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Nilpotent metric Lie algebras of small dimension

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Abstract

In [KO 2] we developed a general classification scheme for metric Lie algebras, i.e. for finite-dimensional Lie algebras equipped with a non-degenerate invariant inner product. Here we determine all nilpotent Lie algebras $\mathfrak{l}$ with $\dim \mathfrak{l}' = 2$ which are used in this scheme. Furthermore, we classify all nilpotent metric Lie algebras of dimension at most 10.

Contents

1 Introduction

In [KO 2] we developed a structure theory for metric Lie algebras, i.e. for Lie algebras with invariant non-degenerate inner product or, equivalently, for simply-connected Lie groups with a bi-invariant pseudo-Riemannian metric. We used this structure theory in order to give a description of the moduli space of all isomorphism classes of indecomposable non-simple metric Lie algebras as

$$\coprod_{(\mathfrak{l},\mathfrak{a})} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 / G_{(\mathfrak{l},\mathfrak{a})}, \tag{1}$$

where the union is taken over all isomorphism classes of pairs $(\mathfrak{l},\mathfrak{a})$ of Lie algebras $\mathfrak{l}$ and semi-simple orthogonal $\mathfrak{l}$-modules $\mathfrak{a}$. Here $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ denotes a certain subset of the second quadratic cohomology $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ (see also Sections 2 and 4 for a definition of these sets) and $G_{(\mathfrak{l},\mathfrak{a})}$ is the automorphism group of the pair $(\mathfrak{l},\mathfrak{a})$.

Moreover, in [KO 2] we gave explicitly the map which assigns an element of (1) to each isomorphism class of indecomposable metric Lie algebras as well as its inverse map. The construction of these maps relies on the fact that for each metric Lie algebra without simple ideals there is a canonical isotropic ideal $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{g}$ such that $\mathfrak{a} := \mathfrak{i}(\mathfrak{g})^{\perp}/\mathfrak{i}(\mathfrak{g})$ is abelian.

The description (1) of the moduli space of isomorphism classes of metric Lie algebras allows a systematic approach to the construction and classification of metric Lie algebras. Of course it is far from being an explicit classification (e.g. a list). A full classification would require that we can determine all Lie algebras $\mathfrak{l}$ for which $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ is not empty for some orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. These Lie algebras are called admissible. However, although admissibility is a strong condition it seems to be hard to give a classification of these Lie algebras. Another problem is the explicit computation of the cohomology sets which includes for example the classification of $GL(l,\mathbb{R})$-orbits of 3-forms on an l-dimensional vector space. Such a classification is known only for $l \le 9$.

However, (1) yields a general classification scheme which can be used to obtain a full classification for metric Lie algebras satisfying suitable additional assumptions. Such assumptions can be, e.g., restrictions on the index of the inner product or on the structure of the Lie algebra. These restrictions give additional conditions for the Lie algebras $\mathfrak{l}$ occuring in (1). Hence, in order to get a classification from (1) one has first to determine all admissible Lie algebras $\mathfrak{l}$ which satisfy these additional conditions and afterwards one has to determine orbit sets of cohomology classes of these Lie algebras. For example, the classification of metric Lie algebras with index p leads to the classification problem for admissible Lie algebras of dimension $\dim \mathfrak{l} \le p$. In [KO 1] and [KO 2] we show how one can solve this problem for small p. In particular, we give a classification of all metric Lie algebras whose invariant inner product is of index two or three.

We see that the classification of admissible Lie algebras within a certain class is a main step in the solution of the original classification problem for metric Lie algebras (with additional properties). In general, the classification of admissible Lie algebras even within a certain class of Lie algebras seems to be complicated. However, often it is much easier than the determination of all Lie algebras of this class.

Let us consider another suitable condition which allows to make (1) more explicit. Namely, let us consider only indecomposable metric Lie algebras whose canonical isotropic ideal $\mathfrak{i}(\mathfrak{g})$ is "almost central" (this means that the codimension of $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{i}(\mathfrak{g})$ in $\mathfrak{i}(\mathfrak{g})$ is small). The case $\mathfrak{i}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ has been studied in [KO 1]. In particular, the general classification scheme has been specialised to the case of metric Lie algebras with maximal isotropic centre. If $\mathfrak{i}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$, then we have to consider only abelian Lie algebras $\mathfrak{l}$ in (1). All abelian Lie algebras are admissible. In the more general case $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{i}(\mathfrak{g})$ with small codimension we are led to the investigation of admissible Lie algebras $\mathfrak{l}$ with small nilpotent radical $R(\mathfrak{l})$. By definition $R(\mathfrak{l})$ is the minimal ideal such that the adjoint representation of $\mathfrak{l}$ on $\mathfrak{l}/R(\mathfrak{l})$ is semi-simple, e.g. $R(\mathfrak{l}) = \mathfrak{l}'$ for nilpotent $\mathfrak{l}$.

In the first part of this paper we solve the classification problem for nilpotent admissible Lie algebras $\mathfrak{l}$ whose nilpotent radical $R(\mathfrak{l}) = \mathfrak{l}'$ is two-dimensional. We can prove that such Lie algebras are direct sums $\mathfrak{g}\oplus\mathbb{R}^k$, where $\mathfrak{g}$ is nilpotent and admissible of dimension at most 6. The precise classification result is stated in Section 3, Proposition 2. Solvable non-nilpotent admissible Lie algebras with two-dimensional nilpotent radical and solvable admissible Lie algebras with one-dimensional nilpotent radical were already classified in [KO 2].

In the second part we apply the general classification scheme for metric Lie algebras to low-dimensional nilpotent metric Lie algebras. We use the classification results from the first part of the paper to determine all nilpotent metric Lie algebras of dimension ≤ 10, see Theorem 1 at the end of this paper.

2 Admissible cohomology classes

In [KO 2] we defined the quadratic cohomology $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ for a Lie algebra $\mathfrak{l}$ and an orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. Let us recall this definition. An *orthogonal* $\mathfrak{l}$*-module* is a tuple $(\rho,\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ (also $\mathfrak{a}$ or $(\rho,\mathfrak{a})$ in abbreviated notation) consisting of a finite-dimensional pseudo-Euclidean vector space and a representation ρ of $\mathfrak{l}$ on $\mathfrak{a}$ satisfying

$$\langle\rho(L)A_1,A_2\rangle_{\mathfrak{a}}+\langle A_1,\rho(L)A_2\rangle_{\mathfrak{a}}=0$$

for all $L\in\mathfrak{l}$ and $A_1,A_2\in\mathfrak{a}$.

For $\mathfrak{l}$ and (any $\mathfrak{l}$-module) $\mathfrak{a}$ we have the standard cochain complex $(C^*(\mathfrak{l},\mathfrak{a}),d)$ and corresponding cohomology groups $H^p(\mathfrak{l},\mathfrak{a})$. If $\mathfrak{a}$ is the one-dimensional trivial representation, then we denote this cochain complex also by $C^*(\mathfrak{l})$.

