Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

On a fully nonlinear Yamabe problem

by

Yuxin Ge and Guofang Wang

Preprint no.: 49 2005

ON A FULLY NONLINEAR YAMABE PROBLEM

YUXIN GE AND GUOFANG WANG

ABSTRACT. We solve the σ_2-Yamabe problem for a non locally conformally flat manifold of dimension $n > 8$.

Dedicated to Professor W. Y. Ding on the occasion of his 60's birthday

1. INTRODUCTION

Let (M, g_0) be a compact, oriented Riemannian manifold with metric g_0 and $[g_0]$ the conformal class of g_0. Let Ric_g and R_g be the Ricci tensor and scalar curvature of g respectively. The Schouten tensor of the metric g is defined as

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right).$$

The Schouten tensor plays an important role in conformal geometry. Recall that there is an important decomposition of Riemann curvature tensor

$$Riem = W_g + S_g \owedge g,$$

where W_g is the Weyl tensor of g. The Weyl tensor $g^{-1} \cdot W_g$ is invariant in a conformal class. Let σ_k be the kth elementary symmetric function. For a symmetric $n \times n$ matrix A, set $\sigma_k(A) = \sigma_k(\Lambda)$, where $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the set of eigenvalues of A. Following Viaclovsky [34], *σ_k-scalar curvatures* of g is defined as

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is locally defined by $(g^{-1} \cdot S_g)^i_j = \sum_k g^{ik}(S_g)_{kj}$. Note that $\sigma_1(g) = \frac{1}{2(n-1)} R_g$. It is an interesting question to find a metric g in a given conformal class $[g_0]$ such that

$$\sigma_k(g) = constant. \tag{1}$$

Since the Schouten tensors S_g and S_{g_0} of conformal metrics $g = e^{-2u} g_0$ and g_0 have the following relation

$$S_g = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0},$$

equation (1) is equivalent to the following fully nonlinear equation

$$\sigma_k\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) = ce^{-2ku}, \tag{2}$$

for some constant c. When $k = 1$, it is the well-known Yamabe equation.

Let

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall j \le k\}$$

be Garding's cone. A metric g is said to be k-positive or simply $g \in \Gamma_k^+$ if $g^{-1} \cdot S_g \in \Gamma_k^+$ for every point $x \in M$. If $g = e^{-2u} g_0$, we say u is k-admissible if g is k-positive. In this paper we consider the following

σ_k-Yamabe problem. Let $g_0 \in \Gamma_k^+$. Find a conformal metric $g \in [g_0] \cap \Gamma_k^+$ such that

$$\sigma_k(g) = constant.$$

The study of the fully nonlinear equations (1) was initiated by Viaclovsky. Since then there is a lot of work concerning these equations. Here, we just mention some work directly related to the existence of the σ_k-Yamabe problem. This problem has been solved in the following cases. When $k = n$, under a sufficient condition, Viaclovsky proved the existence in [36]. When $n = 2k = 4$, which is an important case, Chang-Gursky-Yang solved the problem in [7]. See also [6] and [22]. When the underlying manifold is locally conformally flat, this problem was solved by Guan-Wang [18] and Li-Li [27] independently. See also [4]. Note that when the underlying manifold (M, g_0) is locally conformally flat and $g \in \Gamma_k^+$ with $k \ge n/2$, M is conformally equivalent to a spherical space form [16]. When $k > n/2$, the σ_k-Yamabe problem was solved by Gursky-Viaclovsky in [23]. See also their earlier work [21].

In this paper, we consider the case $k = 2$. In this case, equation (2) is a variational problem, which was observed by Viaclovsky in [34]. This is crucial for our method presented here. Our main result in this paper is

Theorem 1. *Let (M^n, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$. When $n > 8$ and the Weyl tensor $W_{g_0} \neq 0$, then there is a conformal metric $g \in [g_0] \cap \Gamma_2^+$ such that*

$$\sigma_2(g) = constant.$$

Combining the results of [18] and [27], the σ_2-Yamabe problem is solvable if $n > 8$. As the Yamabe problem, there is a well-known difficulty –the loss of compactness of equation (1). Another more difficult problem is the fully nonlinearity of (1). Our result here is an analogue of the result of Aubin [2] for the ordinary Yamabe problem. Even the ideas of proof are quite similar. However the techniques to realize these ideas become more delicate due to the fully nonlinearity.

Set $\mathcal{C}_2 = \{g \in [g_0] \,|\, g \in \Gamma_2^+\}$ and define a Yamabe type constant by

$$Y_2(M, [g_0]) = \begin{cases} \inf\limits_{g \in \mathcal{C}_2} \tilde{\mathcal{F}}_2(g), & \text{if } \mathcal{C}_2 \neq \emptyset, \\ +\infty, & \text{if } \mathcal{C}_2 = \emptyset, \end{cases}$$

where $\tilde{\mathcal{F}}_2(g) = vol(g)^{-\frac{n-4}{n}} \displaystyle\int_M \sigma_2(g)\, dvol(g)$. This is a natural generalization of the Yamabe constant and was considered in [19] in the fully nonlinear context.

We first prove the following proposition.

Proposition 1. *Let (M^n, g_0) be a compact, oriented Riemannian manifold of dimension $n > 4$ with $g_0 \in \Gamma_2^+$. The σ_2-Yamabe is solvable, provided that*

$$Y_2(M, [g_0]) < Y_2(\mathbb{S}^n). \tag{3}$$

The idea to prove the Proposition is a "blow-up" analysis, which is a typical tool in the field of semilinear equations. The observation that the fully nonlinear equation (1) also admits a blow-up analysis was made in [17]. Here we inspire from the Yamabe method (see [3]). We first prove the existence of solutions to a "subcritical" equation (11) for any small $\varepsilon > 0$. To prove the existence of solutions of (11), we use a fully non-linear flow (8). We show that this flow globally converges to a solution u_ε of the subcritical equation (6). In fact, u_ε is a minimizier for a corresponding functional. Then we consider the sequence u_ε as $\varepsilon \to 0$. Using the blow-up analysis developed in [17] and the classification of "bubbles" in [8] or [27], we can show that the sequence u_ε subconverges to a solution of (2) under the condition (3). The flow method to attack the existence of fully nonlinear equations was used by many mathematicians, see for instance [9], [37], [33] and [10]. In the fully nonlinear conformal equations, it was used in [18] and [19].

Then we show

Proposition 2. *Let (M^n, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$. When $n > 8$ and the Weyl tensor $W_{g_0} \neq 0$,*

$$Y_2(M, [g_0]) < Y_2(\mathbb{S}^n).$$

This is a delicate gluing argument. We need to construct suitable test metrics as in [2] and [30] for the ordinary Yamabe problem. A subtle point in the gluing is that all metrics we constructed should lie in Γ_2^+. Recall that in the ordinary Yamabe problem, the test metrics constructed by Aubin and Schoen has negative scalar curvature somewhere. To overcome this difficulty, we adopt a method of Gromov-Lawson in their construction of metrics of positive scalar curvature, see also [29] for metrics of positive isotropic curvature and [14] for metrics of positive k-scalar curvature on locally conformally flat manifolds. We believe that by a similar, but more delicate construction one can prove Proposition 2 for $n = 8$. For $n = 5, 6, 7$, this problem becomes delicate. We will consider these cases later.

By-products of our work for flow (8) are the Poincaré type inequality and Sobolev inequality for the operator $\sigma_2(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$.

Proposition 3. *Let (M, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$ and the dimension $n > 4$. Then there exists a positive constant $\lambda_1 > 0$ depending only on (M, g_0) such that for any C^2 function u with $e^{-2u} g_0 \in \mathcal{C}_2([g_0])$ we have*

$$\int_M \sigma_2(e^{-2u} g_0) dvol(e^{-2u} g_0) \geq \lambda_1 \int e^{4u} dvol(e^{-2u} g_0).$$

Equivalently, for such a function u we have

$$\int_M e^{(4-n)u} \sigma_2(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) dvol(g_0) \geq \lambda_1 \int e^{(4-n)u} dvol(g_0).$$

Theorem 2. *Let* (M, g_0) *be a compact, oriented Riemannian manifold with* $g_0 \in \Gamma_2^+$ *and the dimension* $n > 4$. *Then there exists a positive constant* $C > 0$ *depending only on* (M, g_0) *such that for any* C^2 *function* u *with* $e^{-2u}g_0 \in \mathcal{C}_2([g_0])$ *we have*

$$\int_M \sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0) \geq Cvol(e^{-2u}g_0)^{\frac{n-4}{n}}.$$

Equivalently, for such a function u *we have*

$$\int_M e^{(4-n)u}\sigma_2(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})dvol(g_0) \geq C(\int_M e^{-nu}dvol(g_0))^{\frac{n-4}{n}}.$$

The Sobolev inequality and other geometric inequalities, the Moser-Trudinger inequality and a conformal quermassintegral inequality for $\sigma_k(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})$ for a locally conformally flat manifolds were established in [19]. See also [16] and [12].

The method presented here works for a conformal quotient equation

$$\frac{\sigma_2(g)}{\sigma_1(g)} = c,$$

on a general manifold. See other results for conformal quotient equations in [19], [15] and [23].

The paper is organized as follows. In Section 2, we discuss various fully nonlinear flows and we prove local estimates for these flows in Section 3. In Section 4, we establish the Poincaré and Sobolev inequalities. We prove the global convergence of these fully nonlinear flows and Proposition 1 in Section 5. In Section 6, we prove Proposition 2, and hence Theorem 1.

2. VARIOUS FLOWS AND IDEAS OF PROOF

Consider the following functional

$$\mathcal{F}_k(g) = \int_M \sigma_k(g)\, dvol(g) \tag{4}$$

and its normalization $\tilde{\mathcal{F}}_k$

$$\tilde{\mathcal{F}}_k(g) = vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g)\, dvol(g). \tag{5}$$

When $k = 2$ or the underlying manifold is locally conformally flat, Viaclovsky proved that critical points of $\tilde{\mathcal{F}}_2$ are solutions of (1). Therefore, in these cases, (1) is a variational problem. The case when the underlying manifold is locally conformally flat was studied in [18] and [27], as mentioned in the Introduction. See also [4]. In this paper we only consider the case $k = 2$. Since the case $k = 2$ and $n \leq 4$ was solved in [6], [21] and [23], we focus on the case $k = 2$ and $n > 4$.

Recall that $\mathcal{C}_2 = \{g \in [g_0] \,|\, g \in \Gamma_2^+\}$ and the Yamabe type constant is defined by

$$Y_2(M, [g_0]) = \begin{cases} \inf\limits_{g\in\mathcal{C}_2} \tilde{\mathcal{F}}_2(g), & \text{if } \mathcal{C}_2 \neq \emptyset, \\ \infty, & \text{if } \mathcal{C}_2 = \emptyset. \end{cases}$$

Our main aim of this paper is to show that $Y(M,[g_0])$ is achieved for non locally flat manifolds when $\mathcal{C}_2([g_0])\neq\emptyset$. In order to achieve our aim, we will first consider *subcritical* equations.

$$\sigma_2^{1/2}(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}) = ce^{(\varepsilon-2)u}, \tag{6}$$

for $\varepsilon\in(0,2]$ and the positive constant c. Its corresponding functional is

$$\tilde{\mathcal{F}}_{2,\varepsilon}(g) = V_\varepsilon(g)^{-\frac{n-4}{n-2\varepsilon}}\int_M \sigma_2(g)\,dvol(g), \tag{7}$$

where

$$V_\varepsilon(g) := \int_M e^{2\varepsilon u} dvol(g) = \int_M e^{(2\varepsilon-n)u} dvol(g_0),$$

for $g=e^{-2u}g_0$. It is clear that $V_0(g)=vol(g)$, the volume of g and $V_2(g)=\int e^{(4-n)u}dvol(g)$. Set

$$Y_\varepsilon(M,[g_0]) = \inf_{g\in\mathcal{C}_2}\tilde{\mathcal{F}}_{2,\varepsilon}(g).$$

We will show that $Y_\varepsilon(M,[g_0])$ is achieved at u_ε, which is clearly a solution of (6). To prove this we consider the following fully nonlinear flow

$$\begin{aligned}
2\frac{du}{dt} &= -g^{-1}\cdot\tfrac{d}{dt}g\\
&= \left(h(e^{-2u}\sigma_2^{1/2}(g)) - h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})\right) - s_\varepsilon(g),\\
&= h(\sigma_2^{1/2}(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})) - h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u}) - s_\varepsilon(g),
\end{aligned} \tag{8}$$

with initial value $u(0)=1$, where $r_\varepsilon(g)$ is given by for any $\varepsilon\in[0,2]$

$$r_\varepsilon(g) := \frac{\int_M \sigma_2(g)\,dvol(g)}{\int_M e^{2\varepsilon u}\,dvol(g)}$$

$$s_\varepsilon(g) := \frac{\int_M e^{2\varepsilon u}\left(h(e^{-2u}\sigma_2^{1/2}(g)) - h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})\right)dvol(g)}{\int_M e^{2\varepsilon u}\,dvol(g)}$$

and $h:\mathbb{R}_+\to\mathbb{R}$ is smooth concave function with $h'(t)\geq 1$ for $t\in\mathbb{R}_+$ satisfying

$$h(s) = \begin{cases} 2\log s & \text{if } t\leq 1\\ s & \text{if } t\geq 2.\end{cases}$$

Flow (8) preserves V_ε and non-increases $\mathcal{F}_2$.

