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DERIVATION OF A ROD THEORY FOR MULTIPHASE MATERIALS

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Abstract

A rigorous derivation is given of a rod theory for a multiphase material, starting from three-dimensional nonlinear elasticity. The stored energy density is supposed to be nonnegative and to vanish exactly on a set consisting of two copies of the group of rotations $\text{SO}(3)$. The two potential wells correspond to the two crystalline configurations preferred by the material. We find the optimal scaling of the energy in terms of the diameter of the rod and we identify the limit, as the diameter goes to zero, in the sense of Γ -convergence.

1 Introduction

In this paper we give a rigorous derivation of a one-dimensional theory of elastic thin beams made of a multiphase material. More precisely, let $\Omega_h = (0, L) \times hS$ be the reference configuration of a thin beam, where S is a bounded domain in \mathbb{R}^2 with Lipschitz boundary and h is a (small) positive parameter. Given an elastic deformation $v : \Omega_h \rightarrow \mathbb{R}^3$, we define

$$E^{(h)}(v) := \frac{1}{h^2} \int_{\Omega_h} W(\nabla v(z)) dz \quad (1.1)$$

as the elastic energy (per unit cross-section) associated to v . We suppose that the stored energy density $W : \mathbb{M}^{3 \times 3} \rightarrow \mathbb{R}$ is nonnegative and vanishes exactly on the set

$$K := \text{SO}(3) \cup \text{SO}(3)H,$$

where $\text{SO}(3)$ is the group of rotations, i.e., $\text{SO}(3) := \{R \in \mathbb{M}^{3 \times 3} : R^T R = \text{Id}, \det R = 1\}$, and H is a matrix in $\mathbb{M}^{3 \times 3}$ with $\det H > 0$. This multiple-well structure of W occurs typically in models for materials undergoing solid-solid phase transformations, the two copies of $\text{SO}(3)$ corresponding to the two different crystalline configurations preferred by the material.

In [4] Bhattacharya and James pointed out that low energy deformations in the thin film limit have a very different structure with respect to low energy states in bulk materials. If Id represents the austenite phase and H one of the martensitic variants, then they are usually incompatible in bulk, so that the only zero energy states are the trivial ones. In contrast, the limiting two-dimensional thin film theory allows for interfaces which may be incompatible in the thickness direction. The incompatibility condition becomes even less stringent in the one-dimensional limit (see [11]). In particular, given two deformation gradients Id and RH , it is always possible to construct a sequence of three-dimensional deformations $(v^{(h)})$ such that their gradients converge to a juxtaposition of Id and RH and their energy $E^{(h)}$ is bounded from above by Ch .

The main results of this paper are two. We first show that the scaling $E^{(h)} \sim h$ is optimal (see Theorem 2.4), provided that a phase transition indeed occurs. Then, in Theorems 2.4 and 3.1 we identify the limit, in the sense of Γ -convergence, corresponding to this scaling. The notion of Γ -convergence, introduced by De Giorgi, has proved to be a successful tool for the problem of rigorous derivation of lower dimensional theories starting from three-dimensional elasticity (in the nonlinear setting see, e.g., [1], [8], [9], [12], [16], [17], [18]). For a comprehensive introduction to Γ -convergence we refer to the book [7]. A very interesting alternative approach for the derivation of rod theories which is based on the use of centre manifolds for a rod of infinite length has been developed by Mielke [14, 15].

To state our results more precisely, it is convenient to introduce in (1.1) the following change of variables:

$$z_1 = x_1, \quad z_2 = hx_2, \quad z_3 = hx_3,$$

and to rescale deformations according to $y(x) := v(z(x))$, so that y is a map from $\Omega := (0, L) \times S$ into \mathbb{R}^3 . We introduce the notation

$$\nabla_h y := \left(\partial_1 y \left| \frac{1}{h} \partial_2 y \right| \frac{1}{h} \partial_3 y \right)$$

and the functional

$$I^{(h)}(y) := \int_{\Omega} W(\nabla_h y(x)) \, dx = E^{(h)}(v).$$

On the stored energy function $W : \mathbb{M}^{3 \times 3} \rightarrow [0, +\infty)$ we require the following conditions:

- (i) $W \in C^0(\mathbb{M}^{3 \times 3})$;

- (ii) W is frame-indifferent, i.e., $W(F) = W(RF)$ for every $F \in \mathbb{M}^{3 \times 3}$ and $R \in \text{SO}(3)$;
- (iii) there exist $C_1, C_2 > 0$ such that $C_1 \text{dist}^2(F, K) \leq W(F) \leq C_2 \text{dist}^2(F, K)$ for every $F \in \mathbb{M}^{3 \times 3}$.

The lower bound in (iii) is a natural condition, it states that the energy wells are non-degenerate. The upper bound, however, is rather restrictive. It is, e.g., incompatible with the natural condition that W should blow up for infinite compression, i.e., $W(F) \rightarrow \infty$ as $\det F \rightarrow 0^+$. The upper bound in (iii) is mainly used in the proof of the lower estimate in Theorem 2.4. There we modify a given low energy sequence in order to enforce affine boundary conditions near $\pm\infty$. The bound is used to ensure that this modification has negligible energy. By a slight refinement of this argument we could work with the weaker upper bound $W(F) \leq C \text{dist}^2(F, K) + C \text{dist}^3(F, K)$ but this is still incompatible with blow-up of the energy at infinite compression.

We assume that the two wells are strongly incompatible in the sense of Matos [13] and Šverák [19]. By polar decomposition and a linear change of variables we may assume H symmetric. Hence in a suitable orthonormal basis H is diagonal, i.e., $H = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Since $\det H > 0$, we may suppose that $\lambda_i > 0$. In the following we always suppose that the material has been cut in such a way that the x_1 axis in the reference configuration of the rod corresponds to the first eigenvector of H . For positive definite diagonal matrices the incompatibility condition is

$$\sum_{i=1}^3 (1 - \lambda_i)(1 - \det H / \lambda_i) > 0. \quad (1.2)$$

In Theorem 2.4 we prove that given any sequence of deformations $(y^{(h)}) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ satisfying

$$\frac{1}{h} I^{(h)}(y^{(h)}) \leq C \quad \text{for every } h,$$

there exists a subsequence (not relabeled) such that the deformation gradients $\nabla_h y^{(h)}$ converge weakly in $L^2(\Omega; \mathbb{M}^{3 \times 3})$ to a limit of the form $(\partial_1 y | d_2 | d_3)$, where $y \in W^{1,\infty}((0, L); \mathbb{R}^3)$ and $d_2, d_3 \in L^\infty((0, L); \mathbb{R}^3)$. Moreover, there exists a finite number of disjoint intervals in $(0, L)$ (whose union we call A) such that

$$(\partial_1 y | d_2 | d_3) \in \begin{cases} \text{co}(\text{SO}(3)) & \text{a.e. in } A, \\ \text{co}(\text{SO}(3)H) & \text{a.e. in } (0, L) \setminus A, \end{cases} \quad (1.3)$$

where $\text{co}(M)$ denotes the convex hull of M for any $M \subset \mathbb{M}^{3 \times 3}$. In Theorems 2.4 and 3.1 we show that the Γ -limit of the functionals $\frac{1}{h} I^{(h)}$ can be expressed in terms of the functions y, d_2, d_3 , and of the set A . More precisely, it is proportional to the number of transition points of the matrix $(\partial_1 y | d_2 | d_3)$ between the convex hulls of the two wells.

