Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Some piece-wise smooth Lagrangian fibrations

by

Ricardo Castaño-Bernard and Diego Matessi

Preprint no.: 47 2005

Some piece-wise smooth Lagrangian fibrations.

R. Castaño-Bernard* and D. Matessi†

May 10, 2005

Abstract

Motivated by the work of M. Gross [5] on topological Mirror Symmetry, we describe some piece-wise smooth local Lagrangian models of singular torus fibrations. In order to understand them better we also develop some tools which generalize the notion of action-angle coordinates of smooth fibrations.

1 Introduction.

In this paper we present some results which are part of a current research project of the authors on Lagrangian torus fibrations of symplectic 6-manifolds. The work is primarily motivated by the topological description of Mirror Symmetry due to Mark Gross [5]. Let $f : X \to B$ be a proper C^∞ submersion of a symplectic manifold (X, ω) with $\mathrm{Crit}(f) = \varnothing$ and such that all the fibres of f are connected Lagrangian submanifolds, that is to say, a Lagrangian T^n bundle. It is well known that B carries the structure of integral affine manifold. In other words, B has an atlas $\mathscr{A}$ whose transition maps are integral affine linear transformations $\mathbb{R}^n \rtimes \mathrm{Gl}(n, \mathbb{Z})$.

When $f : X \to B$ has singular fibres –in which case we call f a Lagrangian fibration– B no longer carries an affine structure but an affine structure *with singularities*. Roughly speaking this means that there is a dense open subset $B_0 \subseteq B$ such that $(B_0, \mathscr{A})$ is integral affine. This process can be reversed: it is a standard fact that given an integral affine manifold $(B_0, \mathscr{A})$, there is a family of maximal lattices, $\Lambda \subset T^*B_0$, a symplectic manifold $X(B_0)$ and a short exact sequence:

$$0 \to \Lambda \to T^*B \to X(B_0) \to 0$$

inducing a Lagragian T^n bundle $f_0 : X(B_0) \to B$.

This article is motivated by the following problem. Suppose we are given a compact integral affine manifold with singularities. Consider the associated symplectic manifold $X(B_0)$ foliated by the fibres of the induced Lagrangian T^n bundle. We can ask: is there a compact smooth symplectic manifold $X(B)$ and a Lagragian T^n fibration $f : X(B) \to B$ such that

$$\begin{array}{ccc} X(B_0) & \hookrightarrow & X(B) \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & B \end{array}$$

commutes? A conjecture says that in the case of the fibrations described by Gross this should be true. One of the problems is that one has to allow piece-wise smooth fibrations. Furtheremore, one should obtain symplectic manifolds $X(B)$ symplectomorphic to known examples of Calabi-Yau manifolds –such as the K3 surface or the quintic hypersurface in $\mathbb{P}^4$. A *topological* compactification has been achieved by Gross [5] by means of gluing some

*Max-Plank-Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Germany (email: castano@mis.mpg.de)

†Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, Via Bellini 25/G, I-15100 Alessandria, Italy (email:matessi@unipmn.it)

standard singular local models to the smooth torus bundle. One of the results of this paper is a simple construction of piece-wise smooth local Lagrangian models. Our hope is that these will allow us to achieve a *symplectic* compactification. It is not yet clear how to do this. Clearly, the symplectic topology of the resulting manifold $X(B)$ a priori should depend on the affine structure $\mathscr{A}$ but also on the structure of the singular fibres needed to compactify. The problem is two-fold. First one has to obtain appropiate models of singular Lagrangian fibrations, and it is not clear how regular the maps describing them can be. Then one has to understand how to glue them together, which is not an easy problem if the models are described by maps which are particularly non-smooth, like in our case.

Over the past few years, there has been important progress towards the understanding of affine geometry in connection to Calabi-Yau manifolds and Mirror Symmetry. Gross and Siebert [7], [8] introduce a discrete Legendre transform of affine manifolds and Log structures to understand how to (re)construct a mirror pair from the affine data. An alternative approach is proposed by Kontsevich and Soibelman [13], [14]. See also [10], [11].

The contributions mentioned above deal mostly with the 'complex side' of Mirror Symmetry. In our project we aim at developing a tool box that hopefully will help us to understand the 'symplectic side' which should be, in theory, simpler. In a long series of papers [15], [16], [17] W.-D. Ruan constructed examples of piece-wise C^∞ Lagrangian fibrations of Calabi-Yau manifolds. Ruan's method consists, roughly speaking, on a "gradient flow deformation" of a well known Lagrangian fibration of a singular Calabi-Yau variety to a Lagrangian fibration of a non-singular Calabi-Yau manifold. Our approach, rather than being global, is semi-global. It may be that some of our models are also implicit in Ruan's construction, so isolating them and understanding them better may also cast a bridge between the work of Ruan and Gross's approach.

This paper is organised as follows. In §2 we review Gross's topological compactifications of torus bundles. In §3 we overview briefly the theory of action-angle coordinates for Lagrangian submersions and describe some known examples of affine manifolds with singularities. In §4 we describe some examples of Lagrangian fibrations given by C^∞ maps, resembling the topology of the positive and generic fibrations of §2. These models are already well understood [1], so here we just give a short description. Our results are described in §5 and §6. In §5 we give a simple general method to construct piece-wise smooth S^1-invariant Lagrangian fibrations with interesting singular fibres. These fibrations fail to be smooth along the slice $\mu^{-1}(0)$ of the moment map $\mu : X \to \mathfrak{s}^1$. As a byproduct, these fibrations do not induce affine structures with singularities on the base. The affine structure is "broken" in two pieces, separated by a codimension 1 wall containing the discriminant locus of the fibration. In Example 5.4 we give an explicit fibration with discriminant locus having the shape of an amoeba. This model resembles an example of special Lagrangian fibration proposed by Joyce [12] and topologically is a perturbation of the negative fibration described by Gross. We are also able to perturb this model to a piecewise C^∞ fibration with discriminant locus of mixed codimension one and two. Again, the affine structure of this fibration is broken.

The results of the last section are only sketched, detailed proofs will appear elsewhere. These results aim at a theory of action-angle coordinates for piece-wise C^∞ Lagrangian submersions with S^1-symmetry. In particular they allow us to produce examples of piece-wise C^∞ Lagrangian fibrations by stitching together two honest C^∞ fibrations along the "seam" $\mu^{-1}(0)$. In addition to the affine data induced by each of the two pieces, the stitching data consists of a family of closed 1-forms on a torus satisfying some integrality properties. The family is parametrized by the codimension 1 wall on the base. The piece-wise smooth Examples of §5 are of this type. With such fibrations the computation of the periods fails to provide information on the topological monodromy. This is in contrast with what happens in the smooth situation. One of the results of §6 is that in the case of these stitched Lagrangian fibrations monodromy can be interpreted as a jump in the cohomology class of the family of closed 1-forms.

Aknowledgments: The second author was partially funded by an EPSRC Research Grant, UK. Both authors would like to thank the Abdus Salam International Centre for Theoretical Physics in Trieste and Max Planck Institute in Leipzig for their hospitality.

2 The topology.

In this section we review Gross's work [5] on topological mirror symmetry which is the main motivation for our own work on Lagrangian fibrations. Gross explains how to compactify certain n-torus bundles over n-dimensional manifolds obtaining topological models of Calabi-Yau manifolds. We shall focus only on dimensions $n = 2$ and 3.

Definition 2.1. A *topological T^n fibration* $f : X \to B$ is a continuous, proper, surjective map between smooth manifolds, $\dim X = 2n$, $\dim B = n$, such that for a dense open set $B_0 \subseteq B$ and for all $b \in B_0$ the fibre $X_b = f^{-1}(b)$ is homeomorphic to an n-torus. We call $\Delta := B - B_0$ the discriminant locus of f. We require our topological T^n fibrations to satisfy the following conditions:

1. There is a co-dimension four closed subset $\Sigma \subset X$ such that for each $b \in B$, the set $\Sigma \cap X_b$ consists of a finite union of locally closed submanifolds of dimension at most $n - 2$. Furthermore, $f(\Sigma) = \Delta$;
2. for $n = 2$, Δ is a finite union of points;
3. for $n = 3$, $\Delta = \Delta_d \cup \Delta_g \cup \Delta_a$ where
   (a) Δ_d is a finite union of points;
   (b) Δ_g is a finite disjoint union of 1-manifolds diffeomorphic to an open interval with end points contained in Δ_d forming a 3-valent vertex;
   (c) Δ_a is a finite disjoint union of closed subsets, each connected component being a 3-legged amoeba with ends contained in Δ_g.

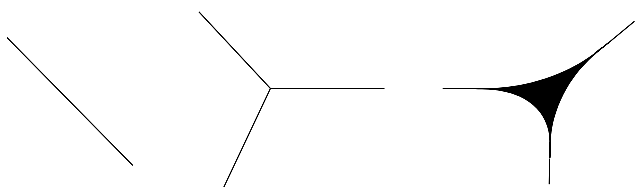

Figure 1: Components of Δ, $n = 3$.

The topological T^3 fibrations in [5] have everywhere codimension two discriminant, a connected trivalent graph to be precise. In this paper we consider a slightly more general situation by allowing Δ to jump dimension, i.e. including a region Δ_a, which may be regarded as a "fattening" of a graph near some vertices. We give explicit examples of fibrations with such discriminant locus later on. For now, we only pay attention to the trivalent graph case.

Definition 2.2. Let $f : X \to B$ be a topological T^n fibration and let $U \subset B$ be an open neighbourhood of $b \in \Delta$ such that $U \cap \Delta$ is homeomorphic to a point, when $n = 2$; or else to an open interval, when $b \in \Delta_g$; to a cone over three points, when $b \in \Delta_d$; or to a 3-legged amoeba, when $b \in \Delta_a$. Let X_{b_0} be a fiber over $b_0 \in U - \Delta$. The image of

$$\mathcal{M}_b : \pi_1(U - \Delta, b_0) \to SL(H_1(X_{b_0}, \mathbb{Z}))$$

is called the *local monodromy group about X_b* (also denoted by $\mathcal{M}_b$). If there is a basis of $H_1(X_{b_0}, \mathbb{Z})$ with respect to which $\mathcal{M}_b$ is generated by unipotent matrices we say that $\mathcal{M}_b$ is *semi-stable*. We shall also say that X_b is a *semi-stable singular fibre.*

The above definition is motivated by the well-known notion of semi-stable singularities of elliptic fibrations.

In [5] Gross proposes a method for constructing semi-stable T^n fibrations with prescribed monodromy. Now we review the construction of these fibrations. For the details we refer the reader to [5] §2. The construction of some of the building blocks rely on the following:

Proposition 2.3. Let Y be a manifold of dimension $2n-1$. Let $\Sigma \subseteq Y$ be an oriented submanifold of codimension three and let $Y' = Y - \Sigma$. Let $\pi' : X' \to Y'$ be a principal S^1-bundle over Y' with Chern class $c_1 = \pm 1$. For each triple (Y, Σ, π') there is a unique compactification $X = X' \cup \Sigma$ extending the topology of X', making X into a manifold and such that

$$\begin{array}{ccc} X' & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \hookrightarrow & Y \end{array}$$

commutes, with $\pi : X \to Y$ proper and $\pi|_\Sigma : \Sigma \to \Sigma$ the identity.

Remark 2.4. One can explicitly describe the above compactification as follows. For any point $p \in \Sigma$ there is a neighbourhood $U \subset Y$ of p such that $U \cong \mathbb{R}^3 \times \mathbb{C}^{n-2}$ and $U \cap \Sigma$ can be identified with $\{0\} \times \mathbb{C}^{n-2}$. By unicity of π, there is a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\cong} & \mathbb{C}^2 \times \mathbb{C}^{n-2} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\bar{\pi}}\big\downarrow \\ U & \xrightarrow{\cong} & \mathbb{R}^3 \times \mathbb{C}^{n-2} \end{array} \qquad (1)$$

where $\bar{\pi}(z_1, z_2, \zeta) = (|z_1|^2 - |z_2|^2, z_1 z_2, \zeta)$, $\zeta \in \mathbb{C}^{n-2}$.

