

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Analytic Aspects of the Toda System: II.
Bubbling behavior and existence of solutions

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Preprint no.: 45

2005



ANALYTIC ASPECTS OF THE TODA SYSTEM: II. BUBBLING BEHAVIOR AND EXISTENCE OF SOLUTIONS

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1. INTRODUCTION

In this paper, we continue to consider the 2-dimensional (open) Toda system (Toda lattice) for $SU(N + 1)$

$$(1.1) \quad -\Delta u_i = \sum_{j=1}^N a_{ij} e^{u_j},$$

for $i = 1, 2, \dots, N$, where $K = (a_{ij})_{N \times N}$ is the Cartan matrix for $SU(N + 1)$ given by

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

As a very natural generalization of the Liouville equation

$$(1.2) \quad -\Delta u = 2e^u,$$

system (1.1) is completely integrable, which is well-known in integrable systems theory. The Liouville equation and the Toda system arise in many physical models. In Chern-Simons theories, the Liouville equation is closely related to Abelian models, while the Toda system is related to non-Abelian models. See for instance the books [10] and [24] and the references therein.

Though the Liouville equation has been extensively studied after Liouville [16], the precise behavior of convergence of its solutions has been rather well understood only in the last two decades, see for example [3, 4, 5, 6, 7, 8, 14, 15, 17]. Such a delicate study leads to many applications in the Abelian Chern-Simons theories and mean field equations.

To well understand the non-Abelian Chern-Simons model, we have to study the analytic aspects of the Toda system. It is natural to ask if we

can generalize delicate analytic results for the Liouville equation to the Toda system. However, the analysis of the Toda system becomes more difficult, because the basic analytic tool, the maximum principle, does not work. In [11], we established a Moser-Trudinger type inequality for the Toda system, while a rough bubbling behavior of solutions to system (1.1) was considered. See also [13]. Its bubbles -entire solutions- were classified in [12]. In this paper, we will prove existence results for solutions to the Toda system in various cases by using methods developed in [8, 17, 6, 20, 19, 14]. And we will give a much more precise bubbling behavior of solutions, which will be very useful in our further study of the Toda system.

Let Σ be a Riemann surface with Gaussian curvature K . We consider the following system

$$(1.3) \quad -\Delta u_i = \sum_{j=1}^N \rho_j a_{ij} \left(\frac{h_j e^{u_j}}{\int_{\Sigma} h_j e^{u_j}} - 1 \right), \quad \text{in } \Sigma \quad 1 \leq i \leq N,$$

for the coefficient matrix $A = (a_{ij})_{N \times N}$, the Cartan matrix of $SU(N+1)$ and $\rho = (\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i > 0$ ($i = 1, 2, \dots, N$) given constants. Here $h_i : \Sigma \rightarrow \mathbb{R}$ is a given positive C^1 function for $i = 1, 2, \dots, N$. System (1.3) is the Euler-Lagrange system of the functional

$$(1.4) \quad J_{\rho}(u) = \frac{1}{2} \sum_{i,j=1}^N \int a^{ij} \nabla u_i \nabla u_j + \sum_{j=1}^N \int \rho_j u_j - \sum_{j=1}^N \rho_j \log \int_{\Sigma} h_j e^{u_j},$$

in $H := (H^1(\Sigma))^N$ where (a^{ij}) is the inverse matrix of A . In [11], we proved that J_{ρ} has a lower bound in H if and only if $\rho_i \leq 4\pi$ for any i , which is the Moser-Trudinger inequality for the Toda system. From this inequality, we know that if $\rho_i < 4\pi$ ($\forall j$) then J_{ρ} iscoercive condition and hence J_{ρ} has a minimizer, which certainly satisfies system (1.3). When one of the ρ_i 's equals 4π , the existence problem becomes subtler. For simplicity of notation, we consider only the case $N = 2$.

Our first result in this case is

Theorem 1.1. *Let Σ be a Riemann surface with Gaussian curvature K and $N = 2$. And let $\rho_1 = 4\pi$ and $\rho_2 \in (0, 4\pi)$. Suppose that*

$$(1.5) \quad \Delta \log h_1(x) + (8\pi - \rho_2) - 2K(x) > 0 \quad \text{for } x \in \Sigma.$$

Then J_ρ has a minimizer $u = (u_1, u_2)$ satisfying

$$(1.6) \quad \begin{aligned} -\Delta u_1 &= 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Omega h_2 e^{u_2}} - 1 \right) \\ -\Delta u_2 &= 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}} - 1 \right), \end{aligned}$$

for $\rho_1 = 4\pi$ and $\rho_2 \in (0, 4\pi)$.

Theorem 1.1 is obtained by using a result in [6], which is a refined result of [8] and [17]. For $\rho_1 = \rho_2 = 4\pi$, we also have a sufficient condition under which (1.6) has a solution. See Theorem 5.2 below.

For the general case, we have the following compactness of the solution space.

Theorem 1.2. *For any compact set $\Lambda_1 \times \Lambda_2 \subset \mathbb{R}_+ \times \mathbb{R}_+$, if there are two positive integers m_1 and m_2 such that $\Lambda_i \subset (4\pi m_i, 4\pi(m_i + 1))$ for $i = 1, 2$, then the solution space of (1.6) for $(\rho_1, \rho_2) \in \Lambda_1 \times \Lambda_2$ is compact.*

The proof of Theorem 1.2 follows from the study of the convergence of the sequence of solutions, which was initiated in [11] for the Toda system. Let x be a blow-up point of the sequence u^k , i.e., there exists a sequence $x_k \rightarrow x$ such that $\max\{u_1^k(x_k), u_2^k(x_k)\} \rightarrow \infty$ as $k \rightarrow \infty$. Define

$$\sigma_i = \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \int_{B_r} e^{u_i^k}.$$

We call (σ_1, σ_2) a blow-up value at x . First it is easy to check that $\sigma_1 + \sigma_2 > 0$. A local Pohozaev argument gives us a relation between σ_1 and σ_2 .

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = 4\pi(\sigma_1 + \sigma_2).$$

See [13] and [11] for the proof. In Proposition 2.5 below, we show that (ρ_1, ρ_2) can only be one of $(4\pi, 0)$, $(0, 4\pi)$, $(4\pi, 8\pi)$, $(8\pi, 4\pi)$ and $(8\pi, 8\pi)$. It is clear that Theorem 1.2 is a direct consequence of Proposition 2.5. In the first two cases, one of the u_i^k does not blow-up, another bubbles more or less like solutions of the Liouville equation, which has been extensively studied in the last decade. The last case contains a typical blow-up phenomenon of solutions of the Toda system. It is one of our main results to obtain a precise description of the bubbling behavior for this case. To describe it, we assume that there is a sequence of solutions $u^k = (u_1^k, u_2^k)$ of

$$(1.7) \quad \begin{cases} -\Delta u_1^k &= 2h_1^k e^{u_1^k} - h_2^k e^{u_2^k} \\ -\Delta u_2^k &= 2h_2^k e^{u_2^k} - h_1^k e^{u_1^k} \end{cases} \quad \text{in } B_2.$$

Here B_r denotes a disk of radius r and center 0 and h_i^k converges in C^1 to a positive C^1 function h_i for $i = 1, 2$. Assume without loss of generality that $h_1(0) = h_2(0) = 1$. Suppose that u^k bubbles off, i.e., $\max_{x \in B_2} \{u_1^k, u_2^k\} \rightarrow \infty$ as $k \rightarrow \infty$. More precisely we assume that

- (1) 0 is the only blow-up point of u^k .
- (2) $\max_{\partial B_2} u_i^k - \min_{\partial B_2} u_i^k \leq c$ for $i = 1, 2$.
- (3) $\int_{B_2} e^{u_i^k} dx \leq c$ for $i = 1, 2$ and any k .

Assume that $\lambda^k = \lambda_1^k := \max_{B_2} u_1^k \geq \max_{B_2} u_2^k =: \lambda_2^k$. Let x_k be the maximum point of u_1^k . Set $v^k = (v_1^k, v_2^k)$ by

$$v_i^k(x) = u_i(\epsilon_k x + x^k) - \lambda^k \quad \text{for } i = 1, 2,$$

where $\epsilon_k = e^{-\frac{1}{2}\lambda^k}$. Clearly v^k satisfies in $\{x \in \mathbb{R}^2 \mid \epsilon_k x + x^k \in B_2\}$

$$\begin{cases} -\Delta v_1^k = 2h_1^k(\epsilon_k x + x^k)e^{v_1^k} - h_2^k(\epsilon_k x + x^k)e^{v_2^k}, \\ -\Delta v_2^k = 2h_2^k(\epsilon_k x + x^k)e^{v_2^k} - h_1^k(\epsilon_k x + x^k)e^{v_1^k}, \end{cases}$$

and $v_i^k \leq 0, v_1^k(0) = 0$. Since we only consider the case that the bubble is a solution of the Toda system, we may further assume that

- (4) $v_2^k(0)$ is bounded from below.

Then, there exists a solution $v^0 = (v_1^0, v_2^0)$ of the Toda system

$$\begin{cases} -\Delta v_1^0 = 2e^{v_1^0} - e^{v_2^0}, \\ -\Delta v_2^0 = 2e^{v_2^0} - e^{v_1^0}, \end{cases}$$

such that $v_i^k - v_i^0$ converges to zero in $C_{loc}^2(\mathbb{R}^2)$.

From the classification result for entire solutions of the Toda system [12], which is a generalization of the classification result for the entire solutions of the Liouville equation obtained by Chen and Li [5], v^0 is obtained from a rational curve from S^2 to $\mathbb{C}P^2$. In particular, we have

$$(1.8) \quad \int_{\mathbb{R}^2} e^{v_1^0} = \int_{\mathbb{R}^2} e^{v_2^0} = 8\pi.$$

Now we state our main theorem.

Theorem 1.3. *Let $u^k = (u_1^k, u_2^k)$ be a sequence of solutions to (1.7). Suppose that h_i^k converges in C^1 to a positive function h_i with $h_i(0) = 1$ for $i = 1, 2$ and that (1)-(4) hold. Then there exist two constants $r_0 > 0$ and $c > 0$ independent of k , such that*

$$(1.9) \quad |u_i^k(x) - \lambda^k - v_i^0(\epsilon_k^{-1}(x - x^k))| < c \quad \text{in } B_{r_0}$$

for $i = 1, 2$.

Theorem 1.3 gives a precise asymptotic behavior of a blow-up sequence of solutions. When $N = 1$, Theorem 1.3 was proved by Y.Y Li [14] by using the method of moving planes and a symmetry of the entire solutions of the Liouville equation. As mentioned above, the method of moving planes, which is based on the maximum principle, does not work for the Toda system. And also, the symmetry used in [14] is not available for the entire solutions of the Toda system in general. Thus the method in [14] could not provide a proof for the Toda system. Instead, here, together with the geometry related to the Toda system, we will employ a more delicate analysis. This is a combination of methods given in [2] and [12].

Our last result is an existence result for a supercritical case with respect to the Moser-Trudinger inequality established in [12].

Theorem 1.4. *Let Σ be a compact Riemann surface of genus greater than 0. Then for any $\rho = (\rho_1, \rho_2)$ with $\rho_i \in (0, 4\pi) \cup (4\pi, 8\pi)$ system (1.6) has a solution.*

The method to prove Theorem 1.4 follows [20]. See also [21, 19, 8] and also [13]. One can also prove the Theorem by computing a topological degree of system (1.6). In fact, the results -especially Theorem 1.3- established in this paper will be used in computing the topological degree for the Toda system. We will pursue this subject in forthcoming papers.