Lemma 1. *For any $\varepsilon\in[0,2]$, the flow (8) preserves the functional V_ε and nonincreases $\mathcal{F}_2$. In fact, we have*

$$\frac{d}{dt}\mathcal{F}_2(g) = -\frac{n-4}{2}\int_M\left(h(e^{-2u}\sigma_2^{1/2}(g)) - h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})\right)(\sigma_2(g) - r_\varepsilon e^{2\varepsilon u})\,dvol(g). \tag{9}$$

Moreover, r_ε is bounded.

Proof. We note that

$$\frac{d}{dt}\mathcal{F}_2(g) = \frac{n-4}{2}\int_M (g^{-1}\cdot \frac{d}{dt}g)\sigma_2(g)\, dvol(g)$$

and

$$\frac{d}{dt}V_\varepsilon(g) = \frac{n-2\varepsilon}{4}\int_M (g^{-1}\cdot \frac{d}{dt}g)e^{2\varepsilon u}\, dvol(g) = 0.$$

See the proof in [18]. It is clear that V_ε is preserved along the flow. On the other hand, a direct computation gives

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(g) &= \frac{n-4}{2}\int_M (g^{-1}\cdot \frac{d}{dt}g)(\sigma_2(g) - r_\varepsilon e^{2\varepsilon u})\, dvol(g)\\ &= -\frac{n-4}{2}\int_M \left(h(e^{-2u}\sigma_2^{1/2}(g)) - h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})\right)(\sigma_2(g) - r_\varepsilon e^{2\varepsilon u})\, dvol(g).\end{aligned} \tag{10}$$

where in the second equality, we have used the fact

$$\int_M (\sigma_2(g) - r_\varepsilon e^{2\varepsilon u})\, dvol(g) = 0.$$

Hence, r_ε is bounded. ■

In fact, flow (8) strictly decreases the functional $\mathcal{F}_2$ except at the solutions of the following equation

$$\sigma_2^{1/2}(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}) = ce^{(\varepsilon-2)u}, \tag{11}$$

for some positive constant c. When $\varepsilon = 0$ equation (11) is just (2). When $\varepsilon = 2$ equation (11) is a corresponding equation for a nonlinear eigenvalue problem, which was considered in [20]. See also Section 4.

Since $g_0 \in \Gamma_2^+$, flow (8) is parabolic near $t = 0$, by the standard implicit function theorem we have the following short-time existence result.

Proposition 4. *For any $g_0 \in C^2(M)$ with $g_0 \in \Gamma_2^+$, there exists a positive constant $T^* \in (0,\infty]$ such that flow (8) exists and is parabolic for $t \in [0,T^*)$, and $\forall T < T^*$,*

$$g \in C^{3,\alpha}([0,T]\times M), \forall 0<\alpha<1, \quad \text{and} \quad g(t) \in \Gamma_2^+.$$

We assume that T^* is the largest number, for which Proposition 4 holds. We first show that the global convergence of flow (8) when $\varepsilon = 2$. The global convergence implies a Poincaré type inequality. Then, using this inequality and the divergence free of the first Newton transformation of the Schouten tensor, which was an observation Viaclovsky, we obtain an optimal Sobolev inequality. By establishing a flow version of local gradient estimates, which was proved in [17], we show that flow (8) globally converges to a solution u_ε of (11) for any $\varepsilon \in (0,2]$. With the help of the local estimate obtained in [17] and a classification in [27] or [8], we show that u_ε subconverges to a solution u_0 of (2), provided that

$$Y_2(M,[g_0]) < Y_2(\mathbb{S}^n). \tag{12}$$

In this case, it is clear that u_0 is the minimum of $\tilde{\mathcal{F}}_2$.

3. Local Estimates

In this section, we will establish a local estimate for solutions of (8), which is a parabolic version of a local estimate for solutions of (2) obtained in [17].

Theorem 3. *Let u be a solution of (8) with $\varepsilon \in [0,2]$ in a geodesic ball $B_r \times [0,T]$ for $T < T^*$ and $r < r_0$, the injectivity radius of M. There is a constant $C > 0$ depending only on (B_r, g_0) such that for any $(x,t) \in B_{r/2} \times [0,T]$*

$$|\nabla u|^2 + |\nabla^2 u| \le C(1 + e^{-(2-\varepsilon)\inf_{(x,t)\in B_r\times[0,T]} u(x,t)}). \tag{13}$$

Proof. The proof follows [17] closely. We only point out the different places. Let $\rho \in C_0^\infty(B_1)$ be a test function defined as in [17]. such that

$$\begin{aligned} \rho &\ge 0, && \text{in } B_1,\\ \rho &= 1, && \text{in } B_{1/2},\\ |\nabla\rho(x)| &\le 2b_0\rho^{1/2}(x), && \text{in } B_1,\\ |\nabla^2\rho| &\le b_0, && \text{in } B_1. \end{aligned} \tag{14}$$

Here $b_0 > 1$ is a constant. Set $H(x,t) = \rho|\nabla u|^2$. Let (x_0,t_0) be the maximum of H in $M \times [0,T]$. Without loss of generality, we assume $t_0 > 0$. We have at (x_0,t_0) that

$$0 \le H_t = 2\rho\sum_l u_l u_{lt}, \tag{15}$$

$$0 = H_j = \rho_j|\nabla u|^2 + 2\rho\sum_l u_l u_{lj}, \tag{16}$$

$$0 \ge (H_{ij}) \tag{17}$$

Let $W = (w_{ij})$ be an $n\times n$ matrix with $w_{ij} = \nabla^2_{ij}u + u_iu_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}$. Here u_i and u_{ij} are the first and second derivatives of u with respect to the background metric g_0. By choosing suitable normal coordinates, we may assume that W is diagonal at (x_0,t_0), and hence we have at (x_0,t_0),

$$\begin{aligned} w_{ii} &= u_{ii} + u_i^2 - \frac{1}{2}|\nabla u|^2 + (S_{g_0})_{ii},\\ u_{ij} &= -u_iu_j - (S_{g_0})_{ij}, \qquad \forall i \ne j. \end{aligned} \tag{18}$$

In view of (14), (16) and (18), we have at (x_0,t_0)

$$|\sum_{l=1}^n u_{il}u_l| \le nb_0\rho^{-1/2}|\nabla u|^2. \tag{19}$$

We may assume that

$$H(x_0,t_0) \ge n^2A_0^2b_0^2,$$

i. e., $\rho^{-1/2} \leq \frac{1}{nA_0b_0}|\nabla u|$, and

$$|\nabla S_{g_0}| + |S_{g_0}| \leq A_0^{-1}|\nabla u|^2,$$

where $A_0 > 1$ is a large, but fixed number to be chosen later, otherwise we are done. Thus, from (19) we have

$$|\sum_{l=1}^n u_{il}u_l| \leq \frac{|\nabla u|^3}{A_0}(x_0, t_0). \tag{20}$$

Set $F = h(\sigma_2^{1/2}(W))$ and

$$F^{ij} = \frac{\partial F(W)}{\partial w_{ij}}.$$

Note that flow (8) is equivalent to $2u_t = F - h(r_\varepsilon^{1/2}e^{(\varepsilon-2)u}) - s_\varepsilon^{1/2}$ and F^{ij} is diagonal at (x_0, t_0). Since matrix (F^{ij}) is positive definite, from (15) and (17) we have

(21)

$$\begin{aligned} 0 &\geq \sum_{i,j} F^{ij}H_{ij} - 2H_t \\ &= \sum_{i,j} F^{ij}\left\{\left(-2\frac{\rho_i\rho_j}{\rho} + \rho_{ij}\right)|\nabla u|^2 + 2\rho\sum_l u_{lij}u_l + 2\rho\sum_l u_{il}u_{jl}\right\} - 4\rho\sum_l u_lu_{lt}. \end{aligned}$$

We need to estimate the term $\sum_{i,j,l} F^{ij}u_{lij}u_l - 2\sum_l u_lu_{lt}$. Since changing the order of derivatives only causes a low order term, we have

(22)

$$\begin{aligned} \sum_{i,j,l} F^{ij}u_{lij}u_l - 2\sum_l u_lu_{lt} &\geq \sum_{i,j,l} F^{ij}u_{ijl}u_l - 2\sum_l u_lu_{lt} - c\sum_i F^{ii}|\nabla u|^2 \\ &\geq \sum_{i,j,l} F^{ij}(w_{ij})_lu_l - \sum_{i,l} F^{ii}(u_i^2 - \frac{1}{2}|\nabla u|^2)_lu_l - 2\sum_l u_lu_{lt} \\ &\quad -c\sum_i F^{ii}|\nabla u|^2 - \sum_{i,l} F^{ii}\nabla_l(S_{g_0})_{ii}u_l \\ &\geq \sum_l (F_l - 2u_{tl})u_l - cA_0{}^{-1}\sum_i F^{ii}|\nabla u|^4 \\ &\geq (\varepsilon-2)h'(r_\varepsilon^{1/2}e^{(\varepsilon-2)u})r_\varepsilon^{1/2}e^{(\varepsilon-2)u}|\nabla u|^2 - cA_0{}^{-1}\sum_i F^{ii}|\nabla u|^4, \end{aligned}$$

where we have used (8) and (20). Here c is a constant independent of u, but it may vary from line to line. The term $\sum_{i,j} F^{ij}(-2\frac{\rho_i\rho_j}{\rho} + \rho_{ij})|\nabla u|^2$ is bounded from below by $-10b_0^2|\nabla u|^2\sum_j F^{jj}$. For the term $F^{ij}u_{il}u_{jl}$ we have the following crucial Lemma.

Lemma 2 ([17]). *There is a constant A_0 sufficient large (depending only on n, and $\|g_0\|_{C^3(B_1)}$), such that,*

$$\sum_{i,j,l} F^{ij}u_{il}u_{jl} \geq A_0^{-\frac{3}{4}}|\nabla u|^4 \sum_{i\geq 1} F^{ii}. \tag{23}$$

Altogether gives us

$$(A_0^{-\frac{3}{4}} - cA_0^{-1})\rho|\nabla u|^4 \sum_i F^{ii} \leq 10b_0^2|\nabla u|^2\rho\sum_i F^{ii} + c\rho(1+e^{(\varepsilon-2)u})|\nabla u|^2. \tag{24}$$

By the Newton-McLaurin inequality and the fact that $h'(t) \geq 1$ for any $t \geq 0$, it is easy to check that

$$\sum_i F^{ii} \geq 1,$$

which, together with (24), proves the local gradient estimate

$$|\nabla u|^2 \leq C(1 + e^{(2-\varepsilon)\inf_{(x,t)\in B_r\times[0,T]} u(x,t)}),$$

for some constant $C > 0$ depending only on (B_r, g_0).