The basic principle to construct topological T^n fibrations can be described as follows. Typically, one starts with a manifold $Y = B \times T^{n-1}$ with $\dim B = n$, a submanifold $\Sigma \subset Y$ and a map $\pi : X \to Y$ as in Proposition 2.3. The trivial T^{n-1} fibration $P : Y \to B$ can be lifted to a T^n fibration $f := P \circ \pi : X \to B$ with discriminant locus $\Delta := P(\Sigma)$. One can readily see that for $b \in \Delta$, the singularities of the fibre X_b occur along $\Sigma \cap P^{-1}(b)$. The set Σ –which is the locus of singular fibres of π– can be regarded as the locus where the vanishing cycles of the fibres of f collapse (cf. Figure 2).

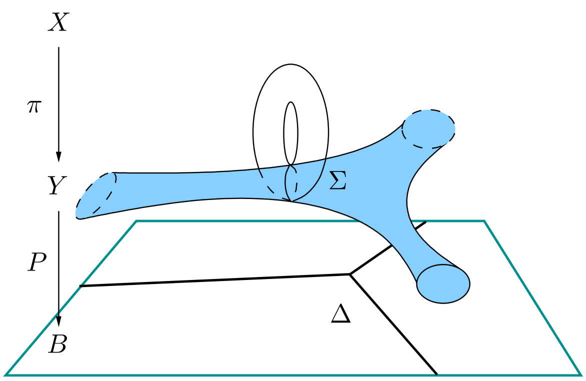

Figure 2: Negative fibration.

Example 2.5. Let $D^* = D - \{0\}$ and let $f_0 : X_0 \to D^*$ be a T^2-bundle with monodromy $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We can use Proposition 2.3 to compactify X_0 as follows. The monodromy invariant cycle, $L \in H_1(f_0^{-1}(b), \mathbb{Z})$, induces a fibre preserving $T(L)$ action, with $T(L) =$

$L \otimes \mathbb{R}/L$. The quotient modulo this action yields an S^1-bundle $\pi_0 : X_0 \to Y_0 = D^* \times S^1$. One can verify that π_0 extends to an S^1-bundle $\pi' : X' \to Y' = D \times S^1 - \{(0,p)\}$, where $p \in S^1$. Furthermore $c_1(\pi') = \pm 1$. Then Proposition 2.3 ensures that X' compactifies to a manifold $X = X' \cup \{pt\}$ and that there is a proper map $\pi : X \to Y = D \times S^1$ extending π'. Defining $P : Y \to D$ as the projection map, we obtain a semi-stable fibration $f = P \circ \pi : X \to D$ extending f_0. The only singular fibre, $f^{-1}(0)$, is homeomorphic to a pinched torus, i.e. homeomorphic to a Kodaira type I_1 singular fibre.

In dimension $n = 3$ there is a local model with discriminant of the type $\Delta = \Delta_g$ –the so-called generic fibration– and two models with discriminant of the type $\Delta = \Delta_g \cup \Delta_d$ –the so-called positive and negative fibrations– which are dual in certain sense. These three models are due to Gross [5]. A model with discriminant of the type $\Delta = \Delta_g \cup \Delta_a$ will be given later on in this article.

Example 2.6 (Generic fibration). Let $B = D \times (0,1)$, where D is a centered disc, and let $Y = T^2 \times B$. Define $\Sigma \subset Y$ to be the cylinder sitting above $\{0\} \times (0,1) \subset B$ defined as follows. Let e_2, e_3 be a basis of $H_1(T^2, \mathbb{Z})$. Let $S^1 \subset T^2$ be the circle whose homology class is represented by e_3. Define $\Sigma = S^1 \times \{0\} \times (0,1)$. Now let $\pi' : X' \to Y' := Y - \Sigma$ be an S^1-bundle with Chern class $c_1 = 1$. Then X' compactifies to a manifold $X = X' \cup \Sigma$ and there is a proper map $\pi : X \to Y$ extending π'. We can now define $f = P \circ \pi : X \to B$ where $P : Y \to B$ is the projection. Then it is clear f is a T^3 fibration with singular fibres homeomorphic to $I_1 \times S^1$ lying over $\Delta := \{0\} \times (0,1)$. One can take e_1, e_2, e_3 as a basis of a regular fibre X_b, where e_1 is an orbit of π. In this basis, e_1 and e_3 are monodromy invariant and the monodromy of f about Δ is represented in this basis by

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Example 2.7 (Negative fibration). Let $Y = T^2 \times B$ with B homeomorphic to a 3-ball. Let $\Delta \subset B$ be a cone over three distinct, non-collinear points. We write $\Delta = \{b_0\} \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$ where b_0 is the vertex of Δ and the Δ_i are the legs of Δ. Fix a basis e_2, e_3 for $H_1(T^2, \mathbb{Z})$. Define $\Sigma \subset T^2 \times B$ to be a pair of pants lying over Δ such that for $i = 1,2,3$, $\Sigma \cap T^2 \times \Delta_i$ is a leg of Σ which is the cylinder generated by $-e_3$, e_2 and $-e_2 + e_3$ respectively. These legs are glued together along a nodal curve or 'figure eight' lying over b_0. Now consider an S^1-bundle $\pi' : X' \to Y' = Y - \Sigma$ with Chern class $c_1 = 1$. This bundle compactifies to $\pi : X \to Y$. Now consider the projection map $P : Y \to B$. The composition $f = P \circ \pi$ is a proper map. The generic fibre of f is a three-torus. For $b \in \Delta$ the fibre $f^{-1}(b)$ is singular along $P^{-1}(b) \cap \Sigma$, which is a circle when $b \in \Delta_i$, or the aforementioned figure eight when $b = b_0$. Thus the fibres over Δ_i are homeomorphic to $I_1 \times S^1$, whereas the central fibre, X_{b_0}, is singular along a nodal curve. A regular fibre can be regarded as the total space of an S^1-bundle over $P^{-1}(b)$. We can take as a basis of $H_1(X_b, \mathbb{Z})$, $e_1(b), e_2(b), e_3(b)$, where e_2 and e_3 are the 1-cycles in $P^{-1}(b) = T^2$ as before and e_1 is a fibre of the S^1-bundle. The cycle $e_1(b)$ vanishes as $b \to \Delta$. In this basis, the monodromy matrices corresponding to loops γ_i about Δ_i with $\gamma_1\gamma_2\gamma_3 = 1$ are

$$T_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Observe that in the above examples there is a fibre-preserving S^1-action, induced by the S^1 bundle π'. One can use the same principle to construct T^2-invariant fibrations starting from suitable compactifications of T^2-bundles:

Example 2.8 (Positive fibration). Let $Y = S^1 \times B$ with B and $\Delta \subset B$ as in Example 2.7. Let $Y' = Y \setminus (\{p\} \times \Delta)$, where $p \in S^1$. Let $L \cong \mathbb{Z}^2$ and define $T(L) = L \otimes_{\mathbb{Z}} \mathbb{R}/L$. Now consider a principal $T(L)$-bundle $\pi' : X' \to Y'$. Under some mild assumptions on π'

(cf. [5] Prop. 2.9), there is a unique manifold X with $X' \subset X$ extending the topology of X' and a proper extension $\pi : X \to Y$ of π'. The composition of π with the projection $Y \to B$ defines a topological T^3-fibration, $f : X \to B$. The fibre of f over $b \in B \setminus \Delta$ is T^3. The fibre over $b \in \Delta_i$ is homeomorphic to $S^1 \times I_1$, whereas the fibre over the vertex $b_0 \in \Delta$ is homeomorphic to $S^1 \times T^2/(\{point\} \times T^2)$. One can check that the monodromy of this model can be represented by the inverse transpose matrices of the previous example.

The above models can be glued together to produce semi-stable T^3 fibrations with interesting properties. We refer the reader to [5] §1 for the details:

Theorem 2.9 (Gross). Let B be a 3-manifold and let $B_0 \subseteq B$ be a dense open set such that $\Delta := B - B_0$ is a trivalent graph, $\Delta_d \cup \Delta_g$. Assume that the vertices of Δ are labeled, i.e. Δ_d decomposes as a union $\Delta_+ \cup \Delta_-$ of positive and negative vertices. Suppose there is a T^3 bundle $f_0 : X(B_0) \to B_0$ such that its local monodromy $\mathcal{M}_b$ is generated by

1. T as in (2), when $b \in \Delta_g$;
2. T_1, T_2, T_3 as in (3), when $b \in \Delta_-$;
3. $(T_1^t)^{-1}, (T_2^t)^{-1}, (T_3^t)^{-1}$, when $b \in \Delta_+$.

Then there is a T^3 fibration $f : X \to B$ with semi-stable singular fibres and a commutative diagram:

$$\begin{array}{ccc} X(B_0) & \hookrightarrow & X \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & B. \end{array}$$

Definition 2.10. The manifold X obtained from $X(B_0)$ as in Theorem 2.9 is called a *topological semi-stable compactification.*

In suitable cases, Theorem 2.9 produces *dual* semi-stable T^3 fibrations of mirror pairs of Calabi-Yau manifolds. In §3 we shall review how to construct a T^3 bundle $X(B_0)$ which compactifies to a smooth manifold X diffeomorphic to the quintic hypersurface in $\mathbb{P}^4$. The compactification of the dual bundle produces a manifold $\check{X}$, which is diffeomorphic to the mirror quintic. Gross uses this construction as a topological evidence that the SYZ duality should indeed explain Mirror Symmetry.

Our work is motivated by a conjecture stating that there should exist *symplectic compactifications* with respect to which the fibres are Lagrangian. We shall explain how this problem can be approached using affine manifolds and Lagrangian fibrations over them.

3 Affine manifolds and Lagrangian fibrations

Definition 3.1. An n-dimensional affine manifold is a pair $(B, \mathscr{A})$ where B is an n-dimensional manifold and $\mathscr{A}$ is an atlas of coordinate charts $\{U_i, \phi_i\}$ whose transition maps are affine, i.e. such that $\phi_i \circ \phi_j^{-1} \in \mathbb{R}^n \rtimes \mathrm{Gl}_n(\mathbb{R})$, for all i, j, $\mathscr{A}$ is also called an affine structure on B. We say that $(B, \mathscr{A})$ is integral affine if $\phi_i \circ \phi_j^{-1} \in \mathbb{R}^n \rtimes \mathrm{Gl}_n(\mathbb{Z})$.

When B is integral affine we can define a maximal integral lattice $\Lambda \subset T^*_B$ by

$$\Lambda|_U = \mathrm{span}_{\mathbb{Z}} \langle du_1, \ldots, du_n \rangle$$

for all U. Therefore to every integral affine manifold $(B, \mathscr{A})$ we can associate the $2n$-dimensional manifold

$$X(B, \mathscr{A}) = T^*_B/\Lambda,$$

which together with the projection $f : X(B, \mathscr{A}) \to B$ forms a T^n fibre bundle. Also notice that the standard symplectic form ω on T^*_B descends to $X(B, \mathscr{A})$ and the fibres of f are Lagrangian. In the next section we explain how affine manifolds arise naturally from Lagrangian fibrations.

Action-angle coordinates.

We review here some classical facts about Lagrangian fibrations which we will use in the next Sections. For details we refer to Duistermaat [2]. Assume we are given a $2n$-dimensional symplectic manifold X with symplectic form ω, a smooth n-dimensional manifold B and a proper smooth submersion $f : X \to B$ whose fibres are connected Lagrangian submanifolds. For every $b \in B$, denote by F_b the fibre of f at b.

Proposition 3.2. In the above situation, for every $b \in B$, T_b^*B acts transitively on F_b. In particular there exists a maximal lattice Λ_b of T_b^*B such that F_b is naturally diffeomorphic to T_b^*B/Λ_b, therefore F_b is an n-torus.

