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CONFORMAL DEFORMATIONS OF THE SMALLEST EIGENVALUE OF THE RICCI TENSOR

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ABSTRACT. We consider deformations of metrics in a given conformal class such that the smallest eigenvalue of the Ricci tensor to be a constant. It is related to the notion of minimal volumes in comparison geometry. Such a metric with the smallest eigenvalue of the Ricci tensor to be a constant is an extremal metric of volume in a suitable sense in the conformal class. The problem is reduced to solve a Pucci type equation with respect to the Schouten tensor. We establish a local gradient estimate for this type of conformally invariant fully nonlinear uniform elliptic equations. Combining it with the theory of fully nonlinear equations, we establish the existence of solutions for this equation.

1. INTRODUCTION

The Ricci curvature tensor of a Riemannian metric plays an important role in comparison geometry for Riemannian manifolds, in particular the lower bounds of Ricci curvature. See for instance [2, 13, 30]. In this paper, we are interested in conformal deformations of the smallest eigenvalue of the Ricci tensor. Let (M^n, g_0) be an n -dimensional compact Riemannian manifold and $[g_0]$ its conformal class. And let Ric_g and R_g be the Ricci curvature tensor and the scalar tensor of a metric g respectively. Define $\min Ric_g(x)$ the smallest eigenvalue of $g^{-1} \cdot Ric_g$ at $x \in M$. Our problem is to find a conformal metric $g = e^{-2u}g_0$ such that

$$(1) \quad \min Ric_g(x) = \text{constant}.$$

It turns out that the problem is equivalent to solving an interesting fully nonlinear uniform elliptic equation. First we recall that the Schouten tensor of the metric g is defined by

$$S_g = \frac{1}{n-2} \left(Ric_g - \frac{R_g}{2(n-1)}g \right).$$

Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$. Assume that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. For an integer $1 \leq p \leq n-1$, define a function $G_p : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$G_p(\Lambda) = (n-p) \sum_{i \leq p} \lambda_i + p \sum_{i > p} \lambda_i.$$

For a symmetric matrix A , $G_p(A) = G_p(\Lambda)$, where Λ is the set of eigenvalues of A . It is easy to check $\min Ric = G_1(g^{-1} \cdot S_g)$.

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We may also ask if there is a conformal metric with a constant $W_p(g) := G_p(g^{-1} \cdot S_g)$. $W_p(g)$ is also an interesting geometric object, which will be called p -Weitzenböck curvature, for it arises from the Weitzenböck formula for p -forms in a locally conformally flat manifold. See the Appendix. We will consider the following general equation

$$(2) \quad W_p(g)(x) = f(x), \quad \forall x \in M.$$

We first treat the case when the background metric has negative curvature, i.e., $W_p(g_0) < 0$.

Theorem 1. *Let (M, g_0) be a compact Riemannian manifold and $1 \leq p \leq n$. Suppose that $W_p(g_0)(x) < 0$ for any $x \in M$, then there is a unique $C^{2,\alpha}$ metric $g \in [g_0]$ for some $\alpha > 0$ such that $W_p(g)(x) = -1$, $\forall x \in M$.*

A geometric consequence of Theorem 1 is the existence of an extremal metric in the conformal class with minimal volume. Although (2) has no variational structure in general, a solution of (2) does achieve the minimum of the minimal volume in a conformal class in this case. The following is a simple consequence, see section 4 for other related results.

Corollary 1. *Suppose that $\min Ric_{g_0}(x) < 0$ for any $x \in M$. Then there is a unique conformal metric $g^* \in [g_0]$ such that $vol(g^*) = \min vol(g)$, where minimum is taken over all $g \in [g_0]$ with $\min Ric_g(x) \geq -1$. The extremal metric g^* is characterized by a unique solution to equation $\min Ric_{g^*}(x) = -1, \forall x \in M$.*

This Corollary is related to the minimal volumes considered by Gromov [12] and LeBrun [23]

We now turn to the case of the positive Ricci curvature.

Theorem 2. *Let (M, g_0) be a compact Riemannian manifold with $Ric_{g_0} > 0$. Then there is a conformal metric $g \in [g_0], g \in C^{2,\alpha}(M)$ for some $\alpha > 0$ such that $\min Ric_g(x) = n - 1$ for all $x \in M$.*

The positivity of p -Weitzenböck curvature for $1 < p \leq n/2$ plays an important role in the investigation of the topological structure of locally conformally flat manifolds in [14] and the Appendix.

Theorem 3. *Let (M, g_0) be an n -dimensional smooth compact locally conformally flat manifold with $W_p(g_0) > 0$ and $p \leq n/2$. If (M, g_0) is not conformally equivalent to the standard n -sphere, then there exists $g \in [g_0], g \in C^{2,\alpha}(M)$ for some $\alpha > 0$ such that*

$$(3) \quad W_p(g)(x) = 1, \quad \forall x \in M.$$

Furthermore, the solution space is compact. That is, there is positive constant $C > 0$, such that

$$(4) \quad \|u\|_{C^{2,\alpha}} \leq C$$

for any $C^{2,\alpha}$ solution $g = e^{-2u}g_0$ of (3).

One of the interesting aspect of this paper is that our geometric problem is deduced to a Pucci type fully nonlinear uniformly elliptic equations with respect to the Schouten tensor. These equations have been studied extensively in Euclidean domains, in particular in connection to stochastic optimization. With the breakthrough of the Krylov-Safonov's Harnack estimate

[22] for non-divergent elliptic equations, it followed the fundamental Evans-Krylov theorem [10, 21] on the Hölder regularity of the second derivatives. The deep results of Caffarelli [5] and Safonov [33, 34] provide further improvement of such Hölder estimates bypassing C^2 estimates. Caffarelli's perturbation theory is very important for our application in conformal geometry. Especially some of our geometric equations are in terms of convex operators, where C^2 estimates are difficult to obtain with standard approaches. The core of Caffarelli's perturbation theory is the C^2 estimates for simple model equations, which were established by him for the concave and convex operators. To apply the Caffarelli perturbation theory for our equations, we need one more crucial estimate, that is a local gradient bound depending only on one side bound of solutions. As in the case for nonlinear equations in conformal geometry, this is a key step for the blow-up analysis. The Bishop-Gromov volume comparison theorem also plays an important role in the proof of Theorem 3.

The paper is organized as follows. In Section 2, we discuss briefly our uniformly elliptic fully nonlinear equations and Caffarelli's results. We prove the local gradient estimates in Section 3. In Section 4 we discuss the case of negative curvature and the minimal volume in a conformal class. We prove Theorem 2 and Theorem 3 in Section 5.

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2. FULLY NONLINEAR UNIFORMLY ELLIPTIC EQUATIONS AND CAFFARELLI'S ESTIMATES

In this section, we will deduce our problem to fully nonlinear uniformly elliptic equations. We will make use of Caffarelli's fundamental $W^{2,p}$ and $C^{2,\alpha}$ estimates in [5] (see also Safonov's work on $C^{2,\alpha}$ estimates in [34]). Some of these results have been subsequently generalized to certain type of equations of form $F(\nabla^2 u, \nabla u, x) = f(x)$ by L. Wang in [39] as well as for the case of parabolic equations. As we will see that equation (2) involves ∇u in a delicate way, we need certain appropriate a priori estimates depending only on one side bound of u (to be more explicit, $e^{-2\inf u}$). This type of local gradient estimates will be established in the next section.

Let (M^n, g_0) be a compact, oriented Riemannian manifold of dimension $n > 2$. Let $[g_0]$ be the conformal class of g_0 . For any $g \in [g_0]$, we denote Ric_g, S_g the Ricci tensor and the Schouten tensor of the metric g respectively. We write

$$\begin{aligned} \mathcal{R}_g^s(x) &= \text{smallest eigenvalue of } g^{-1} Ric_g(x), \\ \mathcal{R}_g^l(x) &= \text{largest eigenvalue of } g^{-1} Ric_g(x), \\ \mathcal{S}_g^s(x) &= \text{smallest eigenvalue of } g^{-1} S_g(x), \\ \mathcal{S}_g^l(x) &= \text{largest eigenvalue of } g^{-1} S_g(x). \end{aligned}$$

It is clear that \mathcal{R}_g^s and \mathcal{R}_g^l do not depend on the choice of coordinates. From the definition of the Schouten tensor

$$S_g = \frac{1}{n-2} \left(Ric_g - \frac{R_g}{2(n-1)} \right),$$

we have the following relations

$$(5) \quad \begin{aligned} \mathcal{R}_g^s(x) &= (n-2)\mathcal{S}_g^s(x) + \text{tr}(S_g(x)), \\ \mathcal{R}_g^l(x) &= (n-2)\mathcal{S}_g^l(x) + \text{tr}(S_g(x)). \end{aligned}$$

