcal{H}} \leq 1$ for $\lambda < 0$, we conclude that by choosing $\lambda \ll 0$, we can make the norm of F arbitrarily small. Hence the operator $\mathbb{1} + F$ is invertible, and we obtain the formula

$$R_\lambda = \sqrt{R^0_\lambda}\,(\mathbb{1}+F)^{-1}\sqrt{R^0_\lambda}\,.$$

We conclude that the operator R_λ is also compact. This gives us a self-adjoint extension of B with a purely discrete spectrum without limit points and finite-dimensional eigenspaces.

We now arrange that the operator B has no kernel. Namely, if on the contrary the operator has a kernel, it is obvious from the definition of B that one of the operators A, H or $H-\omega_0$ has a kernel. Using the separation of variables, we get corresponding radial ODEs with Dirichlet boundary conditions at r_L and r_R. Since non-trivial solutions of these ODEs have discrete zeros, we can by increasing the size of the interval $[r_L, r_R]$ arrange that B has no kernel. Due to the purely discrete spectrum, there is a constant $c > 0$ such that

$$\|B\xi\| \geq \frac{1}{c}\,\|\xi\| \qquad \text{for all } \xi \in \mathcal{D}(B)\,. \tag{8.3}$$

Let $\varepsilon > 0$. For given $\omega_1 > |\omega_0|$ and $n_1 \geq n_0$ we set

$$T_I^{\omega n} = \begin{cases} T_-^{\omega n} & \text{if } n > n_1 \text{ and } \omega \in (\omega_0, 0) \\ T^{\omega n} & \text{if } n \leq n_1 \text{ and } \omega \in [-\omega_1, \omega_1] \\ 0 & \text{otherwise} \end{cases}$$

and $T_+^{\omega n} = T^{\omega n} - T_I^{\omega n}$. Furthermore, we introduce the short notation

$$\int_N d\nu \cdots \equiv \frac{1}{2\pi}\sum_{n\in N}\int_{-\infty}^{\infty} d\omega\, \cdots$$

and omit the superscript ${}^{\omega n}$. With this notation, we can write (8.1) for $t = 0$ in the compact form

$$(\eta, \zeta) \;=\; \int_N \langle \eta, (T_+ + T_I)\, \zeta \rangle_{\mathbb{C}^2} \, d\nu \,.$$

Since in Lemma 6.1 we used pointwise estimates of the $\Psi_a^{\omega n}$, these estimates depend on η and ζ only via their norm in the Hilbert space $\mathcal{H}$. The same is true for the finite number of modes $n \leq n_1$ for ω in the compact set $[-\omega_1, \omega_1]$. We thus have the bound

$$\int_N |\langle \eta, T_I \zeta \rangle_{\mathbb{C}^2}| \, d\nu \;\leq\; c(K, \omega_1, n_1)\, \|\eta\|_{\mathcal{H}}\, \|\zeta\|_{\mathcal{H}} \,. \tag{8.4}$$

Now we estimate the inner product $(\eta, \Psi(t))$ for $\eta \in (C^\infty(\overset{\circ}{K})^2)$ as follows,

$$|(\eta, \Psi(t))| \;\leq\; \left| \int_N e^{-i\omega t}\, \langle \eta, T_I\, \Psi_0 \rangle_{\mathbb{C}^2} \, d\nu \right| + \frac{1}{C} \int_N |\langle \eta, T_+\, (1 + H^2)(1 + \mathcal{A}) \Psi_0 \rangle_{\mathbb{C}^2}| \, d\nu \,, \tag{8.5}$$

where the constant C, given by

$$C \;=\; C(n_1, \omega_1) \;=\; \inf_{n > n_1 \text{ or } |\omega| > \omega_1} (1 + \omega^2)(1 + \lambda_n(\omega)) \,,$$

can be made arbitrarily small by increasing ω_1 and n_1. Since T_+ is positive, we have, using the Schwarz inequality,

$$\int_N \langle \eta, T_+ \zeta \rangle_{\mathbb{C}^2} \, d\nu \;\leq\; \left(\int_N \langle \eta, T_+ \eta \rangle_{\mathbb{C}^2} \, d\nu \right)^{\frac{1}{2}} \left(\int_N \langle \zeta, T_+ \zeta \rangle_{\mathbb{C}^2} \, d\nu \right)^{\frac{1}{2}} \,.$$