We define the product

$$\langle\cdot\wedge\cdot\rangle:\ C^p(\mathfrak{l},\mathfrak{a})\times C^q(\mathfrak{l},\mathfrak{a})\longrightarrow C^{p+q}(\mathfrak{l})$$

by the composition

$$C^p(\mathfrak{l},\mathfrak{a})\times C^q(\mathfrak{l},\mathfrak{a})\stackrel{\wedge}{\longrightarrow}C^{p+q}(\mathfrak{l},\mathfrak{a}\otimes\mathfrak{a})\stackrel{\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}}{\longrightarrow}C^p(\mathfrak{l}).$$

Let p be even. Then the group of quadratic $(p-1)$-cochains is the group

$$\mathcal{C}^{p-1}_Q(\mathfrak{l},\mathfrak{a})=C^{p-1}(\mathfrak{l},\mathfrak{a})\oplus C^{2p-2}(\mathfrak{l})$$

with group operation defined by

$$(\tau_1,\sigma_1)*(\tau_2,\sigma_2)=(\tau_1+\tau_2,\sigma_1+\sigma_2+\tfrac{1}{2}\langle\tau_1\wedge\tau_2\rangle)\,.$$

Now we consider the set

$$\mathcal{Z}^p_Q(\mathfrak{l},\mathfrak{a})=\{(\alpha,\gamma)\in C^p(\mathfrak{l},\mathfrak{a})\oplus C^{2p-1}(\mathfrak{l})\mid d\alpha=0,\ d\gamma=\tfrac{1}{2}\langle\alpha\wedge\alpha\rangle\}$$

of so-called quadratic p-cocycles. The group $\mathcal{C}^{p-1}_Q(\mathfrak{l},\mathfrak{a})$ acts on $\mathcal{Z}^p_Q(\mathfrak{l},\mathfrak{a})$ by

$$(\alpha,\gamma)(\tau,\sigma)=\Big(\alpha+d\tau,\gamma+d\sigma+\langle(\alpha+\tfrac{1}{2}d\tau)\wedge\tau\rangle\Big).$$

and we define the quadratic cohomology set $\mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a}) := \mathcal{Z}^p_Q(\mathfrak{l},\mathfrak{a})/\mathcal{C}^{p-1}_Q(\mathfrak{l},\mathfrak{a})$. As usual, we denote the equivalence class of $(\alpha,\gamma)\in\mathcal{Z}^p_Q(\mathfrak{l},\mathfrak{a})$ in $\mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a})$ by $[\alpha,\gamma]$.

Let us now recall the definition of admissible cohomology classes from [KO 2]. In general, admissible cohomology classes are certain elements of $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ for a Lie algebra $\mathfrak{l}$ and a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. Here we will give the definition of admissibility only for nilpotent Lie algebras. So we have the two following simplifications compared to [KO 2]:

If $\mathfrak{l}$ is a nilpotent Lie algebra, then $H^*(\mathfrak{l},\mathfrak{a})=H^*(\mathfrak{l},\mathfrak{a}^\mathfrak{l})$ holds for any semi-simple $\mathfrak{l}$-module $\mathfrak{a}$ (see [D]). This implies that also $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})=\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}^\mathfrak{l})$ holds for any orthogonal semi-simple $\mathfrak{l}$-module $\mathfrak{a}$.

For a Lie algebra $\mathfrak{l}$ we denote by $\mathfrak{l}^1=\mathfrak{l},\dots,\mathfrak{l}^k=[\mathfrak{l},\mathfrak{l}^{k-1}],\dots$ the lower central series. As usual we often denote $\mathfrak{l}^2$ also by $\mathfrak{l}'$. If $\mathfrak{l}$ is nilpotent, then its k-th nilpotent radical $R_k(\mathfrak{l})$ equals $\mathfrak{l}^{k+1}$.

Definition 1 *Let $\mathfrak{l}$ be a nilpotent Lie algebra and let $(\rho,\mathfrak{a},\langle\cdot\,,\cdot\rangle_\mathfrak{a})$ be a semi-simple orthogonal $\mathfrak{l}$-module. Let m be such that $\mathfrak{l}^{m+2}=0$. Put $\mathfrak{l}_{(0)}=\mathfrak{z}(\mathfrak{l})\cap\ker\rho$ and $\mathfrak{l}_{(k)}=\mathfrak{z}(\mathfrak{l})\cap\mathfrak{l}^{k+1}$ for $k\ge 1$. Take a cohomology class in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ and represent it by a cocycle (α,γ) satisfying $\alpha(\mathfrak{l},\mathfrak{l})\subset\mathfrak{a}^\mathfrak{l}$. Then $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ is called admissible if and only if the following conditions (A_k) and (B_k) hold for all $0\le k\le m$.*

(A_k) *Let $L_0\in\mathfrak{l}_{(k)}$ be such that there exist elements $A_0\in\mathfrak{a}$ and $Z_0\in(\mathfrak{l}^{k+1})^*$ satisfying*

(i) $\alpha(L,L_0)=0$,

(ii) $\gamma(L,L_0,\cdot)=-\langle A_0,\alpha(L,\cdot)\rangle_\mathfrak{a}+\langle Z_0,[L,\cdot]_\mathfrak{l}\rangle$ as an element of $(\mathfrak{l}^{k+1})^*$,

for all $L\in\mathfrak{l}$, then $L_0=0$.

(B_k) *The subspace $\alpha(\ker[\cdot\,,\cdot]_{\mathfrak{l}\otimes\mathfrak{l}^{k+1}})\subset\mathfrak{a}$ is non-degenerate, where $\ker[\cdot\,,\cdot]_{\mathfrak{l}\otimes\mathfrak{l}^{k+1}}$ is the kernel of the map $[\cdot\,,\cdot]:\mathfrak{l}\otimes\mathfrak{l}^{k+1}\to\mathfrak{l}$.*

We denote the set of all admissible cohomology classes in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ by $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$. A Lie algebra $\mathfrak{l}$ is called admissible if there is a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ such that $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp\ne\emptyset$.

3 Nilpotent admissible Lie algebras with 2-dimensional radical

In the following we will often describe a Lie algebra by giving a basis and some of the Lie brackets. In this case we always assume that all other brackets of basis vectors vanish. If we do not mention the basis explicitly, then we assume that all basis vectors appear in one of the bracket relations (on the left or the right hand side).

Using this convention we define

$$\mathfrak{h}(1) \;=\; \{[X_1,X_2]=Y\}$$

$$
\begin{aligned}
\mathfrak{g}_{4,1} &= \{[X_1,Z]=Y,\ [X_1,X_2]=Z\},\\
\mathfrak{g}_{5,2} &= \{[X_1,X_2]=Y,\ [X_1,X_3]=Z\},\\
\mathfrak{g}_{6,4} &= \{[X_1,X_2]=Y,\ [X_1,X_3]=Z,\ [X_3,X_4]=Y\},\\
\mathfrak{g}_{6,5} &= \{[X_1,X_2]=Y,\ [X_1,X_3]=Z,\ [X_2,X_4]=Z,\ [X_3,X_4]=-Y\}.
\end{aligned}
$$

Note that $\mathfrak{g}_{6,4}$ and $\mathfrak{g}_{6,5}$ are not isomorphic since $\dim[X_2,\mathfrak{g}_{6,4}]=1$ but $\dim[L,\mathfrak{g}_{6,5}]=2$ for all $L\in\mathfrak{g}_{6,5}$.