If $g = e^{-2u}g_0$, there is a transformation formula between two Schouten tensors

$$(6) \quad S_g = \nabla^2 u + \nabla u \otimes \nabla u - \frac{|\nabla u|^2 g_0}{2} + S_{g_0},$$

where all covariant derivatives are with respect to g_0 . (The same convention will be used in the rest of this paper, unless it is stated otherwise). Locally, denoting S_{ij} the Schouten tensor of g_0 under the frame, we have

$$(S_g)_{ij} = u_{ij} + u_i u_j - \frac{|\nabla u|^2}{2} \delta_{ij} + S_{ij}.$$

For any symmetric matrix A , we denote $\lambda_s(A)$ and $\lambda_l(A)$ to be the smallest and largest eigenvalues of A respectively. And we denote $\sigma_1(A) = \text{tr}(A)$. We obtain the equations for constant \mathcal{R}_g^s and \mathcal{R}_g^l respectively:

$$(7) \quad F_*(u) =: ((n-2)\lambda_s + \sigma_1)(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) = e^{-2u} \mathcal{R}_g^s,$$

$$(8) \quad F^*(u) =: ((n-2)\lambda_l + \sigma_1)(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) = e^{-2u} \mathcal{R}_g^l.$$

Hence F_* and F^* are uniformly elliptic with ellipticity constants 1 and $n-1$. It is clear that F_* is concave and F^* is convex. We also note that F^* and F_* are homogeneous of degree 1.

There are other similar fully nonlinear equations arising in the Weitzenböck formula for p -form on local conformally flat manifolds. Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$ be the set of eigenvalues of a symmetric matrix A . For an integer $1 \leq p \leq n/2$, define a function $G_p : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$G_p(A) = G_p(\Lambda) = \min \left\{ (n-p) \sum_{k \leq p} \lambda_{i_k} + p \sum_{k > p} \lambda_{i_k} \right\},$$

where min is over all permutations of $1, 2, \dots, n$. We define $W_p(g)$ the p -Weitzenböck curvature of g by

$$W_p(g) = G_p(g^{-1} \cdot S_g).$$

It is easy to check $\mathcal{R}_g^s = W_1(g)$. The p -Weitzenböck curvature is as much interest as the scalar curvature, at least for locally conformally flat manifold. For example, from the Weitzenböck formula one can easily show that a locally conformally flat manifold with positive p -Weitzenböck curvature has vanishing q -cohomology group for $n/2 - p \leq q \leq n/2 + p$ (for $p \leq n/2$).

We may also ask whether we can find a conformal metric with constant W_p . The corresponding equation is

$$(9) \quad G_p \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0} \right) = \text{constant} \cdot e^{-2u}.$$

Equation (9) is also uniformly elliptic. It is easy to see that (9) is concave for $p \leq \frac{n}{2}$. From the transformation formula (6), one can check that Equation (9) is also conformally invariant. Hence, in general there is no compactness for equation (9).

We also define for $p \geq n/2$ a function $G_p : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$G_p(A) = G_p(\Lambda) = \max \left\{ (n-p) \sum_{k \leq p} \lambda_{i_k} + p \sum_{k > p} \lambda_{i_k} \right\},$$

where max is over all permutations of $1, 2, \dots, n$. We still define $W_p(g) = G_p(g^{-1} \cdot S_g)$. It is clear that $\mathcal{R}_g^l = W_{n-1}(g)$. We can also consider equation (9) for $p \geq n/2$. For $p \geq n/2$ equation (9) is still uniformly elliptic, but it is convex. Since G_p is homogeneous of order 1, we may rewrite equation (2) as the following equivalent equation by setting $v = e^u$

$$(10) \quad G_p(\nabla^2 v + S_{g_0} v) = p(n-p) \frac{|\nabla v|^2}{v} + \frac{f}{v}.$$

The function G_p is only Lipschitz, when $p \neq n/2$. (Remark that when $p = n/2$, equation (9) is equivalent to the Yamabe equation.) One can find a sequence of smooth functions $\{F_k\}$ such that F_k uniformly converges to G_p in any compact domain of \mathbb{R}^n and homogeneous 1 outside the unit ball in \mathbb{R}^n , i.e., $F_k(x) = |x|F(\frac{x}{|x|})$ for $|x| \geq 1$. Furthermore, for $p < n/2$ ($p > n/2$) F_k is concave (convex).

One may consider a more general class of conformal equations. Let \mathcal{S} be the space of symmetric 2-tensors on M . Let $F : \mathcal{S} \rightarrow \mathbb{R}$ a real continuous function. We consider the following general equation

$$(11) \quad F \left(e^{-2u} (\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) \right) = f(x),$$

for some function $f : M \rightarrow \mathbb{R}$. F is *uniformly elliptic* with constants λ_0 and Λ_0 if there exists two positive constants λ_0 and Λ_0 such that for any $W \in \mathcal{S}$

$$\lambda_0 \|N\| \leq F(W + N) - F(W) \leq \Lambda_0 \|N\| \quad \forall N \geq 0,$$

here by $N \geq 0$ means that N is nonnegative definite and $\|N\| = \sup_{|v|=1} |Nv|$. If F is uniformly elliptic, we call equation (11) a uniformly elliptic fully nonlinear equation with ellipticity constants λ_0 and Λ_0 . There are many typical uniformly elliptic fully nonlinear equations. Our equation (9) is similar to the Pucci equation, see [5]. Let \mathcal{M}^\pm be the Pucci's extremal operators, namely for two given constant $0 < \lambda_0 < \Lambda_0$ and $W \in \mathcal{S}$

$$\begin{aligned} \mathcal{M}^-(W) &= \lambda_0 \sum_{e_i > 0} e_i + \Lambda_0 \sum_{e_i < 0} e_i, \\ \mathcal{M}^+(W) &= \Lambda_0 \sum_{e_i > 0} e_i + \lambda_0 \sum_{e_i < 0} e_i, \end{aligned}$$

where $e_i = e_i(W)$ are the eigenvalues of W . One can also consider

$$(12) \quad \begin{aligned} \mathcal{M}^-(e^{-2u}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})) &= 1, \\ \mathcal{M}^+(e^{-2u}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})) &= 1. \end{aligned}$$

\mathcal{M}^- is concave, while \mathcal{M}^+ is convex.

We now deduce $C^{2,\alpha}$ estimates for equation (2) from the work of Caffarelli [5] assuming the gradient bound and a lower bound of u . The crucial part in Caffarelli's perturbation theory for uniformly elliptic fully nonlinear equation $F(\nabla^2 u, x) = f(x)$ is the $C^{1,1}$ interior estimates for $F(\nabla^2 u, x_0) = \text{constant}$. He obtained such fundamental estimates for concave or convex operator F (note that here concavity and convexity of F can always be switched by $\tilde{F}(\Lambda) = -F(-\Lambda)$). Though Caffarelli proved these estimates in [5] for equations with flat metric, his arguments work under general Riemannian metrics. And the generalization of Caffarelli's estimates by L. Wang [39] to uniformly elliptic equations of form $F(\nabla^2 u, \nabla u, u, x)$ gives the following $C^{2,\alpha}$ estimates for equations of type (9).

Theorem 4 ($C^{2,\alpha}$ -estimates). *Suppose F is a uniformly elliptic concave operator with elliptic constants λ_0, Λ_0 . Let B_1 be a unit disk in a compact Riemannian manifold M and $f, h \in C^1(B_1)$. Suppose $g = e^{-2u}g_0$ with $|\nabla u|_{B_1} \leq A$ is a solution of equation*

$$(13) \quad F\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) = e^{-2u}f(x), \quad x \in B_1,$$

then there exist $\alpha > 0$ and $C > 0$ depending only on $\lambda_0, \Lambda_0, A, \|u\|_{C^0(B_1)}$ and g_0 such that

$$(14) \quad \|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C.$$

In fact, we may directly apply Caffarelli's estimates [5] to obtain $C^{2,\alpha}$ estimates for equations of type (10). Let B_1 be a unit disk in a compact Riemannian manifold M and $f, h \in C^1(B_1)$. Suppose $g = e^{-2u}g_0 = v^{-2}g_0$ with $|\frac{\nabla v}{v}|_{B_1} \leq A$ is a solution of equation

$$(15) \quad F(\nabla^2 v(x) + v(x)S_{g_0}(x)) = h(x)\frac{|\nabla v(x)|^2}{v(x)} + \frac{f(x)}{v(x)}, \quad x \in B_1.$$

Since F is concave, by Theorem 6.6 in [6], the equation

$$F(\nabla^2 v + S_{g_0}(x_0)v) = \text{constant},$$

has $C^{1,1}$ interior estimates for any $x_0 \in B_1$. It follows from Theorem 7.1 in [6] that equation (15) has interior $W^{2,p}$ estimate for any $n < p < \infty$ since $|\frac{\nabla v}{v}| \leq A$. This in turn gives $C^{1,\beta}$ a priori bound for the solution v of equation (15) for all $0 < \beta < 1$. Finally estimate (14) for $v = e^u$ follows from Theorem 8.1 in [6] since the right hand side of equation (15) is a C^β function now.