Now we show the local estimates for second order derivatives. Since $e^{-2u}g_0 \in \Gamma_2^+$, to bound $|\nabla^2 u|$ we only need bound Δu from above. This is a well-known fact, see for instance [17]. Set

$$G = \rho(\Delta u + |\nabla u|^2),$$

where ρ is defined as above. Let (y_0, t_0) be a maximum point of G in $M \times [0, T]$. Without loss of generality, we assume $G(y_0, t_0) > 1 + 2\max H(x)$ and $t_0 > 0$, where $H = \rho|\nabla u|^2$. Hence we have

$$0 < \rho\Delta u(y_0) \leq G(y_0) \leq 2\rho\Delta u(y_0).$$

At (y_0, t_0), we have

$$\begin{aligned}
&(25) \qquad 0 \;\leq\; G_t = \rho\sum_l (u_{llt} + 2u_lu_{lt}),\\
&(26) \qquad 0 \;=\; G_j = \frac{\rho_j}{\rho}G + \rho\sum_{l\geq 1}(u_{llj} + 2u_lu_{lj}), \qquad \text{for any } j,\\
&(27) \qquad 0 \;\geq\; G_{ij} = \frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G + \rho\sum_{l\geq 1}(u_{llij} + 2u_{li}u_{lj} + 2u_lu_{lij}).
\end{aligned}$$

Recall that $F^{ij}=\frac{\partial}{\partial w_{ij}}F$ is non-negative definite. Hence, we have

$$\begin{aligned}
0 \ \geq\ & \sum_{i,j\geq 1} F^{ij}G_{ij}-2G_t\\
\geq\ & \sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G+\rho\sum_{i,j,l\geq 1}F^{ij}(u_{ijll}+2u_{li}u_{lj}+2u_lu_{lij})\\
& -2\rho\sum_l(u_{llt}+2u_lu_{lt})-C\rho\sum_i(|u_{ii}|+|u_i|)\sum_{i,j}|F^{ij}|,
\end{aligned}$$

where the last term comes from the commutators related to the curvature tensor of g_0 and its derivatives. First, from the definition of ρ, we have

$$\sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}G\geq -C\sum_{i,j\geq 1}|F^{ij}|\frac{1}{\rho}G.$$

By the concavity of $\sigma_2^{1/2}$, we have

$$\begin{aligned}
\sum_{i,j,l\geq 1}F^{ij}u_{ijll} &= \sum_{i,j,l\geq 1}F^{ij}w_{ijll}-\sum_{i,j,l\geq 1}F^{ij}(u_iu_j-\frac{1}{2}|\nabla u|^2(g_0)_{ij}+(S_{g_0})_{ij})_{ll}\\
&\geq \sum_l F_{ll}-\sum_{i,j,l\geq 1}F^{ij}(u_iu_j-\frac{1}{2}|\nabla u|^2(g_0)_{ij}+(S_{g_0})_{ij})_{ll}
\end{aligned}\tag{28}$$

We also have

$$\begin{aligned}
\sum_{i,j,l}F^{ij}u_lu_{lij} &= \sum_{i,j,l}F^{ij}u_l(w_{ij})_l-\sum_{i,j,l}F^{ij}u_l(u_iu_j-\frac{1}{2}|\nabla u|^2(g_0)_{ij}+(S_{g_0})_{ij})_l\\
&\quad +\sum_{i,j,l}F^{ij}u_l(u_{lij}-u_{ijl})\\
&= \sum_l F_lu_l-\sum F^{ij}u_l(u_iu_j-\frac{1}{2}|\nabla u|^2(g_0)_{ij}+(S_{g_0})_{ij})_l\\
&\quad +\sum_{i,j,l}F^{ij}u_l(u_{lij}-u_{ijl}).
\end{aligned}\tag{29}$$

Hence, we have

$$\begin{aligned}
\sum_{i,j,l\geq 1}F^{ij}(u_{ijll}+2u_{li}u_{lj}+2u_lu_{lij}) \geq\ & \sum_l(F_{ll}+2F_lu_l)-2\sum_{i,j,l}F^{ij}u_iu_{jll}+\sum_{j,k,l}F^{jj}u_ku_{kll}\\
& -2\sum_{i,j,l}F^{ij}u_l(u_iu_j-\frac{1}{2}|\nabla u|^2(g_0)_{ij}+(S_{g_0})_{ij})_l\\
& +\sum_{i,k,l}F^{ii}(u_{kl})^2-C(1+\frac{G}{\rho})\sum_{i,j}|F^{ij}|.
\end{aligned}\tag{30}$$

From (25) and equation (8), we have

$$\rho\sum_l (F_{ll}+2F_l u_l) \geq 2\rho\sum_l (u_{tll}+2u_l u_{tl}) - C(2-\varepsilon)G(1+e^{(\varepsilon-2)u}). \tag{31}$$

The term $-2\sum_{i,j,l} F^{ij}u_i u_{jll} + \sum_{j,k,l} F^{jj}u_k u_{kll}$ can be controlled as in [17] with the help of (26). And the other terms in (30) can easily be estimated. On the other hand, it follows from the positivity of (F^{ij}) that

$$\sum_{i,j}|F^{ij}| \leq C\sum_i F^{ii}$$

This completes the proof of the Theorem. ■

From the local estimates, we have

Corollary 1. *If "bubble" occurs, i.e.,* $\inf_{M\times[0,T^*)} u = -\infty$, *then there is a positive constant* $c_0 > 0$ *such that*

$$\lim_{\delta\to 0}\lim_{t\to T^*} V_\varepsilon(g, B_\delta) > c_0.$$

4. A Poincaré inequality and a Sobolev inequality

The Sobolev inequality is a very important analytic tool in many problems arising from analysis and geometry. It plays a crucial role in the resolution of the Yamabe problem, which was solved completely by Yamabe [38], Trudinger [32], Aubin [2] and Schoen [30]. See various optimal Sobolev inequalities in [24]. In this section we are interested in a similar type inequality for the class of a fully nonlinear conformal operators $\sigma_k(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})$. In [19], the Sobolev inequality was generalized to the operator $\sigma_k(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})$ for $k < n/2$, if the underlying manifold is locally conformally flat. Namely,

Theorem 4 ([17])**.** *Let* (M^n, g_0) *be a compact, oriented Riemannian manifold with* $g_0 \in \Gamma_k^+$ *and* $k < n/2$. *Assume that* (M, g_0) *is locally conformally flat, then there exists a positive constant* $C > 0$ *depending only on* n, k *and* (M, g_0) *such that for any* C^2 *function* u *with* $e^{-2u}g_0 \in \mathcal{C}_k([g_0])$ *we have*

$$\int_M \sigma_k(e^{-2u}g_0)dvol(e^{-2u}g_0) \geq Cvol(e^{-2u}g_0)^{\frac{n-2k}{n}}. \tag{32}$$

Equivalently, for such a function u *we have*

$$\int_M e^{(2k-n)u}\sigma_k(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})dvol(g_0) \geq C(\int_M e^{-nu}dvol(g_0))^{\frac{n-2k}{n}}. \tag{33}$$

When $k = 1$, inequality (34) is just the Sobolev inequality. The proof of Theorem 4 uses a Yamabe type flow. See also the work of [12].

In this section, we establish the Sobolev inequality for $k = 2$ without the flatness condition.

Theorem 5. *Let (M, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$ and the dimension $n > 4$. Then there exists a positive constant $C > 0$ depending only on (M, g_0) such that for any C^2 function u with $e^{-2u}g_0 \in \mathcal{C}_2([g_0])$ we have*

$$\int_M \sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0) \geq Cvol(e^{-2u}g_0)^{\frac{n-4}{n}}. \tag{34}$$

Equivalently, for such a function u we have

$$\int_M e^{(4-n)u}\sigma_2(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})dvol(g_0) \geq C(\int_M e^{-nu}dvol(g_0))^{\frac{n-4}{n}}. \tag{35}$$

First we prove a Poincaré type inequality, which will be used in the proof of our Sobolev inequality. The usual Poincaré type inequality is associated to the first eigenvalue problem. In our case, there is a nonlinear eigenvalue problem, which was studied in [20].

Proposition 5. *Let (M, g_0) be a compact manifold with $g_0 \in \Gamma_k^+$. Then there is a function u with $e^{-2u}g_0 \in \Gamma_k^+$ satisfying*

$$\sigma_k(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}) = \lambda_1 > 0. \tag{36}$$

Moreover the constant λ_1 is unique and the solution is unique up to a constant.

An elliptic method was used in the proof, which was motivated by a method introduced in [28]. See also [37] for a Hessian operator. In view of Proposition 5, one may guess that

$$\int e^{2ku}\sigma_k(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})dvol(e^{-2u}g_0) \geq \lambda_1 \int e^{2ku}dvol(e^{-2u}g_0), \tag{37}$$

for any u with $e^{-2u}g_0 \in \Gamma_k^+$. It is easy to see that when $k = 1$ inequality (37) holds. In fact it is the Poincaré inequality. In this section, we show that (37) holds for $k = 2$ by flow (8) with $\varepsilon = 2$.

Proposition 6. *Let (M, g_0) be a compact, oriented Riemannian manifold with $g_0 \in \Gamma_2^+$ and the dimension $n > 4$. Then for any C^2 function u with $e^{-2u}g_0 \in \mathcal{C}_2([g_0])$ we have*

$$\int_M \sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0) \geq \lambda_1 \int e^{4u}dvol(e^{-2u}g_0). \tag{38}$$

Equivalently, for such a function u we have

$$\int_M e^{(4-n)u}\sigma_2(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})dvol(g_0) \geq \lambda_1 \int e^{(4-n)u}dvol(g_0). \tag{39}$$

Proof. To prove the Proposition, we consider flow (8) with $\varepsilon = 2$. We want to show that the flow converges globally to a solution obtained in Proposition 5. By Theorem 3, we have

$$|\nabla^2 u| + |\nabla u|^2(x, t) \leq C, \tag{40}$$

where C is a constant independent of $(x, t) \in M \times [0, T^*)$. Since the flow preserves the functional V_2, in view of (40) we have that $|u| \leq C$, for some constant $C > 0$. Now following the method in [18] we can show that

$$\sigma_2(g) > c_0,$$

for some constant c_0 independent of t. See the proof in the next section. Hence, this flow exists globally and is uniformly elliptic. By the result of Krylov, $g(t) \in C^{4+\alpha,2+\alpha}$. Since the flow satisfies (9), one can show that for any sequence of $\{t_i\}$ with $t_i \to \infty$ there is a subsequence, still denoted by $\{t_i\}$, such that $g(t_i)$ converges strongly to g^*, which satisfies (36). On the other hand, $V_2(g^*) \equiv V_2(g(t))$. By the uniqueness in Proposition 5, one can show that the flow globally converges to g^*. Since the flow preserves V_2 and decreases $\mathcal{F}_2$, we have

$$\mathcal{F}_2(g) \geq \mathcal{F}_2(g^*),$$

for any $g \in \mathcal{C}_2$. This is the Poincaré inequality that we want to prove. ■