The last section is devoted to the study of the limit functional. In particular, we remark that, if no force terms or boundary conditions are considered, the limit problem becomes trivial. Indeed, given a deformation $y \in W^{1,\infty}((0,L);\mathbb{R}^3)$ with $|\partial_1 y| \leq \max\{1, \lambda_1\}$, one can always find $d_2, d_3 \in L^\infty((0,L);\mathbb{R}^3)$ such that $(\partial_1 y | d_2 | d_3)$ belongs to the convex hull of only one well, so that the energy associated to y is zero. If $|\partial_1 y| > \max\{1, \lambda_1\}$, the energy is infinite, since the constraint (1.3) can never be satisfied. Instead, if force terms are present, it may be energetically convenient to have transition points between the convex hulls of the two wells, as shown in Remark 4.2.

2 Compactness and lower bound

The following estimate will be crucial in the proof of the compactness result of Theorem 2.4 below. For the proof we refer to [5].

Theorem 2.1 *Let U be a bounded Lipschitz domain in \mathbb{R}^n , $n \geq 2$, and let $K := \text{SO}(n) \cup \text{SO}(n)H$, where $H = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_i > 0$ such that $\sum_{i=1}^n (1 - \lambda_i)(1 - \det H/\lambda_i) > 0$. Then there exists a positive constant $C(U, H)$ with the following property: for each $u \in W^{1,2}(U; \mathbb{R}^n)$ there is an associated matrix $F \in K$ such that*

$$\|\nabla u - F\|_{L^2(U)} \leq C(U, H) \|\text{dist}(\nabla u, K)\|_{L^2(U)}.$$

Theorem 2.1 generalizes the rigidity estimate for a single well problem [8, Thm. 3.1] to the case of two strongly incompatible wells. We note also that Theorem 2.1 is invariant under uniform scaling and translation of the domain; e.g., the same value of $C(U, H)$ serves for $\lambda U + a$, and the rescaled function $\lambda u((x - a)/\lambda)$ may be associated with the same $F \in K$.

For any $R \in \text{SO}(3)$ we introduce the two following quantities, which represent the cost associated to a transition of the deformation from one well to the other.

$$\begin{aligned} \gamma_H^-(R) &:= \inf \left\{ \int_{(-M, M) \times S} W(\nabla v(x)) dx : M > 0, v \in W_{loc}^{1,2}(\mathbb{R} \times \overline{S}), \right. \\ &\quad \left. \nabla v = H \text{ a.e. in } (-\infty, -M) \times S, \nabla v = R \text{ a.e. in } (M, +\infty) \times S \right\}, \\ \gamma_H^+(R) &:= \inf \left\{ \int_{(-M, M) \times S} W(\nabla v(x)) dx : M > 0, v \in W_{loc}^{1,2}(\mathbb{R} \times \overline{S}), \right. \\ &\quad \left. \nabla v = R \text{ a.e. in } (-\infty, -M) \times S, \nabla v = H \text{ a.e. in } (M, +\infty) \times S \right\}. \end{aligned}$$

Remark 2.2 In general one cannot guarantee that $\gamma_H^-(R) = \gamma_H^+(R)$ for every $R \in \text{SO}(3)$. The equality is true if the material satisfies isotropy conditions. Assume for instance that the stored energy density is transversely isotropic, i.e.,

$$W(FR) = W(F) \text{ for every } F \in \mathbb{M}^{3 \times 3} \text{ and } R \in \mathcal{G},$$

where

$$\mathcal{G} := \left\{ R \in \text{SO}(3) : \exists R' \in \text{SO}(2) \text{ such that } R = \begin{pmatrix} 1 & 0 \\ 0 & R' \end{pmatrix} \right\}.$$

In this case it is easy to show that $\gamma_H^-(R) = \gamma_H^+(R)$ for every $R \in \text{SO}(3)$.

Using the growth conditions on W and frame indifference, we show in the next proposition that $\gamma_H^\pm(R)$ are actually independent of R .

We will use the notation $x' = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$.

Proposition 2.3 *Assume W satisfies conditions (i)-(iii). Let $\gamma_H^\pm := \gamma_H^\pm(\text{Id})$. Then $\gamma_H^\pm(R) = \gamma_H^\pm$ for every $R \in \text{SO}(3)$.*

PROOF. – It is enough to prove the statement for γ_H^- , the proof for γ_H^+ being completely analogous.

Let $R, Q \in \text{SO}(3)$. Let $M > 0$ and $v \in W_{loc}^{1,2}(\mathbb{R} \times \bar{S})$ be such that

$$\nabla v = H \quad \text{a.e. in } (-\infty, -M) \times S, \quad (2.1)$$

$$\nabla v = R \quad \text{a.e. in } (M, +\infty) \times S. \quad (2.2)$$

For every $k \in \mathbb{N}$ let $R_k : \mathbb{R} \rightarrow \text{SO}(3)$ be a smooth function satisfying $R_k(x_1) = R$ for $x_1 \leq 0$ and $R_k(x_1) = Q$ for $x_1 \geq k$. Then, consider the function $u_k : \mathbb{R} \times S \rightarrow \mathbb{R}^3$ defined by

$$u_k(x) := \int_0^{x_1} R_k(s) e_1 ds + R_k(x_1) \begin{pmatrix} 0 \\ x' \end{pmatrix}.$$

Note that u_k is smooth and its gradient satisfies

$$\nabla u_k(x) = R_k(x_1) + \partial_1 R_k(x_1) \begin{pmatrix} 0 \\ x' \end{pmatrix} \otimes e_1; \quad (2.3)$$

so, in particular, we have that

$$\nabla u_k = R \quad \text{in } (-\infty, 0) \times S, \quad (2.4)$$

$$\nabla u_k = Q \quad \text{in } (k, +\infty) \times S. \quad (2.5)$$

Now let $v_k : \mathbb{R} \times S \rightarrow \mathbb{R}^3$ be the function defined by

$$v_k(x) := \begin{cases} v(x) & \text{for } x \in (-\infty, M) \times S, \\ u_k(x_1 - M, x') + c_k & \text{for } x \in [M, +\infty) \times S. \end{cases}$$

By (2.2) and (2.4) we can choose the constant c_k such that the function v_k belongs to the space $W_{loc}^{1,2}(\mathbb{R} \times \bar{S})$. Moreover, by (2.1) we have that $\nabla v_k = H$ a.e. in $(-\infty, -M) \times S$, while by (2.5) we have that $\nabla v_k = Q$ in $(M + k, +\infty) \times S$.

Therefore, $M+k$ and v_k are admissible for the infimum problem defining $\gamma_H^-(Q)$; hence,

$$\begin{aligned} \gamma_H^-(Q) &\leq \int_{(-M, M+k) \times S} W(\nabla v_k(x)) dx \\ &= \int_{(-M, M) \times S} W(\nabla v(x)) dx + \int_{(0, k) \times S} W(\nabla u_k(x)) dx, \end{aligned} \quad (2.6)$$

where the last equality follows from the fact that $\nabla v_k = \nabla v = H$ a.e. in $(-\infty, -M) \times S$ by (2.1). Now, we claim that

$$\lim_{k \rightarrow +\infty} \int_{(0, k) \times S} W(\nabla u_k(x)) dx = 0. \quad (2.7)$$

If this is true, then we can pass to the limit in (2.6), as $k \rightarrow +\infty$, obtaining

$$\gamma_H^-(Q) \leq \int_{(-M, M) \times S} W(\nabla v(x)) dx$$

for every $M > 0$ and $v \in W_{loc}^{1,2}(\mathbb{R} \times \overline{S})$ which are admissible for the infimum problem defining $\gamma_H^-(R)$. Therefore, we have $\gamma_H^-(Q) \leq \gamma_H^-(R)$, and hence the thesis.