In Section 2, we first recall basic facts about the convergence of solutions to the Toda system and we then show that the possible blow-up values of the Toda system are isolated. We recall the geometric interpretations of solutions to the Toda system in Section 3. Such a solution corresponds to a flat connection and its singularity to the holonomy of its corresponding flat connection. We will give a more precise bubbling behavior, Theorem 1.3, in Section 4. The proof is a fine combination of arguments presented in [2] and [12]. In Section 5, we prove Theorem 1.1 by applying a result in [6]. In Section 6, we give an existence result for (1.1) with the Dirichlet boundary for a supercritical case, based on a method of Struwe [20] and the Moser-Trudinger inequality established in [11]. We give an asymptotic behavior of singularities of solutions in Appendix.

Acknowledgement. A part of the paper was carried out while the third author was visiting the Center of Theoretical Science in Taiwan (CTS). He would like to thank the CTS for warm hospitality.

2. BUBBLING BEHAVIORS

The convergence of solutions of Toda system was studied in [11]. Here for the convenience of the reader, we first recall some basic facts.

Proposition 2.1. *Let Ω be a bounded smooth domain in \mathbb{R}^2 and $u^k = (u_1^k, u_2^k)$ be a sequence of solutions of the following system*

$$(2.1) \quad \begin{cases} -\Delta u_1^k &= 2h_1^k e^{u_1^k} - h_2^k e^{u_2^k}, & \text{on } \Omega, \\ -\Delta u_2^k &= 2h_2^k e^{u_2^k} - h_1^k e^{u_1^k}, & \text{on } \Omega, \end{cases}$$

with

$$(2.2) \quad \int_{\Omega} e^{u_1^k} < C, \text{ and } \int_{\Omega} e^{u_2^k} < C.$$

Set

$$(2.3) \quad S_j = \{x \in \Sigma \mid \exists \text{ a sequence } y^k \rightarrow x \text{ such that } u_j^k(y^k) \rightarrow +\infty\}.$$

Then, one of the following possibilities happens: (after taking subsequences)

- (1) u_i^k is bounded in $L_{loc}^{\infty}(\Omega) \times L_{loc}^{\infty}(\Omega)$.
- (2) For some $j \in \{1, 2\}$, u_i^k in $L_{loc}^{\infty}(\Omega)$, but $u_j^k \rightarrow -\infty$ uniformly on any compact subset of Ω for $j \neq i$.
- (3) For some $i \in \{1, 2\}$, $S_i \neq \emptyset$, but $S_j = \emptyset$, for $j \neq i$. In this case, $u_i^k \rightarrow -\infty$ on any compact subset of $\Omega \setminus S_i$, and either, u_j^k is bounded in $L_{loc}^{\infty}(\Omega)$, or $u_j^k \rightarrow -\infty$ on any compact subset of Ω .
- (4) $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Moreover, u_j^k is either bounded or $\rightarrow -\infty$ on any compact subset of $\Omega \setminus (S_1 \cup S_2)$ for $j = 1, 2$.

Proof. The proof is given in [11]. The idea follows closely [3]. \square

Remark 2.2. *One can prove that when $S_1 \neq \emptyset$, $S_2 \neq \emptyset$ and $S_1 \setminus S_2 \neq \emptyset$, $u_1^k \rightarrow -\infty$ uniformly in any compact set of $\Omega \setminus \{S_1 \cup S_2\}$. See [3, 11]. However, the proof of Proposition 2.5 below implies that when $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, both $u_1^k \rightarrow -\infty$ uniformly in any compact set of $\Omega \setminus \{S_1 \cup S_2\}$. This is an improvement of (4).*

In this paper, we do not distinguish convergence and subconvergence. We may assume that there exist two nonnegative bounded measures μ_1 and μ_2 such that

$$e^{u_i^k} \psi \rightarrow \int \psi d\mu_i \text{ as } k \rightarrow \infty,$$

for every smooth function ψ with support in Ω and $i = 1, 2$. A point $x \in \Omega$ is called a γ -regular point with respect to μ_j if there is a function

$\psi \in C_c(\Omega)$, $0 \leq \psi \leq 1$, with $\psi = 1$ in a neighborhood of x such that

$$\int_{\Omega} \psi d\mu_j < \gamma.$$

We define

$$\Omega_j(\gamma) = \{x \in \Omega \mid x \text{ is not a } \gamma\text{-regular point with respect to } \mu_j\}.$$

One can show $\Omega_1(\gamma)$ and $\Omega_2(\gamma)$ are finite. And $\Omega_j(\gamma)$ is independent of γ for small $\gamma < 2\pi$, see [11]. Furthermore, we have

$$(2.4) \quad S_i = \Omega_i(\gamma), \quad \text{for } \gamma < 2\pi.$$

Proposition 2.1 implies that the blow-up points are isolated. Let 0 be an isolated blow-up point of the sequence u^k , i.e., there exists a sequence $x_k \rightarrow 0$ such that $\max\{u_1^k(x_k), u_2^k(x_k)\} \rightarrow \infty$ as $k \rightarrow \infty$. Define

$$\sigma_i = \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \int_{B_r} e^{u_i^k}.$$

We call (σ_1, σ_2) a blow-up value. If one of σ_1 and σ_2 is zero, then another is 4π . This can also be obtained from the following local Pohozaev identity.

Lemma 2.3. *We have*

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = 4\pi(\sigma_1 + \sigma_2)$$

Proof. See the proof in [11] and [13]. □

Hence we assume that $\sigma_i > 0$ for $i = 1, 2$. From (2.4) we know that $\sigma_i \geq 2\pi$ for $i = 1, 2$. This can be improved in the following

Lemma 2.4. *We have $\sigma_i \geq 4\pi$ for $i = 1, 2$.*

Proof. A proof of this Lemma is given in [13]. Here for convenience of the reader, we give a proof. Assume by contradiction that $\sigma_1 < 4\pi$. From Lemma 2.3, we have $\sigma_2 < 8\pi$. Let x_k be the maximum point of u_1^k and $\lambda_k = \max u_i^k$ the maximum value of u_1^k . Since $\sigma_1 > 0$, we know that $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. Consider

$$\tilde{u}_i^k(x) = u_i^k(e^{-\lambda_k/2}x + x_k) - \lambda_k,$$

in $\Omega_k := \{x \in \mathbb{R}^2 \mid e^{-\lambda_k/2}x + x_k \in B_{r_0}\}$ for a fixed $r_0 > 0$. It is clear that $(\tilde{u}_1^k, \tilde{u}_2^k)$ satisfies (2.1) in Ω_k and Ω_k converges to \mathbb{R}^2 as $k \rightarrow \infty$. Now we have three possibilities:

- (i) \tilde{u}_2^k blows up. Namely there is a point $y \in \mathbb{R}^2$ and a sequence y_k such that $y_k \rightarrow y$ and $\tilde{u}_2^k(y_k) \rightarrow +\infty$ as $k \rightarrow \infty$.
- (ii) \tilde{u}_2^k uniformly converges to $-\infty$ in any compact set of \mathbb{R}^2 .

(iii) \tilde{u}_2^k is bounded from above and there is a point x such that $\tilde{u}_2^k(x) \geq -C$ for some constant $C > 0$ independent of k .

We first consider case (i). We define \tilde{S}_2 to be the blowup set of \tilde{u}_2^k . It is clear that \tilde{S}_2 is finite. In fact, one can show that the number of \tilde{S}_2 is 1.

Since $\tilde{u}_1^k(x) \leq 0$, by using the equation for \tilde{u}_2^k alone, we can apply a result in [13] to conclude that for each blow-up point $y \in \tilde{S}_2$, the local mass around y would converge to 4π . Thus

$$4\pi\#\tilde{S}_2 \leq \int_{\Omega_k} e^{\tilde{u}_2^k} dx \leq \sigma_2 \leq 8\pi.$$

Hence $\#\tilde{S}_2 = 1$.

Suppose $\tilde{S}_2 = \phi$. Then both \tilde{u}_1^k and \tilde{u}_2^k converge to v_1 and v_2 , which are a solution of the Toda system (1.1) with $N = 2$. But the mass formulas (1.8) implies $\sigma_2 \geq 8\pi$, which is a contradiction to the assumption $\sigma_2 < 8\pi$.

Now suppose $\tilde{S}_2 = \{p\}$. Then it is easy to see \tilde{u}_1^k converges to u in $\mathbb{R}^2 \setminus \{p\}$, and u satisfies

$$-\Delta u = 2e^u - 4\pi\delta_p.$$

Let $v = u - 2 \log|x - p|$. It is clear that v satisfies

$$-\Delta v = 2|x - p|^2 e^v.$$

A result given in [23] (see also [2]) gives that

$$2 \int_{\mathbb{R}^2} e^u = \int_{\mathbb{R}^2} |x|^2 e^v = 16\pi,$$

which implies that $\sigma_1 \geq 8\pi$. This contradicts $\sigma_1 < 4\pi$.

In case (ii), one can show that \tilde{u}_1^k converges in $C_{loc}^2(\mathbb{R}^2)$ to a u , which is an entire solution of (1.2) with finite energy, the classification result of Chen-Li in [5] gives us

$$\lim \int_{B_{r_0}} e^{u_1^k} = \lim \int_{\Omega_k} e^{\tilde{u}_1^k} \geq 4\pi,$$

which contradicts $\sigma_1 < 4\pi$. In case (iii), we can show that $(\tilde{u}_1^k, \tilde{u}_2^k)$ converges to an entire solution of the Toda system (1.1) with finite energy. Now the classification result for the Toda system [11] gives us that

$$\lim \int_{B_{r_0}} e^{u_1^k} = \lim \int_{\Omega_k} e^{\tilde{u}_1^k} \geq 8\pi,$$

again, a contradiction. \square

From Lemma 2.3, the dimension of the set of possible blow-up values is less than or equal to one. We will show that the possible values of (σ_1, σ_2) are in fact isolated. It might be not difficult to see that $(4\pi, 0)$ and $(0, 4\pi)$ are possible values. And $(8\pi, 8\pi)$ is also a possible blow-up value. On \mathbb{R}^2 , the solution space of the Toda system is noncompact and the blow-up value is $(8\pi, 8\pi)$. Now we have

Proposition 2.5. *The blow-up value of the Toda system (2.1) can only be one of $(4\pi, 0)$, $(0, 4\pi)$, $(4\pi, 8\pi)$, $(8\pi, 4\pi)$ and $(8\pi, 8\pi)$.*

Proof. We only need to exclude the case that one of σ_1 and σ_2 is greater than 8π . Assume by contradiction that $\sigma_2 > 8\pi$. In view of Lemma 2.3, we have $\sigma_2 < 12\pi$ and $4\pi < \sigma_1 < 8\pi$.

Choose r_k such that

$$(2.5) \quad \int_{B_{r_k}} e^{u_2^k} = 8\pi.$$

Since $\sigma_2 > 8\pi$, it is easy to check that $r_k \rightarrow 0$ as $k \rightarrow \infty$. Consider

$$\tilde{u}_i^k(x) = u_i^k(2r_k x) - 2 \log 2r_k,$$

in $B_{r_k^{-1}}$. Let \tilde{S}_i be the blow-up set of \tilde{u}_i^k .