Proposition 1 *The Lie algebras $\mathfrak{h}(1)$, $\mathfrak{g}_{4,1}$, $\mathfrak{g}_{5,2}$, $\mathfrak{g}_{6,4}$, $\mathfrak{g}_{6,5}$ are admissible.*

Proof. In [KO 2] we proved that $\mathfrak{h}(1)$ is admissible. In Propositions 5 and 6 we will see that $\mathfrak{g}_{4,1}$ and $\mathfrak{g}_{5,2}$ are also admissible. Let us verify now that the statement holds for $\mathfrak{g}_{6,4}$ and $\mathfrak{g}_{6,5}$. First we consider $\mathfrak{l}=\mathfrak{g}_{6,4}$. Take $\mathfrak{a}=\mathbb{R}^{2,2}$ and $\rho=0$. Let A_1,A_2,A_3,A_4 be a Witt basis of $\mathfrak{a}$, i.e. $\langle A_1,A_3\rangle=\langle A_2,A_4\rangle=1$ and $\langle A_i,A_j\rangle=0$ for the remaining pairs $1\le i\le j\le 4$. We define $\alpha\in C^2(\mathfrak{l},\mathfrak{a})$ by

$$
\begin{aligned}
\alpha(X_1,Y)=-\alpha(X_4,Z)&=A_1, &\quad \alpha(X_3,Y)&=\alpha(X_2,Z)=A_2\\
\alpha(X_3,Z)&=A_3, &\quad \alpha(X_1,Z)&=A_4\\
\alpha(X_2,Y)=\alpha(X_4,Y)&=0, &\quad \alpha(X_i,X_j)&=\alpha(Y,Z)=0.
\end{aligned}
$$

Then it is easy to see that $d\alpha=0$ and $\langle\alpha\wedge\alpha\rangle=0$. Hence $(\alpha,0)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$. Let us now show that $[\alpha,0]$ is admissible. Take $L_0=cY+dZ\in\mathfrak{l}_{(k)}\subset\operatorname{span}\{Y,Z\}$. Then $\alpha(L_0,\mathfrak{l})=0$ would imply $\alpha(cY+dZ,X_1)=-cA_1-dA_4=0$, hence $c=d=0$, and therefore $L_0=0$. Thus Condition (A_k) is satisfied. It remains to check Conditions (B_k) for $k=0,1$ since $\mathfrak{l}^3=0$. These are satisfied because of $\alpha(\ker[\cdot\,,\cdot]_{\mathfrak{l}\otimes\mathfrak{l}^{k+1}})=\mathfrak{a}$ for $k=0,1$.

Now we consider $\mathfrak{l}=\mathfrak{g}_{6,5}$. Let $(\rho,\mathfrak{a})$ be as above and define $\alpha\in C^2(\mathfrak{l},\mathfrak{a})$ by

$$
\begin{aligned}
\alpha(X_1,Y)=\alpha(X_4,Z)&=A_1, &\quad \alpha(X_3,Y)&=\alpha(X_2,Z)=A_2\\
\alpha(X_2,Y)=-\alpha(X_3,Z)&=A_3, &\quad \alpha(X_4,Y)&=-\alpha(X_1,Z)=A_4\\
&\alpha(X_i,X_j)=\alpha(Y,Z)=0.
\end{aligned}
$$

In the same way as above one verifies $[\alpha,0]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$. $\square$

Proposition 2 *If $\mathfrak{l}$ is an admissible nilpotent Lie algebra with $\dim\mathfrak{l}'=2$, then $\mathfrak{l}$ is isomorphic to one of the (admissible) Lie algebras*

$$\mathfrak{h}(1)\oplus\mathfrak{h}(1)\oplus\mathbb{R}^k,\ \mathfrak{g}_{4,1}\oplus\mathbb{R}^k,\ \mathfrak{g}_{5,2}\oplus\mathbb{R}^k,\ \mathfrak{g}_{6,4}\oplus\mathbb{R}^k,\ \mathfrak{g}_{6,5}\oplus\mathbb{R}^k.$$

Proof. Let us verify that all these Lie algebras are admissible. First notice that $\mathbb{R}^k$ is admissible. Indeed, let $X_1,\dots,X_k$ be a basis of $\mathbb{R}^k$ and take $\mathfrak{a}$ and $\alpha\in C^2(\mathfrak{l},\mathfrak{a})$ such that $\alpha(X_i,X_j)=A_{ij}$ for $1\le i<j\le k$ and $\{A_{ij}\}_{1\le i<j\le k}$ is an orthonormal basis of $\mathfrak{a}$. Then $[\alpha,0]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$. If we use now Proposition 1 and the fact that direct sums of admissible Lie algebras are admissible the assertion follows.

Now we prove that each admissible nilpotent Lie algebra $\mathfrak{l}$ with $\dim\mathfrak{l}'=2$ is isomorphic to one of the mentioned Lie algebras. We distinguish between two cases: $\mathfrak{l}'\not\subset\mathfrak{z}(\mathfrak{l})$ (case I) and $\mathfrak{l}'\subset\mathfrak{z}(\mathfrak{l})$ (case II).

Case I: $\mathfrak{l}' \not\subset \mathfrak{z}(\mathfrak{l})$

The representation of $\mathfrak{l}$ on $\mathfrak{l}'$ is nilpotent and non-trivial. Hence we may choose a basis Y, Z of $\mathfrak{l}'$ and a vector X_1 in $\mathfrak{l} \setminus \mathfrak{l}'$ such that

$$[X_1, Z] = Y\,. \tag{2}$$

In particular, since $\mathfrak{l}$ is nilpotent this implies

$$[\mathfrak{l}, Y] = 0\,. \tag{3}$$

Using this we can see that $[X_1, \mathfrak{l}]$ is not contained in $\mathbb{R}\cdot Y$. Indeed, $[X_1, \mathfrak{l}] \subset \mathbb{R}\cdot Y$ would imply $[X_1, \mathfrak{l}'] = [X_1, [\mathfrak{l}, \mathfrak{l}]] = [[X_1, \mathfrak{l}], \mathfrak{l}] \subset [Y, \mathfrak{l}] = 0$, which contradicts $Y = [X_1, Z] \in [X_1, \mathfrak{l}']$. We conclude that there is a vector $X_2 \in \mathfrak{l} \setminus \mathfrak{l}'$ such that

$$[X_1, X_2] = Z\,. \tag{4}$$

Here we may assume

$$[X_2, Z] = 0\,. \tag{5}$$

By (2) and (4) it is possible to choose a vector space decomposition

$$\mathfrak{l} = \mathrm{span}\{X_1, X_2\} \oplus V \oplus \mathfrak{l}'$$

of $\mathfrak{l}$ such that $[X_1, V] = 0$ and $[X_2, V] \subset \mathbb{R}\cdot Y$. In particular, this implies $[X_1, [V, V]] = [[X_1, V], V] = 0$, hence

$$[V, V] \subset \mathbb{R}\cdot Y$$

by (2). Moreover, $[X_1, V] = 0$ and $[X_2, V] \subset \mathbb{R}\cdot Y$ together with (4) gives

$$[V, Z] = [V, [X_1, X_2]] = [X_1, [V, X_2]] \subset [X_1, \mathbb{R}\cdot Y] = 0.$$

Now we distinguish between the cases $[X_2, V] \neq 0$ and $[X_2, V] = 0$.