Proof of Theorem 5. Let $g = e^{-2u} g_0$. We have

$$2\sigma_2 = \sum_{i,j} T^{ij} S_{ij},$$

where $T(g)^{ij} = \sigma_1(g) g^{ij} - S(g)^{ij}$ is the so-called the first Newton transformation. We will use the following formulas

$$S(g)_{ij} = u_{ij} + u_i u_j - \frac{1}{2}|\nabla u|^2_{g_0} (g_0)_{ij} + S(g_0)_{ij} \tag{41}$$

and

$$\widetilde{\nabla}^2_{ij} u = u_{ij} + 2u_i u_j - |\nabla u|^2_{g_0} (g_0)_{ij}, \tag{42}$$

where $\widetilde{\nabla}$ are the derivatives w. r. t g. Thus,

$$2\sigma_2(g) = \sum_{i,j} T(g)^{ij} \widetilde{\nabla}^2_{ij} u - \sum_{i,j} T(g)^{ij} u_i u_j + \frac{n-1}{2}\sigma_1(g)|\widetilde{\nabla} u|^2_g + \sum_{i,j} T(g)^{ij} S(g_0)_{ij}. \tag{43}$$

here we have used $\mathrm{tr} T(g) = (n-1)\sigma_1(g)$. Note that

$$\sum_{i,j} T(g)^{ij} S(g_0)_{ij} > 0, \tag{44}$$

thanks to Garding's inequality

$$\sum_{i,j} T(g)^{ij} S(g_0)_{ij} \geq 2e^{2u} \sigma_2^{1/2}(g) \sigma_2^{1/2}(g_0).$$

Due to an observation of Viaclovsky, $\sum_i \widetilde{\nabla}_i T(g)^{ij} = 0$, we have

$$\begin{aligned} 2\int \sigma_2(g) &= -\int \sum_{i,j} T(g)^{ij} u_i u_j dvol(g) + \frac{n-1}{2} \int \sigma_1(g) |\widetilde{\nabla} u|^2_g dvol(g) \\ &\quad + \int \sum_{i,j} T(g)^{ij} S(g_0)_{ij} dvol(g). \end{aligned} \tag{45}$$

Recall that $T(g)=\sigma_1(g)g-S(g)$. We have

$$
\begin{aligned}
-\int\sum_{i,j}T(g)^{ij}u_iu_jdvol(g) &= -\int\sigma_1(g)|\widetilde{\nabla}u|_g^2dvol(g)+\int\sum_{i,j}S(g)^{ij}u_iu_jdvol(g)\\
&= -\int\sigma_1(g)|\widetilde{\nabla}u|_g^2dvol(g)+\int\sum_{i,j}\widetilde{\nabla}^{ij}uu_iu_jdol(g)\\
&\quad -\frac{1}{2}\int|\widetilde{\nabla}u|_g^4dvol(g)+\int\sum_{i,j}S(g_0)^{ij}u_iu_jdvol(g).
\end{aligned}
\tag{46}
$$

and

$$
\begin{aligned}
\int\sum_{i,j}\widetilde{\nabla}^{ij}uu_iu_jdvol(g) &= \frac{1}{2}\int\sum_i\widetilde{\nabla}^i(|\widetilde{\nabla}u|_g^2)u_idvol(g)\\
&= -\frac{1}{2}\int|\widetilde{\nabla}u|_g^2\mathrm{tr}(\widetilde{\nabla}^2u)dvol(g)\\
&= -\frac{1}{2}\int\sigma_1(g)|\widetilde{\nabla}u|_g^2+\frac{n-2}{4}\int|\widetilde{\nabla}u|_g^4\\
&\quad +\int\frac{1}{2}\sigma_1(g_0)|\widetilde{\nabla}u|_g^2e^{2u}dvol(g)
\end{aligned}
\tag{47}
$$

Hence

$$
\begin{aligned}
-\int\sum_{i,j}T(g)^{ij}u_iu_jdvol(g) &= -\frac{3}{2}\int|\widetilde{\nabla}u|_g^2+\frac{n-4}{4}\int|\widetilde{\nabla}u|_g^4\\
&\quad +\int\sum_{i,j}S(g_0)^{ij}u_iu_j+\frac{1}{2}\int\sigma_1(g_0)|\widetilde{\nabla}u|_g^2e^{2u},
\end{aligned}
\tag{48}
$$

where all integrals are w.r.t g. (45) and (48) give us

$$
\begin{aligned}
2\int\sigma_2(g)dvol(g) &= \frac{n-4}{2}\int\sigma_1(g)|\widetilde{\nabla}u|_g^2dvolg+\frac{n-4}{4}\int|\widetilde{\nabla}u|_g^4dvol(g)\\
&\quad +\int\sum_{i,j}T^{ij}S(g_0)_{ij}dvol(g)+\int\sum_{i,j}S(g_0)^{ij}u_iu_jdvol(g)\\
&\quad +\frac{1}{2}\int\sigma_1(g_0)|\widetilde{\nabla}u|_g^2e^{2u}dvol(g)
\end{aligned}
\quad .
\tag{49}
$$

Finally, we obtain

$$\begin{aligned} 2\int \sigma_2(g)dvol(g) &= \frac{n-4}{2}\int \sigma_1(g)|\nabla u|^2_{g_0}e^{2u}dvolg + \frac{n-4}{4}\int |\nabla u|^4_{g_0}e^{4u}dvol(g) \\ &\quad + \int \sum_{i,j} T^{ij}S(g_0)_{ij}dvol(g) + \int \sum_{i,j} S(g_0)^{ij}u_iu_jdvol(g) \\ &\quad + \frac{1}{2}\int \sigma_1(g_0)|\nabla u|^2_{g_0}e^{4u}dvol(g). \end{aligned} \tag{50}$$

Recall (44) and positivity of $\sigma_1(g)$ and $\sigma_1(g_0)$. Using the estimates

$$\sum_{i,j} S(g_0)^{ij}u_iu_j \geq -c|\nabla u|^2_{g_0}e^{4u} \geq -\frac{n-4}{8}|\nabla u|^4_{g_0}e^{4u} - \frac{2c^2}{n-4}e^{4u} \tag{51}$$

we deduce

$$2\int \sigma_2(g)dvol(g) \geq \frac{n-4}{8}\int |\nabla u|^4_{g_0}e^{4u}dvol(g) - c\int e^{4u}dvol(g). \tag{52}$$

In view of the Poincaré inequality (38), the Sobolev inequality (34) follows from (52). ■

We remark that a similar method was used to obtain Sobolev inequalities on locally conformally flat manifolds by Gonzales in [12].

5. Global convergence of flow (8) when $\varepsilon > 0$

Proposition 7. *For any $\varepsilon \in (0, 2]$, flow (8) converges globally to u_ε, which satisfies (11).*

Proof. For any $t \in [0, T^*)$, set

$$m(t) = \min_{(x,s)\in M\times[0,t]} u(x,s).$$

If $\inf_{t\in[0,T^*)} m(t) > -\infty$, then by estimates given in Section 3, we have a uniform bound of $|\nabla u|^2 + |\nabla^2 u|$. Since flow (8) preserves the the functional V_ε, we have a uniform C^2 bound. Now we claim that there is a constant $c > 0$ such that

$$F(x,t) \geq c > 0, \qquad \text{for any } (x,t) \in M \times [0, T^*). \tag{53}$$

Recall that $F = \sigma_2^{1/2}(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})$. We will prove the claim at the end of the proof. (53) implies that flow (8) is uniformly elliptic in $M \times [0, T^*)$. Hence, by Krylov's result, u has a uniform bound for higher order derivatives, which implies first that $T^* = \infty$, the global existence. The global convergence of (8) with $\varepsilon \in (0, 2]$ follows now closely the argument presented in [18], which, in turn, follows closely the argument given in [31] and [1]. Therefore, to prove the Proposition, we only need to exclude that

$$\inf_{t\in[0,T^*)} m(t) = -\infty. \tag{54}$$

We assume by contradiction that $\inf_{t\in[0,T^*)} m(t) = -\infty$. Let T_i be a sequence tending to T^* with $m(T_i) \to -\infty$ as $i \to \infty$. Let $(x_i, t_i) \in M \times [0, T_i]$ with $u(x_i, t_i) = m(T_i)$. Fix

$\delta \in (\frac{2}{5}, \frac{1}{2})$, we consider $r_i = \frac{\varepsilon}{2}|m(T_i)|e^{(1-\delta\varepsilon)m(T_i)}$. Clearly, we have $r_i \to 0$. It follows from Theorem 3 that for sufficiently large i

$$\begin{aligned} u(x,t_i) &\leq m(T_i) + |\nabla u| r_i \\ &\leq m(T_i) + Ce^{(\frac{\varepsilon}{2}-1)m(T_i)}\frac{\varepsilon}{2}|m(T_i)|e^{(1-\delta\varepsilon)m(T_i)} \\ &= m(T_i) + C\frac{\varepsilon}{2}|m(T_i)|e^{\varepsilon(\frac{1}{2}-\delta)m(T_i)} \\ &\leq (1-\kappa)m(T_i), \qquad \forall x \in B(x_i, r_i), \end{aligned}$$

for some $\kappa \in (0, (\delta - \frac{2}{n})\varepsilon)$. Note that $\delta - \frac{2}{n} > 0$, for $n \geq 5$. Therefore, we obtain

$$\begin{aligned} \int_{B(x_i,r_i)} e^{2\varepsilon} dvol(g) &\geq \int_{B(x_i,r_i)} e^{(2\varepsilon-n)m(T_i)(1-\kappa)} dvol(g_0) \geq Ce^{(2\varepsilon-n)m(T_i)(1-\kappa)} r_i^n \\ &\geq C\left(\frac{|m(T_i)|\varepsilon}{2}\right)^n \to \infty. \end{aligned}$$

where we have used $n \geq 5$. Hence, this fact contradicts the boundedness of V_ε.

Now we remain to prove Claim (53). For any $0 < T < T^*$, let us consider a function $H : M \times [0,T]$ defined by $H = \log(e^{-\varepsilon u}\sigma_2^{1/2}(g)) - e^{-u}$. Recall Now we first compute the evolution equation for $\sigma^{1/2}$. A direct computation, see for instance Lemma 2 in [18], gives

$$\begin{aligned} \frac{d}{dt}\sigma_2 &= 2\sigma_2 g \cdot \frac{d}{dt}(g^{-1}) + \mathrm{tr}\{T_1(S_g)g^{-1}\frac{d}{dt}S_g\} \\ &= 4\sigma_2(g)u_t + \mathrm{tr}\{T_1(S_g)g^{-1}\tilde{\nabla}_g^2(u_t)\} \end{aligned}$$

Without loss of generality, we assume that the minimum of H is achieved at $(x_0, t_0) \in M \times (0,T]$. We will show that there is a constant $c_0 > 0$ independent of T such that

$$\sigma_2(g)(x_0,t_0) > c_0. \tag{55}$$

Since $|u|$ has a uniform bound, without loss of generality we may assume that at (x_0, t_0)

$$e^{-\varepsilon u}\sigma_2^{1/2}(g) < 1.$$

Recall that $h(t) = 2\log t$ for $t < 1$. Let us use $O(1)$ denote terms with uniform bound. Using the fact that $\|u\|_{C^2} \leq C$, we have near (x_0, t_0)

$$\begin{aligned} \frac{d}{dt}H &= \frac{1}{2\sigma_2(g)}\mathrm{tr}\{T_1(S_g)g^{-1}\tilde{\nabla}_g^2(u_t)\} + (e^{-u} + 2 - \varepsilon)u_t \\ &= \frac{1}{2\sigma_2(g)}\mathrm{tr}\{T_1(S_g)g^{-1}\tilde{\nabla}_g^2 \log(e^{-\varepsilon u}\sigma_2^{1/2}(g))\} + (e^{-u} + 2 - \varepsilon)u_t \\ &= \frac{1}{2\sigma_2(g)}\mathrm{tr}\{T_1(S_g)g^{-1}\tilde{\nabla}_g^2(H)\} + \frac{1}{2\sigma_2(g)}\mathrm{tr}\{T_1(S_g)g^{-1}\tilde{\nabla}_g^2(e^{-u})\} \\ &\quad + (e^{-u} + 2 - \varepsilon)u_t \end{aligned} \tag{56}$$

Let $F = \log \sigma_2(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$ and $F^{ij} = \frac{\partial}{\partial w_{ij}} F$. Since (x_0, t_0) is the minimum of H in $(x_0, t_0) \in M \times [0, T]$, at this point, we have

$$\frac{dH}{dt} \le 0,$$

$$0 = H_l = \frac{1}{2} \sum_{ij} F^{ij} w_{ijl} + (e^{-u} + 2 - \varepsilon) u_l = 0 \quad \forall l$$

and

$$(H_{ij}) \text{ is non-negative definite.}$$

Note that

$$(\tilde{\nabla}_g^2)_{ij} H = H_{ij} + u_i H_j + u_j H_i - \sum_l u_l H_l \delta_{ij} = H_{ij},$$

at (x_0, t_0), where H_j and H_{ij} are the first and second derivatives with respect to the back-ground metric g_0.