If $\tilde{S}_2 = \emptyset$ and $\tilde{S}_1 \neq \emptyset$, then \tilde{u}_2^k converges in $C_{loc}^2(\mathbb{R}^2 \setminus \tilde{S}_1)$ to a solution of

$$(2.6) \quad -\Delta v_0 = 2e^{v_0} - 4\pi \sum_{p \in \tilde{S}_1} \delta_p.$$

It is clear that the number of \tilde{S}_1 is 1. Otherwise, $\sigma_1 \geq 8\pi$, which is a contradiction to our assumption. Hence as above we have

$$\int_{\mathbb{R}^2} e^{v_0} = 8\pi.$$

However by (2.5), $8\pi = \int_{\mathbb{R}^2} e^{v_0} > \int_{B_2} e^{v_0} \geq \lim \int_{B_1} e^{\tilde{u}_2^k} = 8\pi$, a contradiction. Thus $\tilde{S}_1 = \emptyset$. If $\tilde{S}_2 = \tilde{S}_1 = \emptyset$, then as discussed in the proof of Lemma 2.4, either \tilde{u}_2^k converges to an entire solution \tilde{v}_0 of the Liouville equation and \tilde{u}_1^k converges uniformly to $-\infty$ in any compact domain of \mathbb{R}^2 , or $(\tilde{u}_1^k, \tilde{u}_2^k)$ converges to an entire solution of the Toda system. The latter case is clearly a contradiction to $\sigma_1 < 8\pi$ and the former leads to

$$4\pi = \int_{\mathbb{R}^2} e^{\tilde{v}_0} > \int_{B_2} e^{\tilde{v}_0} \geq \lim_{k \rightarrow \infty} \int_{B_{r_k^{-1}}} e^{\tilde{u}_2^k} = 8\pi,$$

a contradiction again. Hence $\tilde{S}_2 \neq \emptyset$. By rescaling a factor of close to 1 if necessary, we may assume that $\{x \in \mathbb{R}^2 \mid |x| = 1\} \cap (\tilde{S}_1 \cup \tilde{S}_2) = \emptyset$,

$$(2.7) \quad \tilde{u}_i^k(x) \rightarrow -\infty, \quad \text{locally in } \mathbb{R}^2 \setminus (\tilde{S}_1 \cup \tilde{S}_2) \text{ as } k \rightarrow \infty,$$

for $i = 1, 2$ and

$$(2.8) \quad \int_{\{|x| \leq 1\}} e^{u_2^k} \rightarrow 8\pi \quad \text{as } k \rightarrow \infty.$$

Here (2.7) holds by Remark 2.2. Now we use the Kelvin transformation to define

$$v_i^k(x) = \tilde{u}_i^k \left(\frac{x}{|x|^2} \right) - 4 \log |x|.$$

Since the Toda system is conformally invariant, $v^k = (v_1^k, v_2^k)$ satisfies the Toda system in $\mathbb{R}^2 \setminus B_{r_k}$. We consider v^k on $\Omega_k := B_1 \setminus B_{r_k}$. The boundary of Ω_k consists of two components

$$\partial_1 \Omega_k = \{|x| = 1\} \quad \text{and} \quad \partial_2 \Omega_k = \{|x| = r_k\}.$$

On $\partial_1 \Omega_k$, $v_i^k \rightarrow -\infty$ as $k \rightarrow \infty$. Let $\mu_k = \max_{\Omega_k} v_2^k$. Since $\sigma_2 - 8\pi > 0$, it is clear that $\mu_k \rightarrow \infty$ as $k \rightarrow \infty$. Let $y_k \in \overline{\Omega}_k$ such that $v_2^k(y_k) = \mu_k$. We claim that

$$(2.9) \quad \frac{|y_k|}{r_k} \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Suppose that $|y_k|/r_k \leq d$ uniformly for some constant $d > 1$. We consider \tilde{u}^k in $B_{r_k^{-1}}$. We first have by integrating by parts, for any $r \geq 1$,

$$(2.10) \quad -\frac{d}{dr} \bar{u}_2^k(r)r = \frac{1}{2\pi} \left(2 \int_{B_r} \tilde{h}_1^k e^{\tilde{u}_2^k} - \int_{B_r} \tilde{h}_1^k e^{\tilde{u}_1^k} \right) \geq \frac{1}{2\pi} (16\pi - \sigma_1) > 4(1 + \epsilon_0)$$

for some constant $\epsilon_0 > 0$. Here $\bar{u}_i^k(r) = \frac{1}{2\pi r} \int_{|x|=r} \tilde{u}_i^k d\sigma$ is the average of \tilde{u}_i^k over $\{|x| = r\}$.

Let $f(r) = \bar{u}_2^k(r) + 4 \log r$. From above we have $f'(r) \leq -\epsilon_0/r$. Hence, we have for $r \geq 1$

$$(2.11) \quad \bar{u}_2^k(r) + 4 \log r \leq f(1) = \bar{u}_2^k(1) =: c_k.$$

From above, we know $c_k \rightarrow -\infty$ as $k \rightarrow \infty$. Now by the definition of y_k , we have for any $|x| \geq 1$

$$(2.12) \quad \begin{aligned} \tilde{u}_2^k(x) + 4 \log |x| &= v_2^k \left(\frac{x}{|x|^2} \right) \leq v_2^k(y_k) \\ &= \tilde{u}_2^k \left(\frac{y_k}{|y_k|^2} \right) - 4 \log |y_k|. \end{aligned}$$

Recall that there is a constant $c_d > 0$, independent of k , such that

$$(2.13) \quad |u_2^k(x) - u_2^k(y)| < c_d,$$

for $|x| \geq d^{-1}$, $|y| \geq d^{-1}$. Thus

$$(2.14) \quad |\tilde{u}_2^k(x) - \tilde{u}_2^k(y)| \leq c_d$$

for $|x| \geq r_k^{-1}d^{-1}$ and $|y| \geq r_k^{-1}d^{-1}$. (2.14), together with $\frac{r_k}{|y_k|} \geq d^{-1}$, implies that

$$(2.15) \quad \left| \bar{u}_2^k\left(\frac{y_k}{|y_k|^2}\right) - \tilde{u}_2^k\left(\frac{y_k}{|y_k|^2}\right) \right| < c_d.$$

(2.15), together with (2.11) and (2.12), implies that for $|x| \geq 1$,

$$\begin{aligned} \tilde{u}_2^k(x) + 4 \log |x| &\leq \tilde{u}_2^k\left(\frac{y_k}{|y_k|^2}\right) - 4 \log |y_k| \\ &\leq \bar{u}_2^k\left(\frac{y_k}{|y_k|^2}\right) - 4 \log |y_k| + c_d \\ &\leq c_k + c_d \rightarrow -\infty \text{ as } k \rightarrow +\infty. \end{aligned}$$

Therefore, we have

$$\int_{1 \leq |x| \leq \frac{1}{r_k}} e^{\tilde{u}_2^k} \leq e^{c_k + c_d} \int_{|x| \geq 1} |x|^{-4} = O(1)e^{c_k} \rightarrow 0$$

as $k \rightarrow \infty$, which implies that $\sigma_2 = 8\pi$, a contradiction. Thus, (2.9) is proved.

Consider a new rescaled $\tilde{v}^k = (\tilde{v}_1^k, \tilde{v}_2^k)$ defined by

$$\tilde{v}_i^k(x) = v_i^k(e^{-\mu_k/2}x + y_k) - \mu_k$$

in

$$\tilde{\Omega}_k = \{x \in \mathbb{R}^2 \mid e^{-\mu_k/2}x + y_k \in \Omega_k\}.$$

From (2.9), we have two possibilities:

- (i) $|y_k|e^{\mu_k/2} \rightarrow \infty$ as $k \rightarrow \infty$.
- (ii) $|y_k|e^{\mu_k/2}$ is uniformly bounded.

For case (i), we know that $\tilde{\Omega}_k \rightarrow \mathbb{R}^2$. Arguing as in the proof of Lemma 2.4, we can show that \tilde{v}_2^k converges in a suitable topology to an entire solution to an equation like (2.6) or $(\tilde{v}_1^k, \tilde{v}_2^k)$ to an entire solution of the Toda system. In both cases, we have

$$\lim \int_{\Omega_k} e^{\tilde{u}_2^k} \geq 4\pi,$$

which, it turn, implies that $\sigma_2 \geq 12\pi$, a contradiction. For case (ii), we assume that $y_k e^{\mu_k/2} \rightarrow p \in \mathbb{R}^2$. In view of (2.9), we have that $\tilde{\Omega}_k$ converges to $\mathbb{R}^2 \setminus \{p\}$. In this case, one can show as above that \tilde{v}_2^k

converges to a function \tilde{v}_0 in $C_{loc}^2(\mathbb{R}^2 \setminus (\{p\} \cup \hat{S}_1))$, where \hat{S}_1 is the set of blow-up points of \tilde{v}_1^k . One can check that \tilde{v}_0 satisfies

$$(2.16) \quad -\Delta \tilde{v}_0 = 2e^{\tilde{v}_0} - \alpha 4\pi \delta_p - 4\pi \sum_{x \in \hat{S}_1} \delta_x.$$

Since $\tilde{v}_2^k \leq 0$, we have $\alpha \geq 0$. It is clear that the number of \hat{S}_1 is less than 2 because $\sigma_1 < 8\pi$. If $\hat{S}_1 = \emptyset$, together with $\alpha \geq 0$, then we have $\int_{\mathbb{R}^2} e^{\tilde{v}_0} \geq 4\pi$, which implies that $\sigma_2 \geq 12\pi$, a contradiction. Now assume that $\hat{S}_1 = \{q\}$. We will show in the appendix that $\int_{\mathbb{R}^2} e^{\tilde{v}_0} > 4\pi$, a contradiction again. This completes the proof of the Proposition. \square

3. GEOMETRIC INTERPRETATIONS

Before we give a proof of the precise asymptotic behavior of solutions of the Toda system, we would like to recall the geometric interpretation of solutions of the Toda system (1.1).

Let Ω be a simply connected domain and $u = (u_1, u_2, \dots, u_N)$ a solution of (1.1) on Ω . Define $\tilde{w}_0, \tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_N$ by

$$(3.1) \quad u_i = 2\tilde{w}_i - 2\tilde{w}_{i-1} \quad \text{for } i \in I \text{ and } \sum_{i=0}^N \tilde{w}_i = 0.$$

It is clear that

$$\tilde{w}_0 = -\frac{1}{2(N+1)} \sum_{i=1}^N (N-i+1)u_i.$$

Now set $w_i = \tilde{w}_i - i \log 4$ for $i = 1, 2, \dots, N$. It is easy to check that w_0, w_1, \dots, w_N satisfies

$$(3.2) \quad \begin{cases} 2(w_0)_{z\bar{z}} &= e^{2(w_1-w_0)} \\ 2(w_1)_{z\bar{z}} &= -e^{2(w_1-w_0)} + e^{2(w_2-w_1)} \\ \dots & \dots \dots \\ 2(w_N)_{z\bar{z}} &= -e^{2(w_N-w_{N-1})}. \end{cases}$$

(3.2) is equivalent to an integrability condition

$$(3.3) \quad \mathcal{U}_{\bar{z}} - \mathcal{V}_z = [\mathcal{U}, \mathcal{V}]$$

of the following two equations

$$(3.4) \quad \phi^{-1} \cdot \phi_z = \mathcal{U}$$

and

$$(3.5) \quad \phi^{-1} \cdot \phi_{\bar{z}} = \mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} (w_0)_z & & & \\ & (w_1)_z & & \\ & & \dots & \\ & & & (w_N)_z \end{pmatrix} + \begin{pmatrix} 0 & e^{w_1-w_0} & & \\ & 0 & & \\ & & \dots & e^{w_N-w_{N-1}} \\ & & & 0 \end{pmatrix}$$

and

$$\mathcal{V} = - \begin{pmatrix} (w_0)_{\bar{z}} & & & \\ & (w_1)_{\bar{z}} & & \\ & & \dots & \\ & & & (w_N)_{\bar{z}} \end{pmatrix} - \begin{pmatrix} 0 & & & \\ e^{w_1-w_0} & 0 & & \\ & & \dots & \\ & & & e^{w_N-w_{N-1}} & 0 \end{pmatrix}$$

Set

$$(3.6) \quad \alpha = -(\mathcal{U}dz + \mathcal{V}d\bar{z}).$$

Then, α is a one-form valued in $su(N+1)$. With the help of the Frobenius Theorem, we obtain a map $\phi : \Omega \rightarrow SU(N+1)$ such that

$$\alpha = \phi^{-1} \cdot d\phi.$$

As a connection on the trivial bundle $\Omega \times \mathbb{C}^{N+1} \rightarrow \Omega$, α (or $d + \alpha$) is flat, i.e., α satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

When Ω is not simply connected, we cannot apply the Frobenius theorem directly to obtain ϕ . Let $\Omega = B^* = B \setminus \{0\}$ be the punctured disk. We introduce the *holonomy* (see [18]) of an $SU(N+1)$ connection α on the bundle $\Omega \times \mathbb{C}^{N+1} \rightarrow \Omega$ along around 0. Let (r, θ) be the polar coordinates. Decompose α as $\alpha = \alpha_r dr + \alpha_\theta d\theta$. α_r and α_θ are $su(N+1)$ -valued. For any given $r \in (0, 1)$, the following initial value problem,

$$\frac{d\phi_r}{d\theta} + \alpha_\theta \phi_r = 0, \quad \phi_r(0) = Id,$$

has a unique solution $\phi_r(\theta) \in SU(N+1)$. Here Id is the identity matrix. If the connection $d + \alpha$ has no singularity at 0, the conjugacy class of $\phi_r(2\pi)$ is trivial. If α is a flat connection on D^* , then the conjugacy class of $\phi_r(2\pi)$ is independent of r . In this case, denote the conjugacy class of $\phi_r(2\pi)$ by h_α . h_α is called the *holonomy* of α .