Proof. To every $\alpha \in T_b^*B$ we can associate a vector field v_α on F_b by

$$\iota_{v_\alpha}\omega = f^*\alpha.$$

Let ϕ_α^t be the flow of v_α with time $t \in \mathbb{R}$. Then we define the action θ_α of α on F_b by

$$\theta_\alpha(p) = \phi_\alpha^1(p),$$

where $p \in F_b$. One can check that such an action is well defined and transitive. Then, Λ_b defined as

$$\Lambda_b = \{\lambda \in T_b^*B \mid \theta_\lambda(p) = p, \text{ for all } p \in F_b\}.$$

is a closed subgroup of T_b^*B, i.e. a lattice. From the properness of F_b it follows that Λ_b is maximal (in particular homomorphic to $\mathbb{Z}^n$) and that F_b is diffeomorphic to T_b^*B/Λ_b. ☐

We denote $\Lambda = \cup_{b\in B}\Lambda_b$. An interesting way to compute the lattice Λ is the following. Given a point $b_0 \in B$ and contractible neighbourhood U of b_0, for every $b \in U$, $H_1(F_b, \mathbb{Z})$ is naturally identified with $H_1(F_{b_0}, \mathbb{Z})$. Choose a basis $\gamma_1, \dots, \gamma_n$ of $H_1(F_{b_0}, \mathbb{Z})$. Given a vector field v on U, denote by $\tilde{v}$ a lift of v on $f^{-1}(U)$. We can define the following 1-forms $\lambda_1, \dots, \lambda_n$ on B:

$$\lambda_j(v) = -\int_{\gamma_j} \iota_{\tilde{v}}\omega. \tag{4}$$

It is a classical fact that the 1-forms λ_j are closed 1-forms and they generate the lattice Λ. If $\sigma : B \to X$ is a section, we can define the map $\Theta : T^*B/\Lambda \to X$ by $\Theta(b, \alpha) = \theta_\alpha(\sigma(b))$. This map is a diffeomorphism and it is a symplectomorphism if σ is a Lagrangian fibration. A choice of functions a_j such that $\lambda_j = da_j$ defines coordinates $a = (a_1, \dots, a_n)$ on U called *action coordinates*. In particular a covering $\{U_i\}$ of B by contractible open sets and a choice of action coordinates a_i on each U_i defines an integral affine structure on B. The lattice Λ defined above coincides with the lattice coming from this affine structure.

A less invariant approach –but useful for explicit computations– can be described as follows. Let $(b_1, \dots, b_n)$ be local coordinates on $U \subseteq B$ and let $f_j = b_j \circ f$. Then $f_1, \dots, f_n$ is an integrable Hamiltonian system. Let $\Phi_{\eta_j}^t$ be the flow of the Hamiltonian vector field η_j of f_j. Let σ be a Lagrangian section of f over U. In canonical coordinates the map $\Theta : T^*U/\Lambda \to f^{-1}(U)$ is the following

$$\Theta : (b, t_1 db_1 + \ldots + t_n db_n) \mapsto \Phi_{\eta_1}^{t_1} \circ \cdots \circ \Phi_{\eta_n}^{t_n}(\sigma(b)). \tag{5}$$

One may easily verify that

$$\Lambda_b = \{(b, t_1 db_1 + \cdots + t_n db_n) \in T_b^*U \mid \Phi_{\eta_1}^{t_1} \circ \cdots \circ \Phi_{\eta_n}^{t_n}(\sigma(b)) = \sigma(b)\}.$$

When $(b_1, \dots, b_n)$ are action coordinates, the coordinates $(b_1, \dots, b_n, t_1, \dots, t_n)$ and the map Θ are called *action-angle coordinates*.

Remark 3.3. Observe that (4) allows us to read the (topological) monodromy of $f : X \to B$ from the mondoromy of the locally constant sheaf Λ.

Affine manifolds with singularities.

When a Lagrangian fibration has singular fibres, the base of the fibration is no longer an affine manifold but an affine manifold *with singularities,* i.e. we have to remove the set of critical values from B before defining the affine structure. We are thus motivated to give the following

Definition 3.4. An (integral) affine manifold with singularities is a triple $(B, \Delta, \mathscr{A})$, where B is a topological n-dimensional manifold, $\Delta \subset B$ a set which is locally a finite union of locally closed submanifolds and $\mathscr{A}$ is an (integral) affine structure on $B_0 = B - \Delta$.

Example 3.5. Let $X = \mathbb{C}^2 - \{z_1 z_2 + 1 = 0\}$ and let ω be the restriction to X of the standard symplectic form on $\mathbb{C}^2$. One can easily check that the following map onto $\mathbb{R}^2$ is a Lagrangian fibration $f : X \to \mathbb{R}^2$,

$$f(z_1, z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log |z_1 z_2 + 1| \right). \tag{6}$$

The only singular fibre is $f^{-1}(0)$, which has the topology of $S^1 \times S^1$ after $\{x\} \times S^1$ is collapsed to a point, i.e. this fibration coincides topologically with the model in Example 2.5. These type of singular Lagrangian fibrations are called of *focus-focus* type.

Let $\arg : \mathbb{C}^* \to \mathbb{R}$ be the multivalued function $\rho e^{i\theta} \mapsto \theta$. Denote $D = \{b \in \mathbb{C} \mid |b| < 1\}$ and $D^* = D - \{0\}$. It has been shown [18] that there are coordinates $b = (b_1, b_2)$ on $\mathbb{R}^2$, with values in D, a smooth function $q : D \to \mathbb{R}$ and a choice of generators of $H_1(F_b, \mathbb{Z})$ with respect to which the periods λ_1 and λ_2 of the fibration (6) can be written as

$$\begin{aligned} \lambda_1 &= -\log |b| \; db_1 + \arg b \; db_2 + dq \\ \lambda_2 &= 2\pi \, db_2. \end{aligned}$$

Clearly λ_1 is multivalued and blows up as $b \to 0$. The lattice

$$\Lambda = \mathrm{span}_{\mathbb{Z}} \langle \lambda_1, \lambda_2 \rangle$$

has monodromy given by

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{7}$$

Taking primitives of λ_1 and λ_2 on suitable contractible open subsets of D^* gives the affine structure with singularities $(D, \Delta, \mathscr{A})$, with $\Delta = \{0\}$.

Example 3.6. This example is taken from [6] §19.3. Let Ξ be the 4-symplex in $\mathbb{R}^3$ spanned by

$$P_0 = (-1,-1,-1,-1), \; P_1 = (4,-1,-1,-1), \; P_2 = (-1,4,-1,-1), \\ P_3 = (-1,-1,4,-1), \quad P_4 = (-1,-1,-1,4).$$

Let $B = \partial\Xi$. Denote by Σ_j the open 3-face of B opposite to the point P_j and by F_{ij} the closed 2-face separating Σ_i and Σ_j. Each F_{ij} contains 21 integral points (including those on its boundary). These form the vertices of a triangulation of F_{ij}. By joining the barycenter of each triangle with the barycenter of its sides we form a 3-valent graph as in Figure 3. Define the set Δ to be the union of all such graphs in each 2-face. Denote by I the set of integral points of B. We can form a covering of $B_0 = B - \Delta$ by taking the open 3-faces Σ_j and small open neighbourhoods U_Q inside B_0 of $Q \in I$. A coordinate chart ϕ_i on Σ_i can be obtained from its affine embedding in $\mathbb{R}^4$. If we denote again by R_Q the linear space spanned by $Q \in I$, as a chart on U_Q we take the projection $\phi_Q : U_Q \to \mathbb{R}^4/R_Q$. Let $X(B_0, \mathscr{A})$ be the bundle T^*B_0/Λ as defined at the beginning of this section. Then a computation shows that the monodromy of this bundle computed around each vertex is given either by the matrices (3) or by their inverse transpose matrices. In fact the vertices of Δ which are contained in the interior of each 2-face are of negative type and those which are contained in the 1-faces are of positive type.

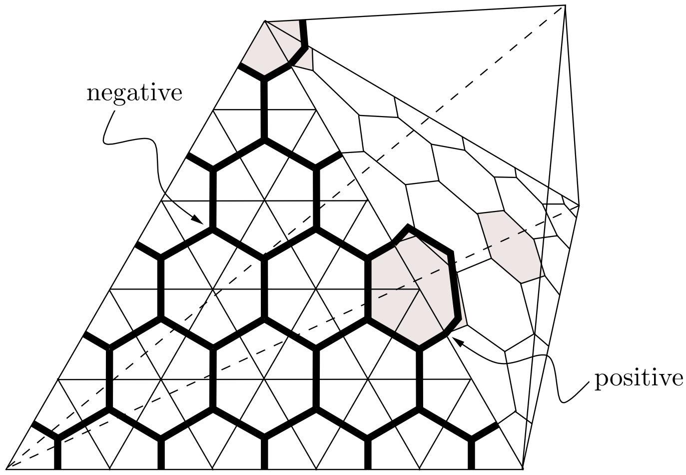

Figure 3: Affine S^3 with singularities.

This is a very interesting example for the following result

Theorem 3.7 (Gross [5]). Let $(B, \Delta, \mathscr{A})$ be the integral affine manifold with singularities described in Example 3.6. Then the bundle $X(B_0, \mathscr{A}) = T^*B_0/\Lambda$ admits a semi-stable topological compactification $X(B_0, \mathscr{A}) \hookrightarrow X$. Moreover X is diffeomorphic to a non-singular quintic hypersurface in $\mathbb{P}^4$.

In fact Gross proves more. If $\check{\Lambda}$ is the dual lattice in TB_0, then he proves that also $\check{X}(B_0, \mathscr{A}) = TB_0/\check{\Lambda}$ admits a semi-stable compactification $\check{X}$ which is diffeomorphic to the mirror of the quintic. It is also important to mention that in recent work Gross and Siebert [7] generalize a great deal Example 3.6 and describe an interesting form of mirror symmetry of affine manifolds with singularities.

We insist that the compactifications in Theorem 3.7 are topological. A much more difficult problem is to carry out these compactifications in the complex and symplectic categories. The main difficulty to achieve a symplectic compactification is the existence of local models of singular *Lagrangian* fibrations.

4 Lagrangian models: positive and generic fibrations.

Lagrangian fibrations resembling the topology of the positive and generic models of §2 are well understood [1]. Now we give some explicit examples.

Example 4.1 (Lagrangian positive fibration). Let $X = \mathbb{C}^3 - \{1 + z_1 z_2 z_3 = 0\}$ with canonical coordinates z_1, z_2, z_3 and the standard symplectic structure. Consider the T^2-action on X given by $(z_1, z_2, z_3) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{-i(\theta_1+\theta_2)} z_3)$. We obtain $f : X \to \mathbb{R}^3$ given by $f = (f_1, f_2, f_3)$ where

$$f_1 = \log |1 + z_1 z_2 z_3|, \; f_2 = |z_1|^2 - |z_2|^2, \quad f_3 = |z_1|^2 - |z_3|^2.$$

The reader may verify that $\mu = (f_2, f_3)$ is the moment map of the T^2 action and f_1 is T^2-invariant so the fibres of f are Lagrangian. The critical locus of f is $Crit(f) = \bigcup_{ij}\{z_i = z_j = 0\}$ and its discriminant locus is $\Delta = \{b_1 = 0, b_2 = b_3 \geq 0\} \cup \{b_1 = b_2 = 0, b_3 \leq 0\} \cup \{b_1 = b_3 = 0, b_2 \leq 0\}$, i.e. a cone over three points with vertex at $0 \in \mathbb{R}^3$. One can verify that this fibration coincides, topologically, with the one in Example 2.8.

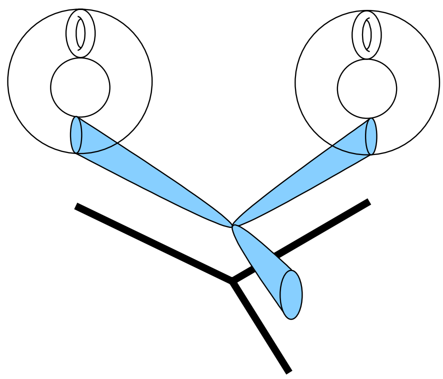

Figure 4: Positive fibration.