Applying this inequality in the last term of (8.5), we obtain

$$\begin{aligned} \left(\int_N |\langle \eta, T_+\, (1 + H^2)(1 + \mathcal{A}) \Psi_0 \rangle_{\mathbb{C}^2}| \, d\nu \right)^2 \;&\leq\; c(\Psi_0) \int_N \langle \eta, T_+\, \eta \rangle_{\mathbb{C}^2} \, d\nu \\ =\; c(\Psi_0) \left((\eta, \eta) \;-\; \int_N \langle \eta, T_I\, \eta \rangle_{\mathbb{C}^2} \, d\nu \right) \;&\leq\; c(\Psi_0) \left(|(\eta, \eta)| + \|\eta\|^2_{\mathcal{H}} \right) \,, \end{aligned}$$

where in the last step we used (8.4). Hence by choosing ω_1 and n_1 sufficiently large, we can arrange that the second summand in (8.5) is smaller than $\varepsilon(|(\eta, \eta) + \|\eta\|^2_{\mathcal{H}})^{\frac{1}{2}}$. The first term in (8.5) consists of the sum of the angular modes $n > n_1$ and $n \leq n_1$. For the sum over $n > n_1$, we can again apply Lemma 6.1 keeping in mind that the dependence on η and ζ is controlled by their norms. Possibly by further increasing n_1 we can arrange that this contribution is bounded by $\varepsilon \|\eta\|_{L^2}$. For the remaining finite sum $n \leq n_1$ we simply apply the Riemann-Lebesgue lemma. We conclude that, for large t,

$$|(\eta, \Psi(t))| \;\leq\; \varepsilon \left(\|\eta\|_{L^2(K, d\mu)} + |(\eta, \eta)|^{\frac{1}{2}} \right) \qquad \text{for all } \eta \in C^\infty_0(\overset{\circ}{K})^2 \,.$$

We rewrite this inequality in the Hilbert space $\mathcal{H}$,

$$<\eta, B\, \Psi(t)>^2_{\mathcal{H}} \;\leq\; 2\varepsilon^2 \left(\|\eta\|^2_{\mathcal{H}} + <\eta, B\, \eta>_{\mathcal{H}} \right) \,.$$

By continuity, this inequality holds for any $\eta \in \mathcal{H}$. Evaluating this inequality for the sequence η_k obtained by projecting $\Psi(t)$ on the eigenspaces with eigenvalues $\leq k$,

$$\eta_k \;=\; \chi_{(-\infty, k]}(B)\, \Psi(t) \,,$$

and taking the limit $k \to \infty$, we obtain the inequality

$$<\Psi(t), B\,\Psi(t)>_{\mathcal{H}}^2 \;\leq\; 2\varepsilon^2 \left(\|\Psi(t)\|_{\mathcal{H}}^2 + <\Psi(t), B\,\Psi(t)>_{\mathcal{H}} \right) \,.$$

Since the term $<\Psi(t), B\,\Psi(t)>_{\mathcal{H}}$ vanishes only if $\Psi(t) = 0$, we may divide by this term. Using (8.3), we conclude that

$$\frac{1}{c}\,\|\Psi(t)\|^2 \;\leq\; 2\varepsilon^2 \left(\frac{\|\Psi(t)\|_{\mathcal{H}}^2}{<\Psi(t), B\Psi(t)>_{\mathcal{H}}} + 1 \right) \;\leq\; 2\varepsilon^2\,(c+1)$$

and thus, for sufficiently large t,

$$\|\Psi(t)\|_{\mathcal{H}} \;\leq\; \varepsilon\,(2c(c+1))^{\frac{1}{2}} \,.$$

We conclude that $\Psi(t)$ converges to zero in $L^2(K)$.

Applying the same argument to the initial data $H^n \Psi_0$, we conclude that the partial derivatives of $\Psi(t)$ also decay in $L^2(K)$. The Sobolev embedding $H^{2,2}(K) \hookrightarrow L^\infty(K)$ gives the claim. ■

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