Proposition 3.1. *Let $u = (u_1, u_2, \dots, u_N)$ be a solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$ with*

$$u_i(x) = \mu_i \log |x| + O(1), \quad \text{near } 0,$$

where $\mu_i > -2$ for $i \in I = \{1, 2, \dots, N\}$. Then the flat connection α defined by (3.6) has holonomy

$$(3.7) \quad h_\alpha = \begin{pmatrix} e^{2\pi i \beta_0} & 0 & \cdots & 0 \\ & e^{2\pi i \beta_1} & 0 & \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & e^{2\pi i \beta_N} \end{pmatrix},$$

where $\beta_0, \beta_1, \dots, \beta_N$ are determined by

$$(3.8) \quad \beta_i - \beta_{i-1} = -\frac{1}{2}\mu_i \quad (i \in I) \quad \text{and} \quad \sum_{j=0}^N \beta_j = 0.$$

For the proof, see [12].

Now by using Proposition 3.1, we can generalize a result in [12]. Given an N -tuple $(\mu_1, \mu_2, \dots, \mu_N)$ with $\mu_i > -2$ for any i and let $u = (u_1, u_2, \dots, u_N)$ be a C^2 solution of the following system

$$(3.9) \quad -\Delta u_i = \sum_{j=1}^N a_{ij} |x|^{\mu_j} e^{u_j}, \quad \text{in } \mathbb{R}^2$$

with

$$(3.10) \quad \int_{\mathbb{R}^2} |x|^{\mu_i} e^{u_i} < \infty.$$

Let

$$m_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{\mu_i} e^{u_i} \quad \text{and} \quad \gamma_i = \sum a_{ij} m_j.$$

The potential analysis gives that

$$(3.11) \quad \begin{aligned} u_i &= -\gamma_i \log |x| + a_i + O(|x|^{-1}), & \text{near } \infty, \text{ and} \\ \nabla u_i &= -\gamma_i \frac{x}{|x|^2} + O(|x|^{-1}) & \text{near } \infty, \end{aligned}$$

for some constants a_i and

$$(3.12) \quad \gamma_i - \mu_i > 2,$$

for all i . For the convenience of the reader, we will give the proofs of (3.11) and (3.12) in the appendix of the paper.

Proposition 3.2. *If $\mu_i \leq 0$ for any i , then we have*

$$\gamma_i = 2(2 + \mu_i), \quad \text{for any } i.$$

Proof. Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N)$ with $\tilde{u}_i = u_i + \mu_i \log |z|$. It is clear that \tilde{u} satisfies (1.1) in $\mathbb{R}^2 \setminus \{0\}$. Let α be the flat connection (3.6) obtained from the solution \tilde{u} . We have

$$\begin{aligned} \tilde{u}_i(x) &= \mu_i \log |x| + O(1), & \text{near } 0 \\ \tilde{u}_i(x) &= -(\gamma_i - \mu_i) \log |x| + O(1), & \text{near } \infty. \end{aligned}$$

Let β_i be determined by $\{\mu_i, i \in I\}$ using (3.8). By Proposition 3.1, the holonomy of α at 0 is $h_\alpha(0)$ given by (3.7). Now we compute the the holonomy of α at ∞ , $h_\alpha(\infty)$. To compute it, we use the Kelvin transformation to consider

$$v_i(x) = \tilde{u}_i\left(\frac{\bar{x}}{|x|^2}\right) - 4 \log |x|, \quad i \in I.$$

Clearly, $v = (v_1, v_2, \dots, v_N)$ satisfies (1.1) on $\mathbb{R}^2/\{0\}$ and

$$v_i = (\gamma_i - \mu_i - 4) \log |x| + O(1), \quad \text{near } 0.$$

Let α' be its corresponding flat connection. It is obvious that $h_\alpha(\infty) = h_{\alpha'}(0)$, which can be obtained again by (3.7) by replacing μ_i by $\{\gamma_i - \mu_i - 4\}$, i.e., $h_{\alpha'}(0)$ is a matrix of form (3.7) with β'_i decided by

$$\beta'_i - \beta'_{i-1} = -\frac{1}{2}(\gamma_i - \mu_i - 4) \quad \text{and} \quad \sum \beta'_i = 0.$$

Now the key fact is $h_\alpha(0) = h_\alpha(\infty)$, which implies that

$$\beta_i - \beta'_i = 1 \pmod{\mathbb{Z}}, \quad \text{for } i = 0, 1, 2, \dots, N.$$

Hence, $\gamma_i - 2\mu_i = 2n_i$ for some $n_i \in \mathbb{Z}$. By (2.11), we have $n_i > 1$ for any i . We now need a global Pohozaev identity, which is given in Proposition 3.3 below. By (3.13), we have

$$\sum a^{ij} \gamma_i (2n_i - 4) = 0.$$

Since $a^{ij} > 0$ for any i, j (see [12]) and $n_i > 1$, the previous formula implies that $n_i = 2$ for any i . This proves the Proposition. \square

Remark 3.1. *To remove the condition $\mu_i \leq 0$ for any i , one may have to relate the solutions of the Toda system to other geometric objects. For instance, a generalization of projective connections considered in [22] might be a good candidate.*

Proposition 3.3. γ_i satisfies

$$(3.13) \quad \sum a^{ij} \gamma_i (\gamma_j - 2(2 + \mu_j)) = 0,$$

where (a^{ij}) is the inverse matrix of the Cartan matrix (a_{ij}) .

Proof. First for a given $R > 0$, multiplying (1.1) by $x \cdot \nabla u_k$ and integrating over B_R , we have a Pohozaev identity

$$(3.14) \quad \begin{aligned} & \frac{1}{2} \int_{B_R} \nabla u_k \nabla u_i + \frac{1}{2} \int_{B_R} x \cdot \nabla (\nabla_j u_k) \nabla_j u_i \\ &= \frac{1}{2} \int_{\partial B_R} R \frac{\partial u_k}{\partial n} \frac{\partial u_i}{\partial n} + \sum a_{ij} \int_{B_R} |x|^{\mu_j} e^{u_j} x \cdot \nabla u_k \end{aligned}$$

From (3.14), we have

$$(3.15) \quad \begin{aligned} & \frac{1}{2} \sum a^{ki} \int_{B_R} \nabla u_k \nabla u_i + \frac{1}{2} \sum a^{ki} \int_{B_R} x \cdot \nabla (\nabla_j u_k) \nabla_j u_i \\ &= -\frac{1}{2} \sum a^{ki} \int_{\partial B_R} R \frac{\partial u_k}{\partial n} \frac{\partial u_i}{\partial n} + \sum \int_{B_R} |x|^{\mu_j} e^{u_j} x \cdot n u_j. \end{aligned}$$

It is easy to compute that

$$(3.16) \quad \begin{aligned} \frac{1}{2} \sum a^{ki} \int_{B_R} x \cdot \nabla (\nabla_j u_k) \nabla_j u_i &= \frac{1}{4} \int_{B_R} x \cdot \nabla (\nabla_j u_k \nabla_j u_i) \\ &= -\frac{1}{2} \int_{B_R} \sum a^{ki} \nabla u_k \nabla u_i \\ &\quad + \frac{1}{4} \sum a^{ik} \int_{\partial B_R} R \nabla u_k \nabla u_i. \end{aligned}$$

Inserting (3.16) into (3.15), we get

$$\begin{aligned} & \sum \int_{B_R} (2 + \mu_j) |x|^{\mu_j} e^{u_j} - \sum \int_{\partial B_R} R |x|^{\mu_j} e^{u_j} \\ &= \frac{1}{4} \sum a^{ik} \left(2 \int_{\partial B_R} \frac{\partial u_k}{\partial n} \frac{\partial u_i}{\partial n} - \int_{\partial B_R} \nabla u_k \nabla u_i \right) \end{aligned}$$

Now using (3.11) in the above formula and taking the limit $R \rightarrow \infty$, we have proved the proposition. \square

A geometric proof of the Proposition can be obtained by using another geometric object, projective connections with singularities, which was used by Troyanov [22] to classify conical metrics of constant curvature. One can check that a quadratic differential $\eta = f dz^2$, where $f : \mathbb{R}^2 / \{0\} \rightarrow \mathbb{C}$ is given by

$$f = \sum_{j,k=1}^N a^{jk} \left\{ (\tilde{u}_k)_{zz} - \frac{1}{2} (\tilde{u}_j)_z \cdot (\tilde{u}_k)_z \right\}$$

is a projective connection with regular singularities at 0 and ∞ . By counting and comparing the weights of the singularities, one can get the global Pohozaev identity.

4. A PRECISE BUBBLING BEHAVIOR

In this section, we prove Theorem 1.3. Let $u^k = (u_1^k, u_2^k)$ be a solution of

$$(4.1) \quad \begin{cases} -\Delta u_1^k &= 2h_1^k e^{u_1^k} - h_2^k e^{u_2^k} \\ -\Delta u_2^k &= 2h_2^k e^{u_2^k} - h_1^k e^{u_1^k} \end{cases} \quad \text{in } B_2$$

satisfying conditions given in Theorem 1.3. Set

$$\sigma_i = \lim_{k \rightarrow +\infty} \int_{B_1} e^{u_i^k}.$$

From Proposition 2.5, we have $\sigma_1 = \sigma_2 = 8\pi$.

In order to show the idea of our proof, we first prove Theorem 1.3 in the case that $h_1^k = h_2^k = 1$. Then we will point out which steps should be modified for the general case.

Proof of Theorem 1.3 in the case that $h_1^k = h_2^k = 1$. We divide the proof into several steps.

Step 1. From the above discussion, (1.9) is valid in $B_{\epsilon_k R}$ for any fixed large number $R > 0$. Hence, we only need to prove that (1.9) is valid in $(\epsilon_k R, r_0)$ for some $r_0 > 0$. By assumption (4), we have

$$|u_2^k(x_1^k) - \lambda^k| < c,$$

for some constant $c > 0$ independent of k . In the sequel, c will denote a positive constant independent of k , which may vary from line to line.