From the positivity of (F^{ij}), we have

$$\begin{aligned}
0 &\ge H_t - \frac{1}{2} \sum_{i,j} F^{ij} H_{ij} \\
&= \frac{1}{2\sigma_2(g)} \mathrm{tr}\{T_{k-1}(S_g) \tilde{\nabla}_g^2 (e^{-u})\} + (e^{-u} + 2 - \varepsilon) u_t \\
&= \frac{1}{2} \sum_{i,j} F^{ij} \{(e^{-u})_{ij} + u_i (e^{-u})_j + u_j (e^{-u})_i - u_l (e^{-u})_l \delta_{ij}\} \\
&\quad + (e^{-u} + 2 - \varepsilon) u_t \\
&= \frac{1}{2} e^{-u} \sum_{i,j} F^{ij} \{-u_{ij} - u_i u_j + |\nabla u|^2 \delta_{ij}\} + (e^{-u} + 2 - \varepsilon) u_t \\
&= \frac{1}{2} e^{-u} \sum_{i,j} F^{ij} \{-w_{ij} + S(g_0)_{ij} + \frac{1}{2} |\nabla u|^2 \delta_{ij}\} + (e^{-u} + 2 - \varepsilon) u_t \\
&\ge \frac{1}{2} e^{-u} \sum_{i,j} F^{ij} \{-w_{ij} + S(g_0)_{ij}\} + (e^{-u} + 2 - \varepsilon) u_t \\
&= \frac{1}{2} e^{-u} \sum_{i,j} F^{ij} S(g_0)_{ij} + O(1) \log \sigma_2(g) + O(1) \\
&\quad - \tfrac{1}{2} (e^{-u} + 2 - \varepsilon)(h(r_\varepsilon^{1/2}(g) e^{(\varepsilon - 2)u}) + s_\varepsilon(g)).
\end{aligned} \tag{57}$$

Here we have used $\sum_{i,j} F^{ij} w_{ij} = \frac{1}{\sigma_2(g)} \frac{\partial \sigma_2(g)}{\partial w_{ij}} w_{ij} = 2$. Since $g_0 \in \Gamma_2^+$, by Garding's inequality [11],

$$\sum_{i,j} F^{ij} S(g_0)_{ij} = \sum_{i,j} \frac{1}{\sigma_2(g)} \frac{\partial \sigma_2(g)}{w_{ij}} S(g_0)_{ij} \ge 2 e^{2u} \frac{\sigma_2^{1/2}(g_0)}{\sigma_2^{1/2}(g)}. \tag{58}$$

On the other hand, one can check $h(r_\varepsilon^{1/2}(g)e^{(\varepsilon-2)u})+s_\varepsilon(g)$ is bounded from above. Now from (57) and (58), we have

$$\begin{aligned} 0 &\geq e^u\frac{\sigma_2^{1/2}(g_0)}{\sigma_2^{1/2}(g)}+O(1)\log\sigma_2(g)+O(1)\\ &\geq \frac{c_1}{\sigma_2^{1/2}(g)}+c_2\log\sigma_2(g)-c_3, \end{aligned}$$

for positive constants c_1, c_2 and c_3 independent of T. Clearly, this inequality implies that there is a constant $c_0>0$ independent of T such that (55) holds. Namely

$$\sigma_2(g)\geq c_0,$$

at point (x_0,t_0). Hence, we have for any point $(x,t)\in M\times[0,T]$

$$\begin{aligned} \log(e^{-\varepsilon u(x,t)}\sigma_2^{1/2}(g)(x,t))-e^{-u(x,t)} &= H(x,t)\geq H(x_0,t_0)\\ &= \log(e^{-\varepsilon u(x_0,t_0)}\sigma_2^{1/2}(g)(x_0,t_0))-e^{-u(x_0,t_0)}\\ &\geq \log C_1-e^C, \end{aligned}$$

provided $e^{-\varepsilon u(x,t)}\sigma_2^{1/2}(g)(x,t)<1$. It follows that $\sigma_2(g)(x,t)\geq C_1e^{-e^C}>0$. This finishes the proof of the Proposition.

■

Proof of Proposition 1. By local estimates established in [17] (in fact a similar local estimates as in Theorem 3 hold), we can use the argument given in the proof of Proposition 7 to show that the set of solutions of (11) with the bounded $\mathcal{F}_2$ and $V_\varepsilon(e^{-2u}g_0)=1$ is compact for $\varepsilon\in(0,2]$. Hence, Proposition 7 implies that Y_ε is achieved by a function u_ε, which clearly is a solution of (11). We may assume that u_ε satisfies $V_\varepsilon(e^{-2u_\varepsilon}g_0)=1$ and

$$\sigma_2(\nabla^2u+du\otimes du-\frac{|\nabla u|^2}{2}g_0+S_{g_0})=ce^{2(\varepsilon-2)u}, \tag{59}$$

where $c=Y_\varepsilon$. For any fixed metric g, the function $\tilde{\mathcal{F}}_{2,\varepsilon}(g)$ is continuous on ε so that Y_ε is semi-continuous from above on ε. On the other hand, it follows from the Hölder's inequality, Y_ε is semi-continuous from below on ε. Hence, Y_ε is continuous and we have

$$\lim_{\varepsilon\to0}Y_\varepsilon=Y_2(M,[g_0])<Y_2(\mathbb{S}^n).$$

If $\inf u_\varepsilon$ has a uniform lower bound, then the estimates established in [17] implies that $\|u_\varepsilon\|_{C^2}$ is uniformly bounded. By the result of Evans-Krylov, $\|u_\varepsilon\|_{C^{2,\alpha}}$ is uniformly bounded for any $\alpha\in(0,1)$. Hence u_ε, by taking a subsequence, converges strongly in $C^{2,\alpha}$ to u_0, which is a solution of (1). Moreover, u_0 is a minimizer. Now suppose $\underline{\lim}_{\varepsilon\to0}\inf u_\varepsilon=-\infty$. Let $(x_\varepsilon)\in M$ such that $u_\varepsilon(x_\varepsilon)=\min_{x\in M}u_\varepsilon(x)$. We consider a new function

$$v_\varepsilon(y)=u(\exp_{x_\varepsilon}\delta_\varepsilon y)-u_\varepsilon(x_\varepsilon)$$

and defined on $B_{\delta_\varepsilon^{-1}}$ with a pull-back metric $g_\varepsilon := (exp_{x_\varepsilon}\delta_\varepsilon\cdot)^* g_0$, where $\delta_\varepsilon = e^{(1-\varepsilon/2)u_\varepsilon(x_\varepsilon)}$. Since $u_\varepsilon(x_\varepsilon) \to -\infty$, $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. And one can check that $B_{\delta_\varepsilon^{-1}}$ tends to $\mathbb{R}^n$ with g_ε tending to the standard Euclidean metric in any compact set in $\mathbb{R}^n$ for any C^k norm. We can check v_ε satisfies the same equation (59) on $B_{\delta_\varepsilon^{-1}}$ with replace S_{g_0} by S_{g_ε}. By the local estimates in [17], (v_ε) is uniformly bounded for the C^2 norm on any fixed compact set. From result of Evans-Krylov, it follows the uniform $C^{2,\alpha}$ norm bound of (v_ε) on ε on on any fixed compact set. Thus, (v_ε) is a compact family for C^2 norm. Hence, v_ε converges in any compact domain of $\mathbb{R}^n$ to an entire solution u of the following equation on $\mathbb{R}^n$

$$\sigma_2\left(\nabla^2 u + du\otimes du - \frac{1}{2}|\nabla u|^2 g_{\mathbb{R}^n}\right) = c_0 e^{-4u}, \tag{60}$$

with $c_0 = Y_2(M,[g_0])$. We claim $\int_{\mathbb{R}^n} e^{-nu} dvol(g_{\mathbb{R}^n}) \le 1$. To see this, we state

$$\begin{aligned}\int_{B_{\delta_\varepsilon^{-1}}} e^{(2\varepsilon-n)v_\varepsilon} dvol(g_\varepsilon) &= \delta_\varepsilon^{-n} e^{(n-2\varepsilon)u_\varepsilon(x_\varepsilon)} \int_{B(x_\varepsilon,1)} e^{(2\varepsilon-n)u_\varepsilon} dvol(g_0)\\ &= e^{(n/2-2)\varepsilon u_\varepsilon(x_\varepsilon)} \int_{B(x_\varepsilon,1)} e^{(2\varepsilon-n)u_\varepsilon} dvol(g_0) \le V_\varepsilon(e^{-2u_\varepsilon} g_0) = 1\end{aligned}$$

Letting $\varepsilon \to 0$, the claim yields. By the classification of (60) given in [27] or [8], we have $c_0 \ge Y_2(\mathbb{S}^n) > Y_2(M,[g_0])$, which contradicts $c_0 = Y_2(M,[g_0])$. ■

6. Existence

In this section, we will construct a conformal metric $\tilde g$ such that $\tilde{\mathcal{F}}_2(\tilde g) < Y_2(\mathbb{S}^n)$ and $\tilde g \in \Gamma_2^+$. Our construction is inspired from the Aubin's work [2] and that of Schoen [30]. The basic idea is to construct some suitable data which blows up about one point. But the more delicate point in our case is to keep the conformal metric in the admissible class Γ_2^+ as in [19]. For this purpose, we fix a point $P \in M$. Assume $n \ge 5$. It follows from the work by Lee-Parker that there exists a conformal metric g_1 on $\mathcal{M}$ such that in a normal coordinate system for g_1 at P

$$R = O(r^2), \tag{61}$$

$$\Delta R = -\frac{1}{6}|W(P)|^2, \tag{62}$$

$$\mathrm{Ric}(P) = 0, \tag{63}$$

$$\sqrt{\det g_1} = 1 + O(r^5), \tag{64}$$

where $r = |x|$. We denote

$$g_v = v^{-2} g_1, \tag{65}$$

where

$$v(x) = \begin{cases} \lambda + r^2, & \text{if } x \in B(0,r_0),\\ \lambda + r_0^2, & \text{else.}\end{cases}$$

We will establish the basic estimates.