Case I.1: $[X_2, V] \neq 0$

Claim. A Lie algebra $\mathfrak{l}$ which satisfies the conditions of case I.1 is not admissible.

Proof. By (2) – (5) and our choice of V we find a basis $X_3, \ldots, X_l$ of V such that

$$\mathfrak{l} = \{\, [X_1, X_2] = Z,\ [X_1, Z] = Y,\ [X_2, X_3] = Y,\ [X_i, X_j] = y_{ij}Y,\ i, j \geq 3\,\}$$

for suitable $y_{ij} \in \mathbb{R}$. Assume that $\mathfrak{l}$ is admissible. Then we can choose a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ and $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ such that $[\alpha, \gamma]$ is admissible. As explained above we may assume $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. Hence $d\alpha = 0$ implies

$$\begin{aligned}
0 &= \alpha([X_2, X_3], Z) = \alpha(Y, Z) && (6)\\
0 &= \alpha([X_1, X_2], X_3) + \alpha([X_2, X_3], X_1) = \alpha(Z, X_3) + \alpha(Y, X_1) && (7)\\
0 &= \alpha([X_1, Z], X_j) = \alpha(Y, X_j), \quad j \geq 2\,. && (8)
\end{aligned}$$

Because of $\langle \alpha \wedge \alpha \rangle = 2d\gamma$ we have

$$\begin{aligned}\langle \alpha(X_1, X_3), \alpha(Y, Z)\rangle + \langle \alpha(X_3, Y), \alpha(X_1, Z)\rangle + \langle \alpha(Y, X_1), \alpha(X_3, Z)\rangle \\ = d\gamma(X_1, X_3, Y, Z) = -\gamma([X_1, Z], X_3, Y) = -\gamma(Y, X_3, Y) = 0\end{aligned}$$

and by (6) – (8) this yields $\langle \alpha(Y, X_1), \alpha(Y, X_1)\rangle = 0$. Summarizing we obtain

$$\alpha(Y, Z) = 0, \quad \langle \alpha(Y, X_1), \alpha(Y, X_1)\rangle = 0, \quad \alpha(Y, X_j) = 0,\ j \geq 2\,. \tag{9}$$

Now let us consider Condition (B_2). Since $\mathfrak{l}^3 = \mathbb{R} \cdot Y \subset \mathfrak{z}(\mathfrak{l})$ it is satisfied if and only if the space $\alpha(\mathfrak{l}, Y)$ is non-degenerate. Now (9) implies that (B_2) is satisfied if and only if $\alpha(Y, \mathfrak{l}) = 0$. But if $\alpha(Y, \mathfrak{l}) = 0$, then Condition (A_2) is not satisfied. Indeed, $L_0 = Y \neq 0$, $A_0 = 0$, $Z_0 = 0$ obviously satisfy $(A_2)\,(i)$ and $(A_2)\,(ii)$ since $\mathfrak{l}^3 = \mathbb{R} \cdot Y$ is one-dimensional. Thus we obtain a contradiction and $\mathfrak{l}$ is not admissible. □

Case I.2: $[X_2, V] = 0$

Claim. An admissible Lie algebra $\mathfrak{l}$ which satisfies the conditions of case I.2 is isomorphic to $\mathfrak{g}_{4,1} \oplus \mathbb{R}^k$.

Proof. By (2) – (5) and our choice of V we find a basis $X_3, \ldots, X_l$ of V such that

$$\mathfrak{l} = \{\, [X_1, X_2] = Z,\ [X_1, Z] = Y,\ [X_i, X_j] = y_{ij} Y,\ i, j \geq 3 \,\}\,.$$

for suitable $y_{ij} \in \mathbb{R}$, $i, j \geq 3$. Suppose $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ is admissible and $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. The cocycle conditions

$$d\alpha(X_1, X_2, Y) = 0,\ d\alpha(X_i, X_j, X_1) = 0,\ d\alpha(X_1, X_2, Z) = 0,\ d\alpha(X_1, X_i, Z) = 0$$

for $i, j \geq 3$ yield

$$\alpha(Y, Z) = 0,\ y_{ij}\alpha(Y, X_1) = 0,\ \alpha(Y, X_2) = 0,\ \alpha(Y, X_i) = 0,\ i, j \geq 3,$$

respectively. Assume that $y_{ij} \neq 0$ for some $i, j \geq 3$. Then $\alpha(Y, \mathfrak{l}) = 0$ follows. In this case $L_0 = Y \in \mathfrak{z}(\mathfrak{l}) \cap \mathfrak{l}^3$, $A_0 = 0$ satisfy $(A_2)\,(i)$ and $(A_2)\,(ii)$ since $\mathfrak{l}^3 = \mathbb{R} \cdot Y$ is one-dimensional. Since $Y \neq 0$ we see that (A_2) is not satisfied, a contradiction. Consequently, $y_{ij} = 0$ for all $i, j \geq 3$, which proves the claim. □

Case II: $\mathfrak{l}' \subset \mathfrak{z}(\mathfrak{l})$

Lemma 1 *There exists a 3-dimensional subspace* $\bar{\mathfrak{l}}$ *of* $\mathfrak{l}$ *such that* $[\bar{\mathfrak{l}}, \bar{\mathfrak{l}}] = \mathfrak{l}'$. *Moreover, we can choose a basis* X_1, X_2, X_3 *of* $\bar{\mathfrak{l}}$ *and a basis* Y, Z *of* $\mathfrak{l}'$ *such that*

$$[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_2, X_3] = 0. \tag{10}$$

Proof. Since $\dim \mathfrak{l}' = 2$ we can choose vectors $L_1, \ldots, L_4$ such that $\mathfrak{l}' = \mathrm{span}\{[L_1, L_2], [L_3, L_4]\}$. We consider $\mathfrak{l}_1 = \mathrm{span}\{L_1, L_2\}$ and $\mathfrak{l}_2 = \mathrm{span}\{L_3, L_4\}$. If $[\mathfrak{l}_1, \mathfrak{l}_2] \neq 0$, then we may assume $[L_1, L_3] \neq 0$. In this case at least one of the pairs $[L_1, L_3], [L_1, L_2]$ and $[L_1, L_3], [L_3, L_4]$ consists of two linearly independent vectors and we can choose $\bar{\mathfrak{l}}$ correspondingly. If $[\mathfrak{l}_1, \mathfrak{l}_2] = 0$, then e.g. $\bar{\mathfrak{l}} = \mathrm{span}\{L_1, L_3, L_2 + L_4\}$ satisfies $[\bar{\mathfrak{l}}, \bar{\mathfrak{l}}] = \mathfrak{l}'$.