Example 4.2 (Lagragian generic fibration). Let $X' = \mathbb{C}^2 - \{1 + z_1 z_2 = 0\}$ and let $X = X' \times \mathbb{C}^*$ with the standard symplectic structure. Define $f : X \to \mathbb{R}^3$ by $f = (f_1, f_2, f_3)$ where

$$f_1 = |z_1|^2 - |z_2|^2, \quad f_2 = \log|1 + z_1 z_2|, \quad f_3 = \log|z_3|.$$

The singular fibres of f are lying over $\Delta = \{(0, 0, r) \mid r \in \mathbb{R}\}$. The reader may verify that the above gives a Lagrangian fibration with the topology of the fibration of Example 2.6.

Remark 4.3. There other ways of constructing Lagrangian fibrations as above. In fact, there is an infinite number of Lagrangian positive (respectively generic) fibrations which are not fibre-preserving symplectomorphic to the one described in Example 4.1 (respectively Example 4.2). Details of this can be found in [1].

The affine structures.

Now we shall describe the singular affine structures induced by the above models. For the details we refer the reader to [1].

Proposition 4.4. Let $f : (X, \omega) \to B$ be a Lagrangian T^3 fibration with singular fibres of generic type along $\Delta \subset B$. There are coordinates (U, b_1, b_2, b_3) around $b_0 \in \Delta$, with $U \cong D^2 \times D^1$ and a basis of $H_1(f^{-1}(b), \mathbb{Z})$, $b \in B - \Delta$ such that, in this basis, the period lattice of f is $\Lambda = \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ where

$$\lambda_1 = \lambda_0 + dA, \qquad \lambda_2 = 2\pi db_2, \qquad \lambda_3 = db_3 \tag{8}$$

where $A \in C^\infty(B)$ and $\lambda_0 = -\log|b_1 + ib_2| db_1 + \operatorname{Arg}(b_1 + ib_2) db_2$. The monodromy of $\mathcal{F}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

Sketch of the Proof. Let η_i be the Hamiltonian vector field on X corresponding to the function $b_i \circ f$, $i = 1, 2, 3$. The vector fields η_i are tangent to the regular fibres. One can define generators of $H_1(f^{-1}(b), \mathbb{Z})$ as a composition of suitably chosen integral curves of the vector fields η_i. Then one uses formula (4) to compute the periods explicitly. For the details we refer the reader to [1] Proposition 3.9. $\square$

As in the focus-focus case (cf. Example 3.5), one can choose suitable branches of λ_0 and define action coordinates on these branches obtaining an affine manifold with singularities, $(B, \Delta, \mathscr{A})$, with the monodromy of $\mathscr{A}$ being generated by (9).

The case of Lagrangian fibrations of positive type is completely analogous. We have (cf. [1] Theorem 4.19):

Proposition 4.5. Let $f : (X, \omega) \to B$ be a Lagrangian fibration of positive type. Then there are local coordinates b_1, b_2, b_3 on B and a basis of $H_1(f^{-1}(b), \mathbb{Z})$ such that the corresponding period 1-forms are:

$$\lambda_1 = \lambda_0 + dH, \quad \lambda_2 = 2\pi db_2, \quad \tau_3 = 2\pi db_3$$

where H is a smooth function on B and λ_0 is multi-valued 1-form blowing up at $\Delta \subset B$, where $\Delta = \{b_1 = 0, b_2 = b_3 \geq 0\} \cup \{b_1 = b_2 = 0, b_3 \leq 0\} \cup \{b_1 = b_3 = 0, b_2 \leq 0\}$. In the basis $\lambda_1, \lambda_2, \lambda_3$ of Λ and for suitable generators of $\pi_1(B - \Delta)$ satisfying $\gamma_1\gamma_2\gamma_3 = I$, the monodromy representation of f is generated by the matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{10}$$

5 Some piece-wise smooth examples

It is now a commonly accepted fact (cfr. [4], [12], [16]) that to produce Lagrangian fibrations of the type described in Section 2, one has to allow also piece-wise smooth Lagrangian fibrations. The reason is that there may not exist a smooth model of negative Lagrangian fibration. Examples of piece-wise smooth Lagrangian fibrations of Calabi-Yau manifolds were constructed by Ruan [15, 16, 17]. Here we wish to present a simple way to produce local models of piece-wise smooth Lagrangian fibrations. We suspect that models similar to our own are also implicit in Ruan's fibrations, but we have been unable to verify this. The method described here is inspired by ideas of Gross [4], Goldstein [3] and Joyce [12].

Lagrangian fibrations with torus symmetry.

Let (X, ω) be a symplectic $2n$-manifold and let $\mu : (X, \omega) \to \mathfrak{t}^*$ be the moment map of a Hamiltonian T^k-action. Let $t \in \mu(X)$ and let $\pi_t : \mu^{-1}(t) \to X_t$ be the projection modulo the T^k action. When t is a regular value of μ, X_t is a smooth manifold and the symplectic form ω descends to a symplectic form ω_t on X_t. When t is a critical value of μ, X_t may be a singular space and ω_t will be only defined on the smooth part of X_t. The space (X_t, ω_t) is the Marsden-Weinstein reduced space at t.

Remark 5.1. We shall denote by

$$\omega_{\mathbb{C}^m} = \frac{i}{2} \sum_k dz_k \wedge d\overline{z}_k$$

the standard symplectic structure on $\mathbb{C}^m$ and ω_0 will denote the reduced symplectic form of the reduced space X_t at time $t = 0$.

Goldstein [3] and Gross [4] used reduced spaces to construct T^k-invariant (special) Lagrangian fibrations. The following is a particular case of [4] Thm 1.2:

Proposition 5.2. Let T^k act effectively on X, $k \leq n-1$. Suppose that there is a continuous map $G : X \to M$ to an $n-k$-dimensional manifold M such that $G(T \cdot x) = G(x)$ for all $T \in T^k$. Suppose that for t in a dense subset of $\mu(X)$ the induced maps $G_t : X_t \to M$ have fibres that are Lagrangian with respect to ω_t. Then $f : X \to \mu(X) \times M$ given by:

$$f = (\mu, G) \tag{11}$$

defines a T^k-invariant Lagrangian fibration.

When the T^k-action has fixed points, the construction of Proposition 5.2 will produce fibrations with interesting singular fibres. Later, in this paper we will show some examples.

The reduced geometry.

Consider the following S^1 action on $\mathbb{C}^3$:

$$e^{i\theta}(z_1, z_2, z_3) = (e^{i\theta} z_1, e^{-i\theta} z_2, z_3). \tag{12}$$

This action is Hamiltonian with respect to $\omega_{\mathbb{C}^3}$. Clearly it is singular along the surface $\Sigma = \{z_1 = z_2 = 0\}$. The corresponding moment map is:

$$\mu(z_1, z_2, z_3) = \frac{|z_1|^2 - |z_2|^2}{2}. \tag{13}$$

The only critical value of μ is $t = 0$ and $\mathrm{Crit}(\mu) = \Sigma \subset \mu^{-1}(0)$.

Now consider the map

$$\begin{aligned} \bar{\pi} : \mathbb{C}^3 &\rightarrow \mathbb{R} \times \mathbb{C}^2 && (14)\\ (z_1, z_2, z_3) &\mapsto (\mu, z_1 z_2, z_3). && (15) \end{aligned}$$

Notice that $\bar{\pi}$ is the same map as in Remark 2.4. Therefore $\bar{\pi}$ restricted to $\mathbb{C}^3 - \Sigma$ is an S^1-bundle onto $(\mathbb{R} \times \mathbb{C}^2) - \bar{\pi}(\Sigma)$ with Chern class $c_1 = 1$. Let π_t be the restriction to $\mu^{-1}(t)$ of the map

$$(z_1, z_2, z_3) \mapsto (z_1 z_2, z_3).$$

Then π_t can be used to identify the reduced space $\mu^{-1}(t)/S^1$ with $\mathbb{C}^2$. Under this identification, i.e. letting the coordinates $u_1 = z_1 z_2$ and $u_2 = z_3$, the reduced Kähler form ω_t can be written as:

$$\omega_t = \frac{i}{2} \left(\frac{1}{2\sqrt{t^2 + |u_1|^2}} du_1 \wedge d\overline{u}_1 + du_2 \wedge d\overline{u}_2 \right). \tag{16}$$

Away from $t = 0$, the reduced spaces are smooth manifolds.

On the other hand, at $t = 0$ the reduced form $\omega_0 = \omega_t|_{t=0}$ blows up along the plane $\pi_0(\Sigma) = \{u_1 = 0\}$, so the reduced space (X_0, ω_0) is singular. However, it was observed by Guillemin and Sternberg in [9], that it can be smoothed out, i.e. it can be identified with $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$. Indeed, the identification is given by the following

$$\Gamma_0 : (u_1, u_2) \mapsto \left(\frac{u_1}{\sqrt{|u_1|}}, u_2 \right). \tag{17}$$

One can see that Γ_0 is continuous, smooth away from $u_1 = 0$ and such that $\Gamma_0^* \omega_{\mathbb{C}^2} = \omega_0$. One can do more: one can identify all the reduced spaces with $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$ at once. In fact consider the map

$$\Gamma_t : (u_1, u_2) \mapsto \left(\frac{u_1}{\sqrt{|t| + \sqrt{t^2 + |u_1|^2}}}, u_2 \right) \tag{18}$$

One can verify that Γ_t is a symplectomorphism between $(\mathbb{C}^2, \omega_t)$ and the standard symplectic space $\mathbb{C}^2$. However the problem is that this identification, although continuous and smooth for fixed t, is not smooth in t when $t = 0$. In fact one can show that it cannot be otherwise.

A construction

We now illustrate a general method to construct piece-wise smooth Lagrangian fibrations using Proposition 5.2 and the observations about the reduced geometry. Let $\mathrm{Log} : (\mathbb{C}^*)^2 \rightarrow \mathbb{R}^2$ be defined by

$$\mathrm{Log}(v_1, v_2) = (\log |v_1|, \log |v_2|). \tag{19}$$

Clearly the map Log is a Lagrangian fibration, with respect to the standard symplectic form. Moreover it is a trivial T^2-bundle over $\mathbb{R}^2$. Let $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a symplectomorphism of the standard $\mathbb{C}^2$. Let X_t be the open and dense subsets of $(\mathbb{C}^2, \omega_t)$ defined by

$$X_t = \Gamma_t^{-1} \circ \Phi^{-1}((\mathbb{C}^*)^2).$$

Then examples of maps $G_t : X_t \to \mathbb{R}^2$ of Proposition 5.2 can be defined by

$$G_t = \operatorname{Log} \circ \Phi \circ \Gamma_t.$$

This clearly makes sense also when $t = 0$. It is also clear that, for all fixed $t \in \mathbb{R}$, G_t is a Lagrangian fibration with respect to the reduced symplectic form (16). We summarize this in the following

Proposition 5.3. Let X_t and G_t be as above. Define a map Q by

$$Q(t, u_1, u_2) = (t, G_t(u_1, u_2)). \tag{20}$$

Then Q is defined on the dense open subset Y of $\mathbb{R} \times \mathbb{C}^2$ defined by

$$Y = \{(t, u_1, u_2) \in \mathbb{R} \times \mathbb{C}^2 \mid (u_1, u_1) \in X_t\}.$$

If $\bar{\pi}$ is as in (14), let

$$X = (\bar{\pi})^{-1}(Y)$$

with the standard symplectic form induced from $\mathbb{C}^3$. Then the map $f : X \to \mathbb{R}^3$ given by

$$f = Q \circ \bar{\pi}$$

is a piece-wise smooth Lagrangian fibration of X which fails to be smooth on the 5-dimensional subspace $\mu^{-1}(0) \cap X$.