Let us prove (2.7). By frame indifference and the growth condition from above on W we have

$$\begin{aligned} \int_{(0, k) \times S} W(\nabla u_k(x)) dx &= \int_{(0, k) \times S} W(R_k^T(x_1) \nabla u_k(x)) dx \\ &\leq C_2 \int_{(0, k) \times S} \text{dist}^2(R_k^T(x_1) \nabla u_k(x), K) dx \\ &\leq C_2 \int_{(0, k) \times S} |R_k^T(x_1) \nabla u_k(x) - Id|^2 dx \\ &\leq C_2 \int_{(0, k) \times S} \left| R_k^T(x_1) \partial_1 R_k(x_1) \begin{pmatrix} 0 \\ x' \end{pmatrix} \right|^2 dx, \end{aligned} \quad (2.8)$$

where the last inequality follows from (2.3). Using the boundedness of R_k , we obtain the following estimate:

$$\begin{aligned} \int_{(0, k) \times S} \left| R_k^T(x_1) \partial_1 R_k(x_1) \begin{pmatrix} 0 \\ x' \end{pmatrix} \right|^2 dx &\leq C \int_0^k |R_k^T(x_1) \partial_1 R_k(x_1)|^2 dx_1 \\ &\leq \frac{C}{k} |R - Q|^2. \end{aligned}$$

Now the claim (2.7) follows from (2.8) and the previous inequality. \square

In the following we denote by \mathcal{A} the class of subsets of $(0, L)$ which are finite unions of disjoint open intervals. Given $A \in \mathcal{A}$, we call $\partial^- A$ the set of points

$a \in (0, L) \cap \partial A$ such that $(a - \varepsilon, a) \subset (0, L) \setminus A$ and $(a, a + \varepsilon) \subset A$ for some $\varepsilon > 0$. We call $\partial^+ A$ the set of points $a \in (0, L) \cap \partial A$ such that $(a - \varepsilon, a) \subset A$ and $(a, a + \varepsilon) \subset (0, L) \setminus A$ for some $\varepsilon > 0$.

Now we can state and prove the main result of this section.

Theorem 2.4 *Assume W satisfies conditions (i)-(iii) and (1.2). Let $(y^{(h)}) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ be a sequence such that*

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \int_{\Omega} W(\nabla_h y^{(h)}(x)) dx \leq c. \quad (2.9)$$

Then there exist a subsequence (not relabeled) such that

$$\nabla_h y^{(h)} \rightharpoonup (\partial_1 y | d_2 | d_3) \text{ weakly in } L^2(\Omega; \mathbb{M}^{3 \times 3}),$$

where the functions y , d_2 , and d_3 are independent of x_2 and x_3 and satisfy $y \in W^{1,\infty}((0, L); \mathbb{R}^3)$ and $d_2, d_3 \in L^\infty((0, L); \mathbb{R}^3)$. Moreover, there exists $A \in \mathcal{A}$ such that

$$(\partial_1 y | d_2 | d_3) \in \begin{cases} \text{co}(\text{SO}(3)) & \text{a.e. in } A, \\ \text{co}(\text{SO}(3)H) & \text{a.e. in } (0, L) \setminus A. \end{cases} \quad (2.10)$$

Finally, for each such subsequence we have

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} \int_{\Omega} W(\nabla_h y^{(h)}(x)) dx \geq \gamma_H^- \mathcal{H}^0(\partial^- A) + \gamma_H^+ \mathcal{H}^0(\partial^+ A). \quad (2.11)$$

PROOF. – We split the proof in two parts.

1) *Compactness of the sequence $(y^{(h)})$.*

Using the growth condition from below on W we deduce from (2.9) that, up to subsequences, $(\nabla_h y^{(h)})$ is uniformly bounded in $L^2(\Omega; \mathbb{M}^{3 \times 3})$. Therefore, there exists a subsequence (not relabeled) converging to some $(\partial_1 y | d_2 | d_3)$ weakly in $L^2(\Omega; \mathbb{M}^{3 \times 3})$.

We now divide $(0, L)$ into subintervals of length $\sim h$. We then can apply the rigidity estimate in Theorem 2.1 to the Cartesian product of a subinterval and the cross-section S . This yields a good approximation of $\nabla_h y^{(h)}$ by a map $F^{(h)} : (0, L) \rightarrow K$ which is constant on each subinterval. Applying the rigidity estimate to the union of two neighbouring subintervals we will see that an energetic cost of order h arises whenever $F^{(h)}$ switches from $SO(3)$ to $SO(3)H$ or vice versa. This implies a uniform upper bound on the number of switching points and hence strong convergence of the sets A^h where $F^{(h)} \in SO(3)$.

To implement this strategy in detail let $k_h := [L/h]$, where $[x]$ denotes the largest integer less than or equal to x . We consider a partition of $[0, L]$ in k_h subintervals of length $\tau_h := L/k_h$ and we apply Theorem 2.1 to the function $v^{(h)}(z) = y^{(h)}(z_1, \frac{z_2}{h}, \frac{z_3}{h})$ first restricted to the set $(a, a + 2h) \times hS$ for every $a \in$

$[0, L - \tau_h) \cap \tau_h \mathbb{N}$ and then, restricted to the set $(L - 2h, L) \times hS$. This provides for every $h > 0$ and for every $a \in [0, L) \cap \tau_h \mathbb{N}$ a constant $F^h(a) \in K$ such that

$$\int_{(a, a+\tau_h) \times S} |\nabla_h y^{(h)}(x) - F^h(a)|^2 dx \leq C \int_{(a, a+2h) \times S} \text{dist}^2(\nabla_h y^{(h)}(x), K) dx \quad (2.12)$$

for every $a \in [0, L - \tau_h) \cap \tau_h \mathbb{N}$, and

$$\int_{(L-\tau_h, L) \times S} |\nabla_h y^{(h)}(x) - F^h(L - \tau_h)|^2 dx \leq C \int_{(L-2h, L) \times S} \text{dist}^2(\nabla_h y^{(h)}(x), K) dx. \quad (2.13)$$

By interpolation we define a piecewise constant map $F^{(h)} : [0, L) \rightarrow K$ such that $F^{(h)}(x_1) = F^h(a)$ for every $x_1 \in [a, a + \tau_h)$. Summing the inequalities (2.12) and (2.13), we obtain

$$\int_{\Omega} |\nabla_h y^{(h)}(x) - F^{(h)}(x_1)|^2 dx \leq C \int_{\Omega} \text{dist}^2(\nabla_h y^{(h)}(x), K) dx. \quad (2.14)$$