We can choose linearly independent vectors X_1, X_2, X_3, Y, Z of $\mathfrak{l}$ such that $[X_1, X_2] = Y$ and $[X_1, X_3] = Z$. If $[X_2, X_3] = yY + zZ$, then $\bar{X}_2 := X_2 - zX_1$ and $\bar{X}_3 := X_3 + yX_1$ satisfy $[X_1, \bar{X}_2] = Y$, $[X_1, \bar{X}_3] = Z$ and $[\bar{X}_2, \bar{X}_3] = 0$. $\square$

Lemma 2 *Let X_1, X_2, X_3, Y, Z be as in Lemma 1. If $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ and $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$, then we have*

(i) $\alpha(Y, Z) = 0$;

(ii) $\alpha(Y, L) = 0$ *for all* $L \in \mathfrak{l}$ *satisfying* $[L, X_1] = [L, X_2] = 0$;

(iii) $\alpha(Z, L) = 0$ *for all* $L \in \mathfrak{l}$ *satisfying* $[L, X_1] = [L, X_3] = 0$;

(iv) $\langle \alpha(U_1, L_1), \alpha(U_2, L_2) \rangle = \langle \alpha(U_1, L_2), \alpha(U_2, L_1) \rangle$ *for all* $U_1, U_2 \in \mathfrak{l}'$ *and* $L_1, L_2 \in \mathfrak{l}$.

Proof. Assertions *(i)*, *(ii)*, *(iii)* follow from the cocycle condition for α, from $\mathfrak{l}' \subset \mathfrak{z}(\mathfrak{l})$ and from the special conditions on $L \in \mathfrak{l}$ in *(ii)* and *(iii)*, respectively:

$$
\begin{array}{rclclcl}
\alpha(Y, Z) & = & \alpha([X_1, X_2], Z) & = & \alpha([Z, X_2], X_1) + \alpha([X_1, Z], X_2) & = & 0 \\
\alpha(Y, L) & = & \alpha([X_1, X_2], L) & = & \alpha([L, X_2], X_1) + \alpha([X_1, L], X_2) & = & 0 \\
\alpha(Z, L) & = & \alpha([X_1, X_3], L) & = & \alpha([L, X_3], X_1) + \alpha([X_1, L], X_3) & = & 0.
\end{array}
$$

As for assertion *(iv)* we first observe that

$$
d\gamma(U_1, U_2, L_1, L_2) = -\gamma([L_1, L_2], U_1, U_2) = 0, \tag{11}
$$

where the first equality follows from $U_1, U_2 \in \mathfrak{z}(\mathfrak{l})$ and the second equality follows from $[L_1, L_2] \in \mathfrak{l}'$ and $\dim \mathfrak{l}' = 2$. Combining now (11) with the cocycle condition for (α, γ) we obtain

$$
\langle \alpha(U_1, U_2), \alpha(L_1, L_2) \rangle + \langle \alpha(U_2, L_1), \alpha(U_1, L_2) \rangle + \langle \alpha(L_1, U_1), \alpha(U_2, L_2) \rangle = 0.
$$

Since *(i)* implies $\alpha(U_1, U_2) = 0$ the first term vanishes and the assertion follows. $\square$

Lemma 3 *Let $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ be admissible and choose α such that $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. Let $\bar{\mathfrak{l}}$ be as in Lemma 1. If $K \in \mathfrak{l}'$ satisfies $\alpha(K, \bar{\mathfrak{l}}) = 0$, then $\alpha(K, \mathfrak{l}) = 0$.*

Proof. Choose X_1, X_2, X_3, Y, Z as in Lemma 1. For $L_1, L_2 \in \mathfrak{l}$ we have

$$
\begin{array}{rcl}
\langle \alpha(K, L_1), \alpha(Y, L_2) \rangle & = & \langle \alpha(K, L_1), \alpha([X_1, X_2], L_2) \rangle \\
& = & \langle \alpha(K, L_1), \alpha([L_2, X_2], X_1) \rangle + \langle \alpha(K, L_1), \alpha([X_1, L_2], X_2) \rangle \\
& = & \langle \alpha(K, X_1), \alpha([L_2, X_2], L_1) \rangle + \langle \alpha(K, X_2), \alpha([X_1, L_2], L_1) \rangle \\
& = & 0,
\end{array}
$$

where we first used the cocycle condition for α and then Lemma 2. Similarly, we have

$$\langle \alpha(K, L_1), \alpha(Z, L_2)\rangle = \langle \alpha(K, L_1), \alpha([X_1, X_3], L_2)\rangle = 0.$$

This implies $\alpha(K, L_1) \perp \alpha(\mathfrak{l}', \mathfrak{l})$. Now (B_1) yields $\alpha(K, L_1) = 0$ for all $L_1 \in \mathfrak{l}$. $\square$

Lemma 4 *Let $\mathfrak{l}$ be admissible and let $\bar{\mathfrak{l}}$ be as in Lemma 1. Then $[L_1, L_2] = 0$ holds for all $L_1, L_2 \in \mathfrak{l}$ satisfying $[L_1, \bar{\mathfrak{l}}] = [L_2, \bar{\mathfrak{l}}] = 0$.*

Proof. We choose a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ such that there is an admissible cohomology class $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$. We may assume $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. From $d\alpha = 0$ and $[\bar{\mathfrak{l}}, L_1] = [\bar{\mathfrak{l}}, L_2] = 0$ we obtain $\alpha([L_1, L_2], X_i) = 0$. Lemma 3 now implies

$$\alpha([L_1, L_2], \cdot) = 0. \tag{12}$$

Take X_1, X_2, X_3, Y, Z as in Lemma 1. Using $[L_1, L_2] \in \mathfrak{l}' \subset \mathfrak{z}(\mathfrak{l})$ we see that

$$d\gamma([L_1, L_2], X_1, X_2, L_i) = -\gamma(Y, [L_1, L_2], L_i)$$

for $i = 1, 2$. On the other hand (12) gives

$$2d\gamma([L_1, L_2], X_1, X_2, L_i) = \langle \alpha \wedge \alpha\rangle([L_1, L_2], X_1, X_2, L_i) = 0$$

for $i = 1, 2$. Hence we have

$$\gamma(Y, [L_1, L_2], L_i) = 0 \tag{13}$$

for $i = 1, 2$ and similarly we obtain

$$\gamma(Z, [L_1, L_2], L_i) = 0 \tag{14}$$

for $i = 1, 2$. Assume now that $[L_1, L_2] \neq 0$. By (13), (14) and $[L_1, L_2] \in \mathrm{span}\{Y, Z\}$ we obtain $\gamma(Y, Z, L_i) = 0$ for $i = 1, 2$. This yields

$$d\gamma(U, L_1, L_2, L) = -\gamma([L_1, L_2], U, L) \tag{15}$$

for all $L \in \mathfrak{l}$ and $U \in \mathfrak{l}'$. On the other hand, Lemma 2, *(ii)*, *(iii)* gives $\alpha(U, L_1) = \alpha(U, L_2) = 0$ and therefore

$$d\gamma(U, L_1, L_2, L) = \langle \alpha(U, L), \alpha(L_1, L_2)\rangle. \tag{16}$$