It is clear that all the singular fibres of f must lie in $\mu^{-1}(0) \cap X$. In fact the singular fibres are all the lifts of fibres of G_0 in X_0 which intersect the surface $\Sigma = \{u_1 = 0\} \cap X_0$. The topology of the singularity depends on the topology of this intersection. The discriminant locus of the fibration is therefore the set $\Delta \subset \mathbb{R}^3$ given by

$$\Delta = \{0\} \times (\operatorname{Log} \circ \Phi \circ \Gamma_0(\Sigma \cap X_0)).$$

Given a point $b = (0, b_1, b_2) \in \Delta$, the fibre $f^{-1}(b)$ looks like $S^1 \times G_0^{-1}(b_1, b_2)$ after the circles over all points in $G_0^{-1}(b_1, b_2) \cap \Sigma$ have been collapsed to points (cfr. Figure 6).

Examples

Define the piece-wise smooth map $\gamma : \mathbb{C}^2 \to \mathbb{C}$ by

$$\gamma(z_1, z_2) = \begin{cases} \frac{z_1 z_2}{|z_1|}, & \text{when } t \geq 0 \\ \\ \frac{z_1 z_2}{|z_2|}, & \text{when } t < 0. \end{cases} \tag{21}$$

Then one can easily see that for all $(z_1, z_2, z_3) \in \mu^{-1}(t)$, the map $\Gamma_t \circ \pi_t$ is given by

$$\Gamma_t \circ \pi_t : (z_1, z_2, z_3) \mapsto (\gamma(z_1 z_2), z_3).$$

This is useful when computing the explicit examples below. We point out that in the construction of Proposition 5.3 the topology of the fibration depends on how we choose the symplectomorphism Φ.

Example 5.4 (The amoeba). Take as a symplectomorphism Φ the linear map

$$\Phi(u_1, u_2) = \frac{1}{\sqrt{2}} \left(u_1 - u_2, u_1 + u_2 - \sqrt{2} \right). \tag{22}$$

Then the fibration resulting from Proposition 5.3 can be written explicitly in the coordinates of the total space, in fact

$$f(z_1, z_2, z_3) = (\mu, \log \frac{1}{\sqrt{2}} |\gamma - z_3|, \log \frac{1}{\sqrt{2}} |\gamma + z_3 - \sqrt{2}|). \tag{23}$$

The discriminant locus of this fibration is depicted in Figure 5. This is not difficult to see. In fact $\Phi \circ \Gamma_0$ sends the surface Σ to the plane in $\mathbb{C}^2$ given by

$$\Sigma' = \{v_1 + v_2 + 1 = 0\}.$$

Then the discriminant locus is

$$\Delta = \{0\} \times \mathrm{Log}(\Sigma'),$$

which has the shape in Figure 5. The images under Log of algebraic hyper-surfaces in $(\mathbb{C}^*)^2$ are known in the literature as amoebas. Figure 5 is the amoeba of Σ'.

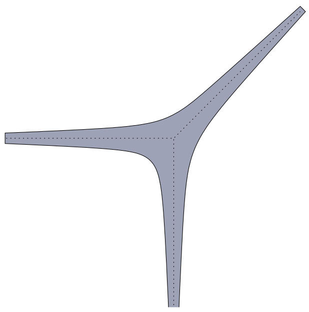

Figure 5: Amoeba of $v_1 + v_2 + 1 = 0$

One can see that the fibres of Log over a point (b_1, b_2) intersect Σ' in two distinct points when (b_1, b_2) is in the interior of the amoeba. These two points come together to a double point as (b_1, b_2) approaches the boundary of the amoeba. If p_1 and p_2 are two points on T^2 (which may coincide), then the singular fibres of f look like $S^1 \times T^2$ after $S^1 \times \{p_j\}$ is collapsed to a point. This model has the same topology as one conjectured by Joyce [12] for special Lagrangian fibrations. Also the singularities are modeled on those of an explicit example by Joyce of a singular special Lagrangian fibration with non-compact fibres.

In view of Proposition 2.3 and Remark 2.4, the total space of the amoeba fibration is diffeomorphic to X as in Example 2.7, although the fibrations differ. In Example 2.7 the discriminant locus is of codimension two (a graph), in the amoeba the discriminant has codimension one. In both cases the singularities of the fibres occur along the intersection of the critical surface Σ with the fibres of P (in Example 2.7) or of Q (as in (20)) in the amoeba. But the intersections happen in a different way. In Example 2.7 they occur either along circles, or along a figure eight. In the amoeba they occur as pairs of isolated points which come together as a base point approaches the boundary of the amoeba. Intuitively, the amoeba may be interpreted as a perturbation of Example 2.7 . In some sense, the singularities in the amoeba are more generic. A schematic description of the amoeba fibration is depicted in Figure 6. It can be compared with Figure 2. We remark that in the amoeba fibration the monodromy around the legs is the same as the monodromy of Example 2.7, i.e. it is represented by the matrices (3).

In two dimensions we have the following:

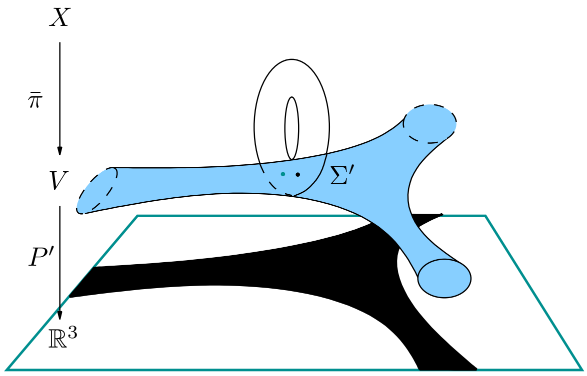

Figure 6: Negative fibration with amoeba-like discriminant.

Example 5.5 (Stitched focus-focus). With a similar construction in dimension two we can obtain the following piece-wise smooth fibration

$$f(z_1, z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log|\gamma(z_1, z_2) + 1| \right). \tag{24}$$

where γ is as in (21). It is clearly well defined on $X = \{(z_1, z_2) \in \mathbb{C}^2 \mid \gamma(z_1, z_2) + 1 \neq 0\}$. Observe that f has the same topology as a smooth focus-focus fibration. The only singular fibre is $f^{-1}(0)$ and it is a pinched torus. The fibration fails to be smooth on $\mu^{-1}(0)$.

There is an analogous model in three dimensions:

Example 5.6 (The leg). This is another three dimensional example. Consider the following affine symplectomorphism of $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$

$$\Phi : (u_1, u_2) \mapsto (-u_2, u_1 - 1). \tag{25}$$

The surface Σ is sent by $\Phi \circ \Gamma_0$ to $\Sigma' = \{v_2 + 1 = 0\}$. The amoeba of Σ' is just a straight line. The resulting fibration f becomes

$$f(z_1, z_2, z_3) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log|z_3|, \log|\gamma(z_1, z_2) - 1| \right). \tag{26}$$

The discriminant locus is $\{0\} \times \mathbb{R} \times \{0\} \subset \mathbb{R}^3$. The fibration is a piece-wise smooth version of the generic fibration in Example 4.2.

Example 5.7 (The amoeba with a thin leg). We now construct an example which interpolates Example 5.4 and Example 5.6. Take the following Hamiltonian function

$$H_0 = \frac{\pi}{4} \operatorname{Im}(u_1 \overline{u}_2)$$

and let η_{H_0} be the Hamiltonian vector field associated to H_0. If Φ_s is the flow generated by η_{H_0}, then the Hamiltonian symplectomorphism associated to H_0 is defined to be $\Phi_{H_0} = \Phi_1$. One computes that in our case

$$\Phi_{H_0} : (u_1, u_2) \mapsto \frac{1}{\sqrt{2}}(u_1 - u_2, u_1 + u_2).$$

It maps $\{u_1 = 0\}$ to $\{v_1 + v_2 = 0\}$. We now want a symplectomorphism which acts like Φ_{H_0} in a small ball centered at the origin and like the identity outside a slightly bigger ball. So choose a cut-off function $k : \mathbb{R}_{\geq 0} \to [0, 1]$ such that, for some $\epsilon > 0$,

$$k(t) = \begin{cases} 1 \text{ when } 0 < t \leq \epsilon \\ 0 \text{ when } t \geq 2\epsilon \end{cases} \tag{27}$$

and define the Hamiltonian

$$H = k(|u_1|^2 + |u_2|^2)H_0.$$

The Hamiltonian symplectomorphism Φ_H associated to H satisfies

$$\Phi_H(u_1, u_2) = \begin{cases} \mathrm{Id}_{\mathbb{C}^2}, & \text{when } |u_1|^2 + |u_2|^2 \geq 2\epsilon \\ \\ \frac{1}{\sqrt{2}}(u_1 - u_2, u_1 + u_2), & \text{when } |u_1|^2 + |u_2|^2 \leq \epsilon. \end{cases}$$

Now let Ψ be the affine symplectomorphism

$$\Psi : (v_1, v_2) \mapsto \frac{1}{\sqrt{2}}(v_1 - v_2, v_1 + v_2 - \sqrt{2}).$$

and finally, define $\Phi = \Psi \circ \Phi_H$. It is clear that

$$\Phi(u_1, u_2) = \begin{cases} \Psi, & \text{when } |u_1|^2 + |u_2|^2 \geq 2\epsilon \\ \\ (-u_2, u_1 - 1), & \text{when } |u_1|^2 + |u_2|^2 \leq \epsilon. \end{cases}$$

Notice that Φ acts like in (25) on the ball of radius $\sqrt{\epsilon}$ around the origin and like in (22) outside a larger ball. We use this Φ to construct a fibration f from Proposition 5.3. One can then see that $\Sigma' = \Phi \circ \Gamma_0(\Sigma)$ is a surface such that $\mathcal{A} = \mathrm{Log}(\Sigma')$ is a three-legged amoeba with a leg pinched down to a straight line. The discriminant locus of f is then $\Delta = \{0\} \times \mathcal{A} \subset \mathbb{R}^3$. Of course, f fails to be smooth on the slice $\mu^{-1}(0)$.

Using the above method, one can choose a symplectomorphism Φ twisting the surface $\{u_1 = 0\}$ suitably and obtain an amoeba with three thin legs (cf. Figure 7).

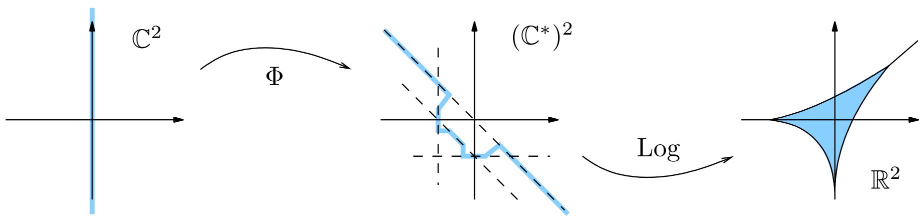

Figure 7: Amoeba with thin legs.

Definition 5.8. A fibration as in Example 5.7 with three thin legs will be called a *Lagrangian negative fibration.*

The total space of a topological negative fibration coincides with that of a Lagrangian negative fibration but the fibrations themselves are different. In view of Proposition 5.3, a Lagrangian negative fibration is piece-wise C^∞. More precisely, a Lagrangian negative fibration is a union of two honest C^∞ fibrations meeting along $\mu^{-1}(0)$. A very similar phenomenon occurs in special Lagrangian geometry [12].