Step 2. By the Green representation, we have for $|x| < r_0$,

$$(4.2) \quad \begin{aligned} u_1^k(x) - \bar{m}_i^k &= \int_{B_{2r_0}} G(x, y) \left(2e^{u_1^k(y)} - e^{u_2^k(y)} \right) dy + O(1) \\ &= \frac{1}{2\pi} \int_{B_{r_0}(x_\epsilon)} \log \frac{1}{|x - y|} \left(2e^{u_1^k(y)} - e^{u_2^k(y)} \right) dy + O(1), \end{aligned}$$

where $\bar{m}_i^k = \inf_{\partial B_2} u_i^k(x)$. In this paper, $O(1)$ denotes a term bounded by a constant independent of k . Hence, at x_k we have

$$(4.3) \quad \lambda^k - \bar{m}_i^k = \frac{1}{2\pi} \int_{B_{r_0}(x_\epsilon)} \log \frac{1}{|x_\epsilon - y|} \left(2e^{u_1^k(y)} - e^{u_2^k(y)} \right) dy + O(1).$$

(4.2) and (4.3) imply

$$(4.4) \quad \begin{aligned} v_1^k(x) &= u_1^k(x_k + \epsilon_k x) - u_1^k(x_k) \\ &= \frac{1}{2\pi} \int_{B_{r_0} \epsilon_k^{-1}} \log \frac{|x|}{|y-x|} \left(2e^{v_1^k(y)} - e^{v_2^k(y)} \right) dy + O(1). \end{aligned}$$

Similarly, we have

$$u_2^k(x_k + \epsilon_k x) - u_2^k(x_k) = \frac{1}{2\pi} \int_{B_{r_0} \epsilon_k^{-1}} \log \frac{|y|}{|y-x|} \left(2e^{v_2^k(y)} - e^{v_1^k(y)} \right) dy + O(1).$$

Since $|u_2^k(x_k) - \lambda^k| \leq c$, we have

$$(4.5) \quad v_2^k(x) = \frac{1}{2\pi} \int_{B_{r_0} \epsilon_k^{-1}} \log \frac{|y|}{|y-x|} \left(2e^{v_2^k(y)} - e^{v_1^k(y)} \right) dy + O(1).$$

Set $w_1^k = \frac{1}{3}(2v_1^k + v_2^k)$ and $w_2^k = \frac{1}{3}(v_1^k + 2v_2^k)$. It is clear that

$$\begin{aligned} w_1^k &= \frac{1}{2\pi} \int_{B_{r_0} \epsilon_k^{-1}} \log \frac{|y|}{|y-x|} e^{v_1^k(y)} dy + O(1) \\ w_2^k &= \frac{1}{2\pi} \int_{B_{r_0} \epsilon_k^{-1}} \log \frac{|y|}{|y-x|} e^{v_2^k(y)} dy + O(1). \end{aligned}$$

A standard potential analysis (see for instance [2]) gives

$$(4.6) \quad \left| w_i^k + \frac{\sigma_i^k}{2\pi} \log |x| \right| \leq \delta \log |x| + O(1),$$

for a small $\delta > 0$ and $|x|$ large enough, where σ_i^k is the local mass defined by

$$\sigma_i^k = \int_{B_{r_0}} e^{u_i^k}.$$

Thus, we have

$$(4.7) \quad |v_i^k - m_i^k \log |x|| \leq 3\delta \log |x| + O(1),$$

where

$$m_1^k = \frac{2\sigma_1^k - \sigma_2^k}{2\pi} \quad \text{and} \quad m_2^k = \frac{2\sigma_2^k - \sigma_1^k}{2\pi}.$$

As in [2], we choose a small constant $\delta_3 > 0$ such that for $\log(\frac{1}{\epsilon_k}) \leq |x| \leq \frac{1}{\epsilon_k}$,

$$(4.8) \quad |\tilde{m}_i^k(x) + m_i^k| = O(1)(\log \epsilon_k^{-1})^{-1},$$

where

$$\tilde{m}_i^k(x) = \int_{|y| \leq \delta_3 |x|} e^{v_i^k(y)} dy.$$

This can be done, since from (4.7) we have

$$(4.9) \quad \int_{|y| \geq \delta_3 \log \epsilon_k^{-1}} e^{v_i^k(y)} dy \leq c \int_{|y| \geq \delta_3 \log \epsilon_k^{-1}} |y|^{-m_i^k + 3\delta} dy = O(1)(\log \epsilon_k^{-1})^{-3/2},$$

by noting that $m_i^k \rightarrow 4$ as $k \rightarrow \infty$. It is also easy to check that

$$\int_{|y| \geq \delta_3 |x|} \log \frac{|y|}{|x-y|} e^{v_i^k(y)} dy \leq O(1)(\log \epsilon_k^{-1})^{-1},$$

for $|x| \geq \log \epsilon_k^{-1}$. Therefore, we have

$$\begin{aligned} w_i^k &= \frac{1}{2\pi} \int_{|y| \leq \delta_3 |x|} \log \frac{|y|}{|y-x|} e^{v_i^k(y)} dy + O(1) \\ &= \frac{1}{2\pi} \int_{|y| \leq \delta_3 |x|} \log \frac{|y|}{|y-x|} e^{v_i^k(y)} dy + O(1) \\ &= \tilde{m}_i^k \log |x| + O(1), \end{aligned}$$

for $\log \epsilon_k^{-1} \leq |x| \leq \frac{1}{\epsilon_k}$. Hence, we have

$$(4.10) \quad |v_i^k(x) + m_i^k \log |x|| < c,$$

for $\log \epsilon_k^{-1} \leq |x| \leq \frac{1}{\epsilon_k}$. Similarly, we can show that

$$(4.11) \quad \left| \nabla v_i^k(x) + m_i^k \frac{x}{|x|^2} \right| < c$$

holds for $\log \epsilon_k^{-1} \leq |x| \leq \frac{1}{\epsilon_k}$.

By (4.10)-(4.11), we have

$$(4.12) \quad \begin{aligned} u_i^k(x) &= m_i^k \log \frac{1}{|x|} + (2 - m_i^k) \log \frac{1}{\epsilon_k} + O(1) \\ \nabla u_i^k(x) &= -m_i^k \frac{x}{|x|^2} + O(1) \end{aligned}$$

in $B_{r_0} \setminus B_{\epsilon_k \log \epsilon_k^{-1}}$.

Step 3. We want to show that

$$m_i^k = 4\pi + O(1)(\log \frac{1}{\epsilon_k})^{-2}$$

for large k . Here we will use the geometric properties of the Toda system, described in Section 3.

Let \tilde{w}_0, \tilde{w}_1 and \tilde{w}_2 defined by

$$(4.13) \quad 2(\tilde{w}_i - \tilde{w}_{i-1}) = v_i^k \quad (i = 1, 2) \text{ and } \sum_{i=0}^2 \tilde{w}_i = 0.$$

Let $w_1 = \tilde{w}_1 - \log 4$ and $w_2 = \tilde{w}_2 - 2 \log 4$. Since u_1, u_2 satisfy the Toda system, we already know that

$$(4.14) \quad d\alpha + \frac{1}{2}[\alpha, \alpha] = 0,$$

where α is defined by

$$\alpha := \mathcal{U}dz + \mathcal{V}d\bar{z}.$$

Here

$$\mathcal{U} = \begin{pmatrix} (w_0)_z & e^{w_1-w_0} & 0 \\ 0 & (w_1)_z & e^{w_2-w_1} \\ 0 & 0 & (w_2)_z \end{pmatrix}$$

and

$$\mathcal{V} = - \begin{pmatrix} (w_0)_{\bar{z}} & 0 & 0 \\ e^{w_1-w_0} & (w_1)_{\bar{z}} & 0 \\ 0 & e^{w_2-w_1} & (w_2)_{\bar{z}} \end{pmatrix}.$$

It is clear that α is a one-form valued in $su(3)$. Equation (4.14) means that the connection $d + \alpha$ is flat. Now we calculate the holonomy of $d + \alpha$ along

$$\{|x| = (\log \frac{1}{\epsilon_k})^s\}$$

where $s > 0$ will be fixed later. Decompose α by

$$\alpha = \alpha_r dr + \alpha_\theta d\theta.$$

By definition, the holonomy h_α of the connection α is the conjugacy class of $g_r(2\pi)$, where $g_r(\theta)$ is the unique solution of

$$(4.15) \quad \begin{aligned} \frac{dg_r}{d\theta} + \alpha_\theta g_r &= 0, \\ g_r(0) &= 0. \end{aligned}$$

From (4.12), it is easy to check that

$$(4.16) \quad \begin{aligned} \alpha_\theta &= \begin{pmatrix} -i\beta_0^k & b_1 i e^{i\theta} & 0 \\ b_1 i e^{-i\theta} & -i\beta_1^k & b_2 i e^{-i\theta} \\ 0 & b_2 i e^{i\theta} & -i\beta_2^k \end{pmatrix} \\ &+ O(1)\epsilon_k \log \frac{1}{\epsilon_k} I \end{aligned}$$

at $|x| = (\log \frac{1}{\epsilon_k})^s$, where

$$b_i = O(1) \left(\log \frac{1}{\epsilon_k}\right)^{(1-\frac{m_i^k}{2})s}.$$

Since $m_i > 2$ for all i , we choose $s > 0$ such that $s(\min_i m_i^k - 2) > 4$.

Now we can compute the holonomy h_α of α along $\{|x| = (\log \frac{1}{\epsilon_k})^s\}$

$$(4.17) \quad \begin{pmatrix} e^{-2\pi i \beta_0^k} & & \\ & e^{-2\pi i \beta_1^k} & \\ & 0 & e^{-2\pi i \beta_2^k} \end{pmatrix} + O(1) \left(\log \frac{1}{\epsilon_k}\right)^{-2} I.$$

Since the holonomy h_α must be trivial, we have

$$\beta_i^k = n_i + O(1) \left(\log \frac{1}{\epsilon_k}\right)^{-2}$$

for some integer n_i , which implies

$$(4.18) \quad m_i^k = 2n_i' + O(1) \left(\log \frac{1}{\epsilon_k}\right)^{-2}$$

for some integer n_i' . From Lemma 2.3 we know $m_i^k \rightarrow 4$ as $k \rightarrow \infty$. Hence, we have

$$m_i^k = 4 + O(1) \left(\log \frac{1}{\epsilon_k}\right)^{-2},$$

which, together with (4.10), implies

$$(4.19) \quad \begin{aligned} v_i^k &= -m_i^k \log |x| + O(1) \\ &= -4 \log |x| + O(1) \left(\log \frac{1}{\epsilon_k}\right)^{-2} \log |x| + O(1) \\ &= -4 \log |x| + O(1), \end{aligned}$$

for $\log \epsilon_k^{-1} \leq |x| \leq \frac{1}{\epsilon_k}$.

Step 4. We have proved that (4.19) holds in $\log \epsilon_k^{-1} \leq |x| \leq \frac{1}{\epsilon_k}$. By Step 1, we have, for any large $R > 0$, v_i^k converges to v_i^0 uniformly for $|x| \leq R$. By a classification result for entire solutions to the Toda system [12], we have

$$|v_i^0 + 4 \log |x|| \leq c$$

for $|x| \geq R$, where $c > 0$ is a constant independent of R . Therefore,

$$(4.20) \quad |v_i^k + 4 \log |x|| \leq 2c$$

for $|x| = R$ and large k . Choose R large such that

$$e^{v_i^k} \leq |x|^{-3}$$

for $|x| \geq R$ and $i = 1, 2$, and define

$$w_\pm(x) = -4 \log |x| \pm (c_1 - c_1 |x|^{-\frac{1}{2}}).$$

It is clear that

$$\Delta w_\pm = \mp \frac{1}{4} c_1 |x|^{-\frac{5}{2}}.$$

By choosing R and c_1 large, we have, from the maximum principle, that

$$w_-(x) \leq v_i^k(x) \leq w_+(x)$$

for $R \leq |x| \leq \epsilon_k^{-1}$. Now we complete the proof of Theorem 1.3 in the case $h_1^k = h_2^k = 1$. \square

Proof of Theorem 1.3 in the general case. Assume without loss of generality that $h_i^k(x) = h_i(x)$ and $h_i(0) = 1$ for $i = 1, 2$. It is clear that we only need to prove Step 3, i.e.,

$$(4.21) \quad m_i^k = 4\pi + O(1)(\log \frac{1}{\epsilon})^{-2},$$

where

$$m_1^k = \frac{2\sigma_1^k - \sigma_2^k}{2\pi} \quad \text{and} \quad m_2^k = \frac{2\sigma_2^k - \sigma_1^k}{2\pi}.$$

When h_i is not constant, (4.1) is not an integrable system. However, the same idea as in Step 3 still works.