From (15) and (16) we get

$$\gamma(L, [L_1, L_2], \cdot) = \langle \alpha(L_1, L_2), \alpha(L, \cdot)\rangle$$

as an element of $(\mathfrak{l}')^*$. Hence Condition $(A_1)(ii)$ is satisfied for $L_0 = [L_1, L_2]$, $A_0 = -\alpha(L_1, L_2)$, $Z_0 = 0$. Since also $(A_1)(i)$ holds by (12) and $[\alpha, \gamma]$ is admissible we get $[L_1, L_2] = 0$, which is a contradiction. $\square$

Lemma 5 *If $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ is admissible and $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$, then $\alpha([L, \mathfrak{l}], \mathfrak{l}) = 0$ holds for all $L \in \mathfrak{l}$ satisfying $[L, \bar{\mathfrak{l}}] = 0$.*

Proof. Let $L \in \mathfrak{l}$ satisfy $[L, \bar{\mathfrak{l}}] = 0$. By Lemma 3 it suffices to prove that $\alpha([L, L'], X) = 0$ holds for all $L' \in \mathfrak{l}$ and $X \in \bar{\mathfrak{l}}$. Since $[X, L] = 0$ the cocycle condition for α gives

$$d\alpha(L, L', X) = -\alpha([L, L'], X) - \alpha([L', X], L) = 0.$$

The assertion now follows since $\alpha([L', X], L) = 0$ by Lemma 2 (ii), (iii). □

Lemma 6 *There exists a basis $X_1, X_2, X_3, \ldots, X_l, Y, Z$ of $\mathfrak{l}$ such that*

$$[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_2, X_3] = 0 \tag{17}$$

$$[X_1, X_4] = 0,\ [X_2, X_4] = \lambda Z,\ \lambda \in \{0, 1\}, \tag{18}$$

$$[X_1, X_j] = [X_2, X_j] = 0,\ j \geq 5. \tag{19}$$

Let $X_1, X_2, X_3, \ldots, X_l, Y, Z$ be such a basis and let j_0 be such that

$$[X_1, X_j] = [X_2, X_j] = [X_3, X_j] = 0$$

for all $j \geq j_0$. Then we have $[X_r, X_s] = 0$ for all $r, s \geq j_0$.

Proof. We choose $\bar{\mathfrak{l}}$, a basis X_1, X_2, X_3 of $\bar{\mathfrak{l}}$ and a basis Y, Z of $\mathfrak{l}'$ as in Lemma 1. Because of $[X_1, X_2] = Y$ and $[X_1, X_3] = Z$ one can find a complementary vector space W of $\mathrm{span}\{X_1, X_2, X_3, Y, Z\}$ in $\mathfrak{l}$ such that $[X_1, W] = 0$ and $[X_2, W] \subset \mathbb{R} \cdot Z$ (choose a basis of an arbitrary complement and change each basis vector by a suitable linear combination of X_1, X_2 and X_3). If $[X_2, W] = 0$, then we can choose an arbitrary basis $X_4, \ldots, X_l$ of W and (18) and (19) are satisfied for $\lambda = 0$. If $[X_2, W] \neq 0$ then we can choose a basis $X_4, \ldots, X_l$ of W such that $[X_2, X_4] = Z$ and $[X_2, X_j] = 0$ for $j = 1, \ldots, l$.

The second statement follows from Lemma 4. □

Now we fix a basis of $\mathfrak{l}$ which satisfies the conditions of Lemma 6.

Case II.1: $\lambda = 0$

Let $X_1, X_2, X_3, \ldots, X_l, Y, Z$ be a basis of $\mathfrak{l}$ satisfying (17), (18) and (19) with $\lambda = 0$. We define $W := \mathrm{span}\{X_4, \ldots, X_l\}$.

Case II.1.1: $[X_3, W] = 0$

Claim. An admissible Lie algebra $\mathfrak{l}$ which satisfies the conditions of case II.1.1 is isomorphic to $\mathfrak{g}_{5,2} \oplus \mathbb{R}^k$,

Proof. In this case we have $[X_1, W] = [X_2, W] = [X_3, W] = 0$ by assumption and $[W, W] = 0$ by Lemma 4. □

Case II.1.2: $\dim[X_3, W] = 1$

Claim. An admissible Lie algebra $\mathfrak{l}$ which satisfies the conditions of case II.1.2 is isomorphic to $\mathfrak{g}_{6,4} \oplus \mathbb{R}^k$ or to $\mathfrak{h}(1) \oplus \mathfrak{h}(1)$.

Proof. We may assume $[X_3, X_4] = cY + dZ \neq 0$, $c, d \in \mathbb{R}$ and $[X_3, X_r] = 0$ for $r \geq 5$. Let us first consider the case $d \neq 0$. Replacing X_3, X_4, and Z by

$$X_3' := dX_3 + cX_2,\ X_4' := 1/d \cdot X_4,\ \text{and}\ Z' := cY + dZ,$$

respectively, we see that we may assume $c = 0$ and $d = 1$. Hence we have a basis $X_1, X_2, X_3, \ldots, X_l, Y, Z$ of $\mathfrak{l}$ satisfying (17), (18), (19), $[X_3, X_4] = Z$ and $[X_3, X_r] = 0$ for $r \geq 5$. We will prove that the admissibility of $\mathfrak{l}$ implies $[X_4, X_r] = 0$ for $r \geq 5$. Assume first that there is a vector $X \in \mathrm{span}\{X_5, \ldots, X_l\}$ such that $[X_4, X] = aY + bZ$, $a \neq 0$ and $b \neq 0$. Let $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ be admissible and choose α such that $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. Then the cocycle condition for α yields

$$\begin{aligned}
d\alpha(X_1, X_2, X_3) &= -\alpha(Y, X_3) + \alpha(Z, X_2) \ = \ 0\\
d\alpha(X_1, X_2, X_4) &= -\alpha(Y, X_4) \ = \ 0\\
d\alpha(X_1, X_3, X_4) &= -\alpha(Z, X_4) - \alpha(Z, X_1) \ = \ 0\\
d\alpha(X_2, X_3, X_4) &= -\alpha(Z, X_2) \ = \ 0.
\end{aligned}$$