3. LOCAL GRADIENT ESTIMATES

In this section, we establish a local gradient estimates for solutions $g = e^{-2u}g_0$ of equation (2) in the case $p \leq n/2$, depending only on lower bound of u .

Theorem 5 (Local gradient estimates). *Suppose F is concave and uniformly elliptic with ellipticity constants λ_0, Λ_0 . Let B_1 be a unit disk in a compact Riemannian manifold M and u a C^2 solution of the following equation*

$$(16) \quad F\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) = e^{-2u}f(x), \quad x \in B_1$$

for a C^1 function $f : B_1 \rightarrow \mathbb{R}$. Then there is a constant $C > 0$ depending only on $\lambda_0, \Lambda_0, g_0$ such that

$$(17) \quad |\nabla u|^2(x) \leq C(1 + \|f\|_{C^1(B_1)}e^{-2\inf_{B_1} u}), \quad \text{for any } x \in B_{1/2}.$$

Combining Theorem 5 and Theorem 4, we deduce the following.

Corollary 2 (Local $C^{2,\alpha}$ -estimates). *Let B_1 be a unit disk in a compact Riemannian manifold M and $f \in C^1(B_1)$. Suppose u is a solution of equation (16), then there is a constant $C > 0$ depending only on $\lambda_0, \Lambda_0, g_0, \|f\|_{C^1(B_1)}, \inf_{B_1} u$, such that*

$$(18) \quad \|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C.$$

In particular, (18) is true for any solution of (2) when $p \leq \frac{n}{2}$.

We note that Corollary 2 guarantees that we can use the so-called blow-up analysis to study the fully nonlinear equation (2). We remark that the local gradient estimates (17) does not true for $p = n - 1$. The local gradient estimates were first obtained for σ_k conformal invariant equations in [15] and for $\frac{\sigma_k}{\sigma_1}$ conformal invariant equations in [17], though in general fully nonlinear equations have no local estimates. The operator G_p we are considering here is only Lipschitz, which we will deal with by a smoothing argument. Actually we can prove the local gradient estimates for a more general class of uniformly elliptic fully nonlinear conformal equations.

We first prove the local gradient estimates for C^2 uniformly elliptic operator F . Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^2 symmetric function and consider the following equation

$$(19) \quad F\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) = \tilde{f},$$

for some C^1 function \tilde{f} . We denote the left hand side of (19) by $F(W)$ and set

$$F^{ij} = \frac{\partial F}{\partial w_{ij}},$$

where w_{ij} is the entry of the matrix W . As mentioned above, $F(W) = F(\Lambda)$, where Λ is the set of eigenvalues of W .

Proposition 1. *Let B_1 be a unit disk in a compact Riemannian manifold M and u a C^3 solution of (19) for a C^1 function $\tilde{f} : B_1 \rightarrow \mathbb{R}$. Let $F : \mathcal{S} \rightarrow \mathbb{R}$ be a C^2 function satisfying*

- (1) F is an uniformly elliptic with ellipticity constants λ_0 and Λ_0
- (2) F is concave.

Then for any $\rho \in C_0^2(B_1)$ with $0 \leq \rho(x) \leq 1$, there is a constant $C > 0$ depending only on $\lambda_0, \Lambda_0, \|\rho\|_{C^2(B_1)}^2$ and g_0 such that

$$(20) \quad \max_{B_1}\{\rho(x)|\nabla u|^2(x)\} \leq C(1 + \max_{B_1}\{\rho(x)|\nabla \tilde{f}(x)|\}).$$

Proof of Proposition 1. The Proof follows closely the argument given in [15] and [16]. As in [15], we first reduce the proof of the Lemma to the following claim.

Claim. *There is a constant A_0 depending, such that*

$$(21) \quad \sum_{i,j} F^{ii} \tilde{u}_{ij}^2 \geq A_0^{-\frac{5}{8}} \sum_i F^{ii} |\nabla u|^4,$$

where $\tilde{u}_{ij} = u_{ij} + S_{ij}$.

For convenience of the reader, we sketch the reduction. Let ρ be a test function $\rho \in C_0^2(B_1)$. We may assume

$$(22) \quad |\nabla \rho(x)| \leq 2b_0 \rho^{1/2}(x) \quad \text{and} \quad |\nabla^2 \rho| \leq b_0, \quad \text{in } B_1,$$

for $b_0 > 1$. Set $H = \rho |\nabla u|^2$. Our aim is to bound $\max_{B_1} H$. Let $x_0 \in B_1$ be a maximum point of H and assume that W is a diagonal matrix at the point x_0 by choosing a suitable normal coordinates around x_0 . Set $\lambda_i = w_{ii}$ and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Since W is diagonal at x_0 , we have at x_0

$$(23) \quad w_{ii} = u_{ii} + u_i^2 - \frac{1}{2} |\nabla u|^2 + S_{ii}, \quad u_{ij} = -u_i u_j - S_{ij}, \quad \forall i \neq j,$$

where S_{ij} are entries of S_{g_0} . We may assume that

$$H(x_0) \geq b_0^2 A_0^2,$$

for some large, but fixed constant $A_0 > 0$ which will be fixed later. We may also assume that

$$(24) \quad |S_{g_0}|(x_0) \leq A_0^{-1} |\nabla u|^2(x_0).$$

Otherwise, we are done. The fact that the derivatives of H at x_0 vanish imply

$$(25) \quad \left| \sum_{l=1}^n u_{il} u_l \right| (x_0) \leq \frac{|\nabla u|^3}{A_0}(x_0) \quad \text{for any } i.$$

Applying the maximum principle to H , we have

$$(26) \quad 0 \geq F^{ij} H_{ij} = F^{ij} \left\{ \left(-2 \frac{\rho_i \rho_j}{\rho} + \rho_{ij} \right) |\nabla u|^2 + 2\rho u_{ij} u_l + 2\rho u_{il} u_{jl} \right\}.$$

The first term in the left hand side of (26) is bounded from below by $-10nb_0\Lambda_0|\nabla u|^2$. By using equation (19) and inequality (25), the second term can be bounded by

$$(27) \quad \begin{aligned} \sum_{i,j,l} F^{ij} u_{ij} u_l &\geq \sum_l F_l u_l - 2 \sum_{i,l} F^{ii} u_{il} u_l u_i + \sum_{i,l} F^{ii} u_{jl} u_j u_l - C |\nabla u|^2 \sum_i F^{ii} \\ &\geq -|f|^2 - 2n \sum_i F^{ii} \frac{|\nabla u|^4}{A_0} - C |\nabla u|^2, \end{aligned}$$

where $C > 0$ depends only on g_0 and Λ_0 . See also (2.20) in [15]. It is easy to see that the third term is bounded by the Claim. Hence if the Claim is true, from (26) we have

$$(28) \quad 0 \geq -C|\nabla u|^2 - \rho|\nabla \tilde{f}|^2 - \rho \sum_i F^{ii} \frac{|\nabla u|^4}{A_0} + \rho A_0^{-\frac{5}{8}} \sum_i F^{ii} |\nabla u|^4.$$

Multiplying (28) by ρ , we have

$$0 \leq \sum_i F^{ii} (A_0^{-\frac{5}{8}} - A_0^{-1}) H^2 - CH - \rho^2 |\nabla \tilde{f}|^2,$$

from which we have (20).

Now we prove the Claim. By (23), we have

$$(29) \quad \begin{aligned} \sum_{i,l} F^{ii} \tilde{u}_{il}^2 &= \sum_i F^{ii} \tilde{u}_{ii}^2 + \sum_{i \neq l} F^{ii} u_i^2 u_l^2 \\ &= \sum_i F^{ii} \{ \tilde{u}_{ii}^2 + u_i^2 (|\nabla u|^2 - u_i^2) \} \\ &= \sum_i F^{ii} (w_{ii}^2 - 2u_i^2 w_{ii} + w_{ii} |\nabla u|^2 + \frac{|\nabla u|^4}{4}). \end{aligned}$$

Set $\delta_0 = A_0^{-1/4} < 0.1$. We divide the set $I = \{1, 2, \dots, n\}$ as in [15] into two parts:

$$I_1 = \{i \in I \mid u_i^2 \geq \delta_0 |\nabla u|^2\} \quad \text{and} \quad I_2 = \{i \in I \mid u_i^2 < \delta_0 |\nabla u|^2\}.$$

It is clear that I_1 is non-empty.