Lemma 3. *Assume*

$$A = {g_1}^{-1}(\frac{\nabla^2_{g_1} v}{v} - \frac{1}{2}\frac{|\nabla_{g_1} v|^2}{v^2} g_1 + S_{g_1}),$$

where

$$S_{g_1} = \frac{1}{n-2}(\mathrm{Ric}_{g_1} - \frac{R}{2(n-1)} g_1).$$

Then we have

$$\mathrm{tr}(A) = \frac{2n\lambda}{(\lambda+r^2)^2} + \frac{O(r^5)}{\lambda+r^2} + \frac{R}{2(n-1)} \tag{66}$$

and

$$\mathrm{tr}(A^2) = \frac{4n\lambda^2}{(\lambda+r^2)^4} + \frac{2R\lambda}{(n-1)(\lambda+r^2)^2} - \frac{\mathrm{Ric}(\nabla_{g_1} v, \nabla_{g_1} v)}{(\lambda+r^2)^2} + O(r). \tag{67}$$

Proof. We know

$$\sigma_2(g_v) = v^4 \sigma_2({g_1}^{-1}(\frac{\nabla^2_{g_1} v}{v} - \frac{1}{2}\frac{|\nabla_{g_1} v|^2}{v^2} g_1 + S_{g_1})). \tag{68}$$

Thus, we have

$$\begin{aligned} \mathrm{tr}(A) &= \frac{\Delta_{g_1} v}{v} - \frac{n}{2}\frac{|\nabla_{g_1} v|^2}{v^2} + \mathrm{tr}({g_1}^{-1} S_{g_1}) \\ &= \frac{\Delta_{g_1} v}{v} - \frac{n}{2}\frac{|\nabla_{g_1} v|^2}{v^2} + \frac{R}{2(n-1)}, \end{aligned}$$

where

$$\Delta_{g_1} = \frac{1}{\sqrt{\det g_1}} \sum_{i,j} \frac{\partial}{\partial x^i}(\sqrt{\det g_1}\, {g_1}^{ij} \frac{\partial}{\partial x^j}).$$

Recall

$$|\nabla_{g_1} v|^2 = 4r^2. \tag{69}$$

In view of (64), we obtain

$$\Delta_{g_1} v = 2n + O(r^5) \tag{70}$$

so that

$$\begin{aligned} \mathrm{tr}(A) &= \frac{2n + O(r^5)}{\lambda + r^2} - \frac{n}{2}\frac{4r^2}{(\lambda+r^2)^2} + \frac{R}{2(n-1)} \\ &= \frac{2n\lambda}{(\lambda+r^2)^2} + \frac{O(r^5)}{\lambda+r^2} + \frac{R}{2(n-1)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathrm{tr}(A^2) \quad &= \quad \frac{|\nabla^2_{g_1} v|^2}{v^2} + \frac{n|\nabla_{g_1} v|^4}{4v^4} + \mathrm{tr}(({g_1}^{-1} S_{g_1})^2) \\ &\quad -\frac{|\nabla_{g_1} v|^2 \Delta_{g_1} v}{v^3} + 2\mathrm{tr}(\frac{{g_1}^{-1}\nabla^2_{g_1} v\ {g_1}^{-1} S_{g_1}}{v}) - \frac{|\nabla_{g_1} v|^2}{v}\mathrm{tr}({g_1}^{-1} S_{g_1}). \end{aligned}$$

We can estimate

$$\frac{n|\nabla_{g_1}v|^4}{4v^4} = \frac{4nr^4}{(\lambda+r^2)^4}, \tag{71}$$

$$\mathrm{tr}(({g_1}^{-1}S_{g_1})^2) = O(r^2), \tag{72}$$

$$-\frac{|\nabla_{g_1}v|^2\Delta_{g_1}v}{v^3} = -\frac{8nr^2}{(\lambda+r^2)^3}+O(r), \tag{73}$$

$$-\frac{|\nabla_{g_1}v|^2}{v^2}\mathrm{tr}({g_1}^{-1}S_{g_1}) = -\frac{2r^2R}{(n-1)(\lambda+r^2)^2}, \tag{74}$$

$${g_1}^{-1}\nabla^2_{g_1}v = 2I+O(r^2), \tag{75}$$

$$\mathrm{tr}({g_1}^{-1}\nabla^2_{g_1}v\ {g_1}^{-1}S_{g_1}) = 2\mathrm{tr}({g_1}^{-1}S_{g_1})+O(r^3) = \frac{R}{n-1}+O(r^3), \tag{76}$$

$$2\mathrm{tr}(\frac{{g_1}^{-1}\nabla^2_{g_1}v\ {g_1}^{-1}S_{g_1}}{v}) = \frac{2R}{(n-1)(\lambda+r^2)}+O(r). \tag{77}$$

To handle $\frac{|\nabla^2_{g_1}v|^2}{v^2}$, we recall the Bochner's formula

$$\langle\nabla(\Delta v),\nabla v\rangle = -|\nabla^2_{g_1}v|^2+\frac{1}{2}\Delta(|\nabla_{g_1}v|^2)-\mathrm{Ric}(\nabla_{g_1}v,\nabla_{g_1}v) \tag{78}$$

so that

$$\begin{aligned}|\nabla^2_{g_1}v|^2 &= -\langle\nabla(\Delta v),\nabla v\rangle+\frac{1}{2}\Delta(|\nabla_{g_1}v|^2)-\mathrm{Ric}(\nabla_{g_1}v,\nabla_{g_1}v)\\ &= -\langle\nabla(2n+O(r^5)),\nabla v\rangle+\frac{1}{2}\Delta(4r^2)-\mathrm{Ric}(\nabla_{g_1}v,\nabla_{g_1}v)\\ &= 4n-\mathrm{Ric}(\nabla_{g_1}v,\nabla_{g_1}v)+O(r^5).\end{aligned} \tag{79}$$

Now combining (71) to (77) and (79), we deduce

$$\begin{aligned}\mathrm{tr}(A^2) &= \frac{4n}{(\lambda+r^2)^2}+\frac{4nr^4}{(\lambda+r^2)^4}-\frac{8nr^2}{(\lambda+r^2)^3}+\frac{2R}{(n-1)(\lambda+r^2)}-\frac{2Rr^2}{(n-1)(\lambda+r^2)^2}\\ &\quad-\frac{\mathrm{Ric}(\nabla_{g_1}v,\nabla_{g_1}v)}{(\lambda+r^2)^2}+O(r)\\ &= \frac{4n\lambda^2}{(\lambda+r^2)^4}+\frac{2R\lambda}{(n-1)(\lambda+r^2)^2}-\frac{\mathrm{Ric}(\nabla_{g_1}v,\nabla_{g_1}v)}{(\lambda+r^2)^2}+O(r).\end{aligned}$$

■

Lemma 4. *Assume $\beta\in(\frac{1}{2},\frac{1}{4})$. Then we have $\sigma_1(g_v)>0$ and $\sigma_2(g_v)>0$ in $B(0,\lambda^\beta)$. Moreover, if we suppose $n\geq 9$, there holds*

$$\begin{aligned}\int_{B(0,\lambda^\beta)}\sigma_2(g_v)dvol(g_v) &= \lambda^{-\frac{n}{2}+2}\{2n(n-1)B+C\Delta R(0)\lambda^2\\ &\quad+O\left(\lambda^{\frac{5}{2}}+\lambda^{n(\frac{1}{2}-\beta)}+\lambda^{2+(n-8)(\frac{1}{2}-\beta)}\right)\}\end{aligned} \tag{80}$$

and

$$\int_{B(0,\lambda^\beta)} dvol(g_v) = \lambda^{-\frac{n}{2}} \left[B + O\left(\lambda^{5/2} + \lambda^{n(1/2-\beta)} \right) \right], \tag{81}$$

where the constants B, C *are given by*

$$B = \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^n} dx, \tag{82}$$

$$C = \int_{\mathbb{R}^n} \left(\frac{|x|^2}{2n(1+|x|^2)^{n-2}} + \frac{2|x|^4}{n(n+2)(1+|x|^2)^{n-2}} \right) dx > 0. \tag{83}$$

Proof. It follows directly from (66) and (67)

$$\sigma_1(g_v) = v^2 \left(\frac{2n\lambda}{(\lambda+r^2)^2} + \frac{R}{2(n-1)} + O(r^3) \right),$$

$$\sigma_2(g_v) = \frac{v^4}{2} \left[\frac{4n(n-1)\lambda^2}{(\lambda+r^2)^4} + \frac{2\lambda R}{(\lambda+r^2)^2} + \frac{\mathrm{Ric}(\nabla_{g_1} v, \nabla_{g_1} v)}{(\lambda+r^2)^2} + O(r) \right]. \tag{84}$$

Thus, the first part of lemma is clear. On the other hand, we obtain

$$\begin{aligned} &\int_{B(0,\lambda^\beta)} \sigma_2(g_v) dvol(g_v) \\ = &\int_{B(0,\lambda^\beta)} \frac{1}{(\lambda+r^2)^n} \{2n(n-1)\lambda^2 + R\lambda(\lambda+r^2)^2 \\ &+ \frac{1}{2}\mathrm{Ric}((\nabla_{g_1} v, \nabla_{g_1} v)(\lambda+r^2)^2 + O(r(\lambda+r^2)^4)\}(1+O(r^5))dx. \end{aligned} \tag{85}$$

We can calculate

$$\begin{aligned} \mathrm{Ric}(\nabla_{g_1} v, \nabla_{g_1} v) &= 4\sum_{i,j} R_{ij}(x) x^i x^j \\ &= 4\sum_{i,j} (R_{ij}(0) + \sum_k R_{ij,k}(0) x^k + \sum_{k,l} \frac{1}{2} R_{ij,kl}(0) x^k x^l) x^i x^j + O(r^5) \\ &= 2\sum_{i,j,k,l} R_{ij,kl}(0) x^k x^l x^i x^j + O(r^5). \end{aligned}$$

It is known that (see [2])

$$\frac{1}{r^4 w_{n-1}} \int_{S(r)} \sum_{i,j,k,l} R_{ij,kl}(0) x^k x^l x^i x^j d\Omega = \frac{2}{n(n+2)} \Delta R(0)$$

and

$$R(x) = \frac{1}{2} \sum_{i,j} R_{,ij}(0) x^i x^j + O(r^3),$$

where $S(r)$ is the geodesic sphere of radius equal to r, $d\Omega$ is the volume element on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and w_{n-1} is the volume of the unit sphere $\mathbb{S}^{n-1}$. Therefore,

$$\begin{aligned}
&\int_{B(0,\lambda^\beta)} \sigma_2(g_v) dvol(g_v) \\
= &\int_{B(0,\lambda^\beta)} \frac{2n(n-1)\lambda^2}{(\lambda + r^2)^n} + \frac{\lambda r^2 \Delta R(0)}{2n(\lambda + r^2)^{n-2}} dx \\
&+ \int_{B(0,\lambda^\beta)} \left(\frac{2r^4 \Delta R(0)}{n(n+2)(\lambda + r^2)^{n-2}} + \frac{O(r)}{(\lambda + r^2)^{n-4}} \right) dx \\
= &\lambda^{-n/2+2} \int_{B(0,\lambda^{\beta-1/2})} \left(\frac{2n(n-1)}{(1+r^2)^n} + a(x)\lambda^2 \Delta R(0) \right) dx + O(\lambda^{-n/2+4+1/2}),
\end{aligned}$$

where

$$a(x) = \frac{|x|^2}{2n(1+|x|^2)^{n-2}} + \frac{2|x|^4}{n(n+2)(1+|x|^2)^{n-2}}.$$

Thus, (80) yields. Similarly, we can estimate

$$\begin{aligned}
\int_{B(0,\lambda^\beta)} dvol(g_v) &= \int_{B(0,\lambda^\beta)} v^{-n} \sqrt{\det g_1} dx \\
&= \int_{B(0,\lambda^\beta)} \frac{1 + O(r^5)}{(\lambda + r^2)^n} dx \\
&= \lambda^{-n/2} \int_{B(0,\lambda^{\beta-1/2})} \frac{dx}{(1+r^2)^n} + O(\lambda^{-n/2+5/2}) \\
&= \lambda^{-n/2} \left[B + O\left(\lambda^{5/2} + \lambda^{n(1/2-\beta)} \right) \right].
\end{aligned}$$

Therefore, we finish the proof. ■

Lemma 5. *Let g_1 as above and $\gamma \in (0,2)$ be given. Assume $n \geq 9$. For sufficiently small $\delta > 0$ such that $\lambda^{1/4} >> \delta >> \lambda^{1/2}$, there exists a constant $1 > \delta_1 > \delta$ and a function $u : B_{\delta_1} \to \mathbb{R}$ satisfying :*

(0) $\delta_1^{\frac{n-4}{2}} = (\frac{2}{\gamma} - 1)\lambda^{-1}\delta^{\frac{n}{2}}(1 + o(1))$,
(1) *The metric $\tilde{g} = e^{-2u} g_1$ has positive 2-curvature,*
(2) $u = \log(\lambda + |x|^2) + b_0$ *for* $|x| \leq \delta$,
(3) $u = \gamma \log |x|$ *for* $|x| \geq \delta_1$,
(4) $vol(B_{\delta_1} \backslash B_\delta, \tilde{g}) \leq C \left(\frac{\delta^{\frac{n+4-n\gamma}{2(2-\gamma)}}}{\lambda} \right)^{2n(2-\gamma)/(n-4)}$,
(5) $\int_{B_{\delta_1} \backslash B_\delta} \sigma_2(\tilde{g}) dvol(\tilde{g}) \leq C \delta^{4+n(1-\gamma)} \lambda^{-3+2\gamma}$,

where b_0 satisfies (95) below.