It is convenient to introduce the sets

$$J_h := \left\{ a \in [0, L) \cap \tau_h \mathbb{N} : F^{(h)}(x_1) \in \text{SO}(3) \text{ for } x_1 \in [a, a + \tau_h) \right\},$$

$$A_h := \bigcup_{a \in J_h} [a, a + \tau_h),$$

so that we can write $F^{(h)}$ as

$$F^{(h)} = R^{(h)}(\chi_{A_h} \text{Id} + (1 - \chi_{A_h})H),$$

where $R^{(h)} : [0, L) \rightarrow \text{SO}(3)$ is piecewise constant. We claim that the number of points of $\partial A_h \cap (0, L)$ is uniformly bounded with respect to h . Indeed, if $a \in \partial A_h \cap (0, L)$, then either $F^{(h)}(x_1) = R^{(h)}(x_1)$ for $x_1 \in [a - \tau_h, a)$ and $F^{(h)}(x_1) = R^{(h)}(x_1)H$ for $x_1 \in [a, a + \tau_h)$, or $F^{(h)}(x_1) = R^{(h)}(x_1)H$ for $x_1 \in [a - \tau_h, a)$ and $F^{(h)}(x_1) = R^{(h)}(x_1)$ for $x_1 \in [a, a + \tau_h)$. Assume the latter (the proof for the former being completely analogous) and apply Theorem 2.1 to $v^{(h)}$ restricted to the set $(a - 2h, a + 2h) \times hS$. This provides $\tilde{F}^h \in K$ such that

$$\int_{(a-2h, a+2h) \times S} |\nabla_h y^{(h)}(x) - \tilde{F}^h|^2 dx \leq C \int_{(a-2h, a+2h) \times S} \text{dist}^2(\nabla_h y^{(h)}(x), K) dx. \quad (2.15)$$

Assume $\tilde{F}^h \in \text{SO}(3)H$. Then,

$$\begin{aligned} \tau_h |\tilde{F}^h - R^{(h)}(a)|^2 &= \int_{(a, a+\tau_h) \times S} |\tilde{F}^h - F^{(h)}(x_1)|^2 dx \\ &\leq 2 \int_{(a, a+\tau_h) \times S} |\nabla_h y^{(h)}(x) - \tilde{F}^h|^2 dx + 2 \int_{(a, a+\tau_h) \times S} |\nabla_h y^{(h)}(x) - F^{(h)}(x_1)|^2 dx \\ &\leq C \int_{(a-2h, a+2h) \times S} \text{dist}^2(\nabla_h y^{(h)}(x), K) dx, \end{aligned} \quad (2.16)$$

where the last inequality follows from (2.15) and (2.12). Since $|\tilde{F}^h - R^{(h)}(a)| \geq \text{dist}(\text{SO}(3), \text{SO}(3)H) =: \delta$, inequality (2.16) implies that

$$\tau_h \delta^2 \leq C \int_{(a-2h, a+2h) \times S} \text{dist}^2(\nabla_h y^{(h)}(x), K) dx.$$

In the case $\tilde{F}^h \in \text{SO}(3)$ the same inequality can be proved by comparing \tilde{F}^h with $F^{(h)}$ on $(a - \tau_h, a)$.

Summing over all $a \in \partial A_h \cap (0, L)$, we have

$$\begin{aligned} \tau_h \delta^2 \mathcal{H}^0(\partial A_h \cap (0, L)) &\leq C \int_{\Omega} \text{dist}^2(\nabla_h y^{(h)}(x), K) dx \\ &\leq C \int_{\Omega} W(\nabla_h y^{(h)}(x)) dx. \end{aligned}$$

Using the bound in (2.9) and the definition of τ_h we deduce that

$$\mathcal{H}^0(\partial A_h \cap (0, L)) \leq C \delta^{-2} h \tau_h^{-1} \leq C \delta^{-2}, \quad (2.17)$$

whence the claim.

From (2.17) it follows that the sequence (χ_{A_h}) converges, up to subsequences, to χ_A strongly in $L^1(0, L)$, where A is a finite union of disjoint intervals.

Since the sequence $(F^{(h)})$ is uniformly bounded in $L^\infty((0, L); \mathbb{M}^{3 \times 3})$, it converges, up to subsequences, to some $F \in L^\infty((0, L); \mathbb{M}^{3 \times 3})$ in the weak* topology of $L^\infty((0, L); \mathbb{M}^{3 \times 3})$. From (2.14) and (2.9) it follows that the weak limit $(\partial_1 y | d_2 | d_3)$ of $\nabla_h y^{(h)}$ agrees with F . Thus the matrix $(\partial_1 y | d_2 | d_3)$ does not depend on the variables x_2, x_3 and belongs to $L^\infty((0, L); \mathbb{M}^{3 \times 3})$. Moreover, since $(\chi_{A_h} F^{(h)})$ converges weakly to $\chi_A F$, property (2.10) holds true.

2) *Liminf inequality*

Since $\partial^- A$ and $\partial^+ A$ contain a finite number of points, we can write them as

$$\partial^- A = \{\alpha_i : i = 1, \dots, n\}, \quad \partial^+ A = \{\beta_j : j = 1, \dots, n'\},$$

where $|n - n'| \leq 1$.

Since $\chi_{A_h} \rightarrow \chi_A$ and since the number of points in ∂A_h is uniformly bounded, there exist $\varepsilon > 0$, $\alpha_i^h \rightarrow \alpha_i$ and $\beta_j^h \rightarrow \beta_j$ such that

$$\begin{aligned} (\alpha_i^h - 2\varepsilon, \alpha_i^h - \varepsilon) &\subset (0, L) \setminus A_h, & (\alpha_i^h + \varepsilon, \alpha_i^h + 2\varepsilon) &\subset A_h, \\ (\beta_j^h - 2\varepsilon, \beta_j^h - \varepsilon) &\subset A_h, & (\beta_j^h + \varepsilon, \beta_j^h + 2\varepsilon) &\subset (0, L) \setminus A_h. \end{aligned} \quad (2.18)$$

Moreover, ε can be chosen so small that all the intervals $(\alpha_i^h - 2\varepsilon, \alpha_i^h + 2\varepsilon)$ and $(\beta_j^h - 2\varepsilon, \beta_j^h + 2\varepsilon)$ are mutually disjoint. It thus suffices to establish the lower bound for one of these intervals.

Set

$$v^{(h)}(z_1, z_2, z_3) := \frac{1}{h} y^{(h)}(\alpha_i^h + h z_1, z_2, z_3), \quad L_h := \frac{\varepsilon}{h}.$$

Then $\nabla v^{(h)} = \nabla_h y^{(h)}$ and by (2.14), (2.9), (2.18), and the definition of A_h we have

$$\int_{(-2L_h, -L_h) \times S} \text{dist}^2(\nabla v^{(h)}, SO(3)H) dz + \int_{(L_h, 2L_h) \times S} \text{dist}^2(\nabla v^{(h)}, SO(3)) dz \leq C.$$

Thus Proposition 2.5 below yields

$$\begin{aligned} \frac{1}{h} \int_{(\alpha_i^h - 2\varepsilon, \alpha_i^h + 2\varepsilon) \times S} W(\nabla_h y^{(h)}) dx &= \int_{(-2L_h, 2L_h) \times S} W(\nabla v^{(h)}) dz \\ &\geq \gamma_H^- - C \frac{h}{\varepsilon}. \end{aligned}$$