Case 1. There is j_0 satisfying

$$(30) \quad \tilde{u}_{j_0 j_0}^2 \leq \delta_0^2 |\nabla u|^4 \quad \text{and} \quad u_{j_0}^2 < \delta_0 |\nabla u|^2.$$

We may assume that $j_0 = n$. We have $|w_{nn} + \frac{|\nabla u|^2}{2}| = |\tilde{u}_{nn} + u_n^2| < 2\delta_0 |\nabla u|^2$ by (30). From (25) and (23), we have

$$\left| w_{ii} - \frac{|\nabla u|^2}{2} \right| = |u_{ii} + u_i^2 - |\nabla u|^2 + S_{ii}| \leq 3\delta_0^2 |\nabla u|^2,$$

for any $i \in I_1$.

Using these estimates, we repeat the derivation of equation (2.38) in [15] to obtain

$$(31) \quad \sum_{i,l} F^{ii} \tilde{u}_{il}^2 \geq \tilde{F}^1 \frac{|\nabla u|^4}{4} - \tilde{F}^1 |\nabla u|^4 + F^{nn} \frac{|\nabla u|^4}{4} + (1 - 32\delta_0^2) \frac{|\nabla u|^4}{4} \sum_i F^{ii},$$

where $\tilde{F}^1 := \max_{i \in I_1} F^{ii}$. Recall that I_1 is necessarily non-empty. We assume $1 \in I_1$ with $F^{11} = \tilde{F}^1$. The concavity of F implies that

$$(32) \quad F^{nn} \geq F^{11},$$

for $w_{11} > w_{nn}$. Hence, from (31) we have

$$\begin{aligned} \sum_{i,l} F^{ii} \tilde{u}_{il}^2 &\geq -16\delta_0^2 |\nabla u|^4 + (1 - 32\delta_0^2) \frac{|\nabla u|^4}{4} \sum_{i=2}^{n-1} F^{ii} \\ &\geq \left(1 - 32\delta_0^2 - 64\delta_0^2 \frac{\Lambda_0}{(n-2)\lambda_0}\right) \frac{|\nabla u|^4}{4} \sum_{i=2}^{n-1} F^{ii} \\ &\geq \left(1 - 32\delta_0^2 - 64\delta_0^2 \frac{\Lambda_0}{(n-2)\lambda_0}\right) \frac{(n-2)\lambda_0}{n\Lambda_0} \frac{|\nabla u|^4}{4} \sum_{i=2}^{n-1} F^{ii}, \end{aligned}$$

Case 2. There is no $j \in I$ satisfying (30).

For this case, the proof is the same as in [16]. We repeat it here for completeness.

We may assume that there is i_0 such that $\tilde{u}_{i_0 i_0}^2 \leq \delta_0^2 |\nabla u|^4$, otherwise the claim is automatically true. Assume $i_0 = 1$. As in Case 4 in [15], we have $u_1^2 \geq (1 - 2\delta_0) |\nabla u|^2$ and $\tilde{u}_{jj}^2 + u_j^2 (|\nabla u|^2 - u_j^2) \geq \delta_0^2 |\nabla u|^4$ for $j > 1$. From equation (29), we have

$$\begin{aligned} \sum_{i,l} F^{ii} \tilde{u}_{il}^2 &= \sum_i F^{ii} \{ \tilde{u}_{ii}^2 + u_i^2 (|\nabla u|^2 - u_i^2) \} \\ &\geq \sum_{i \geq 2} F^{ii} (\tilde{u}_{ii}^2 + u_i^2 (|\nabla u|^2 - u_i^2)) \\ &\geq \delta_0^2 |\nabla u|^4 \sum_{i \geq 2} F^{ii} \geq C \delta_0^2 |\nabla u|^4 \sum_{i \geq 1} F^{ii}. \end{aligned}$$

The latter inequality follows from the uniformly ellipticity of F . This finishes the proof the Claim and hence the Proposition. \blacksquare

We have a direct corollary.

Corollary 3. *Suppose F is a C^2 concave and uniformly elliptic operator with ellipticity constants λ_0, Λ_0 . Let B_1 be a unit disk in a compact Riemannian manifold M and u a C^2 solution of equation*

$$(33) \quad F \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0} \right) = e^{-2u} f(x), \quad x \in B_1$$

for a C^1 function $f : B_1 \rightarrow \mathbb{R}$. Then there is a constant $C > 0$ depending only on $\lambda_0, \Lambda_0, g_0$ such that

$$(34) \quad |\nabla u|^2(x) \leq C(1 + \|f\|_{C^1(B_1)} e^{-2 \inf_{B_1} u}), \quad \text{for any } x \in B_{1/2}.$$

Proof. We pick $\rho \in C_0^2(B_1)$ such that $\rho(x) = 1, \forall x \in B_{\frac{1}{2}}$ and $0 \leq \rho(x) \leq 1, \forall x \in B_1$. (34) follows directly from (20) with $\tilde{f} = e^{-2u} f$. \blacksquare

In what follows in the next sections, we will only need Corollary 3 as we will work on smooth operator F . We note that estimates (34) and (18) are independent of the smoothness of F . Theorem 5 can also be proved by certain appropriate approximations.

A sketch proof of Theorem 5. Since $u \in C^2$, u is in fact $C^{2,\alpha}$ by the Evans-Krylov theorem. We may find two sequences of smooth functions $\{u_k\}$ and $\{f_k\}$, such that $u_k \rightarrow u$ in $C^{2,\alpha}(\bar{B}_1)$, $f_k \rightarrow f$ in $C^{0,1}(\bar{B}_1)$, and

$$F \left(\nabla^2 u_k + du_k \otimes du_k - \frac{|\nabla u_k|^2}{2} g_0 + S_{g_0} \right) \geq e^{-2u_k} f_k.$$

We now construct a sequence of smooth concave $F_k : \mathcal{S} \rightarrow \mathbb{R}$ such that F_k converges to F uniformly in compacts of \mathcal{S} and F_k is uniformly elliptic with ellipticity constants $\frac{\lambda_0}{2}$ and $2\lambda_0$. We may assume

$$F_k(\lambda) \geq F(\lambda), \quad \forall |\lambda| \leq \sup_{B_1} \left| \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0} \right| + 1.$$

By the symmetry of F_k , $\frac{\partial F_k(1, \dots, 1)}{\partial \lambda_i} = \frac{\partial F_k(1, \dots, 1)}{\partial \lambda_j}$ for all i, j . Set $\frac{\partial F_k(1, \dots, 1)}{\partial \lambda_j} = A$. Let $R_0(x)$ be the scalar curvature of g_0 , and let \tilde{u} be the solution of $\Delta \tilde{u}_k = \frac{f_k e^{-2u_k}}{A} - R_0 + n - \frac{F_k(1, \dots, 1)}{A}$ in B_1 with $\tilde{u}_k = u_k$ on ∂B_1 . We consider the following Dirichlet problem

$$(35) \quad \begin{aligned} F_k \left(\nabla^2 v_k + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2} g_0 + S_{g_0} \right) &= f_k e^{-2u_k}, & \text{in } B_1, \\ v_k &= u_k, & \text{on } \partial B_1. \end{aligned}$$

By the concavity of F_k ,

$$F_k(\lambda) \leq A\sigma_1(\lambda) - nA + F_k(1, \dots, 1).$$

We have

$$\begin{aligned} A \left(\Delta v_k + R_0 - n + \frac{F_k(1, \dots, 1)}{A} \right) &\geq A \left(\Delta v_k - \frac{n-2}{2} |\nabla v_k|^2 + R_0 - n + \frac{F_k(1, \dots, 1)}{A} \right) \\ &\geq f_k e^{-2u_k} \\ &\geq F_k \left(\nabla^2 v_k + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2} g_0 + S_{g_0} \right). \end{aligned}$$

In turn, we have $v_k \leq \tilde{u}_k$ in B_1 . On the other hand, we have

$$\begin{aligned} F_k \left(\nabla^2 u_k + du_k \otimes du_k - \frac{|\nabla u_k|^2}{2} g_0 + S_{g_0} \right) &\geq F \left(\nabla^2 u_k + du_k \otimes du_k - \frac{|\nabla u_k|^2}{2} g_0 + S_{g_0} \right) \\ &= F_k \left(\nabla^2 v_k + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2} g_0 + S_{g_0} \right). \end{aligned}$$

This gives $v_k \geq u_k$ in B_1 . From this, we obtain a C^0 bound of v_k and a bound of $|\nabla v_k|$ at the boundary ∂B_1 . Using the same proof of Proposition 1, we can obtain a bound of $|\nabla v_k|$ on \bar{B}_1 (simply let $H(x) = |\nabla v_k|^2$ and estimate at the maximum point if it is not on the boundary). At this end, we have a uniform C^1 bound of v_k . The standard barrier construction ω_{\pm} similar to the one in Step 3 in Chapter 9 of [6] (page 91), with the modified operator $\tilde{F}(\omega_{\pm}) = F_k(\nabla^2 \omega_{\pm} +$

$dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2}g_0 + S_{g_0}$), will give a C^2 bound near boundary. The global C^2 estimate follows easily along the lines of proof in Proposition 3.1 in [15] (see also proof of Lemma 3 in [14]). Higher regularity estimates follow from the Krylov Theorem [21]. We note that C^2 and higher regularity bounds of v_k may depend on higher smoothness assumptions on u_k and f_k , but the interior $C^{2,\alpha}$ estimates of v_k depend only on $\|u_k\|_{C^{2,\alpha}(\bar{B}_1)}$ and $\|f_k\|_{C^1(\bar{B}_1)}$ by Theorem 4. In any case, we can establish the existence of the Dirichlet problem (35) by using the method of continuity for equation

$$F_t \left(\nabla^2 v_k + dv_k \otimes dv_k - \frac{|\nabla v_k|^2}{2}g_0 + S_{g_0} \right) = tf_k e^{-2u_k} + (1-t)f_k^*,$$

where $F_t(\lambda) = tF_k(\lambda) + (1-t)\sigma_1(\lambda)$ and $f_k^* = \sigma_1 \left(\nabla^2 u_k + du_k \otimes du_k - \frac{|\nabla u_k|^2}{2}g_0 + S_{g_0} \right)$. Now by Proposition 1, we have for any $\rho \in C_0^2(B_1)$ with $0 \leq \rho \leq 1$, there exists C independent of k and u_k such that

$$(36) \quad \max_{B_1} \{ |\rho(x) \nabla v_k|^2(x) \} \leq C(1 + (\max_{B_1} \{ |\rho(x) f_k(x)| |\nabla u_k(x)| \}) + \max_{B_1} \{ \rho(x) |\nabla f_k(x)| \}) e^{-2u_k(x)}.$$

From our estimates, $\|v_k\|_{C^1(\bar{B}_1)} \leq C$, where C is a constant independent of k , since $\|u_k\|_{C^{2,\alpha}(\bar{B}_1)}$ and $\|f_k\|_{C^1(\bar{B}_1)}$ are uniformly bounded. By Theorem 4, $v_k \rightarrow v_0$ (after passing a subsequence) in $C^{2,\alpha}(B_1)$, v_k converges to $v_0 = u$ by the uniqueness. Therefore, there is a constant $C > 0$ depending only on λ_0, Λ_0 , and the geometry of B_1 such that

$$\max_{B_{\frac{1}{2}}} |\nabla u|^2(x) \leq C(1 + \|f\|_{C^1(B_1)} e^{-2 \inf_{B_1} u(x)}).$$

This finishes the proof of Theorem 5. ■

A direct consequence of Theorem 5 is the following

Corollary 4. *Let B_1 be a unit disk in a Riemannian manifold (M, g_0) and $p < n/2$. There exists a small constant $\varepsilon_0 > 0$ depending only on (B_1, g_0) such that for any sequence of solutions u_i of (2) in B_1 with*

$$\int_{B_1} e^{-nu_i} d\text{vol}(g_0) \leq \varepsilon_0,$$

either

- (1) *there is a subsequence u_{i_j} uniformly converging to $+\infty$ in any compact subset of B_1 , or*
- (2) *there is a subsequence u_{i_j} converging strongly in $C_{loc}^{2,\alpha}(B_1)$.*

Proof. It follows the same lines of proof in [36] or [15]. ■

4. NEGATIVE CURVATURE CASE

In this section, we discuss the negative curvature case, where the geometry is rich. By [11] ($n = 3$) and [27] (general dimension $n \geq 3$), every higher dimensional manifold has a metric with negative Ricci tensor. It is clear that such a metric also has negative W_p for any $1 \leq p \leq n - 1$. Hence every higher dimensional manifold has a metric with negative W_p . The

conformal deformation will yield interesting geometric information about the extremal metrics in a given conformal class.

Proof of Theorem 1. First we take the sequence of smooth F_k considered in the previous section. For each F_k we consider the following equation

$$(37) \quad F_k \left(e^{2u} (\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) \right) = -1.$$

For large k , from the condition of the Theorem, we have

$$F_k(S_{g_0})(x) < 0, \quad \forall x \in M.$$

We first prove the existence of solutions to equation (37). Here we use the method of continuity. Let us consider the following equation

$$(38) \quad F_t(u) := F_k \left(e^{2u} (\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) \right) + t - (1-t)F_k(S_{g_0}) = 0$$

and define $J = \{t \rightarrow [0, 1] \mid (38) \text{ has a solution for } t\}$. It is clear that $0 \in J$. First, we prove the openness of J . Let $t_0 \in J$. By the maximum principle, we know that there is only one solution u of (38) for $t = t_0$. Let L be its linearization. We want to show that L is invertible. By the maximum principle again, we know that the kernel of L is trivial. Note that L might be not self-adjoint. To show the invertibility of L , we need to show that the cokernel of L is also trivial. However, one can readily check that the Fredholm index of L is zero, and hence the cokernel of L is trivial. Now the openness follows from the implicit function theorem.

Then we show the closeness. Let x_0 and x_1 be the minimum and maximum of u respectively. By the maximum principle, we have

$$(39) \quad e^{2u(x_0)} \geq \frac{t - (1-t)F_k(S_{g_0})(x_0)}{-F_k(S_{g_0})(x_0)}$$

and

$$(40) \quad e^{2u(x_1)} \leq \frac{t - (1-t)F_k(S_{g_0})(x_1)}{-F_k(S_{g_0})(x_1)}.$$

Hence, we have C^0 bound of u independent of t .

By a global estimates proven in Proposition 2 below and Theorem 4, we have the closeness. Hence we have a solution u_k of (37) with the bound

$$(41) \quad \frac{1}{-\min F_k(S_{g_0})(x_0)} \leq e^{2u_k} \leq \frac{1}{-\max F_k(S_{g_0})(x_0)}.$$

In viewing of (41), we use again the global estimates and Theorem 4 to obtain a $C^{2,\alpha}$ uniform bound of u_k for some $\alpha > 0$. Hence u_k converges (by taking a subsequence) to u . It is clear that $e^{-2u}g_0$ satisfies (2). \blacksquare

Proposition 2. *Let u be a solution of*

$$(42) \quad F(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) = f e^{-2u}$$

with C^1 function f . Suppose that F is uniformly elliptic and is homogeneous of degree 1. Assume that u has C^0 bound. Then u has a C^1 bound and a $C^{2,\alpha}$ bound.

Proof. Since we already have full C^0 bound, the proposition can be proved using standard Pogorelov type of trick, for example, as in [19]. Let $v = e^u$ and consider the following equivalent form of (42)

$$(43) \quad F \left(\frac{\nabla^2 v}{v} - \frac{|\nabla v|^2}{v^2} g_0 + S_{g_0} \right) = f v^{-2}.$$

Without loss of generality, we may assume that $v \leq 1$. Set $H = e^{2\phi(v)} |\nabla v|^2$. Here ϕ will be fixed later. Let x_0 be a maximum point of H . At x_0 , we have

$$(44) \quad \sum_k (2v_k v_{ki} + 2\phi'(v) v_i |\nabla v|^2) = 0, \quad \text{for any } i.$$

Without loss of generality, we may assume that $v_1 = |\nabla v|$, $v_i = 0$ for any other i and that v_{ij} is diagonal at x_0 . Hence, (44) is equivalent to $v_{11} = -\phi'(v) |\nabla v|^2$. Set $w_{ij} = v_{ij} - \frac{|\nabla v|^2}{2v} \delta_{ij} + v S_{ij}$. The maximum principle, together with (44), implies

$$(45) \quad 0 \geq \sum F^{ij} (v_{kj} v_{ki} + v_k v_{kij} + \phi'(v) v_{ij} |\nabla v|^2 + v_i v_j \phi''(v) |\nabla v|^2 + 2\phi'(v) v_i v_{ki} v_k)$$

By (44), it is easy to check that

$$\begin{aligned} \sum F^{ij} v_{ki} v_{kj} &\geq F^{11} v_{11} v_{11} = F^{11} (\phi'(v))^2 \frac{|\nabla v|^4}{v}, \\ \sum F^{ij} v_k v_{kij} &= \sum F^{ij} v_k v_{ijk} + \sum F^{ij} v_k v_m R_{ijk}^m \\ &\geq \sum F^{ij} v_k (w_{ij} + \frac{|\nabla v|^2}{2v} \delta_{ij} + v S_{ij})_k - C \max v |\nabla v|^2 \\ &\geq 2v_k (f v^{-1})_k + \sum F^{ii} v_1 (-\frac{v_1^3}{2v^2} - \phi'(v) \frac{|\nabla v|^3}{v}) - C \max v |\nabla v|^2 \\ &\geq 2f_k v_k - 2v^{-1} |\nabla v|^2 - (\frac{1}{2v} + \phi'(v)) \sum F^{ii} \frac{|\nabla v|^4}{v} - C \max v |\nabla v|^2, \end{aligned}$$

$$\begin{aligned} \phi'(v) \sum F^{ij} v_{ij} |\nabla v|^2 &= \phi'(v) \sum F^{ij} (w_{ij} + \frac{|\nabla v|^2}{2v} \delta_{ij} - v S_{ij}) |\nabla v|^2 \\ &\geq \phi'(v) f v^{-1} |\nabla v|^2 + \phi'(v) \sum F^{ii} \frac{|\nabla v|^4}{2v} - C \max v |\nabla v|^2, \end{aligned}$$

$$\phi''(v) \sum F^{ij} v_i v_j |\nabla v|^2 = \phi''(v) F^{11} |\nabla v|^4,$$

$$2\phi'(v) \sum F^{ij} v_j v_k v_{ki} = 2\phi'(v) F^{11} v_1^2 v_{11} = -2(\phi'(v))^2 F^{11} |\nabla v|^4.$$