Define \tilde{w}_i and w_i as above. We define a connection $\alpha = -(\mathcal{U}dz + \mathcal{V}d\bar{z})$ on the trivial bundle, where

$$\mathcal{U} = \begin{pmatrix} (w_0)_z & f_1^k e^{w_1 - w_0} & 0 \\ 0 & (w_1)_z & f_2^k e^{w_2 - w_1} \\ 0 & 0 & (w_2)_z \end{pmatrix}$$

and

$$\mathcal{V} = - \begin{pmatrix} (w_0)_{\bar{z}} & 0 & 0 \\ f_1^k e^{w_1 - w_0} & (w_1)_{\bar{z}} & 0 \\ 0 & f_2^k e^{w_2 - w_1} & (w_2)_{\bar{z}} \end{pmatrix}.$$

Here $f_i^k = h_i^{1/2}(\epsilon^k x + x^k)$. Note that now $v^k = (v_1^k, v_2^k)$ satisfies in $\Omega_k := \{x \in \mathbb{R}^2 \mid \epsilon_k x + x^k \in B_2\}$

$$(4.22) \quad \begin{cases} -\Delta v_1^k = 2h_1(\epsilon_k x + x^k)e^{v_1^k} - h_2(\epsilon_k x + x^k)e^{v_2^k}, \\ -\Delta v_2^k = 2h_2(\epsilon_k x + x^k)e^{v_2^k} - h_1(\epsilon_k x + x^k)e^{v_1^k}, \end{cases}$$

Now the connection $d + \alpha$ is not flat. But it satisfies

$$F = d\alpha + \frac{1}{2}[\alpha, \alpha] = \begin{pmatrix} 0 & (f_1^k)_{\bar{z}} e^{w_1 - w_0} & 0 \\ -(f_1^k)_z e^{w_1 - w_0} & 0 & (f_2^k)_{\bar{z}} e^{w_2 - w_1} \\ 0 & -(f_2^k)_z e^{w_2 - w_1} & 0 \end{pmatrix} dz \wedge d\bar{z}.$$

Here $F = F_{r\theta} dr \wedge d\theta$ is the curvature of the connection $d + \alpha$.

Let $g_r(\theta)$ be the solution of (4.15) for this connection $d + \alpha$. We denote the conjugacy class of $g_r(2\pi)$ by h_α^r . Now h_α^r may depend on r .

Computing as in Step 3, we have (4.16) and

$$(4.23) \quad h_\alpha^r = \begin{pmatrix} e^{-2\pi i \beta_0^k} & 0 & 0 \\ 0 & e^{-2\pi i \beta_1^k} & 0 \\ 0 & 0 & e^{-2\pi i \beta_2^k} \end{pmatrix} + O(1) \left(\log \frac{1}{\epsilon_k}\right)^{-2} I,$$

at $r = \left(\log \frac{1}{\epsilon_k}\right)^s$.

Now we consider gauge transformations. Let $\phi : \mathbb{R}^2 \rightarrow SU(N+1)$. From the connection α , we can get a new connection by

$$\tilde{\alpha} = \phi^{-1} d\phi + \phi^{-1} \alpha \phi.$$

For the new connection, we compute h_α^r .

Lemma 4.1 ([18]). *We have $h_{\tilde{\alpha}}^r = h_\alpha^r$.*

Using this Lemma, we can compute h_α^r by using a suitable equivalent connection. This new connection is chosen as follows. By solving in $\Omega_\epsilon \setminus B_R$

$$\begin{aligned} \frac{d\phi}{dr} + \alpha_r \phi &= 0 \\ \phi(0, \theta) &= I, \end{aligned}$$

we obtain a new connection, $\tilde{\alpha} = \phi^{-1} d\phi + \phi^{-1} \alpha \phi$. Decompose the new connection as above $\tilde{\alpha} = \tilde{\alpha}_\theta d\theta + \tilde{\alpha}_r dr$. Clearly, we have $\tilde{\alpha}_r = 0$. Hence, we have

$$(4.24) \quad \frac{d\tilde{\alpha}_\theta}{dr} = \tilde{F}_{r\theta},$$

where $\tilde{F} = \tilde{F}_{r\theta} dr \wedge d\theta$ is the curvature of the new connection. It is well-known that $\tilde{F} = \phi^{-1} F \phi$.

Now we estimate $\tilde{F}_{r\theta}$. We claim that

$$(4.25) \quad e^{v_1^k} \leq C|x|^{-3}, \quad \text{for } |x| \geq R,$$

for some fixed $R > 0$. The claim will be proved at the end of the proof. From (4.25), it is clear that

$$|(f_1^k)_z e^{w_1 - w_0}| \leq C\epsilon^k e^{\frac{1}{2}v_1^k} \leq C\epsilon^k |x|^{-3/2},$$

for $|x| \leq R$. We have similar estimates for the other entries. Therefore, we have

$$|\tilde{F}_{r\theta}| = |F_{r\theta}| \leq C\epsilon^k |x|^{-1/2} \quad \text{for } |x| \geq R.$$

On B_R we have $|\tilde{F}_{r\theta}| = |F_{r\theta}| \leq C\epsilon^k$. Therefore, from (4.24) we have

$$|\tilde{\alpha}_\theta(r, \theta)| \leq C\epsilon \left(\log \frac{1}{\epsilon^k}\right)^s I, \quad \text{for } r = \left(\log \frac{1}{\epsilon^k}\right)^s.$$

Now it is clear to see that

$$h_{\alpha}^r = O(1)\epsilon(\log \frac{1}{\epsilon^k})^s I,$$

which, together with (4.23) and Lemma 4.1, implies

$$\beta_i^k = n_i + O(1)(\log \frac{1}{\epsilon^k})^{-2},$$

and hence

$$m_i^k = 4 + O(1)(\log \frac{1}{\epsilon^k})^{-2}.$$

This finishes the proof of Step 3 and the Theorem for the general case.

Now it remains to check Claim (4.25). The idea is similar to Step 4. We define for $|x| \geq R$ a function w

$$w = -3 \log |x| + (c_1 - c_1 |x|^{-\frac{1}{2}}).$$

We have $\Delta w = -\frac{1}{4}c_1|x|^{-\frac{5}{2}}$. Since $m_i^k \rightarrow 4$ as $k \rightarrow \infty$, we may assume $m_i^k > 3$ for large k . Applying the maximum principle, we get the claim. \square

5. EXISTENCE: A CRITICAL CASE

Let Σ be a Riemann surface with Gaussian curvature K . In this section and the next section, we will study the existence of solutions of

$$(5.1) \quad -\Delta u_i = \sum_{j=1}^N \rho_j a_{ij} \left(\frac{h_j e^{u_j}}{\int_{\Sigma} h_j e^{u_j}} - 1 \right), \quad \text{in } \Omega \quad 1 \leq i \leq N,$$

for the coefficient matrix $A = (a_{ij})_{n \times n}$, the Cartan matrix of $SU(N+1)$ and $\rho = (\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i > 0$ ($i = 1, 2, \dots, N$) given constants. System (5.1) is the Euler-Lagrange system of the functional

$$(5.2) \quad J_{\rho}(u) = \frac{1}{2} \sum_{i,j=1}^N \int a^{ij} \nabla u_i \nabla u_j + \sum_{j=1}^N \int \rho_j u_j - \sum_{j=1}^N \rho_j \log \int_{\Sigma} h_j e^{u_j},$$

in $H := (H^1(\Sigma))^N$. When $\rho_i < 4\pi$, it is a subcritical case with respect to the Moser-Trudinger inequality established in [11]. By the Moser-Trudinger inequality, the functional J_{ρ} is coercive and hence has a minimizer. When $\rho_i \leq 4\pi$ and some of the ρ_i 's are equal to 4π , the functional J_{ρ} still has a lower bound, but it does not satisfy the coercive condition. It is a critical case and the existence problem becomes more subtle. Even in the case that $N = 1$, the existence problem is a difficult problem, see [8, 17, 6]. Here we will use results in [6] to obtain a sufficient condition under which the functional J_{ρ} achieves its minimum. For simplicity we only consider the case $N = 2$.

Proposition 5.1. *For a fixed $\rho_2 \in (0, 4\pi)$, define*

$$\mathcal{X} = \{\text{Solutions of (1.6) for } \rho_1 \in (0, 4\pi] \text{ and } \rho_2\}.$$

If (1.5) holds, then \mathcal{X} is a compact set.

Proof. If \mathcal{X} is not compact, then we may assume that there is a sequence $\{\rho_k\}$ with $\rho_k < 4\pi$ and $\rho_k \rightarrow 4\pi$ as $k \rightarrow \infty$, and a sequence of solutions $u^k = (u_1^k, u_2^k)$ satisfying (1.6) with $\rho_1 = \rho_k$ and ρ_2 such that

$$\max\{\max u_1^k(x), \max u_2^k(x)\} \rightarrow \infty$$

as $k \rightarrow \infty$. Since $\rho_2 < 4\pi$, by Proposition 2.5, we know that $\max u_1^k \rightarrow \infty$ as $k \rightarrow \infty$ and u_2^k is uniformly bounded from above. Let w^k be the unique solution of

$$(5.3) \quad -\Delta w^k = \rho_2 \left(\frac{h_2 e^{u_2^k}}{\int_{\Sigma} h_2 e^{u_2^k}} - 1 \right) =: f_k$$

with $\int w^k = 0$. Since u_2^k is uniformly bounded from above, w^k is uniformly bounded in $H^{2,q}$ for any $q < \infty$. Hence w^k converges to w_0 in $C^{1,\alpha}$ for some $\alpha > 0$.