The same estimate holds for the set $(\beta_j^h - 2\varepsilon, \beta_j^h + 2\varepsilon) \times S$, provided that γ_H^- is replaced with γ_H^+ . Since the relevant intervals are disjoint this yields (2.11) and the proof of Theorem 2.4 is finished. \square

Proposition 2.5 *There exists $C \in \mathbb{R}$ such that for all $L \geq 3$ and all $v \in W^{1,2}((-2L, 2L) \times S, \mathbb{R}^3)$ the following implication holds: if*

$$\int_{(-2L, -L) \times S} \text{dist}^2(\nabla v, SO(3)H) dz + \int_{(L, 2L) \times S} \text{dist}^2(\nabla v, SO(3)) dz \leq C_0,$$

then

$$\int_{(-2L, 2L) \times S} W(\nabla v) dz \geq \gamma_H^- - C \frac{C_0}{L}.$$

PROOF. – There exists $j \in \{L, L+1, \dots, 2L-1\}$ such that

$$\int_{(j, j+1) \times S} \text{dist}^2(\nabla v, SO(3)) dz \leq \frac{C_0}{L}.$$

The rigidity estimate [8, Thm. 3.1] and the Poincaré inequality imply that there exists $R \in SO(3)$ and $c \in \mathbb{R}^3$ such that

$$\int_{(j, j+1) \times S} (|v - c|^2 + |\nabla v - R|^2) dz \leq C \frac{C_0}{L}. \quad (2.19)$$

Let $\varphi \in C^\infty(\mathbb{R})$ be a cut-off function, i.e., $\varphi = 1$ on $(-\infty, 0)$, $\varphi = 0$ on $(1, +\infty)$, $0 \leq \varphi \leq 1$, and set

$$w(x) = \varphi(x_1 - j)v(x) + (1 - \varphi(x_1 - j))(Rx + c) \quad \text{for } x_1 \in [0, +\infty) \times S.$$

Then

$$|\nabla w - R| \leq C|\nabla v - R| + C|v - (Rx + c)| \quad \text{for } x_1 \in [j, j+1] \times S, \quad (2.20)$$

and $w = v$ on $(0, j] \times S$, while $w = Rx + c$ on $[j + 1, +\infty) \times S \supset [2L, +\infty) \times S$. In view of (2.19) and (2.20) and the upper growth condition (iii) on W we deduce that

$$\int_{(0, \infty) \times S} W(\nabla w) dz \leq \int_{(0, j) \times S} W(\nabla v) dz + C \frac{C_0}{L}.$$

Modifying v similarly on $(-2L, -L) \times S$ we obtain a function w such that

$$w = \begin{cases} R'Hx + c' & \text{on } (-\infty, -2L] \times S, \\ Rx + c & \text{on } [2L, \infty) \times S \end{cases}$$

and

$$\int_{(-2L, 2L) \times S} W(\nabla w) dz \leq \int_{(-2L, 2L) \times S} W(\nabla v) dz + C \frac{C_0}{L}.$$

By definition of γ_H^- and Proposition 2.3 the left-hand side is bounded from below by γ_H^- and this finishes the proof. \square

3 Upper bound

We recall that \mathcal{A} denotes the class of subsets of $(0, L)$ which are finite unions of disjoint open intervals.

Theorem 3.1 *Assume W satisfies (i)-(iii) and (1.2). Let $y \in W^{1, \infty}((0, L); \mathbb{R}^3)$, $d_2, d_3 \in L^\infty((0, L); \mathbb{R}^3)$ have the following property: there exists $A \in \mathcal{A}$ such that*

$$(\partial_1 y | d_2 | d_3) \in \begin{cases} \text{co}(\text{SO}(3)) & \text{a.e. in } A, \\ \text{co}(\text{SO}(3)H) & \text{a.e. in } (0, L) \setminus A. \end{cases} \quad (3.1)$$

Then, there exists a sequence $(y^{(h)}) \subset W^{1, 2}(\Omega; \mathbb{R}^3)$ such that

$$\nabla_h y^{(h)} \rightharpoonup (\partial_1 y | d_2 | d_3) \text{ weakly in } L^2(\Omega; \mathbb{M}^{3 \times 3}), \quad (3.2)$$

and

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \int_{\Omega} W(\nabla_h y^{(h)}(x)) dx \leq \gamma_H^- \mathcal{H}^0(\partial^- A) + \gamma_H^+ \mathcal{H}^0(\partial^+ A). \quad (3.3)$$

PROOF. – We first consider the case where the matrix $F := (\partial_1 y | d_2 | d_3)$ is piecewise constant with values in K . In this case there exist $0 = a_0 < a_1 < \dots < a_{n+1} = L$ and $F_i \in K$, $i = 0, \dots, n$, such that

$$F = \sum_{i=0}^n \chi_{(a_i, a_{i+1})} F_i.$$

With this notation the set $\partial^- A$ is given by all points $a_i \in (0, L)$ such that $F_{i-1} \in \text{SO}(3)H$ and $F_i \in \text{SO}(3)$, while the set $\partial^+ A$ is given by all $a_i \in (0, L)$ such that $F_{i-1} \in \text{SO}(3)$ and $F_i \in \text{SO}(3)H$.

Let (σ_h) be a sequence of positive numbers such that $h \ll \sigma_h \ll 1$, i.e., $\sigma_h \rightarrow 0$ and $h\sigma_h^{-1} \rightarrow 0$. For every h positive we first define $y^{(h)}$ outside a neighbourhood of the points a_i in the following way:

$$y^{(h)}(x) := \begin{cases} F_0 \begin{pmatrix} x_1 \\ hx' \end{pmatrix} & \text{if } x \in [0, a_1 - \sigma_h) \times S, \\ F_i \begin{pmatrix} x_1 \\ hx' \end{pmatrix} + c_i^{(h)} & \text{if } x \in [a_i + \sigma_h, a_{i+1} - \sigma_h) \times S, \ i = 1, \dots, n-1, \\ F_n \begin{pmatrix} x_1 \\ hx' \end{pmatrix} + c_n^{(h)} & \text{if } x \in [a_n + \sigma_h, L) \times S, \end{cases}$$

where the constants $c_i^{(h)}$, $i = 1, \dots, n$ will be chosen later in a suitable way.

In order to define $y^{(h)}$ on the sets $[a_i - \sigma_h, a_i + \sigma_h) \times S$, let us consider a point a_i with $i = 1, \dots, n$.

Suppose first that $a_i \in \partial^- A$. Then from Proposition 2.3 it follows that for every $\eta > 0$ there exist $M_i > 0$ and $v_i \in W_{loc}^{1,2}(\mathbb{R} \times \overline{S})$ such that

$$\nabla v_i = F_{i-1} \quad \text{a.e. in } (-\infty, -M_i) \times S, \quad \nabla v_i = F_i \quad \text{a.e. in } (M_i, +\infty) \times S, \quad (3.4)$$

and

$$\int_{(-M_i, M_i) \times S} W(\nabla v_i) dx \leq \gamma_H^- + \eta. \quad (3.5)$$

Similarly, if $a_i \in \partial^+ A$, then for every $\eta > 0$ there exist $M_i > 0$ and $v_i \in W_{loc}^{1,2}(\mathbb{R} \times \overline{S})$ such that (3.4) is satisfied and

$$\int_{(-M_i, M_i) \times S} W(\nabla v_i) dx \leq \gamma_H^+ + \eta. \quad (3.6)$$

In both cases we define $y^{(h)}$ on the set $[a_i - \sigma_h, a_i + \sigma_h) \times S$ as

$$y^{(h)}(x) := hv_i \left(\frac{x_1 - a_i}{h}, x' \right) + d_i^{(h)},$$

where the constant $d_i^{(h)}$ will be chosen later.