Periods of a Lagrangian negative fibration

Let $f : X \to \mu(X) \times M$ be a Lagrangian fibration constructed as in Proposition 5.2. Let us also assume that the action if free so that μ has no critical points. It is natural to ask what is the relation between the periods of f and the periods of the reduced fibrations $G_t : X_t \to M$. Let $U \subset \mu(X) \times M$ be a contractible open subset. Given $t \in \mu(X)$ denote $U_t = U \cap (\{t\} \times M)$. A basis $\{\gamma_1, \ldots, \gamma_n\}$ of $H_1(f^{-1}(U), \mathbb{Z})$ can always be chosen so that the following holds. The first $\gamma_1, \ldots, \gamma_k$ are induced, via the T^k action, by some basis $e_1, \ldots, e_k$ of $\mathfrak{t}$, the Lie algebra of T^k. Moreover, for every $t \in \mu(X)$ with U_t non-empty, $\{\pi_{t*}\gamma_{k+1}, \ldots, \pi_{t*}\gamma_n\}$ form a basis of $H_1(G_t^{-1}(U_t), \mathbb{Z})$. We have the following simple but useful result:

Lemma 5.9. Given a basis $\{\gamma_1, \ldots, \gamma_n\}$ of $H_1(f^{-1}(U), \mathbb{Z})$ chosen as above, let $\lambda_1, \ldots, \lambda_n$ be the periods of f with respect to $\gamma_1, \ldots, \gamma_n$ and let $\xi_1^t, \ldots, \xi_{n-k}^t$ be the periods of G_t with respect to $\pi_{t*}\gamma_{k+1}, \ldots, \pi_{t*}\gamma_n$. Then we have

$$\lambda_j = e_j, \quad j = 1, \ldots, k \tag{28}$$

$$\lambda_{k+j}|_{U_t} = \xi_j^t, \quad j = 1, \ldots, n-k, \tag{29}$$

where $e_1, \ldots, e_k$ are interpreted as 1-forms on $\mu(X) \subseteq \mathfrak{t}^*$.

Proof. The proof of (28) is left to the reader. Let v be a vector field on U_t and let v'' be a lift of v with respect to f. Then $v' = \pi_{t*}v$ is a lift of v with respect to G_t. When $j = 1, \ldots, n-k$, we have

$$\xi_j^t(v) = -\int_{\pi_{t*}\gamma_{k+j}} \iota_{v'}\omega_t = -\int_{\gamma_{k+j}} \iota_{v''}\omega = \lambda_{k+j}(v), \tag{30}$$

which proves (29). □

Now let $f : X \to \mathbb{R}^3$ be the amoeba fibration of Example 5.4 or a Lagrangian negative fibration. Remember that Δ is the discriminant locus. We compute the periods of f over the following contractible open subset U of $\mathbb{R}^3 - \Delta$:

$$U = (\mathbb{R}^3 - \Delta) - (\{b_1 = 0, b_2 \geq 0\} \cup \{b_1 = 0, b_3 \geq 0\})$$

Proposition 5.10. Let $f : X \to \mathbb{R}^3$ be an amoeba fibration or a negative fibration and $U \subset \mathbb{R}^3$ be as above. Then there is a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of $H^1(f^{-1}(U), \mathbb{Z})$ with respect to which the periods $\lambda_1, \lambda_2, \lambda_3$ are

$$\begin{aligned} \lambda_1 &= db_1, \\ \lambda_2 &= \alpha_1\, db_1 - e^{2b_2}\, db_2, \\ \lambda_3 &= \alpha_2\, db_1 - e^{2b_3}\, db_3, \end{aligned}$$

where α_1 and α_2 are functions on U.

Proof. Given $t \in \mathbb{R}$, let $U_t = U \cap \{b_1 = t\}$. When $t \neq 0$, $U_t = \mathbb{R}^2$ and when $t = 0$, U_0 is a subset of $\{b_1 = 0\}$ not containing Δ. The reduced Lagrangian fibrations G_t are given by

$$G_t = \mathrm{Log} \circ \Phi \circ \Gamma_t.$$

Observe that $\Phi \circ \Gamma_t$ is a C^∞ symplectomorphism between $(\mathbb{C}^2, \omega_t)$ and $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$. Given $b = (b_2, b_3) \in \mathbb{R}^2$,

$$\mathrm{Log}^{-1}(b) = \{(e^{b_2 + i\theta_1}, e^{b_3 + i\theta_2}),\ (\theta_1, \theta_2) \in T^2\}.$$

We can choose a basis ζ_1', ζ_2' of $H_1(\mathrm{Log}^{-1}(b), \mathbb{Z})$ represented by $\zeta_1' = \{\theta_2 = 0\}$ and $\zeta_2' = \{\theta_1 = 0\}$. Now let $\zeta_i \in H_1(G_t^{-1}(b), \mathbb{Z})$ be such that $(\Phi \circ \Gamma_t)_*(\zeta_i) = \zeta_i'$. Clearly ζ_1, ζ_2 form a basis of $H_1(G_t^{-1}(b), \mathbb{Z})$. From the fact that $(\Phi \circ \Gamma_t)^*\omega_{\mathbb{C}^2} = \omega_t$, one can see that the periods ξ_1^t and ξ_2^t of G_t with respect to ζ_1 and ζ_2 are equal to the periods of Log with respect to ζ_1' and ζ_2'. After an easy computation we therefore obtain

$$\xi_i^t = -e^{2b_i} db_i \tag{31}$$

We can choose a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(f^{-1}(U), \mathbb{Z})$ so that γ_1 is represented by an S^1-orbit and $\pi_{t*}\gamma_{1+i} = \zeta_i$. Lemma 5.9 and (31) imply the result. □

Remark 5.11. Notice that from the fact that periods are always closed, we find that the α_j's appearing in the expression of λ_2 and λ_3 must depend only on b_1. It follows that in no way the periods could show a monodromy given by matrices (3), as we would expect if the fibration were smooth (cfr. Remark 3.3). In the next section we discuss how to read monodromy in the case of these type of fibrations.

6 Stitching Lagrangian fibrations

In these last two sections we wish to report on some current research of the authors aiming at a better understanding of the examples of piece-wise smooth Lagrangian fibrations described in the previous sections. We briefly outline the main ideas of this research mainly through the discussion of some examples. Proofs and technical details will appear elsewhere in a joint work in preparation. We define and study a certain class of piece-wise-smooth Lagrangian fibrations which include the examples of the previous sections. We call these fibrations stitched Lagrangian fibrations. Before giving any formal definition we recall the motivating examples.

Example 6.1 (Stitched focus-focus, revisited). We take a closer look at the fibration $f : X \to \mathbb{R}^2$ of Example 5.5. Let $\mu = \frac{|z_1|^2-|z_2|^2}{2}$ be the moment map of the usual S^1 action. Then the total space X can be divided in two halves:

$$X^+ = \left\{ \frac{z_1 z_2}{|z_1|} + 1 \neq 0 \right\} \cap \{\mu \geq 0\}$$

and

$$X^- = \left\{ \frac{z_1 z_2}{|z_2|} + 1 \neq 0 \right\} \cap \{\mu \leq 0\},$$

which are separated by $Z = \mu^{-1}(0) = \partial X^+ = \partial X^-$. As we pointed out f is piece-wise smooth on X and fails to be smooth on Z. In fact f restricted to $X^\pm - f^{-1}(0)$ is locally the restriction of a smooth function on $\mathbb{C}^2$. Moreover f has the same topology of a smooth focus-focus fibration. The fibre over $0 \in \mathbb{R}^2$ is a pinched torus. Denote the components of f restricted to $X^\pm$ by $f = (f_1, f_2^\pm)$. Let η_1 and $\eta_2^\pm$ be the Hamiltonian vector fields of f_1 and $f_2^\pm$ respectively. Notice that η_1 is smooth on X while $\eta_2^\pm$ extend to smooth vector fields on $Z - f^{-1}(0)$, but in general $\eta_2^+|_{Z-f^{-1}(0)} \neq \eta_2^-|_{Z-f^{-1}(0)}$. A measure of how f fails to be smooth is given by the difference between η_2^+ and η_2^- along Z. Notice that both of them must be tangent to the fibres of f and both commute with η_1. A computation shows that

$$(\eta_2^+ - \eta_2^-)|_{Z-f^{-1}(0)} = a\, \eta_1|_{Z-f^{-1}(0)},$$

where

$$a = \mathrm{Re}\left(\frac{z_1 z_2}{|z_1|^2 z_1 z_2 - |z_1|^3} \right) |_{Z-f^{-1}(0)} \ .$$

Example 6.2 (The amoeba, revisited). Similar observations hold for $f : X \to \mathbb{R}^3$ given in Example 5.4. Recall that the discriminant locus Δ of f has the shape in Figure 5 and it is contained in in the plane $\{b_1 = 0\}$. For the purpose of this discussion we restrict f to the set of smooth fibres $X^\# = X - f^{-1}(\Delta)$. As in the previous example we can divide $X^\#$ in two halves: $X^+ = X^\# \cap \{\mu \geq 0\}$, $X^- = X^\# \cap \{\mu \leq 0\}$, separated by $Z = \mu^{-1}(0) \cap X^\# = \partial X^+ = \partial X^-$. Again f is piece-wise smooth on $X^\#$, but when restricted to $X^\pm$, f is locally the restriction of a smooth map. We denote f restricted to $X^\pm$ by $f^\pm = (f_1, f_2^\pm, f_3^\pm)$. Let η_1 and $\eta_j^\pm$ be the Hamiltonian vector fields of f_1 and $f_j^\pm$, $j = 2, 3$. Since f_2 and f_3 are piece-wise smooth, we expect $\eta_j^+|_Z \neq \eta_j^-|_Z$. A computation shows that, for $j = 2, 3$

$$\eta_j^+|_Z - \eta_j^-|_Z = a_j\, \eta_1|_Z,$$

where

$$a_2 = -\frac{\mathrm{Re}\left((\gamma - z_3) \frac{\bar{z}_1 \bar{z}_2}{|z_1|^3} \right)}{|\gamma - z_3|^2}$$

and

$$a_3 = -\frac{\mathrm{Re}\left((\gamma + z_3 - \sqrt{2}) \frac{\bar{z}_1 \bar{z}_2}{|z_1|^3} \right)}{|\gamma + z_3 - \sqrt{2}|^2}.$$

Notice that the a_j's are S^1 invariant functions.

Stitched Lagrangian fibrations

In view of the examples above we give the following definition.

Definition 6.3. Let (X, ω) be a smooth $2n$-dimensional symplectic manifold. Suppose there is a free Hamiltonian S^1 action on X with moment map $\mu : X \to \mathbb{R}$. Let $X^+ = \{\mu \geq 0\}$, $X^- = \{\mu \leq 0\}$ and $Z = \partial X^+ = \partial X^-$. A map $f : X \to \mathbb{R}^n$ is said to be a *stitched Lagrangian fibration* if there are S^1 invariant maps G^+ and G^-, $G^\pm : X^\pm \to \mathbb{R}^{n-1}$, such that the following holds.

(i) $G^+|_Z = G^-|_Z$ and $G^\pm$ are restrictions to $X^\pm$ of smooth maps on X.

(ii) f can be written as
$$f = \begin{cases} (\mu, G^+) & \text{on } X^+, \\ (\mu, G^-) & \text{on } X^- \end{cases}$$
and f restricted to $X^\pm$ is a proper submersion with connected Lagrangian fibres. Denote f restricted to $X^\pm$ by $f^\pm$.

(iii) Let $G_j^\pm$ be the components of $G^\pm$ and let η_1 and $\eta_j^\pm$ be the Hamiltonian vector fields of μ and $G_j^\pm$ respectively, $j = 2, \ldots, n$. Then there are S^1 invariant functions a_j, $j = 2, \ldots, n$ on Z such that
$$(\eta_j^+ - \eta_j^-)|_Z = a_j \, \eta_1|_Z. \tag{32}$$

We call Z the **seam**.

We remark that the definition excludes the possibility that a stitched Lagrangian fibration has singular fibres. From the theory of action-angle coordinates it is known that all smooth, proper, Lagrangian submersions with connected fibres are locally fibre-wise symplectomorphic. In general this is no longer true for stitched Lagrangian fibrations. Our ideas aim at a generalization of the theory of action angle coordinates to stitched Lagrangian fibrations. Hopefully, this will allow us to understand if and how we can use the examples in the previous section as building blocks of Lagrangian fibrations of Calabi-Yau manifolds or even of more general symplectic manifolds. We show that in the setting of stitched Lagrangian fibrations one not only needs action-angle coordinates but also a family of closed 1-forms on an $(n-1)$-torus satisfying some integrality properties. This family of 1-forms carries some key information needed to describe the non-smoothness happening along Z. We outline here how this family of 1-forms is obtained.