Moreover, Lemma 5 for $\bar{\mathfrak{l}} = \mathrm{span}\{X_1, X_2, X_3\}$, $L = X$ yields $\alpha(aY + bZ, \cdot) = 0$. Since $a \neq 0$ and $b \neq 0$, this equation together with the cocycle conditions above implies $\alpha(\mathfrak{l}', X_i) = 0$ for $i \leq 4$ and Lemma 2, *(ii)*, *(iii)* now gives $\alpha(\mathfrak{l}', \mathfrak{l}) = 0$. In particular we obtain $d\gamma(\mathfrak{l}', \mathfrak{l}, \mathfrak{l}, \mathfrak{l}) = 0$. From this condition we obtain

$$\begin{aligned}
d\gamma(Z, X_1, X_2, X_j) &= -\gamma(Y, Z, X_j) \ = \ 0,\ j \geq 3\\
d\gamma(Y, X_1, X_3, X_4) &= \gamma(Y, Z, X_4) + \gamma(Y, Z, X_1) \ = \ 0\\
d\gamma(Y, X_2, X_3, X_4) &= \gamma(Y, Z, X_2) \ = \ 0,
\end{aligned}$$

hence $\gamma(Y, Z, \cdot) = 0$. But then $[\alpha, \gamma]$ does not satisfy Condition (A_1). This is a contradiction to the admissibility of $[\alpha, \gamma]$. Hence $[X_4, X_r] \in \mathbb{R}Z$ for all $r \geq 5$ or $[X_4, X_r] \in \mathbb{R}Y$ for all $r \geq 5$. If we are in the first case and if there is an $s \geq 5$ such that $[X_4, X_s] \neq 0$, then we may assume $[X_4, X_5] = Z$. If we define $\bar{\mathfrak{l}} = \mathrm{span}\{X_1, X_2, X_3 + X_5\}$, then $\bar{\mathfrak{l}}$, $L_1 = X_4$ and $L_2 = X_5$ satisfy the assumptions of Lemma 4. But $[X_4, X_5] = Z \neq 0$ yields a contradiction. Similarly, if $[X_4, X_r] \in \mathbb{R}Y$ for all $r \geq 5$ and if there is an $s \geq 5$ such that $[X_4, X_s] \neq 0$, then we may assume $[X_4, X_5] = Y$. Consider now $\bar{\mathfrak{l}} := \mathrm{span}\{X_4, X_3, X_5 - X_2\}$. Then $\bar{\mathfrak{l}}$, $L_1 = X_1 + X_4$ and $L_2 = X_2$ satisfy the assumptions of Lemma 4, but $[X_1 + X_4, X_2] = Y \neq 0$, a contradiction. We deduce $[X_4, X_r] = 0$ for $r \geq 5$. We conclude that in case $d \neq 0$ the Lie algebra $\mathfrak{l}$ is isomorphic to

$$\{[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_3, X_4] = Z\} \oplus \mathbb{R}^k.$$

Putting $X_1' := X_1 + X_4$ we see

$$\mathfrak{l} \cong \{[X_1', X_2] = Y,\ [X_3, X_4] = Z\} \oplus \mathbb{R}^k \cong \mathfrak{h}(1) \oplus \mathfrak{h}(1) \oplus \mathbb{R}^k.$$

Now we consider the case $d = 0$. We may assume $c = 1$. Now we have a basis $X_1, X_2, X_3, \ldots, X_l, Y, Z$ of $\mathfrak{l}$ satisfying (17), (18), (19), $[X_3, X_4] = Y$ and $[X_3, X_r] = 0$ for $r \geq 5$. We will prove that $[X_4, X_r] = 0$ holds for $r \geq 5$. Assume that this is not true. Then we have without loss of generality $[X_4, X_5] = aY + bZ \neq 0$. If $b \neq 0$, then we replace X_3 by $X_3' := X_3 + X_5$. Then the basis $X_1, X_2, X_3', X_4, \ldots, X_l, Y, Z$ satisfies (17), (18), (19) with $\lambda = 0$, $[X_3', X_4] = (1-a)Y - bZ$, and $[X_3', X_r] = 0$ for $r \geq 5$. Thus we are in the above case where $d \neq 0$. This implies $[X_4, X_5] = 0$, a contradiction. If $b = 0$, then we may assume $a = 1$. Again $\bar{\mathfrak{l}} = \mathrm{span}\{X_1, X_2, X_3 + X_5\}$, $L_1 = X_4$ and $L_2 = X_5$ satisfy the assumptions of Lemma 6, but $[X_4, X_5] \neq 0$, a contradiction. Therefore we have $[X_4, X_r] = 0$ for $r \geq 5$, thus $\mathfrak{l}$ is isomorphic to

$$\{[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_3, X_4] = Y\} \oplus \mathbb{R}^k \cong \mathfrak{g}_{6,4} \oplus \mathbb{R}^k.$$

□

Case II.1.3: $\dim[X_3, W] = 2$

Claim. A Lie algebra $\mathfrak{l}$ which satisfies the conditions of case II.1.3 is not admissible.

Proof. Obviously we may assume that $X_1, \ldots, X_l, Y, Z$ is a basis of $\mathfrak{l}$ which satisfies (17), (18), (19) with $\lambda = 0$, $[X_3, X_4] = Y$, $[X_3, X_5] = Z$, and $[X_3, X_r] = 0$ for $r > 5$. Moreover, we have $[X_4, X_5] = yY + zZ$ for suitable $y, z \in \mathbb{R}$. Assume first that $z \neq 0$. Let $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ be admissible and choose α such that $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. Then we have

$$\begin{aligned}
d\alpha(X_1, X_2, X_3) &= -\alpha(Y, X_3) + \alpha(Z, X_2) = 0 && (20)\\
d\alpha(X_2, X_3, X_4) &= -\alpha(Y, X_2) = 0 && (21)\\
d\alpha(X_2, X_3, X_5) &= -\alpha(Z, X_2) = 0 && (22)\\
d\alpha(X_1, X_3, X_5) &= -\alpha(Z, X_5) - \alpha(Z, X_1) = 0 && (23)\\
d\alpha(X_1, X_3, X_4) &= -\alpha(Z, X_4) - \alpha(Y, X_1) = 0. && (24)
\end{aligned}$$

By Lemma 2 *(ii)* we have $\alpha(Y, X_j) = 0$ for $j \geq 4$. Together with (20) – (22) this yields $\alpha(Y, X_j) = 0$ for $j \geq 2$. Lemma 2 *(iv)* gives

$$\langle \alpha(Y, X_1), \alpha(Z, X_j) \rangle = \langle \alpha(Y, X_j), \alpha(Z, X_1) \rangle = 0$$

for $j \geq 2$ and therefore also

$$\begin{aligned}
\langle \alpha(Y, X_1), \alpha(Z, X_1) \rangle &= -\langle \alpha(Y, X_1), \alpha(Z, X_5) \rangle = 0\\
\langle \alpha(Y, X_1), \alpha(Y, X_1) \rangle &= -\langle \alpha(Y, X_1), \alpha(Z, X_4) \rangle = 0
\end{aligned}$$

where we used (23) and (24). We obtain $\alpha(Y, X_1) \perp \alpha(\mathfrak{l}', \mathfrak{l})$. Now we use that the admissibility condition (B_1) implies that $\alpha(\mathfrak{l}, \mathfrak{l}')$ is non-degenerate. Hence $\alpha(Y, X_1) = 0$ and, consequently, $\alpha(Y, \mathfrak{l}) = 0$. Now we get $\alpha(Z, X_2) = \alpha(Z, X_4) = 0$ from (22) and (24). Moreover,

$$d\alpha(X_1, X_4, X_5) = \alpha(X_1, yY + zZ) = 0$$

gives $\alpha(Z, X_1) = 0$ because of $z \neq 0$. Thus $\alpha(Z, X_5) = 0$ by (23). Finally,

$$d\alpha(X_3, X_4, X_5) \;=\; -\alpha(Y, X_5) - \alpha(yY + zZ, X_3) + \alpha(Z, X_4) \;=\; 0$$

yields $\alpha(Z, X_3) = 0$. Since, moreover, $\alpha(Z, X_k) = 0$ for $k \geq 6$ by Lemma 2 *(ii)* we obtain also $\alpha(Z, \mathfrak{l}) = 0$.