If $a_i \notin \partial^- A \cup \partial^+ A$, then F_{i-1} and F_i belong both to the same well. If $F_{i-1}, F_i \in \text{SO}(3)$, we can construct a smooth function $P_i : \mathbb{R} \rightarrow \text{SO}(3)$ such that $P_i(0) = F_{i-1}$ and $P_i(1) = F_i$. We set $P_i^{(h)}(x_1) := P_i \left(\frac{x_1 - a_i + \sigma_h}{2\sigma_h} \right)$ and we define for $x \in [a_i - \sigma_h, a_i + \sigma_h) \times S$

$$y^{(h)}(x) := \int_{a_i - \tau_h}^{x_1} P_i^{(h)}(s) e_1 ds + P_i^{(h)}(x_1) \begin{pmatrix} 0 \\ x' \end{pmatrix} + d_i^{(h)}, \quad (3.7)$$

where the constants $d_i^{(h)}$ will be chosen later in a suitable way. If $F_{i-1}, F_i \in \text{SO}(3)H$, we can construct a smooth function $P_i : \mathbb{R} \rightarrow \text{SO}(3)H$ such that $P_i(0) = F_{i-1}$ and $P_i(1) = F_i$ and we define $y^{(h)}$ in $[a_i - \sigma_h, a_i + \sigma_h) \times S$ as in (3.7).

It is easy to see that for h small enough the constants $c_i^{(h)}$ and $d_i^{(h)}$ can be chosen in such a way that $y^{(h)} \in W^{1,2}(\Omega; \mathbb{R}^3)$. Indeed, assume $c_1^{(h)}, \dots, c_{i-1}^{(h)}, d_1^{(h)}, \dots, d_{i-1}^{(h)}$ have been chosen. Then, we define $d_i^{(h)}$ and $c_i^{(h)}$ in the following way. If $a_i \in \partial^-A \cup \partial^+A$, we need to require that

$$F_{i-1} \begin{pmatrix} a_i - \sigma_h \\ hx' \end{pmatrix} + c_{i-1}^{(h)} = hv_i \left(-\frac{\sigma_h}{h}, x' \right) + d_i^{(h)}$$

and

$$F_i \begin{pmatrix} a_i + \sigma_h \\ hx' \end{pmatrix} + c_i^{(h)} = hv_i \left(\frac{\sigma_h}{h}, x' \right) + d_i^{(h)}$$

for a.e. $x' \in S$. Since $\frac{\sigma_h}{h} > M_i$ for h small enough, by (3.4) this is equivalent to

$$d_i^{(h)} = F_{i-1} \begin{pmatrix} a_i \\ 0 \end{pmatrix} + c_{i-1}^{(h)}, \quad c_i^{(h)} = d_i^{(h)} - F_i \begin{pmatrix} a_i \\ 0 \end{pmatrix}.$$

If $a_i \notin \partial^-A \cup \partial^+A$, since $P_i^{(h)}(a_i - \sigma_h) = F_{i-1}$ and $P_i^{(h)}(a_i + \sigma_h) = F_i$, it is enough to take $d_i^{(h)}$ and $c_i^{(h)}$ such that

$$d_i^{(h)} = F_{i-1} \begin{pmatrix} a_i - \sigma_h \\ 0 \end{pmatrix} + c_{i-1}^{(h)}$$

and

$$c_i^{(h)} = d_i^{(h)} - F_i \begin{pmatrix} a_i + \sigma_h \\ 0 \end{pmatrix} + 2\sigma_h \int_0^1 P_i(t) e_1 dt.$$

Let us prove that $(\nabla_h y^{(h)})$ converges to F strongly in $L^2(\Omega; \mathbb{M}^{3 \times 3})$, so that also condition (3.2) is satisfied. Indeed, since $\nabla_h y^{(h)}$ coincides with F outside the sets $(a_i - \sigma_h, a_i + \sigma_h) \times S$, $i = 1, \dots, n$, we have only to show that

$$\lim_{h \rightarrow 0^+} \int_{(a_i - \sigma_h, a_i + \sigma_h) \times S} |\nabla_h y^{(h)} - F|^2 dx = 0 \quad (3.8)$$

for every $i = 1, \dots, n$. If $a_i \in \partial^-A \cup \partial^+A$, performing a change of variables and using (3.4), we have that

$$\begin{aligned} & \int_{(a_i - \sigma_h, a_i + \sigma_h) \times S} |\nabla_h y^{(h)} - F|^2 dx \\ & \leq h \int_{(-\frac{\sigma_h}{h}, 0) \times S} |\nabla v_i - F_{i-1}|^2 dx + h \int_{(0, \frac{\sigma_h}{h}) \times S} |\nabla v_i - F_i|^2 dx \\ & = h \int_{(-M_i, 0) \times S} |\nabla v_i - F_{i-1}|^2 dx + h \int_{(0, M_i) \times S} |\nabla v_i - F_i|^2 dx \leq Ch. \end{aligned} \quad (3.9)$$

If $a_i \notin \partial^-A \cup \partial^+A$, then for every $x \in (a_i - \sigma_h, a_i + \sigma_h) \times S$ we have

$$\begin{aligned} \nabla_h y^{(h)}(x) & = P_i^{(h)}(x_1) + h \partial_1 P_i^{(h)}(x_1) \begin{pmatrix} 0 \\ x' \end{pmatrix} \otimes e_1 \\ & = P_i \left(\frac{x_1 - a_i + \sigma_h}{2\sigma_h} \right) + \frac{1}{2} h \sigma_h^{-1} \partial_1 P_i \left(\frac{x_1 - a_i + \sigma_h}{2\sigma_h} \right) \begin{pmatrix} 0 \\ x' \end{pmatrix} \otimes e_1, \end{aligned} \quad (3.10)$$

whence

$$|\nabla_h y^{(h)}(x)| \leq \|P_i\|_{L^\infty(0,1)} + Ch\sigma_h^{-1} \|\partial_1 P_i\|_{L^\infty(0,1)} \leq C.$$

Therefore, the integral of $|\nabla_h y^{(h)} - F|^2$ on $(a_i - \sigma_h, a_i + \sigma_h) \times S$ is small. In connection with (3.9) this yields (3.8).