Here C is a positive constant depending only on the Riemannian curvature of the background metric and it varies from line to line. Therefore, we have

$$(46) \quad 0 \geq F^{11}(\phi''(v) - (\phi'(v))^2)|\nabla v|^4 + \frac{1}{2v} \sum F^{ii}(-\frac{1}{v} + \phi(v))|\nabla v|^4 - C \max v |\nabla v|^2 \\ + 2f_k v_k - 2v^{-1}|\nabla v|^2 + \phi'(v)fv^{-1}|\nabla v|^2.$$

Choose

$$\phi = -\frac{1}{2} \log t(ct - 2),$$

for a large constant $c > 0$ so that $c \min v > 3$. One can easily check that for any $t \in [\min v, \infty)$

$$\phi''(v) - (\phi'(v))^2 = \frac{1}{t^2(ct - 2)^2} > 0$$

and

$$-(\frac{1}{t} + \phi(t)) = \frac{1}{t(ct - 2)}.$$

In view of (46), we have

$$0 \geq \frac{1}{2(\max v)^2(c \max v - 2)} |\nabla v|^4 F^{ii} - c(f, C) |\nabla v|^2.$$

Now we have a global bound of $|\nabla v|$, which depend only on g_0 , f , $\min v$ and $\max v$. The $C^{2,\alpha}$ bound follows from Theorem 4. \blacksquare

Remark 1. *The condition $W_p(g_0)(x) < 0, \forall x \in M$ in Theorem 1 can be weaken to $W_p(g_0)(x) \leq 0, \forall x \in M$ and $W_p(g_0)(x_0) < 0$ for some $x_0 \in M$. In fact, under the weaker condition, one may produce a metric $g \in [g_0]$ with the stronger condition holds. This can be done using the short time existence of the fully nonlinear flow*

$$u_t = G_p(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})e^{2u}, \quad u|_{t=0} = 0.$$

The short time existence follows from standard nonlinear parabolic theory, and the strict negativity of $W_p(g)$ (which is equal to u_t) follows from the strong maximum principle.

Remark 2. *It is of interest to characterize when the condition in Theorem 1 is true by some conformal geometric quantities. The difficulty here is the lack of variational structure for this type of equations. We note that when $p > \frac{n}{2}$, if the Yamabe constant $Y([g_0])$ of (M, g_0) is non-positive, then the condition in Theorem 1 is satisfied unless (M, g_0) is conformally equivalent to a Ricci flat manifold. This simple observation follows from the solution of the Yamabe problem and the fact that if the scalar curvature vanishing identically, for $p > \frac{n}{2}$, $G_p \leq 0$ and G_p vanishes identically if and only if the metric is Ricci flat.*

Remark 3. *It is also an interesting problem to consider the equation $W_p(g) = -1$ on a complete, non-compact manifold. The arguments in the proof of the existence of Dirichlet problem (35) can be extended to deal with a given boundary condition at the infinity for equation $W_p(g) = -1$ on a complete non-compact negatively curved manifold.*

As a direct consequence of Theorem 1, we have

Corollary 5. *If there is $\tilde{g} \in [g_0]$ with $\mathcal{R}_{\tilde{g}}^s(x) < 0$ for all $x \in M$ then there is a unique $C^{2,\alpha}$ metric $g^* \in [g_0]$ for some $\alpha > 0$ such that $\mathcal{R}_{g^*}^s(x) = -1, \forall x \in M$. Similarly, If there is $\tilde{g} \in [g_0]$ with $\mathcal{R}_{\tilde{g}}^l(x) < 0$ for all $x \in M$, then there is a unique $g_* \in [g_0]$ and $g_* \in C^{2,\alpha}(M)$ for some $\alpha > 0$ such that $\mathcal{R}_{g_*}^l(x) = -1, \forall x \in M$.*

Corollary 5 can be applied to consider minimal volumes in conformal classes. Set

$$\mathcal{C}^- = \{g \in [g_0] | \mathcal{R}_g^s(x) \geq -1, \forall x \in M\},$$

$$\mathcal{C}_- = \{g \in [g_0] | \mathcal{R}_g^l(x) \geq -1, \forall x \in M\},$$

$$\tilde{\mathcal{C}}^- = \{g \in [g_0] | \mathcal{R}_g^s(x) \leq -1, \forall x \in M\},$$

$$\tilde{\mathcal{C}}_- = \{g \in [g_0] | \mathcal{R}_g^l(x) \leq -1, \forall x \in M\}.$$

Define

$$\begin{aligned} \mathcal{V}^s([g_0]) &= \inf_{g \in \mathcal{C}^-} \text{vol}(g), & \mathcal{V}^l([g_0]) &= \inf_{g \in \mathcal{C}_-} \text{vol}(g), \\ \mathbb{V}^s([g_0]) &= \sup_{g \in \tilde{\mathcal{C}}^-} \text{vol}(g), & \mathbb{V}^l([g_0]) &= \sup_{g \in \tilde{\mathcal{C}}_-} \text{vol}(g), \end{aligned}$$

where $\text{vol}(g)$ is the volume of g . It is clear that the definitions given here

Lemma 1. *Let $g_1, g, g_2 \in [g_0]$. If $\mathcal{R}_g^s(x) < 0$, and $\mathcal{R}_{g_1}^s(x) \leq \mathcal{R}_g^s(x) \leq \mathcal{R}_{g_2}^s(x), \forall x \in M$, then $\text{vol}(g_1) \leq \text{vol}(g) \leq \text{vol}(g_2)$, any one of the equalities holds if and only if the metric is equal to g . Similarly, if $\mathcal{R}_g^l(x) < 0$, and $\mathcal{R}_{g_1}^l(x) \leq \mathcal{R}_g^l(x) \leq \mathcal{R}_{g_2}^l(x), \forall x \in M$, then $\text{vol}(g_1) \leq \text{vol}(g) \leq \text{vol}(g_2)$, any one of the equalities holds if and only if the metric is equal to g .*

The Lemma is a simple consequence of the maximum principle applied to equations (7) or (8). From the lemma, we have the following relations

$$\mathcal{V}^s([g_0]) \geq \mathbb{V}^s([g_0]) \geq \mathcal{V}^l([g_0]) \geq \mathbb{V}^l([g_0]).$$

And we can show that the minimal volumes $\mathcal{V}^s([g_0])$ and $\mathcal{V}^l([g_0])$ are achieved.

Corollary 6. *Suppose that $\mathcal{R}_{g_0}^s(x) < 0$ for any $x \in M$. Then there is a unique conformal metric $g^* \in [g_0]$ such that $\text{vol}(g^*) = \mathcal{V}^s([g_0])$ with $\mathcal{R}_{g^*}^s(x) = -1, \forall x \in M$ and*

$$\mathbb{V}^s([g_0]) = \mathcal{V}^s([g_0]) \geq \mathcal{V}^l([g_0]).$$

The equality holds if and only if there is an Einstein metric in $[g_0]$. If $\mathcal{R}_{g_0}^l(x) < 0$ for any $x \in M$, then there is a unique $g^ \in [g_0]$ such that $\text{vol}(g^*) = \mathcal{V}^l([g_0])$ with $\mathcal{R}_{g^*}^l(x) = -1, \forall x \in M$. In this case, we have*

$$\mathcal{V}^s([g_0]) = \mathbb{V}^s([g_0]) \geq \mathcal{V}^l([g_0]) = \mathbb{V}^l([g_0]).$$

For the study of minimal volumes in general Riemannian manifolds, we refer to [12], [4] and [23].

5. POSITIVE CURVATURE CASE

Now we consider conformal classes with metrics of positive Ricci curvature. Let $[g_0]$ be such a conformal class. After a suitable scaling, we may assume that $Ric_{g_0} \geq (n-1)g_0$. Define $[g_0]_+ = \{g \in [g_0] \mid Ric_g \geq (n-1)g\}$ and $V_{\max}(M, [g_0]) = \sup_{g \in [g_0]_+} vol_g(M)$. This definition is motivated by Gursky and Viaclovsky [18]. From the Bishop comparison, we know

$$(47) \quad V_{\max}(M, [g_0]) \leq vol(\mathbb{S}^n),$$

the volume of the unit sphere.