Assume that $f : X \to U$ is a stitched Lagrangian fibration onto a contractible open neighbourhood U of $0 \in \mathbb{R}^n$, with coordinates $b = (b_1, \ldots, b_n)$, so that X is a topologically trivial torus bundle over U. Let $U^+ = U \cap \{b_1 \geq 0\}$ and $U^- = U \cap \{b_1 \leq 0\}$ and $\Gamma = \partial U^+ = \partial U^-$. Let $\sigma : U \to X$ be a continuous section which is smooth and Lagrangian when restricted to $U^\pm$. Then, as explained in Section 3, there are maximal smooth lattices $\Lambda_\pm \subset T^*U^\pm$ and a diagram

$$\begin{array}{ccc} T^*U^\pm/\Lambda_\pm & \xrightarrow{\Theta^\pm} & X^\pm \\ \pi^\pm \Big\downarrow & & \Big\downarrow f^\pm \\ U^\pm & \xrightarrow{\mathrm{Id}} & U^\pm \end{array}$$

where $\Theta^\pm$ is a symplectomorphism and $\pi^\pm$ are the standard projections. Let $\Phi^t_{\eta_1}, \Phi^t_{\eta_2^\pm} \ldots, \Phi^t_{\eta_n^\pm}$ be the flows of $\eta_1, \eta_2^\pm, \ldots, \eta_n^\pm$ respectively. Then

$$\Theta^\pm : (b, \sum_j t_j \, db_j) \mapsto \Phi^{t_1}_{\eta_1} \circ \Phi^{t_2}_{\eta_2^\pm} \circ \ldots \circ \Phi^{t_n}_{\eta_n^\pm}(\sigma(b)), \tag{33}$$

and

$$\Lambda_\pm = \{(b, \sum_j T_j \, db_j) \in T^*U^\pm \mid \Phi^{T_1}_{\eta_1} \circ \Phi^{T_2}_{\eta_2^\pm} \circ \ldots \circ \Phi^{T_n}_{\eta_n^\pm}(\sigma(b)) = \sigma(b)\}$$

Notice that because of the S^1 action, we always have that $db_1 \in \Lambda_\pm$. Now, due to the discrepancy (32) between η_j^+ and η_j^- along Z, Θ^+ and Θ^- behave differently on fibres over points $b \in \Gamma$. In fact let

$$Z^\pm = (\pi^\pm)^{-1}(\Gamma),$$

then we have the diagram

$$\begin{tikzcd} Z^- \arrow[d, "\Theta^-"'] & \\ Z & Z^+ \arrow[l, "\Theta^+"'] \end{tikzcd}$$

and the difference between the two maps is measured by

$$(\Theta^+)^{-1} \circ \Theta^- : Z^- \to Z^+$$

We denote by Q the map $(\Theta^+)^{-1} \circ \Theta^-$. It is not difficult to compute Q explicitly. In fact let $(b_1, \ldots, b_n, t_1, \ldots, t_n)$ and $(b_1, \ldots, b_n, y_1, \ldots, y_n)$ be the canonical coordinates on T^*U^- and T^*U^+ respectively. From its definition, we see that Θ^+ identifies $\eta_1, \eta_2^+ \ldots, \eta_n^+$ with $\partial_{y_1}, \ldots, \partial_{y_n}$, therefore (32) becomes

$$\eta_j^- = \partial_{y_j} - (a_j \circ \Theta^+)\, \partial_{y_1}. \tag{34}$$

Notice that $a_j \circ \Theta^+$ is independent of y_1. Computing the flows of $\eta_1, \eta_2^-, \ldots, \eta_n^-$ in these coordinates is not difficult and it turns out that Q is given by

$$Q : (b, t_1, \ldots, t_n) \mapsto \left(b,\ t_1 - \sum_{j=2}^{n} \int_0^{t_j} a_j \circ \Theta^-(b, t_2, \ldots, t_{j-1}, t, 0, \ldots, 0) dt,\ t_2, \ldots,\ t_n \right).$$

Now let

$$\lambda_1 = db_1.$$

We have $\lambda_1 \in \Lambda_\pm$. Let us denote a basis for $\Lambda_\pm$ by $\{\lambda_1, \lambda_2^\pm, \ldots, \lambda_n^\pm\}$, where

$$\lambda_j^\pm = \sum_{k=1}^{n} T_{jk}^\pm db_k.$$

The S^1 action on the seam Z corresponds to translations along the λ_1 direction on $Z^\pm$. We want to describe Z/S^1. Let

$$\overline{\lambda}_j^\pm = \lambda_j^\pm \mod db_1$$

and let $\overline{\Lambda}^\pm = \mathrm{span}\langle \overline{\lambda}_2^\pm, , \ldots, \overline{\lambda}_n^\pm \rangle_{\mathbb{Z}}$. Then we can identify Z/S^1 with

$$\overline{Z}^\pm = T^*\Gamma / \overline{\Lambda}_\pm.$$

Now notice that the functions $a_j \circ \Theta^-$ descend to functions on $\overline{Z}^-$. Define the 1-form

$$\ell^- = \sum_{j=2}^{n} (a_j \circ \Theta^-)\ dt_j.$$

One can see that the fact that $[\eta_j^-, \eta_k^-] = 0$ for all $j, k = 2, \ldots, n$ is equivalent to the fact that ℓ^- restricts to a closed 1-form on the fibres of $\overline{Z}^-$. Denote by $\overline{t} = (t_2, \ldots, t_n)$ the coordinates on the fibres of $\overline{Z}^-$. Then Q can be re-written as

$$Q : (b, t_1, \overline{t}) \mapsto \left(b, t_1 - \int_0^{\overline{t}} \ell^-, \overline{t} \right), \tag{35}$$

where the integral is a line integral in the covering space $T^*\Gamma$ along a path joining 0 and $\bar{t}$. We would like to understand how Q acts on the periods. The maps $\Theta^{\pm}$ naturally identify $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^n$ with $\Lambda_{\pm}$. In particular the cycle γ_1 represented by the orbits of the S^1 action is identified with $\lambda_1 = db_1$. Suppose we choose a pair of basis of $H_1(X,\mathbb{Z})$ of the type $\{\gamma_1, \gamma_2^+, \ldots, \gamma_n^+\}$ and $\{\gamma_1, \gamma_2^-, \ldots, \gamma_n^-\}$ satisfying the relation

$$\gamma_j^+ = m_j\gamma_1 + \gamma_j^-$$

for integers $m_2, \ldots, m_n$. Let $\{\lambda_1, \lambda_2^{\pm}, \ldots, \lambda_n^{\pm}\}$ be the basis of $\Lambda_{\pm}$ corresponding to $\{\gamma_1, \gamma_2^{\pm}, \ldots, \gamma_n^{\pm}\}$. One can easily see that, at a point $b \in \Gamma$, we must have

$$\lambda_j^+(b) = \lambda_j^-(b) + \left(m_j - \int_{\overline{\lambda}_j^-} \ell^-(b)\right)\lambda_1, \tag{36}$$

This follows from the fact that Q must map $m_j\lambda_1 + \lambda_j^-$ to λ_j^+, since they represent the same cycle inside Z.

Notice that the whole manifold X is obtained by "gluing" T^*U^-/Λ_- and T^*U^+/Λ_+ along their boundaries Z^- and Z^+ via the map Q. Moreover the map Q can be completely recovered from the closed 1-form ℓ^-.

In general one can try to construct stitched-Lagrangian fibrations in the following way. Suppose we have maximal lattices $\Lambda_{\pm}$ inside $T^*U^{\pm}$ generated by closed one forms $\{\lambda_1, \lambda_2^{\pm}, \ldots, \lambda_n^{\pm}\}$, where $\lambda_1 = db_1$. Suppose we are also given a smooth 1-form ℓ^- on $\overline{Z}^-$ which is closed on the fibres and such that for every $b \in \Gamma$, (36) holds. Then the map Q defined by (35) is a well defined, fibre preserving, smooth map $Q : Z^- \to Z^+$. Let $X^+ = T^*U^+/\Lambda_+$ and $X^- = T^*U^-/\Lambda_-$. Then we can form a manifold

$$X = X^+ \cup_Q X^-.$$

This gluing is only topological, but X has a symplectic form on X^+ and on X^-, i.e. the standard ones. Is there a way of extending the gluing to give rise to a smooth symplectic manifold, whose symplectic form coincides with the standard ones when restricted to X^+ and X^-? If the answer is yes, then one can define the stitched Lagrangian fibration to be π^+ on X^+ and π^- on X^-. So far we have been able to give a positive answer in the 4 dimensional case. In the higher dimensional case it looks more delicate. Nevertheless we believe it should be possible.

With stitched Lagrangian fibrations one can allow a more general set of coordinate changes on the base than just the smooth ones. We call these admissible coordinates and they are homeomorphisms $A : U \to U'$ between the base U and another open neighbourhood U' of $0 \in \mathbb{R}^n$ such that $A \circ f$ is again a stitched Lagrangian fibration. In particular A is piece wise-smooth on U and smooth on U^+ and U^-.

Let $\{\lambda_1, \lambda_2^+, \ldots, \lambda_n^+\}$ and $\{\lambda_1, \lambda_2^-, \ldots, \lambda_n^-\}$ be periods on U^+ and U^- respectively, chosen as above. Then it turns out that action coordinates on U^+ chosen with respect to $\{\lambda_1, \lambda_2^+, \ldots, \lambda_n^+\}$ and on U^- chosen with respect to $\{\lambda_1, \lambda_2^-, \ldots, \lambda_n^-\}$ give an admissible change of coordinates on U. This is convenient because then one can assume that $\lambda_j^{\pm} = db_j$. Computed in action-angle coordinates ℓ^- becomes a 1-form on $\Gamma \times \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$. Since action-angle coordinates are unique up to an integral affine change of coordinates, they give a convenient normalization of ℓ^-. In particular, when the base has action coordinates, (36) implies that ℓ^- must satisfy

$$\int_{\overline{\lambda}_j^-} \ell^-(b) = m_j \tag{37}$$

for all $b \in \Gamma$.

Stitched Lagrangian fibrations with monodromy

We now discuss fibrations over non-simply connected open U which have non-trivial monodromy. With smooth fibrations usually monodromy is reflected by the periods, as discussed in previous sections. This may no longer be true in the case of stitched fibrations, as we observed in Remark 5.11. This is due to the non-smoothness along the seam Z. Here, in fact, there is a discrepancy between the pull back of forms with respect to f^+ and with respect to f^-. To understand monodromy one needs to take into consideration the 1-form ℓ^- which, somehow, measures this discrepancy. We illustrate here how this can be done with two examples.