Proof. We want to find a function u with $u'(r) = \frac{\alpha(r)}{r}$. The Schouten tensor of $\tilde g = e^{-2u} g_1$ is

$$\begin{aligned} S(\tilde g)_{ij} &= \nabla^2_{ij} u + \nabla_i u \nabla_j u - \frac{|\nabla u|^2}{2} g_{1ij} + S(g_1)_{ij} \\ &= \frac{2\alpha}{2r^2}\delta_{ij} - \frac{\alpha^2}{2r^2}(g_1)_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2} + S(g_1)_{ij} + O(r^2)\frac{\alpha}{r^2}, \end{aligned} \tag{86}$$

so that

$$S(\tilde g)^i_j \;=\; \frac{2\alpha - \alpha^2}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2} + S(g_1)^i_j + O(r^2)\frac{\alpha}{r^2}, \tag{87}$$

since it follows from Gauss lemma that $\sum_i (g_1)^{ij} x_i = x_j$. We look for a function $\alpha(r) \in (\gamma, 2)$ for all $r \in (\delta, \delta_1)$. Hence one can find a fixed constant $A > 0$ independent of λ such that

$$S(\tilde g)^i_j \geq \frac{2\alpha - \alpha^2 - Ar^2\alpha}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2} \tag{88}$$

and

$$S(\tilde g)^i_j \leq \frac{2\alpha - \alpha^2 + Ar^2\alpha}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2}. \tag{89}$$

Consequently, we obtain

$$\sigma_2(\tilde g) > e^{4u}\frac{(n-1)}{2}\left(\frac{2\alpha - \alpha^2 - Ar^2\alpha}{2r^2}\right)^2\left(n - 4 + 4\frac{r\alpha' - Ar^2\alpha}{2\alpha - \alpha^2 - Ar^2\alpha}\right)$$

and

$$\sigma_1(\tilde g) > e^{2u}\left(\frac{2\alpha - \alpha^2 - Ar^2\alpha}{2r^2}\right)\left(n - 2 + 2\frac{r\alpha' - Ar^2\alpha}{2\alpha - \alpha^2 - Ar^2\alpha}\right).$$

We want to find an α satisfying

$$\alpha = \begin{cases} \dfrac{2r^2}{\lambda + r^2}, & \text{if } |x| \leq \delta, \\ \text{solution of (90)}, & \text{if } |x| \in (\delta, \delta_1), \\ \gamma, & \text{if } |x| \geq \delta_1. \end{cases}$$

Such a function can be found as follows. First we solve the following equation

$$\frac{n-4}{4} + \frac{r\alpha' - Ar^2\alpha}{2\alpha - \alpha^2 - Ar^2\alpha} = 0. \tag{90}$$

Recall this is the Bernoulli differential equation. One can find a general solution of (90) as follows.

$$\frac{1}{\alpha} = r^{\frac{n-4}{2}} e^{-\frac{nAr^2}{2}}\left(\int_1^r \frac{4-n}{4}\frac{1}{t^{\frac{n-2}{2}}} e^{\frac{nAt^2}{2}}dt + c\right).$$

Set

$$H(r) = -\frac{An}{2}\int_1^r \frac{1}{t^{\frac{n-6}{2}}} e^{\frac{nAt^2}{2}}dt.$$

We have

$$\begin{aligned}\alpha &= \frac{2}{1+2a_1r^{\frac{n-4}{2}}e^{-\frac{nAr^2}{2}}+2H(r)r^{\frac{n-4}{2}}e^{-\frac{nAr^2}{2}}}\\ &= \frac{2}{1+2a_1r^{\frac{n-4}{2}}+2G(r)},\end{aligned}$$

where

$$G = a_1r^{\frac{n-4}{2}}(e^{-\frac{nAr^2}{2}}-1)+H(r)r^{\frac{n-4}{2}}e^{-\frac{nAr^2}{2}}.$$

Here the constant a_1 is determined by

$$\alpha(\delta)=\frac{2\delta^2}{\lambda+\delta^2}.$$

We have the estimate

$$a_1=\frac{\lambda}{2\delta^{\frac{n}{2}}}(1+o(1)), \tag{91}$$

since we use the fact $\lambda^{1/4} >> \delta$. Define δ_1 by $\alpha(\delta_1)=\gamma$. We have

$$\delta_1^{\frac{n-4}{2}} = \left(\frac{2}{\gamma}-1\right)\lambda^{-1}\delta^{\frac{n}{2}}(1+o(1)) \tag{92}$$

so that $1 >> \delta_1 >> \delta$. Note that $n \ge 9$. Hence, for all $r \in (\delta,\delta_1)$ we have

$$G(r)=O(1)r^2, \tag{93}$$

so that for all $r\in(\delta,\delta_1)$

$$u(r)=\frac{4}{4-n}\log(r^{\frac{4-n}{2}}+2a_1)+a_2, \tag{94}$$

where

$$a_2=(\gamma-2)\log\delta_1-\frac{4}{4-n}\log\frac{2}{\gamma}+o(1).$$

For $r<\delta$ we have

$$u(r)=\log(\lambda+r^2)+b_0,$$

where

$$b_0=(\gamma-2)\log\delta_1+O(1), \tag{95}$$

where we use $\delta >> \lambda^{1/2}$. In view of (90), we have

$$(2\alpha-\alpha^2+Ar^2\alpha)\left(n-4+4\frac{r\alpha'+Ar^2\alpha}{2\alpha-\alpha^2+Ar^2\alpha}\right)=2nAr^2\alpha=O(1)r^2. \tag{96}$$

We also have for all $r\in(\delta,\delta_1)$

$$\alpha(r)\in(\gamma,2) \tag{97}$$

and

$$2-\alpha+Ar^2=\frac{4a_1r^{\frac{n-4}{2}}}{1+2a_1r^{\frac{n-4}{2}}}+O(r^2). \tag{98}$$

Now we can check that

$$\begin{aligned}
&\int_{B_{\delta_1}\backslash B_\delta} \sigma_2(\tilde g) dvol(\tilde g)\\
&\leq C(n)\int_{B_{\delta_1}\backslash B_\delta} e^{(4-n)u}\left(\frac{2\alpha-\alpha^2+Ar^2\alpha}{2r^2}\right)^2\left(n-4+4\frac{r\alpha'+Ar^2\alpha}{2\alpha-\alpha^2+Ar^2\alpha}\right)dvol(g_1)\\
&\leq O(1)\int_\delta^{\delta_1} e^{(4-n)a_2}(r^{\frac{4-n}{2}}+2a_1)^4\frac{1}{r^2}(2-\alpha+Ar^2)r^{n-1}dr\\
&\leq O(1)\int_\delta^{\delta_1} \delta_1^{(n-4)(2-\gamma)}r^{5-n}a_1r^{\frac{n-4}{2}}(1+2a_1r^{\frac{n-4}{2}})^4dr\\
&\leq O(1)\int_\delta^{\delta_1} \delta_1^{(n-4)(2-\gamma)}a_1r^{3-\frac{n}{2}}dr\\
&\leq O(1)\delta_1^{(n-4)(2-\gamma)}\delta^{4-\frac{n}{2}}a_1 = O(1)\delta^{n(1-\gamma)+4}\lambda^{-3+2\gamma}
\end{aligned}$$

and

$$\begin{aligned}
vol(B_{\delta_1}\backslash B_\delta,\tilde g) &= \int_{B_{\delta_1}\backslash B_\delta} e^{-nu}dvol(g_1)\\
&\leq O(1)\int_\delta^{\delta_1} e^{-na_2}(r^{\frac{4-n}{2}}+2a_1)^{4n/(n-4)}r^{n-1}dr\\
&= O(1)\int_\delta^{\delta_1} \delta_1^{n(2-\gamma)}r^{-1-n}(1+2a_1r^{\frac{n-4}{2}})^{4n/(n-4)}dr\\
&\leq O(1)\delta_1^{n(2-\gamma)}\delta^{-n} = O(1)\left(\frac{\delta^{\frac{n+4-n\gamma}{2(2-\gamma)}}}{\lambda}\right)^{2n(2-\gamma)/(n-4)}.
\end{aligned}$$

Therefore, after smoothing u, we get a desired u. ■

We write $g_0 = e^{-2u_0}g_1$. In the following result, we try to connect the initial metric g_0 to some tube object. More precisely, we prove the following lemma.

Lemma 6. *Let $g_0 \in \Gamma_2^+$ and the geodesic ball $B(0, r_0)$ as above. Assume that $n \geq 5$. For any given $\gamma \in (0, 2)$, then there is a conformal 2-positive metric $\tilde g = e^{-2u}g_1$ on $B(0, r_0)\backslash\{0\}$ satisfying :*

1. *The metric $\tilde g = e^{-2u}g_1$ has positive 2-curvature;*
2. *$u = \gamma \log |x|$ for $|x| \leq r_2$;*
3. *$u = u_0(x) + b_1$ for $|x| \geq r_1$;*

where $r_2 < r_1 < r_0$ and b_1 is a constant.

Proof. We write $u(x) = w(r) + \xi(r)u_0(x)$ where $\xi(r)$ is some cut-off function equals to 1 near of r_0 and to 0 near 0, and w with $w'(r) = \frac{\alpha(r)}{r}$, where α is equel to 0 near r_0. As before, the Schouten tensor of $\tilde{g} = e^{-2u}g_1$ is

$$\begin{aligned} S(\tilde{g})_{ij} &= \nabla^2_{ij}w + \nabla_i w\nabla_j w + \nabla_i w\nabla_j(\xi u_0) + \nabla_i(\xi u_0)\nabla_j w \\ &\quad - \left(\frac{|\nabla w|^2}{2} + \langle \nabla w, \nabla(\xi u_0)\rangle\right)(g_1)_{ij} + S(e^{-2\xi u_0}g_1)_{ij} \end{aligned} \tag{99}$$

so that

$$S(\tilde{g})^i_j = \frac{2\alpha-\alpha^2}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2-2\alpha}{r^2}\right)\frac{x_ix_j}{r^2} + S(e^{-2\xi u_0}g_1)^i_j + O(r+|\nabla(\xi u_0)|)\frac{\alpha}{r}. \tag{100}$$

Fix $\varepsilon \in (0, \frac{2-\gamma}{5})$ and let C_1 bound the term $O(r+|\nabla u_0|)$. Set $r_4 = \min(\frac{r_0}{2}, \frac{1}{2}, \frac{\varepsilon}{2(1+C_1)})$. For some small r_3 to be fixed later, we want to α decrease from γ to 0 in (r_3, r_4) and $\xi \equiv 1$ in (r_3, r_0). In $B_{r_0} \setminus B_{r_3}$, we write $A = S(\tilde{g}) - S(g_0)$. Therefore

$$\sigma_2(\tilde{g}) = e^{4(w+u_0)}\sigma_2(A + S(g_0)).$$

We want $A + S(g_0) \in \Gamma_2^+$ in $B_{r_0} \setminus B_{r_3}$. It is sufficient to want $A \in \Gamma_2^+$. It is clear in $B_{r_4} \setminus B_{r_3}$

$$A \geq \left(\frac{2\alpha-\alpha^2-\varepsilon\alpha}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2-2\alpha}{r^2}\right)\frac{x_ix_j}{r^2}\right).$$