Let $v_1^k = u_1^k + w^k$ and $v_2^k = u_2^k - 2w^k$. It is clear that v_1^k and v_2^k satisfy

$$(5.4) \quad -\Delta v_1^k = 2\rho_k \left(\frac{\tilde{h}_1^k e^{v_1^k}}{\int_{\Sigma} \tilde{h}_1^k e^{v_1^k}} - 1 \right)$$

and

$$-\Delta v_2^k = -\rho_k \left(\frac{\tilde{h}_1^k e^{v_1^k}}{\int_{\Sigma} \tilde{h}_1^k e^{v_1^k}} - 1 \right).$$

with $\tilde{h}_1^k = h_1 e^{-w^k}$. It is clear that $v_1^k + 2v_2^k = 0$. Set $\tilde{v}_1^k = v_1^k - \log \int_{\Sigma} \tilde{h}_1^k e^{v_1^k}$. It is clear that \tilde{v}_1^k satisfies

$$(5.5) \quad -\Delta \tilde{v}_1^k = 2\rho_k \left(\tilde{h}_1^k e^{\tilde{v}_1^k} - 1 \right).$$

In order to apply Theorem 1.1 in [6] to our situation, we need to show that $\tilde{h}_1^k = h_1 e^{-w^k}$ converges to $h_1 e^{-w_0}$ in $C^{2,\alpha}$. Let p be the unique blow-up point of u_1^k . As in [11], one can show that $v_1^k \rightarrow 8\pi G(p, \cdot)$ and $v_2^k = -v_1^k/2 \rightarrow -4\pi G(p, \cdot)$ in $H^{1,q}(\Sigma)$ ($q < 2$) and in $C_{loc}^2(\Sigma \setminus \{p\})$, where $G(x, \cdot)$ is defined by

$$-\Delta G(x, \cdot) = \delta_x - 1$$

and $\int G = 0$. We apply a result of [14] (see also [2]) to (5.5) and obtain that

$$(5.6) \quad |\tilde{v}_1^k(x) - U_k(x)| < c, \quad \text{for } |x - p| < r_0$$

for some constant $c > 0$ and a small constant $r_0 > 0$. Here

$$U_k(x) = \log \frac{e^{\lambda_k}}{\left(1 + \frac{e^{\lambda_k} h_1(p) e^{-w_0(p)}}{8} |x|^2\right)^2}$$

and $\lambda_k = \max \tilde{v}_1^k$. Furthermore, one can prove that

$$(5.7) \quad \left| \lambda_k - \log \int_{\Sigma} \tilde{h}_1^k e^{v_1^k} \right| < c,$$

and

$$(5.8) \quad |\nabla \tilde{v}_1^k(x) - \nabla U_k(x)| < c$$

for some constant $c > 0$ and $|x - p| \leq r_0$. See [4]. From (5.6), (5.7), (5.8) and $u_2^k = -\frac{1}{2}v_1^k + 2w^k$, we have

$$|\nabla e^{u_2^k}| \leq c e^{-\frac{1}{2}\lambda_k} |\nabla e^{-\frac{1}{2}\tilde{v}_1^k}| \leq c_1 (1 + e^{-\frac{1}{2}\lambda_k} |\nabla e^{-\frac{1}{2}U_k}|) \leq c_2.$$

Therefore, the elliptic estimates implies that w^k converges in $C^{2,\alpha}$ to w^0 satisfying

$$(5.9) \quad -\Delta w^0 = \rho_2 \left(\frac{h_2 e^{-4\pi G} e^{2w^0}}{\int_{\Sigma} h_2 e^{-4\pi G} e^{2w^0}} - 1 \right).$$

One can check by applying the Pohozaev identity that the blow-up point p satisfies

$$(5.10) \quad \nabla \log h_1(p) + \nabla \tilde{G}(p) - \nabla w^0(p) = 0,$$

where \tilde{G} is the regular part of the Green function G . Note that $\Delta w^0(p) = \rho_2$. Since w^k converges to w^0 strongly in $C^{2,\alpha}$, we can apply Theorem 1.1 in [7] to (5.5) and obtain that

$$(5.11) \quad 2\rho_k - 8\pi = 2h_1^{-1}(p) e^{w^0(p)} (\Delta \log h_1(p) + 8\pi - \rho_2 - 2K(p)) \lambda_k e^{-\lambda_k} + o(e^{-\lambda_k}),$$

where $\lambda_k = \max \tilde{v}_1^k$, which tends to ∞ . Since $2\rho_k < 8\pi$, (5.11) leads to a contradiction to (1.5). Hence \mathcal{X} is compact. \square

Proof of Theorem 1.1. Choose a sequence $\{\rho^k\}$ with $\rho^k < 4\pi$ and $\rho^k \rightarrow 4\pi$. From above, we know that for $\rho^k = (\rho^k, \rho_2)$ with $\rho_2 < 4\pi$ the functional J_{ρ^k} has a minimizer u^k . By Proposition 5.1, we have that u^k converges to u^0 , which is clearly a minimizer of J_{ρ} with $\rho = (4\pi, \rho_2)$. \square

Theorem 5.2. *Let Σ be a Riemann surface with Gaussian curvature K . Suppose that*

$$(5.12) \quad \min\{\Delta \log h_1(x), \Delta \log h_2(x)\} + 4\pi - 2K(x) > 0.$$

Then J_{ρ} has a minimizer $u = (u_1, u_2)$ satisfying (1.6) for $\rho = (\rho_1, \rho_2) = (4\pi, 4\pi)$.

Proof. Let $\rho_2^k < 4\pi$ be a sequence tending to 4π . Applying Theorem 1.1, we know that J_ρ has a minimizer $u^k = (u_1^k, u_2^k)$ satisfying (1.6) for $\rho = (4\pi, \rho_2^k)$. To prove the Theorem, we only need to show that u^k converges strongly to u^0 . Assume by contradiction that u^k blows up, namely

$$\max\{\max_\Sigma u_1^k, \max_\Sigma u_2^k\} \rightarrow \infty.$$

Let

$$S_i = \{x \in \Sigma \mid \exists \text{ a sequence } \{x_k\} \text{ with } \lim_{k \rightarrow \infty} x_k = x \ \& \ \lim_{k \rightarrow \infty} u_i^k(x_k) = \infty\}.$$

By Proposition 2.5, we know that $S_1 \cap S_2 = \emptyset$. Hence we have two possibilities:

- (1) $S_1 = \{p\}$ and $S_2 = \emptyset$ or $S_2 = \{p\}$ and $S_1 = \emptyset$.
- (2) $S_1 = \{p\}$ and $S_2 = \{q\}$ with $p \neq q$.

For case (1), using the argument given in the proof of Proposition 5.1, we can compute $8\pi - \rho_2^k$ in terms of the blow-up speed as in (5.11) to get a contradiction to condition (5.12). For case (2), we consider a disk D on Σ with $p \in D$ and $q \notin \bar{D}$. $u^k = (u_1^k, u_2^k)$ satisfies on D

$$(5.13) \quad \begin{aligned} -\Delta u_1 &= 8\pi \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Omega h_2 e^{u_2}} - 1 \right) \\ -\Delta u_2 &= 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}} - 1 \right) - 4\pi \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}} - 1 \right). \end{aligned}$$

On the boundary ∂D , we know that there are two sequences $\{a_1^k\}$ and $\{a_2^k\}$ with $a_i^k \rightarrow -\infty$ for $i = 1, 2$ such that $u_i^k - a_i^k$ is bounded. Define

$$-\Delta w^k = \rho_2^k \left(\frac{h_2 e^{u_2^k}}{\int_\Sigma h_2 e^{u_2^k}} - 1 \right)$$

with the Dirichlet boundary condition $w^k(x) = 0$ for $x \in \partial D$. Now consider $v_1^k = u_1^k + w^k - a_1^k$ which satisfies

$$-\Delta v_1^k = 8\pi \left(\frac{\tilde{h}_1^k e^{v_1^k}}{\int \tilde{h}_1^k e^{v_1^k}} - 1 \right),$$

where

$$\tilde{h}_1^k = e^{a_1^k} h_1^k e^{-w^k}.$$

Note that $v_1^k|_{\partial D}$ is bounded. Define v_0^k by

$$\begin{cases} -\Delta v_0^k &= 8\pi & \text{in } D \\ v_0^k &= -v_1^k & \text{on } \partial D \end{cases}$$

and set $\tilde{v}^k = v_1^k + v_0^k$. It is clear that \tilde{v}_1^k satisfies

$$\begin{cases} -\Delta \tilde{v}_1^k = 8\pi \frac{h_k e^{\tilde{v}_1^k}}{\int_{\Sigma} h_k e^{\tilde{v}_1^k}} & \text{in } D \\ \tilde{v}_1^k = 0 & \text{on } \partial D \end{cases}$$

with $h_k = \tilde{h}_1^k e^{-v_0^k}$. It is also clear that

$$\frac{\int_D h_k e^{\tilde{v}_1^k}}{\int_{\Sigma} h_k e^{\tilde{v}_1^k}} \rightarrow 1,$$

as $k \rightarrow \infty$. As in the proof of Proposition 5.1, we can show that h_k converges strongly in $C^{2,\alpha}$. Hence, again we can use the result in [6] for the Dirichlet boundary problem to get that the divergence of \tilde{v}_1^k implies that

$$(5.14) \quad \lim_{k \rightarrow \infty} (\Delta \log h_1 - \Delta v_0^k - \Delta w^k - 2K) = 0$$

at the blow-up point p . Since we have $-\Delta w^k \rightarrow -4\pi$ and $-\Delta v_0^k = 8\pi$, from (5.14) we have

$$\Delta \log h_1 + 4\pi - 2K = 0,$$

which contradicts (5.12). \square

A direct consequence of Theorem 1.1 is

Corollary 5.1. *Let Σ be a closed surface of genus greater than 0. Suppose that Σ has constant curvature and h_1 and h_2 are constants. Then equation (1.6) has a solution for any $\rho = (\rho_1, \rho_2)$ with $\rho_i \leq 4\pi$.*

6. EXISTENCE: A SUPERCRITICAL CASE

In this section we consider the case that $\rho_i \in (0, 8\pi)$, but $\rho_i \neq 4\pi$ for any i . This is a supercritical case, in the sense that the functional (1.4) has no lower bound. The result obtained in this section is not related to the curvature of the underlying surface. Therefore, in this section we consider the existence in a bounded domain $\Omega \subset \mathbb{R}^2$. A similar result as obtained in Theorem 6.1 holds for a closed surface with genus greater than 0, which is Theorem 1.4.

Theorem 6.1. *Let Ω be a smooth bounded domain and $\rho_i \in (0, 4\pi) \cup (4\pi, 8\pi)$ ($i = 1, 2$) two constants. Suppose that the boundary $\partial\Omega$ of Ω*

has at least two components. For any positive function h_i ($i = 1, 2$), there exists a solution $u = (u_1, u_2)$ satisfying

$$(6.1) \quad \begin{aligned} \Delta u_1 &= 2\rho_1 \frac{h_1 e^{u_1}}{\int_{\Omega} h_1 e^{u_1}} - \rho_2 \frac{h_2 e^{u_2}}{\int_{\Omega} h_2 e^{u_2}} \\ \Delta u_2 &= 2\rho_2 \frac{h_2 e^{u_2}}{\int_{\Omega} h_2 e^{u_2}} - \rho_1 \frac{h_1 e^{u_1}}{\int_{\Omega} h_1 e^{u_1}}, \end{aligned}$$

and $u_i = 0$ on $\partial\Omega$ ($i = 1, 2$).

When $N = 1$, the result was proven in [9], see also [19]. Recently, Lucia and Nolasco [13] obtained a non-trivial solution of (6.1) under some conditions when h_i is a constant for $i = 1, 2$.

Our method of the proof closely follows [9], which, in turn, is motivated by [20] (see also [21]). Hence, here we only sketch the main ideas of the proof. When one of the ρ_i is 4π , the existence becomes more subtle. We will consider this case elsewhere.

Proof of Theorem 6.1. When $\rho_i < 4\pi$ ($i = 1, 2$), the Theorem is true for any bounded domain. We first consider the case $\rho_i \in (4\pi, 8\pi)$ ($i = 1, 2$). We divide the proof into several steps.

Step 1. We first define the center of mass of a function $v \in H_0^1(\Omega)$ by

$$m_c(v) = \frac{\int_{\Omega} x e^v}{\int_{\Omega} e^v}.$$

Assume, for the simplicity of the notation, that $\partial\Omega = \partial_+\Omega \cup \partial_-\Omega$ has only two disjoint components. Define a family of functions

$$(6.2) \quad \Gamma : (-\infty, +\infty) \rightarrow H_0^1(\Omega) \times H_0^1(\Omega)$$

with $\Gamma(t) = (\gamma_1(t), \gamma_2(t))$ satisfying

$$(6.3) \quad J_{\rho}(\Gamma(t)) \rightarrow \pm\infty \quad \text{as } t \rightarrow -\infty$$

and

$$(6.4) \quad m_c(\gamma_i) \rightarrow \partial_{\pm}\Omega \quad \text{as } t \rightarrow \pm\infty.$$

The existence of such a family is guaranteed by $\rho_i > 4\pi$ ($i = 1, 2$). Denote the set of all such families by \mathcal{X} and define a minimax value

$$(6.5) \quad \alpha_{\rho} := \inf_{\Gamma \in \mathcal{X}} \sup_t J_{\rho}(\Gamma(t)).$$

Step 2. The minimax value $\alpha_{\rho} > -\infty$.