Proposition 3. *Let (M, g_0) be a compact Riemannian manifold with $Ric_{g_0} \geq (n-1)g_0$. If $V_{\max}(M, [g_0]) < vol(\mathbb{S}^n)$, then there is a conformal metric $g \in [g_0]_+$ with*

$$\mathcal{R}_g^s = n - 1.$$

Proof. Consider the sequence of approximating function F_k as in Section 2 with a normalization condition that $F_k(1, 1, \cdot, 1) = n - 1$. We first want to find a solution to the following equation

$$(48) \quad F_k \left(e^{2u} (\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) \right) = n - 1,$$

for large k . Define

$$V_{\max}^k([g_0]) = \max\{vol(g) \mid g \in [g_0] \text{ with } F_k(g^{-1} \cdot S_g) \geq n - 1\}.$$

It is easy to check that $\lim_{k \rightarrow \infty} V_{\max}^k([g_0]) = V_{\max}([g_0])$. Hence for large k we have

$$(49) \quad V_{\max}^k([g_0]) < vol(\mathbb{S}^n).$$

To show the existence of solution of (48), we consider a deformation, which is similar to a deformation considered by Gursky and Viaclovsky in their study of σ_k -Yamabe problem.

$$(50) \quad \begin{aligned} & F_k \left(\nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g + \psi(t) S_g + (1 - \psi) g \right) \\ &= (n-1)(1-t) \left(\frac{1}{vol(g_0)} \int_M e^{-(n+1)u} \right)^{\frac{2}{n+1}} + (n-1)\psi(t)e^{-2u}, \end{aligned}$$

where $\psi(t) : [0, 1] \rightarrow [0, 1]$ is a C^1 function satisfying $\psi(0) = 0$ and $\psi(t) = 1$ for $t \geq 1/2$. We now prove that there is a solution of (50) when $t = 1$, provided that $V_{\max}^k(M, [g]) < vol(\mathbb{S}^n)$.

When $t = 0$, it is easy to check that (50) has a unique solution $u = 0$ and its corresponding linearization has no nontrivial kernel. Hence its Leray-Schauder degree is non-zero. If the solution space of (50) for any $t \in [0, 1]$ is compact, then using degree theoretic argument, we are done. Assume by contradiction that there is no compactness. Assume without loss of generality that there is a sequence of solutions $g_i = e^{-2u_i} g$ of (1) with $t = 1$ such that u_i does not converge in $C^{2,\alpha}$. In view of Theorem 5 and Corollary 2, we have either

- (a) $\inf_M u_i \rightarrow -\infty$, or
- (b) $\inf_M u_i \rightarrow +\infty$.

The latter is easy to be excluded as follows. At the minimum point of u_i , we have

$$e^{2\inf u_i} F_k(S_{g_0}) = F_k(e^{2\inf u_i} S_0) \leq (n-1),$$

which certainly implies that $\inf u_i$ is bounded from above uniformly. Hence we are left to exclude (a). Let x_i be the minimum point of u_i and assume that $x_i \rightarrow x_0$ as $i \rightarrow \infty$. Consider a scaled function

$$\tilde{u}_i = u(\exp_{x_i} \varepsilon_i x) - \log \varepsilon_i,$$

where $\varepsilon_i = \exp u_i(x_i)$. It is clear that $\tilde{u}_i \geq 0$ satisfies a similar equation on $B(0, \varepsilon_i^{-1} r_0/2) \subset \mathbb{R}^n$ with a scaled metric, where r_0 is the injectivity radius of (M, g) . The set $B(0, \varepsilon_i^{-1} r_0/2)$ with the scaled metric converges to \mathbb{R}^n . By local estimates and local C^2 estimates in [14] for a more general concave case one can show that \tilde{u}_i converges to an entire solution v of

$$(51) \quad F_k(\nabla^2 v + dv \otimes dv - \frac{1}{2} |\nabla v|^2 g_{\mathbb{R}^n}) = (n-1)e^{-2v}$$

and

$$(52) \quad V_{\max}^k(g) \geq \liminf_i \text{vol}(e^{-2u_i} g) \geq \text{vol}(\mathbb{R}^n, e^{-2v} g_{\mathbb{R}^n}),$$

where $g_{\mathbb{R}^n}$ is the standard Euclidean metric. By a classification result of Li-Li in [26], we know that $(\mathbb{R}^n, e^{-2v} g_{\mathbb{R}^n})$ is equivalent to \mathbb{S}^n . Hence $V_{\max}^k([g_0]) \geq \text{vol}(\mathbb{R}^n, e^{-2v} g_{\mathbb{R}^n}) = \text{vol}(\mathbb{S}^n)$. This contradicts (49). Therefore, we have a solution u_k of (50) for large k .

Now we consider the sequence $\{u_k\}$. As above, we can show first that u_k has a uniform upper bound. If u_k has a uniform lower bound, Corollary 2 implies that the sequence $\{u_k\}$ has a uniform $C^{2,\alpha}$ bound. And hence we have a limit u_0 which is a solution we desire. Hence to prove the Proposition, we only need to exclude the case that $\min u_k \rightarrow -\infty$. Assume that we are in this case. By a similar argument presented above, after considering a suitable rescaling we have a limit $C^{2,\alpha}$ function v_∞ satisfying

$$(53) \quad G_{n-1}(\nabla^2 v + dv \otimes dv - \frac{1}{2} |\nabla v|^2 g_{\mathbb{R}^n}) = (n-1)e^{-2v}$$

and

$$(54) \quad V_{\max}^k([g_0]) \geq \liminf_i \text{vol}(e^{-2u_i} g) \geq \text{vol}(\mathbb{R}^n, e^{-2v_\infty} g_{\mathbb{R}^n}).$$

The contradiction follows from the following Lemma. We finish the proof of the Proposition. \blacksquare

Proposition 4. *Let $p < n/2$ and $g = e^{-2u} g_{\mathbb{R}^n}$ be a C^2 function on \mathbb{R}^n such that*

$$G_p(g^{-1} S_g)(x) = c, \forall x \in \mathbb{R}^n,$$

for some constant c . Then $u = 0$ if $c \leq 0$ and $u(x) = \log \frac{\lambda^2 + |x-x_0|^2}{2\lambda\sqrt{\frac{(n-p)p}{c}}}$ if $c > 0$. That is, $(\mathbb{R}^n, e^{-2u} g_{\mathbb{R}^n})$ can be compactified as a standard sphere if $c > 0$.

Proof. The proof follows [26] closely. The only difference is that the operator there is required to be C^1 . Here our operator G_p is Lipschitz only. However G_p is uniformly elliptic and concave. We will show in Lemma 2 below that the Hopf lemma holds for our equation. Then the argument in [26] can be applied to our equation. \blacksquare

Lemma 2. *Let Ω be a bounded domain in \mathbb{R}^n and $p < n/2$. If (9) has two solutions w and v with $w \geq v$ and $w(x_0) = v(x_0)$ for some $x_0 \in \partial\Omega$, then $x_0 \in \partial\Omega$. Furthermore,*

$$\frac{\partial w}{\partial \nu}(x_0) < \frac{\partial v}{\partial \nu}(x_0),$$

unless $w = v$. Here ν is the outer normal of $\partial\Omega$ at x_0 .

Proof. For any function, set

$$A^u = \nabla u + du \otimes ds - \frac{1}{2}|\nabla u|^2 g_0 + S_{g_0}.$$

Since G_p is concave and homogeneous one, we have

$$G_p(A^w - A^v) \leq G_p(A^w) - G_p(A^v) = e^{-2w} - e^{-2v} \leq 0.$$

Let $\tilde{w} = w - v$. Now we can write $G_p(A^w - A^v)$ as follows

$$G_p(A^w - A^v) = a_{ij}(x)\tilde{w}_{ij} + b_i(x)\tilde{w}_i$$

with $\lambda_0 Id \leq (a_{ij}(x)) \leq \Lambda_0 Id$ and $b_i(x)$ bounded for any i . Therefore, we can apply Theorem 5 on page 61 and Theorem 7 on page 65 in [32] to prove the Lemma. \blacksquare

Remark 4. *Proposition 4 does not hold for $p = n - 1$. For example $u = kx_1$ for any $k > 0$ is a solution of*

$$F(u) = G_{n-1}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}) = 0.$$

The same example indicates that without the concavity of F , Proposition 1 and Theorem 5 are not true. It is easy to check that $F(u_k) = 0$. On a domain $\Omega \subset \{x_1 \geq 0\}$, we have $u_k \geq 0$. But $|\nabla u_k| = k \rightarrow \infty$.