Finally we establish (3.3). Since $\nabla_h y^{(h)}(x) \in K$ a.e. in the complement of the sets $(a_i - \sigma_h, a_i + \sigma_h) \times S$, $i = 1, \dots, n$, we have

$$\frac{1}{h} \int_{\Omega} W(\nabla_h y^{(h)}) dx = \frac{1}{h} \sum_{i=1}^n \int_{(a_i - \sigma_h, a_i + \sigma_h) \times S} W(\nabla_h y^{(h)}) dx. \quad (3.11)$$

If $a_i \in \partial^- A \cup \partial^+ A$, then by (3.4), (3.5), and (3.6) we have

$$\begin{aligned} & \frac{1}{h} \int_{(a_i - \sigma_h, a_i + \sigma_h) \times S} W(\nabla_h y^{(h)}) dx \\ &= \int_{(-\frac{\sigma_h}{h}, \frac{\sigma_h}{h})} W(\nabla v_i) dx = \int_{(-M_i, M_i) \times S} W(\nabla v_i) dx \\ &\leq \gamma_H^+ \chi_{\partial^+ A}(a_i) + \gamma_H^- \chi_{\partial^- A}(a_i) + \eta. \end{aligned} \quad (3.12)$$

Using the growth condition from above on W and (3.10), for every $a_i \notin \partial^- A \cup \partial^+ A$ we obtain

$$\begin{aligned} \frac{1}{h} \int_{(a_i - \sigma_h, a_i + \sigma_h) \times S} W(\nabla_h y^{(h)}) dx &\leq \frac{C_2}{h} \int_{(a_i - \sigma_h, a_i + \sigma_h) \times S} \text{dist}^2(\nabla_h y^{(h)}, K) dx \\ &\leq \frac{C_2}{h} \int_{(a_i - \sigma_h, a_i + \sigma_h) \times S} |\nabla_h y^{(h)} - P_i^{(h)}|^2 dx \\ &\leq Ch\sigma_h^{-1} \int_0^1 |\partial_1 P_i(t)|^2 dt. \end{aligned}$$

Since $h\sigma_h^{-1} \rightarrow 0$, we conclude that

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \int_{(a_i - \sigma_h, a_i + \sigma_h) \times S} W(\nabla_h y^{(h)}) dx = 0 \quad (3.13)$$

for every $a_i \notin \partial^- A \cup \partial^+ A$. Combining (3.11), (3.12), and (3.13) we conclude that

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \int_{\Omega} W(\nabla_h y^{(h)}) dx \leq \gamma_H^+ \mathcal{H}^0(\partial^+ A) + \gamma_H^- \mathcal{H}^0(\partial^- A) + n\eta.$$

Consider now the general case. Given $(\partial_1 y | d_2 | d_3)$ satisfying (3.1), we can find a sequence (Q_j) of piecewise constant maps from $(0, L)$ to $\mathbb{M}^{3 \times 3}$ such that for every j

$$Q_j \in \text{SO}(3) \text{ a.e. in } A, \quad Q_j \in \text{SO}(3)H \text{ a.e. in } (0, L) \setminus A,$$

and

$$Q_j \rightharpoonup (\partial_1 y | d_2 | d_3) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3 \times 3}).$$

Using the previous construction, for every j there exists a sequence $(y_j^{(h)})$ such that $(\nabla_h y_j^{(h)})$ converges to Q_j strongly in $L^2(\Omega; \mathbb{M}^{3 \times 3})$, as $h \rightarrow 0^+$, and

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \int_{\Omega} W(\nabla_h y_j^{(h)}) dx \leq \gamma_H^+ \mathcal{H}^0(\partial^+ A) + \gamma_H^- \mathcal{H}^0(\partial^- A) + \frac{1}{j}.$$

In particular, we can construct a decreasing sequence (h_j) converging to 0^+ such that for every j we have

$$\|\nabla_{h_j} y_j^{(h_j)} - Q_j\|_{L^2} < \frac{1}{j}$$

and

$$\frac{1}{h_j} \int_{\Omega} W(\nabla_{h_j} y_j^{(h_j)}) dx \leq \gamma_H^+ \mathcal{H}^0(\partial^+ A) + \gamma_H^- \mathcal{H}^0(\partial^- A) + \frac{1}{j}.$$

Now, it is easy to see that the sequence $(\nabla_{h_j} y_j^{(h_j)})$ converges to $(\partial_1 y | d_2 | d_3)$ weakly in $L^2(\Omega; \mathbb{M}^{3 \times 3})$ and

$$\limsup_{j \rightarrow +\infty} \frac{1}{h_j} \int_{\Omega} W(\nabla_{h_j} y_j^{(h_j)}) dx \leq \gamma_H^+ \mathcal{H}^0(\partial^+ A) + \gamma_H^- \mathcal{H}^0(\partial^- A).$$

This concludes the proof. □

4 Study of the Γ -limit

In this section we study the behaviour of the limit functional in terms of the one-dimensional limit deformations $y \in W^{1,\infty}((0, L); \mathbb{R}^3)$. If there are no additional (radial) forces or boundary conditions then the limit problem becomes trivial and the minimizer of the limit problem involves no phase change (see Theorem 4.1). In the presence of radial forces, however, phase transitions can arise, leading to a nontrivial limit problem, see Remark 4.2 below.

Theorem 4.1 *Assume W satisfies conditions (i)-(iii) and (1.2). Then, the sequence of functionals $(\frac{1}{h} I^h)$ Γ -converges, as $h \rightarrow 0^+$, to the functional*

$$I(y) := \begin{cases} 0 & \text{if } |\partial_1 y| \leq \max\{1, \lambda_1\} \text{ a.e. and } \partial_2 y = \partial_3 y = 0 \text{ a.e.,} \\ +\infty & \text{otherwise,} \end{cases}$$

with respect to weak convergence in $W^{1,2}(\Omega; \mathbb{R}^3)$.

It can be easily seen from the proof that the result still holds if a tangential loading term of the form $-h \int_{\Omega} \sigma_1(x_1) \cdot \partial_1 y$ is added to $I^{(h)}$ and the corresponding term $-\int_{(0,L)} \sigma_1 \cdot \partial_1 y$ is added to the limit functional I .

PROOF. – The proof is split in two parts.

1) *Liminf inequality.*

Let $y^{(h)}$, $y \in W^{1,2}(\Omega; \mathbb{R}^3)$ be such that $y^{(h)} \rightharpoonup y$ weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$. We have to prove that

$$I(y) \leq \liminf_{h \rightarrow 0} \frac{1}{h} I^{(h)}(y^{(h)}). \quad (4.1)$$

We may assume that

$$\liminf_{h \rightarrow 0} \frac{1}{h} I^{(h)}(y^{(h)}) \leq C.$$

Then, by Theorem 2.4 there exist $u \in W^{1,\infty}((0,L); \mathbb{R}^3)$, $d_2, d_3 \in L^\infty((0,L); \mathbb{R}^3)$ such that, up to subsequences, the sequence $(\nabla_h y^{(h)})$ converges to $(\partial_1 u | d_2 | d_3)$ weakly in $L^2(\Omega; \mathbb{M}^{3 \times 3})$. Therefore, $\partial_1 y = \partial_1 u$ a.e., the function y does not depend on the variables x_2, x_3 and belongs to $W^{1,\infty}((0,L); \mathbb{R}^3)$. Moreover, again by Theorem 2.4 there exists a set $A \in \mathcal{A}$ such that

$$(\partial_1 y | d_2 | d_3) \in \begin{cases} \text{co}(\text{SO}(3)) & \text{a.e. in } A, \\ \text{co}(\text{SO}(3)H) & \text{a.e. in } (0,L) \setminus A. \end{cases}$$

This condition implies that $|\partial_1 y| \leq 1$ a.e. in A and $|\partial_1 y| \leq \lambda_1$ a.e. in $(0,L) \setminus A$. Therefore, inequality (4.1) is proved.