This gives

$$\sigma_2(A) > e^{4u}\frac{(n-1)}{2}\left(\frac{2\alpha-\alpha^2-\varepsilon\alpha}{2r^2}\right)^2\left(n-4+4\frac{r\alpha'-\varepsilon\alpha}{2\alpha-\alpha^2-\varepsilon\alpha}\right)$$

and

$$\sigma_1(A) > e^{2u}\left(\frac{2\alpha-\alpha^2-\varepsilon\alpha}{2r^2}\right)\left(n-2+2\frac{r\alpha'-\varepsilon\alpha}{2\alpha-\alpha^2-\varepsilon\alpha}\right).$$

We see that for all small $\delta > 0$,

$$\alpha(r) = \frac{(2-5\varepsilon)\delta}{\delta + r^{\frac{1}{2}-\frac{5}{4}\varepsilon}} \tag{101}$$

solves the equation

$$\frac{1}{4}(2\alpha - \alpha^2 - \varepsilon\alpha) = -r\alpha' + \varepsilon\alpha. \tag{102}$$

We choose some $r_5 < r_4$ and a non increasing function α in (r_5, r_4) such that $\alpha(r_4) = 0$, $\alpha(r_5) > 0$ and $\tilde{g} \in \Gamma_2^+$ in $B_{r_4} \setminus B_{r_5}$ by openness of Γ_2^+. Now we choose a suitable δ in (101) and take a small $r_6 < r_5$ such that $\alpha(r_6) = \gamma$. Then we set $\alpha(r) = \gamma$ for all $r < r_6$ and $r_3 = r_6$. We see that there exists some cut-off function ξ such that $\xi(r) = 1 \forall r > r_7$, $\xi(r) = 0 \forall r < r_8$ and $r^2 S(e^{-2\xi u_0}g_1)$ is small in $B_{r_7} \setminus B_{r_8}$ where $r_8 < r_7 < r_6$. Thus we can choose such suitable cut-off function such that $\tilde{g}$ in Γ_2^+. Now it is sufficient to choose some $r_2 < r_8$ and $r_1 = r_4$. Finally, we obtain the desired u by smoothing it. ■

The construction of such 2-positive metrics is motivated by the method introduced by Gromov-Lawson [13] in their study of metrics of positive scalar curvature. See also for the constructions of other positive metrics in [29] and [14]. Now we can prove the main result in this section.

Theorem 6. *Let (M, g_0) be a compact, oriented Riemannian manifold with $\sigma_2(g_0) > 0$. Assume that $n \geq 9$. Then there exists $\tilde{g} \in [g_0]$ such that*

$$\tilde{g} \in \Gamma_2^+ \tag{103}$$

and

$$\tilde{\mathcal{F}}_2(\tilde{g}) < Y_2(\mathbb{S}^n). \tag{104}$$

Proof. We fix somme $\gamma \in (1, 2)$ and let the geodesic ball $B(0, r_0)$ w.r.t. g_1 as above. We define a conformal metric $\tilde{g}$ as follows. Let $r_2 < r_1 < r_0$ as in Lemma 6 and set $\delta = \lambda^\beta$ with $\beta \in (\frac{1}{4}, \frac{1}{2})$ for any small λ. Find δ_1 as in Lemma 5. Now for any small λ with $\delta_1 < r_2$, define $\tilde{g}$ on B_{δ_1} by Lemma 5 and on $B_{r_0}\backslash B_{r_2}$ by Lemma 6. And on $M\backslash B(0, r_0)$, $\tilde{g} = e^{-2b_1} g_0$, where the constant b_1 is given in Lemma 6. Since on $B_{r_2}\backslash B_{\delta_1}$ the metrics constructed in Lemma 5 and Lemma 6 are the same, $\tilde{g}$ is smooth. From Lemmas 5 and 6, we know (103) holds. In the following, we keep the notations of the geodesic ball with respect to the background metric g_1. By the Lemmas 4, 5 and 6, we can estimate

$$\int_{B_{\delta_1}\backslash B_\delta} \sigma_2(\tilde{g}) dvol(\tilde{g}) \leq C\delta_1^{(n-4)(2-\gamma)} \delta^{4-\frac{n}{2}} a_1, \tag{105}$$

$$\int_{M\backslash B_{\delta_1}} \sigma_2(\tilde{g}) dvol(\tilde{g}) \leq C\delta_1^{(n-4)(2-\gamma)} \delta_1^{4-n}, \tag{106}$$

$$\begin{aligned} \int_{B_\delta} \sigma_2(\tilde{g}) dvol(\tilde{g}) \; = \; & \delta_1^{(n-4)(2-\gamma)} \lambda^{-\frac{n}{2}+2} \left[2n(n-1)B + C\Delta R(0)\lambda^2 \right. \\ & \left. + O\left(\lambda^{\frac{5}{2}} + \lambda^{n(\frac{1}{2}-\beta)} + \lambda^{2+(n-8)(\frac{1}{2}-\beta)}\right)\right], \end{aligned} \tag{107}$$

$$vol(M, \tilde{g}) \geq vol(B_\delta, \tilde{g}) = \delta_1^{n(2-\gamma)} \lambda^{-\frac{n}{2}} \left[B + O\left(\lambda^{5/2} + \lambda^{n(1/2-\beta)}\right)\right]. \tag{108}$$

We choose some $\beta \in (\frac{1}{4}, \frac{n-4}{2n})$ so that we obtain

$$\delta^{4-\frac{n}{2}} a_1 = o(\lambda^{-\frac{n}{2}+4}) \text{ and } \delta_1^{4-n} = o(\lambda^{-\frac{n}{2}+4}).$$

As a consequence, we get

$$\tilde{\mathcal{F}}_2(\tilde{g}) \leq B^{\frac{4-n}{n}} [2n(n-1)B + C\Delta R(0)\lambda^2 + o(\lambda^2)].$$

Recall $2n(n-1)B^{\frac{4}{n}} = Y_2(\mathbb{S}^n)$ and $\Delta R(0) < 0$. Therefore, we deduce (104) provided λ is sufficiently small. Hence, we finish the proof. ∎

Acknowledgement. A part of the work was carried out while the first author was visiting Max Planck Institute for Mathematics in the Sciences, Leipzig and while the second author was visiting Université Paris XII-Val de Marne. We would like to thank both institutions and Jürgen Jost and Frank Pacard for warm hospitality. The second author also would like to thank Pengfei Guan for his many helpful discussions. The main result of the paper was announced at a talk given by the second author in Magdeburg University, Dec. 3, 2004. We learned that Sheng-Trudinger-Wang have announced a similar result.

REFERENCES

[1] B. Andrews, *Monotone quantities and unique limits for evolving convex hypersurfaces,* Internat. Math. Res. Notices **1997** (1997), 1001–1031.

[2] T. Aubin, *Équations différentilles non linéaires et probléme de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976), 269-296.

[3] T. Aubin and Y. Li, *On the best Sobolev inequality*, J. Math. Pures Appl. **78** (1999), 353-387.

[4] S. Brendle and J. Viaclovsky, *A variational characterization for $\sigma_{n/2}$*, Calc. Var. P. D. E. **20** (2004), 399–402.

[5] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian,* Acta Math. **155** (1985), 261–301.

[6] A. Chang, M. Gursky and P. Yang, *An equation of Monge-ampère type in conformal geometry, and four manifolds of positive Ricci curvature*, Ann. of Math.,**155** (2002), 709-787.

[7] A. Chang, Gursky and P. Yang, An a priori estimates for a fully nonlinear equation on Four-manifolds, J. D'Analysis Math., **87** (2002), 151–186

[8] A. Chang, Gursky and P. Yang, *Entire solutions of a fully nonlinear equation,* Lectures on partial differential equations, 43–60, New Stud. Adv. Math., **2**, Int. Press, Somerville, MA, 2003.

[9] K.-S. Chou, (K.S. Tso)*On a real Monge-Ampère functional,* Invent. Math. **101** (1990), 425–448.

[10] K.-S. Chou and X.-J. Wang, *A variational theory of the Hessian equation,* Comm. Pure Appl. Math., **54** (2001), 1029–1064.

[11] L. Garding, *An inequality for hyperbolic polynomials,* J. Math. Mech **8** (1959), 957–965.

[12] M. d. M. González, *Removability of singularities for a class of fully non-linear elliptic equations*, preprint, 2004.

[13] M. Gromov and H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature,* Ann. of Math. (2) **111** (1980), 423–434.

[14] P. Guan, C.-S. Lin and G. Wang, *Schouten tensor and some topological properties,* to appear in: Comm. Anal. and Geom.

[15] P. Guan, C.-S. Lin and G. Wang, *local gradient estimates for conformal quotient equations*, preprint

[16] P. Guan, J. Viaclovsky and G. Wang, *Some properties of the Schouten tensor and applications to conformal geometry,* Trans A. M. S., **355** (2003), 925-933.

[17] P. Guan and G. Wang, *Local estimates for a class of fully nonlinear equations arising from conformal geometry*, Intern. Math. Res. Not. , **2003**, (2003), 1413-1432.

[18] P. Guan and G. Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, J. reine und angew. Math., **557**, (2003), 219-238.

[19] P. Guan and G. Wang, *Geometric inequalities on locally conformally flat manifolds*, Duke Math. J., **124**, (2004), 177-212.

[20] P. Guan and G. Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, ArXiv: math.DG/**0112256** v1 of [18]

[21] M. Gursky and J. Viaclovsky, *Volume comparison and the σ_k-Yamabe problem*, Adv. in Math. **187** (2004), 447-487.

[22] M. Gursky and J. Viaclovsky, *A fully nonlinear equation on 4-manifolds with positive scalar curvature*, J. Diff. Geom., **63**, (2003), 131-154.

[23] M. Gursky and J. Viaclovsky, *Prescribing symmetric functions of the eigenvalues of the Ricci tensor*, preprint, 2004.

[24] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities.* Courant Lecture Notes in Math., 5. Courant Inst. of Math. Sci., New York; Amer. Math. So., Providence, RI, 1999.

[25] N. Krylov, *Nonlinear elliptic and parabolic equations of the second order,* D. Reidel, (1987)

[26] J. Lee and T. Parker, *The Yamabe problem,* Bull. Amer. Math. Soc. (N.S.) **17** (1987), 37–91.

[27] A. Li and Y. Li, *On some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math., **56** (2003), 1416–1464.

[28] P. L. Lions, *Two remarks on the Monge-Amperé*, Ann. Mat. Pura Appl. **142** (1985), 263-275

[29] M. Micallef and M. Wang, *Metrics with nonnegative isotropic curvature,* Duke Math. J., **72** (1993), 649–672.

[30] R. Schoen, *Conformal deformation of a Riemannian metric to constant curvature,* J. Diff. Geome., **20** (1984), 479-495.

[31] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems,* Ann. of Math. **118** (1983), 525–571.

[32] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[33] N. Trudinger and X-J. Wang, *A Poincaré type inequality for Hessian integrals.* Calc. Var. Partial Differential Equations **6** (1998), no. 4, 315–328.

[34] J. Viaclovsky, *Conformal geometry, contact geometry and the calculus of variations*, Duke J. Math. **101** (2000), no. 2, 283-316.

[35] J. Viaclovsky, *Conformally invariant Monge-Ampère equations: global solutions*, Trans. AMS **352** (2000), 4371–4379.

[36] J. Viaclovsky, *Estimates and some existence results for some fully nonlinear elliptic equations on Riemannian manifolds,* Comm. Anal. Geom. **10** (2002), 815-847.

[37] X. J. Wang, *A class of fully nonlinear elliptic equations and related functionals*, Indiana Univ. Math. J. **43** (1994), 25–54.

[38] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds,* Osaka Math. J., **12** (1960), 21–37.

Laboratoire d'Analyse et de Mathématiques Appliquées, CNRS UMR 8050, Département de Mathématiques, Université Paris XII-Val de Marne,, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France
E-mail address: `ge@univ-paris12.fr`

Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22-26, 04103 Leipzig, Germany
E-mail address: `gwang@mis.mpg.de`