The proof of the step follows from an improved Moser-Trudinger inequality under a condition introduced by Aubin.

Lemma 6.2. *Let S_1 and S_2 be two subsets of $\bar{\Omega}$ satisfying $\text{dist}(S_1, S_2) \geq \delta_0 > 0$ and $\gamma_0 \in (0, 1/2)$. For any $\epsilon > 0$, there exists a constant $c = c(\epsilon, \delta_0, \gamma_0) > 0$ such that*

$$J_{(8\pi-\epsilon, 8\pi-\epsilon)}(u) \geq -c$$

holds for all $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ satisfying

$$(6.6) \quad \frac{\int_{S_1} e^{u_i}}{\int_{\Omega} e^{u_i}} \geq \gamma_0 \quad \text{and} \quad \frac{\int_{S_2} e^{u_i}}{\int_{\Omega} e^{u_i}} \geq \gamma_0,$$

for $i = 1, 2$.

Let $\gamma_0 \subset \Omega$ be a closed curve in Ω enclosing the inner boundary of Ω . Each curve starting from $\partial_-\Omega$ and ending at $\partial_+\Omega$ intersects with γ_0 . By Lemma 6.2, we can show that

$$J_\rho(u) > -c,$$

for any $u \in H_0^1(\Omega) \times H_0^1(\Omega)$ with center of mass $m_c(u) \in \gamma_0$. See the argument in [9]. Hence we prove the step.

Step 3. $\alpha_\rho : (4\pi, 8\pi) \times (4\pi, 8\pi)$ is non-increasing in the following sense: if $\rho = (\rho_1, \rho_2)$ and $\rho' = (\rho'_1, \rho'_2)$ with $\rho_1 \leq \rho'_1$ and $\rho_2 = \rho'_2$, then

$$\frac{\alpha_\rho}{\rho_1} \geq \frac{\alpha_{\rho'}}{\rho'_1}.$$

This is easy to check.

Step 4. Now fix ρ_2 and define

$$\Lambda_1 = \Lambda_1(\rho_2) = \{\rho_1 \in (4\pi, 8\pi) \mid \frac{\alpha_{\rho_1, \rho_2}}{\rho_1} \text{ is differentiable at } \rho_1\}.$$

It is clear that Λ_1 is dense, i.e., $\bar{\Lambda}_1 = [4\pi, 8\pi]$. Following a method given by Struwe [20] and [21], see also [8], we can prove that for fixed ρ_2 and $\rho_1 \in \Lambda_1$, α_{ρ_1, ρ_2} is achieved by $u = (u_1, u_2)$, which is a solution of (6.1).

Step 5. Now for any $\rho = (\rho_1, \rho_2) \in (4\pi, 8\pi) \times (4\pi, 8\pi)$, we have a sequence $\rho^k = (\rho_1^k, \rho_2)$ with $\rho_1^k \in \Lambda_1$ and $u^k = (u_1^k, u_2^k)$ satisfying (6.1) with ρ^k .

Since $\rho_2 \neq 4\pi m$, by Proposition 2.5 we see that u^k converges to $u^0 = (u_1^0, u_2^0)$, which is a solution of (6.1). This finishes the proof for the case $\rho_i \in (4\pi, 8\pi)$ ($i = 1, 2$).

When $\rho_1 < 4\pi$ and $\rho_2 \in (4\pi, 8\pi)$, Γ defined by (6.2)-(6.4) does not exist. We need to modify the definition. Consider $\Gamma = (\gamma_1, \gamma_2)$ with (6.2) and

$$(6.7) \quad m_c(\gamma_2) \rightarrow \partial_\pm \Omega \quad \text{as } t \rightarrow \pm\infty.$$

Then the same argument finishes the proof of the Theorem. \square

Theorem 1.4 can be proven in a similar way.

Furthermore, in this case we can compute the topological degree of the solution space as in [7]. First we define the Leray-Schauder degree for system (1.6). For the Leray-Schauder degree for the Liouville type equation, see [14] and [7]. Consider $\rho = (\rho_1, \rho_2)$ with $\rho_i \neq 0 \pmod{4\pi}$. Define an operator

$$T_\rho := \Delta^{-1} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \begin{pmatrix} \frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - 1 \\ \frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - 1 \end{pmatrix}.$$

T_ρ acts on $H_0^1 \times H_0^1$, where $H_0^1 = \{u \in H^1 \mid \int u = 0\}$. Set \mathcal{X}_ρ the solution space of (1.6). From above, we know that \mathcal{X}_ρ is compact. Hence we can define the Leray-Schauder degree for (1.6)

$$d_\rho := \deg(I + T_\rho, B_R, 0),$$

where I is the identity and $B_R := \{u = (u_1, u_2) \in H_0^1 \times H_0^1 \mid \|u_1\| + \|u_2\| \leq R\}$ for a large $R > 0$. It is clear that d_ρ is well-defined. The homotopy invariance of the Leray-Schauder degree implies that d_ρ is independent of h_1 and h_2 . Furthermore Theorem 1.2 implies that $d_\rho = d_{\tilde{\rho}}$ if there are two integers m_1 and m_2 such that $\rho_i \in (4\pi(m_i - 1), 4\pi m_i)$ and $\tilde{\rho}_i \in (4\pi(m_i - 1), 4\pi m_i)$.

Theorem 6.3. *Let Σ be a closed surface of genus g . Then we have for $\rho = (\rho_1, \rho_2)$ with $\rho_1 \in (0, 4\pi)$ and $\rho_2 \in (4\pi, 8\pi)$*

$$d_\rho = 2g - 1.$$

The proof follows from the argument given in the proof of Theorem 1.1 and a result given in [7]. In a forthcoming paper, we will compute the topological degree for the Toda system for the general case.

7. APPENDIX. BEHAVIOR OF SINGULARITIES OF SOLUTIONS WITH FINITE ENERGY

In this Appendix, we consider the asymptotic behavior of singularities of solutions to the Toda system.

Let $u = (u_1, u_2, \dots, u_N)$ be a solution of

$$(7.1) \quad -\Delta u_i = \sum_{j=1}^N a_{ij} e^{u_j}, \quad \text{for } i = 1, 2, \dots, N$$

in a punctured disk $D^* = D \setminus \{0\}$ with

$$(7.2) \quad \int_D e_i^u < \infty.$$

By the potential analysis, one can check that there is a constant γ_i

$$(7.3) \quad \lim_{|x| \rightarrow 0} \frac{u_i}{-\log |x|} = \gamma_i,$$

for each i . (7.2) and (7.3) imply that $\gamma_i \leq 2$ for any i . Furthermore, one can check that

$$(7.4) \quad \lim_{|x| \rightarrow 0} \frac{u_i + \gamma_i \log |x|}{\log |x|} = 0.$$

Proposition 7.1. *For any i , we have*

$$\gamma_i < 2.$$

Proof. Suppose that there is $j \in I$ such that $\gamma_j = 2$. We claim that $\gamma_{j-1} = 2$ or $\gamma_{j+1} = 2$. Assume by contradiction that $\gamma_{j-1} < 2$ and $\gamma_{j+1} < 2$. We assume that neither $j = 1$ nor $j = N$. (For the cases $j = 1$ and $j = N$, the argument is the same.) Consider in D^*

$$-\Delta u_j = -e^{u_{j-1}} + 2e^{u_j} - e^{u_{j+1}} =: f.$$

Since $\gamma_j = 2$, $\gamma_{j-1} < 2$ and $\gamma_{j+1} < 2$, we know that $f(x) > 0$ in a small punctured disk $D_\delta^* = D_\delta \setminus \{0\}$. Set

$$v(x) = -\frac{1}{2\pi} \int_{D_\delta} \log |x-y| f(y)$$

and $w = u_j - v$. It is clear that $\Delta v = -f$ and $\Delta w = 0$. One can check that

$$\lim_{|x| \rightarrow 0} \frac{v}{-\log |x|} = 0,$$

which implies

$$\lim_{|x| \rightarrow 0} \frac{w(x)}{-\log |x|} = \lim_{|x| \rightarrow 0} \frac{u_j - v}{-\log |x|} = 2.$$

Since w is harmonic in D_δ^* , we have $w = -2 \log |x| + w_0$ with a smooth harmonic function w_0 in D_δ . By definition, we know $v > 0$. Thus, we have

$$\int_{D_\delta} e^{u_j} = \int_{D_\delta} e^{w+v} \geq \int_{D_\delta} \frac{1}{|x|^2} e^{w_0} = \infty,$$

a contradiction. A similar argument implies that if $\gamma_j = \gamma_{j+1} = \dots = \gamma_{j+k} = 2$, then either $\gamma_{j-1} = 2$ or $\gamma_{j+k+1} = 2$, by using the equation for $\frac{1}{k}(u_j + u_{j+1} + \dots + u_{j+k})$. Hence we can show $\gamma_i = 2$ for all i . Now we consider $\tilde{u} = \frac{1}{N}(\sum_{i=1}^N u_i)$. It is clear that \tilde{u} satisfies

$$-\Delta \tilde{u} = \frac{1}{N}(e^{u_1} + e^{u_2}) > 0$$

with $\int_D e^{\tilde{w}} < \infty$ and $\lim_{|x| \rightarrow 0} \frac{\tilde{w}}{-\log|x|} = 2$. The same argument as above gets a contradiction. Hence $\gamma_i < 2$ for any i . \square

Proof of (3.11) and (3.12). Now we apply the Proposition to solutions of (3.9) with condition (3.10). Let $u = (u_1, u_2, \dots, u_N)$ be such a solution. The potential analysis gives us

$$(7.5) \quad \lim_{|x| \rightarrow \infty} \frac{u_i}{\log|x|} = -\gamma_i = -\frac{1}{2\pi} \sum_{j=1}^N a_{ij} \int_{\mathbb{R}^2} |x|^{\mu_j} e^{u_j}.$$

We consider $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N)$ with $\int_D e^{\tilde{u}_i} < \infty$ for any i and

$$\tilde{u}_i = u_i \left(\frac{x}{|x|} \right) - (4 + \mu_i) \log|x|.$$

It is easy to check that \tilde{u} satisfies (7.1) in D^* with

$$\lim_{|x| \rightarrow 0} \frac{\tilde{u}_i(x)}{-\log|x|} = 4 + \mu_i - \gamma_i.$$

Applying the Proposition, we have

$$4 + \mu_i - \gamma_i < 2,$$

for any i . This is (3.12). From the above inequalities and the potential analysis, we have

$$\tilde{u}_i = -(4 + \mu_i - \gamma_i) \log|x| + O(1),$$

from which we have the first formula in (3.11). The second formula follows from the potential analysis. \square

Lemma 7.2. *Let $u \in C^2$ be a solution of*

$$(7.6) \quad -\Delta u = 2|x|^{2\alpha}|x - q|^2 e^u$$

with $\alpha \geq 0$ and

$$(7.7) \quad \gamma := \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{2\alpha}|x - q|^2 e^u < \infty.$$

then $\gamma > 2 + \alpha$.

Proof. We show this lemma following closely [23]. Set

$$w = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\log|x - y| - \log(|y| + 1)) 2|y|^{2\alpha}|y - q|^2 dy.$$

It is easy to check by the potential analysis that $u = w + c$ for some constant c and

$$w \geq -2\gamma \log|x| - c_1,$$

for some constant $c_1 > 0$. This, together with the finiteness condition (7.7), implies that $\gamma > 2 + \alpha$. \square

Remark 7.3. *In fact, all solutions of (7.6) and (7.7) can be classified.*

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