2) *Existence of a recovery sequence.*

We have to show that for every $y \in W^{1,2}(\Omega; \mathbb{R}^3)$ there exist a sequence $(y^{(h)})$ such that $y^{(h)} \rightharpoonup y$ weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$ and

$$\limsup_{h \rightarrow 0} \frac{1}{h} I^{(h)}(y^{(h)}) \leq I(y). \quad (4.2)$$

We can assume that $\partial_2 y = \partial_3 y = 0$ and $|\partial_1 y| \leq \max\{1, \lambda_1\}$ a.e. in Ω .

Suppose $\lambda_1 > 1$ and let N be the subset of $(0,L)$ where $|\partial_1 y| > 0$. Then we define

$$\tau_1(x_1) := \begin{cases} \partial_1 y(x_1) / |\partial_1 y(x_1)| & \text{if } x_1 \in N, \\ 0 & \text{if } x_1 \in (0,L) \setminus N. \end{cases}$$

Since τ_1 is measurable, we can construct a pair of measurable functions $\tau_2, \tau_3 : (0,L) \rightarrow \mathbb{R}^3$ such that $\tau_2 = \tau_3 = 0$ a.e. in $(0,L) \setminus N$ and $(\tau_1 | \tau_2 | \tau_3) \in \text{SO}(3)$ a.e. in N . Now, if we set

$$d_2 := \frac{\lambda_2}{\lambda_1} |\partial_1 y| \tau_2, \quad d_3 := \frac{\lambda_3}{\lambda_1} |\partial_1 y| \tau_3,$$

then $d_2, d_3 \in L^\infty((0, L); \mathbb{R}^3)$. Moreover, the matrix $(\partial_1 y | d_2 | d_3)$ satisfies

$$(\partial_1 y | d_2 | d_3) = 0 \text{ a.e. in } (0, L) \setminus N, \quad (4.3)$$

$$(\partial_1 y(x_1) | d_2(x_1) | d_3(x_1)) = \frac{|\partial_1 y(x_1)|}{\lambda_1} (\tau_1(x_1) | \tau_2(x_1) | \tau_3(x_1)) H \text{ a.e. in } N. \quad (4.4)$$

It is easy to see that $0 \in \text{co}(\text{SO}(3)H)$ and therefore, also $tRH \in \text{co}(\text{SO}(3)H)$ for every $0 \leq t \leq 1$ and $R \in \text{SO}(3)$. Since $|\partial_1 y| \leq \lambda_1$ a.e., this fact together with (4.3) and (4.4) implies that the matrix $(\partial_1 y | d_2 | d_3)$ belongs to the convex hull of $\text{SO}(3)H$ a.e. in $(0, L)$. Applying Theorem 3.1 to y, d_2, d_3 with $A = \emptyset$ we find a sequence $y^{(h)}$ converging to y weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$ such that (4.2) is satisfied.

The proof is analogous in the case $\lambda_1 \leq 1$. \square

Remark 4.2 Let $\sigma_2, \sigma_3 \in L^2((0, L); \mathbb{R}^3)$. Consider the functionals

$$\tilde{I}^{(h)}(y) := \int_{\Omega} W(\nabla_h y(x)) dx - \int_{\Omega} (\sigma_2(x_1) \partial_2 y(x) + \sigma_3(x_1) \partial_3 y(x)) dx$$

defined for every $y \in W^{1,2}(\Omega; \mathbb{R}^3)$. Integrating by parts on the cross-section S , one can see that the additional terms in the functional describes a radial force acting along the rod.

Let $(y^{(h)}) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ be a sequence such that

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \tilde{I}^{(h)}(y^{(h)}(x)) dx \leq c.$$

Then, using the growth conditions from below on W and the Hölder inequality, we deduce that, up to subsequences, $(\nabla_h y^{(h)})$ is uniformly bounded in $L^2(\Omega; \mathbb{M}^{3 \times 3})$. This implies in particular that

$$\frac{1}{h} \int_{\Omega} W(\nabla_h y^{(h)}(x)) dx \leq c.$$

Now, by Theorem 2.4 we have that

$$\nabla_h y^{(h)} \rightharpoonup (\partial_1 y | d_2 | d_3) \text{ weakly in } L^2(\Omega; \mathbb{M}^{3 \times 3}), \quad (4.5)$$

with $y \in W^{1,\infty}((0, L); \mathbb{R}^3)$ and $d_2, d_3 \in L^\infty((0, L); \mathbb{R}^3)$. Moreover, there exists $A \in \mathcal{A}$ such that

$$(\partial_1 y | d_2 | d_3) \in \begin{cases} \text{co}(\text{SO}(3)) & \text{a.e. in } A, \\ \text{co}(\text{SO}(3)H) & \text{a.e. in } (0, L) \setminus A. \end{cases} \quad (4.6)$$

Finally, from (2.11) and (4.5) it follows that

$$\liminf_{h \rightarrow 0^+} \frac{1}{h} \tilde{I}^{(h)}(y^{(h)}(x)) \geq \gamma_H^- \mathcal{H}^0(\partial^- A) + \gamma_H^+ \mathcal{H}^0(\partial^+ A) - \int_0^L (\sigma_2 d_2 + \sigma_3 d_3) dx_1. \quad (4.7)$$

Applying Theorem 3.1, it is easy to show the following upper bound. Given $y \in W^{1,\infty}((0, L); \mathbb{R}^3)$ and $d_2, d_3 \in L^\infty((0, L); \mathbb{R}^3)$ satisfying property (4.6) for some $A \in \mathcal{A}$, there exists a sequence $(y^{(h)}) \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that (4.5) holds and

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \tilde{I}^{(h)}(y^{(h)}(x)) \leq \gamma_H^- \mathcal{H}^0(\partial^- A) + \gamma_H^+ \mathcal{H}^0(\partial^+ A) - \int_0^L (\sigma_2 d_2 + \sigma_3 d_3) dx_1. \quad (4.8)$$

In this case it is not true in general that minimizers of the limit functional determined by (4.7) and (4.8) have no transition points, that is, their deformation gradient belongs to the same well almost everywhere. Assume for instance that the entries of the matrix H are of the form $\lambda_2 = 1 + \varepsilon$, $\lambda_3 = 1 - \varepsilon$ for some $\varepsilon \in (0, 1)$. Then the strong incompatibility condition (1.2) is satisfied if $\lambda_1 < \frac{1}{3}$.

Assume that $\sigma_2(x_1) = \sigma_{22}(x_1)e_2$ and $\sigma_3(x_1) = \sigma_{33}(x_1)e_3$, where $\sigma_{22}, \sigma_{33} \in L^2(0, L)$ have the following property: there exists $\alpha \in (0, L)$ such that $\sigma_{22} > 0$ a.e. in $(0, \alpha)$, $\sigma_{22} = 0$ a.e. in (α, L) , while $\sigma_{33} = 0$ a.e. in $(0, \alpha)$, $\sigma_{33} > 0$ a.e. in (α, L) . Then it is easy to see that if

$$\int_0^\alpha \sigma_{22}(x_1) dx_1 > \frac{\gamma_H^-}{\varepsilon}, \quad \int_\alpha^L \sigma_{33}(x_1) dx_1 > \frac{\gamma_H^-}{\varepsilon},$$

it is energetically more convenient to have a transition point at α instead of having deformation gradients lying in the convex hull of only one well.

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