Proposition 5. *Let (M, g_0) be a compact Riemannian manifold with $Ric_{g_0} \geq (n - 1)g_0$. If $V_{\max}(M, [g_0]) = vol(\mathbb{S}^n)$, then (M, g) is conformally equivalent to the standard unit sphere.*

This Proposition is a direct consequence of the following

Proposition 6. *Let (M, g) be a compact Riemannian manifold with $Ric_g \geq n - 1$. If $vol(M)$ is close to ω_n , the volume of \mathbb{S}^n , then, the Yamabe constant of $(M, [g])$, $Y(M, [g])$ is close to $n(n - 1)\omega_n^{2/n}$, the Yamabe constant of \mathbb{S}^n .*

Proof. Let us first recall the well-known Yamabe constant of (M, g) , which is defined by

$$Y(M, [g]) := \inf \left(\int v^{\frac{2n}{n-2}} dvol(g) \right)^{-\frac{n-2}{n}} \left\{ 4\frac{n-1}{n-2} \int |\nabla v|^2 dvol(g) + \int R_g v^2 dvol(g) \right\}$$

By a result of Ilias [20], which is based on a result of Gromov (see also [3]), we have

$$n(n-1)\omega_n^{2/n} \left(\int v^{\frac{2n}{n-2}} \right)^{(n-2)/2} \leq \left(\frac{\omega_n}{vol(g)} \right)^{2/n} \left\{ 4\frac{n-1}{n-2} \int |\nabla v|^2 + n(n-1) \int v^2 \right\},$$

for any $v \in H_1^2(M)$. Note that $R_g \geq n(n-1)$. Therefore, we have

$$Y(M, [g]) \geq \left(\frac{\omega_n}{vol(g)} \right)^{-2/n} n(n-1)\omega_n^{2/n} \geq n(n-1)\omega_n^{2/n} - \delta,$$

for any small $\delta > 0$, provided that $\text{vol}(g)$ is close to ω_n . \blacksquare

By the results of Colding (see [8],[9] and [31]), we know that the condition in Proposition 6 is equivalent to the one of following three other conditions:

- 1) $\text{rad}M$ is close to π ,
- 2) M is Gromov-Hausdorff close to \mathbb{S}^n ,
- 3) the $(n+1)$ th eigenvalue of the Laplacian, $\lambda_{n+1}(M)$, is close to n .

Proof of Proposition 5. $V_{\max}(M, [g_0]) = \text{vol}(\mathbb{S}^n)$ implies by definition that there is a sequence $g_i \in [g_0]_+$ with $\lim_{i \rightarrow \infty} \text{vol}(g_i) = \text{vol}(\mathbb{S}^n)$. Proposition 6 implies that $Y(M, [g_0]) = Y(M, [g_i]) \rightarrow n(n-1)\omega_n^{2/n}$, the Yamabe constant of \mathbb{S}^n . Hence, the Yamabe constant of $(M, [g_0])$ equals to the Yamabe constant of the standard sphere. By the resolution of the Yamabe problem by Aubin [1] and Schoen [37], $(M, [g_0])$ is conformally equivalent to the standard sphere. \blacksquare

Proof of Theorem 2. Theorem 2 follows from Propositions 5 and 3. \blacksquare

Remark 5. *It is interesting to know whether V_{\max} is achieved as in the negative case. One can show that if V_{\max} is achieved by \tilde{g} , then $\mathcal{R}_{\tilde{g}}^s$ is constant.*

Now we prove Theorem 3.

Proof of Theorem 3. It follows the exact same arguments in the proof of Theorem 3 in [14], since that proof works for general uniform elliptic concave equations as well, as we note that $W_p(g) > 0$ implies the positivity of the mean curvature when $p \leq \frac{n}{2}$. We only give a sketch here.

Step 1. We define a deformation

$$(55) \quad f_t(g) := tW_p(g) + (1-t)R_g = 1,$$

where $g = e^{-2u}g_0$. Equation (55) ($\forall t \in [0, 1]$) is still uniformly elliptic and concave.

Step 2. (Harnack inequality) There is a constant $C > 0$ such that for a solution u of (55) in B_{3R} we have

$$(56) \quad \min_{B_R} u + \max_{B_{2R}} u \geq 2 \log R - \log C.$$

Here B_R is the ball of radius R in \mathbb{R}^n . (56) can be proved as in [14] using the method of moving planes.

By scaling argument, we may assume that $R = 1$. Assume by contradiction that (56) is not true. Then there exists a sequence of solutions of (55) in B_3 such that

$$(57) \quad \min_{B_1} u_i + \max_{B_2} u_i < -i.$$

Let $m_i = \min_{B_1} u_i = u_i(\bar{x}_i)$, and let $x_i \in B_1$ with $\bar{B}_{r_i}(x_i) \subset \bar{B}_1$ and $|x_i - \bar{x}_i| = r_i$. Here $r_i = e^{m_i}$. In view of (57), we know that $r_i \rightarrow 0$ as $i \rightarrow \infty$. Consider a new sequence of functions v_i defined by

$$v_i(x) = u_i(x_i + r_i x) - m_i$$

and set $\bar{x}_i = x_i + r_i \bar{y}_i$. It is clear that v_i satisfies (55) in $\{|x| < r_i^{-1}\}$ and $|\bar{y}_i| = 1$. From (57), we extend v_i by the Kelvin transformation to the whole Euclidean space as in (2.8) of [14]. Now applying the method of moving planes as in [14], which in turn follows closely from [7], we can show that v_i converges to 0 in $B_{\frac{1}{2}}(\bar{y}_i)$. This is a contradiction. Note that though we are dealing with the Lipschitz operators, the method of moving planes works by using the fact that f_t in (55) is uniformly elliptic and concave.

Step 3. Consider a solution u of equation (55). First, it is clear that we have that the scalar curvature of $g = e^{-2u}g_0$ is positive. Hence we can apply the result of Schoen-Yau in [38] to embed the universal cover \widetilde{M} of (M, g) into \mathbb{S}^n by a map Φ conformally. Therefore we can use the method of moving planes (again make use of uniform ellipticity and concavity of f_t) to obtain as in [14] that

$$|\nabla u|(x) \leq C, \quad \text{for any } x \in M,$$

for some constant independent of u , provided that (M, g_0) is not equivalent to \mathbb{S}^n . It follows that

$$(58) \quad \max u - \min u \leq C,$$

for some constant independent of u . (58), together with the Harnack inequality (56), implies that

$$\min u \geq C,$$

for some constant independent of u . Hence by Theorem 4, we know that the solution space of equation (55) is compact.

Step 4. From Step 3 we can apply the degree theory. We may use a result of Li in [25], a variation of the original Leray-Schauder theorem [24]. We also refer to Nirenberg's lecture notes [29] on the exposition of the degree theory in nonlinear differential equations. When $t = 0$, the topological degree for equation (55) is -1 , which was proved by Schoen [35]. Since the solution space is compact, the topological degree for equation (55) with $t = 1$ is also -1 . This finishes the proof of the Theorem. ■

Remark 6. *As in Remark 1, the conditions in Theorem 2 and Theorem 3 can be weakened to the assumption that the corresponding curvature of the background metric is nonnegative and positive at some point. The same argument using the short time existence of the corresponding curvature flows as in Remark 1 can produce a metric $g \in [g_0]$ with the positive curvatures.*

6. APPENDIX

In this Appendix, we assume that (M, g) is a compact locally conformally flat manifold. Recall the Weitzenböck formula for p -forms ω

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{R}\omega,$$

where

$$\mathcal{R}\omega = \sum_{j,l=1}^n \omega_j \wedge i(e_l)R(e_j, e_l)\omega.$$

Here e_j is a local basis and $i(\cdot)$ denotes the interior product $\Delta = dd^* + d^*d$ is the Hodge-de Rham Laplacian and $\nabla^*\nabla$ is the (positive) Laplacian. In local coordinates, let $\omega = \omega_1 \wedge \cdots \wedge \omega_p$. Since (M, g) is locally conformally flat, we have

$$(59) \quad \mathcal{R}\omega = \left((n-p) \sum_{i=1}^p \lambda_i + p \sum_{i=p+1}^n \lambda_i \right) \omega,$$

where λ 's are eigenvalues of the Schouten tensor S_g . See for instance [28] or [14]. By the Bochner technique, we have

Proposition 7. *Let (M, g) is a compact locally conformally flat manifold and $1 \leq p \leq n/2$. If its p -Weitzenböck curvature $W_p(g)$ is positive, then the Betti number $b_q = 0$ for $p \leq q \leq n-p$. If its p -Weitzenböck curvature $W_p(g)$ is non-negative and $b_p \neq 0$, then M is a quotient of $H^p \times \mathbb{S}^{n-p}$ if $p > 1$ and a quotient of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ if $p = 1$, where H^p is the hyperbolic space of curvature -1 and \mathbb{S}^{n-p} is the standard sphere of curvature 1.*

For the vanishing of cohomology groups under various conditions, see [28] and [14] and references therein.

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