Example 6.4. Suppose $U \subset \mathbb{R}^2$ is an annulus centered at $0 \in \mathbb{R}^2$ and let $f : X \to U$ be a stitched Lagrangian fibration. As before denote $U^+ = U \cap \{b_1 \geq 0\}$, $U^- = U \cap \{b_1 \leq 0\}$ and $\Gamma = U^+ \cap U^-$. This time Γ is necessarily disconnected. We let $\Gamma_u = \Gamma \cap \{b_2 \geq 0\}$ and $\Gamma_l = \Gamma \cap \{b_2 \leq 0\}$ be the upper and lower parts of Γ respectively. Now let $b \in \Gamma_u$ and let

$$\mathcal{M}_b : \pi_1(U) \to H_1(f^{-1}(b), \mathbb{Z})$$

be the monodromy map at b. Choose as generator e of $\pi_1(U)$ a loop going once around in the anticlockwise direction. Suppose there is a basis $\{\gamma_1, \gamma_2\}$ of $H_1(f^{-1}(b), \mathbb{Z})$ with respect to which

$$\mathcal{M}_b(e) = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}, \tag{38}$$

for some integer $m \neq 0$. Since γ_1 is monodromy invariant it must be represented by the orbits of the S^1 action. As usual let $X^\pm = f^{-1}(U^\pm)$. Since $U - \Gamma_l$ is contractible, we can think of $\{\gamma_1, \gamma_2\}$ as a basis of $H_1(f^{-1}(U - \Gamma_l), \mathbb{Z})$. Consider the natural isomorphisms

$$H_1(f^{-1}(U - \Gamma_l), \mathbb{Z}) \to H_1(X^+, \mathbb{Z}) \to H_1(f^{-1}(U - \Gamma_u), \mathbb{Z})$$

or

$$H_1(f^{-1}(U - \Gamma_l), \mathbb{Z}) \to H_1(X^-, \mathbb{Z}) \to H_1(f^{-1}(U - \Gamma_u), \mathbb{Z})$$

induced by inclusions and restrictions. The first row identifies $\{\gamma_1, \gamma_2\}$ with a basis of $H_1(f^{-1}(U - \Gamma_u), \mathbb{Z})$ which we call $\{\gamma_1, \gamma_2^+\}$, the second one identifies it with another basis which we call $\{\gamma_1, \gamma_2^-\}$. Notice that, from monodromy it follows that we must have

$$\gamma_2^+ = m\gamma_1 + \gamma_2^-.$$

Let λ_2^+ and λ_2^- be periods on U^+ and U^- respectively, both corresponding to γ_2 (or, equivalently, to γ_2^+ and γ_2^- respectively). Assume for simplicity that we have action coordinates (b_1, b_2) on U so that $\lambda_1 = db_1$ and $\lambda_2^\pm = db_2$. We can now apply the considerations of the previous section first to f restricted to $f^{-1}(U - \Gamma_l)$, then to f restricted to $f^{-1}(U - \Gamma_u)$. In the former case we obtain a 1-form ℓ_u^- on $\Gamma_u \times \mathbb{R}/\mathbb{Z}$ and in the later case a 1-form ℓ_l^- on $\Gamma_l \times \mathbb{R}/\mathbb{Z}$. From the integral condition (37) applied to ℓ_u and ℓ_l in our situation, we obtain that the two 1-forms must satisfy

$$\int_{\overline{\lambda_2^-}} \ell_u^-(b) = 0 \ \text{ and } \ \int_{\overline{\lambda_2^-}} \ell_l^-(b) = m, \tag{39}$$

for all b in Γ_u and in Γ_l respectively. What this tells us is that monodromy happens because ℓ_u^- and ℓ_l^- belong to different cohomology classes! A particular case of the situation considered is the stitched focus-focus example. Here we have $U = \mathbb{R}^2 - \{0\}$ and $m = 1$.

A result we have proved is that any pair of one forms ℓ_u^- and ℓ_l^- on $\Gamma_u \times \mathbb{R}/\mathbb{Z}$ and $\Gamma_l \times \mathbb{R}/\mathbb{Z}$, satisfying (39) can be used to symplectically glue T^*U^+/Λ_+ and T^*U^-/Λ_- along their boundary exactly as outlined in the previous section. We thus obtain many examples of a manifold X with a stitched Lagrangian fibration $f : X \to U$ which has monodromy (38).

Example 6.5. We now discuss a three dimensional example. In $\mathbb{R}^3$ consider the three-valent graph

$$\Delta = \{(0, 0, -t),\ t \geq 0\} \cup \{(0, -t, 0),\ t \geq 0\} \cup \{(0, t, t),\ t \geq 0\}$$

and let D be a tubular neighbourhood of Δ. Take $U = \mathbb{R}^3 - D$ and assume we have a stitched Lagrangian fibration $f : X \to U$ with seam $Z = f^{-1}(\{b_1 = 0\} \cap U)$, e.g. like in Example 5.4. Again we let $U^+ = U \cap \{b_1 \geq 0\}$, $U^- = U \cap \{b_1 \leq 0\}$ and $\Gamma = U^+ \cap U^-$. Also let $X^\pm = f^{-1}(U^\pm)$. This time Γ has three connected components

$$\begin{aligned} \Gamma_0 &= \{(0,t,s),\ t,s < 0\} \cap U, \\ \Gamma_1 &= \{(0,t,s),\ t > 0, s < t\} \cap U, \\ \Gamma_2 &= \{(0,t,s),\ s > 0, t < s\} \cap U. \end{aligned}$$

Fix $b \in \Gamma_0$ and suppose that there is a basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(f^{-1}(b), \mathbb{Z})$ and generators e_0, e_1, e_2 of $\pi_1(U)$, with $e_0 e_1 e_2 = 1$, with respect to which the monodromy transformations are

$$\mathcal{M}_b(e_1) = T_1 = \begin{pmatrix} 1 & 0 & -m_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}_b(e_2) = T_2 = \begin{pmatrix} 1 & -m_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{40}$$

and $M_b(e_0) = T_0 = T_1^{-1} T_2^{-1}$, for non zero integers m_1 and m_2. We have that γ_1 is represented by the orbits of the S^1 action. Now, since $U - (\Gamma_1 \cup \Gamma_2)$ is contractible, $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of $H_1(f^{-1}(U - (\Gamma_1 \cup \Gamma_2)), \mathbb{Z})$. Consider the natural isomorphisms

$$H_1(f^{-1}(U - (\Gamma_1 \cup \Gamma_2)), \mathbb{Z}) \to H_1(X^+, \mathbb{Z}) \to H_1(f^{-1}(U - (\Gamma_0 \cup \Gamma_1)), \mathbb{Z})$$

or

$$H_1(f^{-1}(U - (\Gamma_1 \cup \Gamma_2)), \mathbb{Z}) \to H_1(X^-, \mathbb{Z}) \to H_1(f^{-1}(U - (\Gamma_0 \cup \Gamma_1)), \mathbb{Z})$$

induced by inclusions and restrictions. The first row identifies $\{\gamma_1, \gamma_2, \gamma_3\}$ with a basis of $H_1(f^{-1}(U - (\Gamma_0 \cup \Gamma_1)), \mathbb{Z})$, which we call $\{\gamma_1, \gamma_2^+, \gamma_3^+\}$, while the second one identifies it with another basis, which we call $\{\gamma_1, \gamma_2^-, \gamma_3^-\}$. Notice that, from the monodromy map $\mathcal{M}_b(e_1) = T_1$, it follows that we must have

$$\begin{aligned} \gamma_2^+ &= \gamma_2^-, \\ \gamma_3^+ &= -m_1\gamma_1 + \gamma_3^-. \end{aligned}$$

Let $\{\lambda_1, \lambda_2^+, \lambda_3^+\}$ and $\{\lambda_1, \lambda_2^-, \lambda_3^-\}$ be periods on U^+ and U^- respectively, corresponding to the basis $\{\gamma_1, \gamma_2, \gamma_3\}$ of $H_1(f^{-1}(U - (\Gamma_1 \cup \Gamma_2)), \mathbb{Z})$ (or respectively to the basis $\{\gamma_1, \gamma_2^+, \gamma_3^+\}$ and $\{\gamma_1, \gamma_2^-, \gamma_3^-\}$ of $H_1(f^{-1}(U - (\Gamma_0 \cup \Gamma_1)), \mathbb{Z})$). Let us once more assume that we have action coordinates (b_1, b_2, b_3) on U so that $\lambda_1 = db_1$ and $\lambda_j^\pm = db_j$, $j = 2, 3$. We can apply the considerations of the previous section first to f restricted to $f^{-1}(U - (\Gamma_1 \cup \Gamma_2))$ and then to f restricted to $f^{-1}(U - (\Gamma_0 \cup \Gamma_1))$. We obtain a 1-form ℓ_0^- on $\Gamma_0 \times \mathbb{R}^2/\mathbb{Z}^2$ in the former case and a 1-form ℓ_2^- on $\Gamma_2 \times \mathbb{R}^2/\mathbb{Z}^2$ in the latter. From the integral condition (37) applied to ℓ_0^- and ℓ_2^- we obtain that the two 1-forms must satisfy

$$\int_{\overline{\lambda}_2^-} \ell_0^-(b) = \int_{\overline{\lambda}_3^-} \ell_0^-(b) = 0$$

and

$$\int_{\overline{\lambda}_2^-} \ell_2^-(b) = 0 \quad \text{and} \quad \int_{\overline{\lambda}_3^-} \ell_2^-(b) = -m_1,$$

for all b in Γ_0 and in Γ_2 respectively. Similarly one defines a 1-form ℓ_1^- on Γ_1. It will satisfy

$$\int_{\overline{\lambda}_2^-} \ell_1^-(b) = -m_2 \quad \text{and} \quad \int_{\overline{\lambda}_3^-} \ell_1^-(b) = 0.$$

Again, monodromy is understood in terms of the difference in the cohomology class of the three 1-forms ℓ_0^-, ℓ_1^- and ℓ_2^-. Example 5.4 is a special case of this situation, where $m_1 = m_2 = 1$.

One of the goals of our current research is to prove that any triple of three forms ℓ_0^-, ℓ_1^- and ℓ_2^- of the above type can be used to construct stitched Lagrangian fibrations $f : X \to U$, with the given monodromy (40). The method we have in mind is the gluing illustrated in the previous section.

References

[1] R. Castano-Bernard. Symplectic invariants of some families of Lagrangian T^3 fibrations. *J. of Symplectic Geometry*, 2(3):279–308, 2004. ArXiv: math.SG/0401321.

[2] J.J. Duistermaat. On global action-angle coordinates. *Comm.Pure Appl. Math.* ***6***, pages 678–706, 1980.

[3] E. Goldstein. Calibrated Fibrations on Complete Manifolds via Torus Action. arXiv: math.DG/0002097, 2000.

[4] M. Gross. Examples of special Lagrangian fibrations. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 81–109. World Sci. Publishing, River Edge, NJ, 2001. arXiv: math.AG/0012002.

[5] M. Gross. Topological Mirror Symmetry. *Invent. Math.* ***144***, pages 75–137, 2001.

[6] M. Gross, D. Huybrechts, and D. Joyce. *"Calabi-Yau manifolds and related geometries" Lecture notes at a summer school in Nordfjordeid, Norway, June 2001.* Springer Verlag, 2003.

[7] M. Gross and B. Siebert. Affine manifolds, Log structures and Mirror Symmetry. *Turkish J. of Math.* ***27***, pages 33–60, 2003.

[8] M. Gross and B. Siebert. Mirror Symmetry via Logarithmic degeneration data I. *arXiv: math.AG/0309070*, 2003.

[9] V. Guillemin and S. Sternberg. Birational equivalence in the symplectic category. *Inventiones Mathematicae*, 97:485–522, 1989.

[10] C. Haase and I. Zharkov. Integral Affine Structures on Spheres and Torus Fibrations on Calabi-Yau toric hypersurfaces I. *arXiv: AG/0205321*, 2002.

[11] C. Haase and I. Zharkov. Integral Affine Structures on Spheres and Torus Fibrations on Calabi-Yau toric hypersurfaces II. *math.AG/0301222*, 2003.

[12] D. Joyce. Singularities of special Lagrangian fibrations and the SYZ conjecture. *Communications in Analysis and Geometry*, 11(5):859–907, 2003. arXiv: math.DG/0011179.

[13] M. Kontsevich and Y. Soibelman. Affine structures and non-archimedean analytic spaces. arXiv: math.AG/0406564.

[14] M. Kontsevich and Y. Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 203–263. World Sci. Publishing, River Edge, NJ, 2001. arXiv: math.DG/0011041.

[15] W-D. Ruan. Lagrangian torus fibration of quintic hypersurfaces. I. Fermat quintic case. In *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, volume 23 of *AMS/IP Stud. Adv. Math.*, pages 297–332. Amer. Math. Soc., Providence, RI, 2001.

[16] W-D. Ruan. Lagrangian torus fibration of quintic Calabi-Yau hypersurfaces II: technical results on gradient flow construction. *Journal of Symplectic Geometry*, 1:435–521, 2002.

[17] W-D. Ruan. Lagrangian torus fibration of quintic Calabi-Yau hypersurfaces III: symplectic topological SYZ mirror construction for general quintics. *J. Differential Geometry*, 63:171–229, 2003.

[18] San Vu-Ngoc. On semi-global invariants for focus-focus singularities. *Topology*, 42(2):365–380, 2003.