Because of $2d\gamma = \langle \alpha \wedge \alpha \rangle$ we now get

$$\begin{aligned}
d\gamma(Z, X_1, X_3, X_4) &= -\gamma(Y, Z, X_1) \;=\; 0,\\
d\gamma(Y, X_1, X_2, X_3) &= -\gamma(Y, Z, X_2) \;=\; 0,\\
d\gamma(Z, X_1, X_2, X_3) &= -\gamma(Y, Z, X_3) \;=\; 0,\\
d\gamma(Z, X_1, X_2, X_k) &= -\gamma(Y, Z, X_k) \;=\; 0,\ k \geq 4.
\end{aligned}$$

Consequently, $\gamma(Y, Z, \mathfrak{l}) = 0$. This together with $\alpha(\mathfrak{l}', \mathfrak{l}) = 0$ yields a contradiction to admissibility. Hence $z = 0$ and $[X_4, X_5] = yY$. However, now we can apply Lemma 4 to $\bar{\mathfrak{l}} := \mathrm{span}\{X_3, X_4 + yX_2, X_5\}$, $L_1 = X_1 + X_5$, $L_2 = X_2$ and we obtain a contradiction to $[X_1 + X_5, X_2] = Y \neq 0$. $\square$

Case II.2: $\lambda = 1$

Claim. An admissible Lie algebra $\mathfrak{l}$ which satisfies the conditions of case II.2 and which does not have a basis satisfying already the conditions of case II.1 is isomorphic to $\mathfrak{g}_{6,5} \oplus \mathbb{R}^k$.

Let $X_1, \ldots, X_l, Y, Z$ be a basis of $\mathfrak{l}$ which satisfies (17), (18), and (19) with $\lambda = 1$.

Lemma 7 *If* $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_\sharp$ *and* $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$, *then* $\alpha([X_3, X_j], \cdot) = 0$ *for all* $j \geq 5$.

Proof. By Lemma 3 it suffices to prove $\alpha([X_3, X_j], X_i) = 0$ for all $j \geq 5$ and $i = 1, 2, 3$. For $i = 2$ this follows obviously from the cocycle condition for α.

Now we consider $i = 3$. Using $[X_1, X_2] = Y$, the cocycle condition for α and Lemma 2 *(iv)* we see that

$$\begin{aligned}
\langle \alpha([X_3, X_j], X_3), \alpha(Y, L) \rangle &= \alpha([X_3, X_j], L), \alpha(Y, X_3) \rangle\\
&= \langle \alpha([X_3, X_j], L), \alpha([X_1, X_2], X_3) \rangle\\
&= \langle \alpha([X_3, X_j], L), \alpha([X_1, X_3], X_2) \rangle\\
&= \langle \alpha([X_3, X_j], X_2), \alpha([X_1, X_3], L) \rangle \;=\; 0,
\end{aligned}$$

where the last equality follows from the above considerations for $i = 2$. Similarly (using now $[X_2, X_4] = Z$) we obtain

$$\begin{aligned}
\langle \alpha([X_3, X_j], X_3), \alpha(Z, L) \rangle &= \alpha([X_3, X_j], L), \alpha(Z, X_3) \rangle\\
&= \langle \alpha([X_3, X_j], L), \alpha([X_2, X_4], X_3) \rangle\\
&= \langle \alpha([X_3, X_j], L), \alpha([X_3, X_4], X_2) \rangle\\
&= \langle \alpha([X_3, X_j], X_2), \alpha([X_3, X_4], L) \rangle \;=\; 0.
\end{aligned}$$

Now we use that the admissibility condition (B_1) implies that $\alpha(\mathfrak{l}, \mathfrak{l}')$ is non-degenerate. This gives $\alpha([X_3, X_j], X_3) = 0$.

Finally we consider the case $i = 1$. Note first that the cocycle condition for α implies

$$\alpha([X_3, X_j], X_1) = -\alpha([X_1, X_3], X_j) + \alpha([X_1, X_j], X_3) = -\alpha(Z, X_j)\,. \tag{25}$$

Using now Lemma 2, (ii) and (iv) we obtain

$$\langle \alpha([X_3, X_j], X_1), \alpha(Y, L)\rangle = -\langle \alpha(Z, X_j), \alpha(Y, L)\rangle = -\langle \alpha(Z, L), \alpha(Y, X_j)\rangle = 0$$

for all $L \in \mathfrak{l}$. Consequently, $\alpha([X_3, X_j], X_1) \perp \alpha(Y, \mathfrak{l})$. Now we will prove that also $\alpha([X_3, X_j], X_1) \perp \alpha(Z, \mathfrak{l})$ and thus $\alpha([X_3, X_j], X_1) \perp \alpha(\mathfrak{l}', \mathfrak{l})$ holds. By (B_1) this will give $\alpha([X_3, X_j], X_1) = 0$. We observe that

$$\begin{aligned}\langle \alpha([X_3, X_j], X_1), \alpha(Z, L)\rangle = -\langle \alpha(Z, X_j), \alpha(Z, L)\rangle \;&=\; -\langle \alpha(Z, X_j), \alpha([X_2, X_4], L)\rangle \\ = \;\; -\langle \alpha(Z, X_j), \alpha([X_2, L], X_4)\rangle &- \langle \alpha(Z, X_j), \alpha([L, X_4], X_2)\rangle,\end{aligned} \tag{26}$$

where we used (25) and the cocycle condition for α. By Equation (25) and Lemma 2 (iv) we have

$$\begin{aligned}-\langle \alpha(Z, X_j), \alpha([L, X_4], X_2)\rangle \;&=\; \langle \alpha([X_3, X_j], X_1), \alpha([L, X_4], X_2)\rangle \\ &=\; \langle \alpha([X_3, X_j], X_2), \alpha([L, X_4], X_1)\rangle \;=\; 0\end{aligned}$$

since $\alpha([X_3, X_j], X_2) = 0$. Hence the last term in (26) vanishes and we get

$$\langle \alpha([X_3, X_j], X_1), \alpha(Z, L)\rangle = -\langle \alpha(Z, X_j), \alpha([X_2, L], X_4)\rangle = c\langle \alpha(Z, X_j), \alpha(Z, X_4)\rangle$$

for some real number $c \in \mathbb{R}$ since $[X_2, L] \in \mathrm{span}\{Y, Z\}$ and since

$$\langle \alpha(Z, X_j), \alpha(Y, X_4)\rangle = \langle \alpha(Z, X_4), \alpha(Y, X_j)\rangle = 0$$

by Lemma 2 *(ii)* and (iv). Furthermore, we have

$$\langle \alpha(Z, X_j), \alpha(Z, X_4)\rangle = \langle \alpha(Z, X_j), \alpha([X_1, X_3], X_4)\rangle = -\langle \alpha(Z, X_j), \alpha([X_3, X_4], X_1)\rangle\,.$$

Since we already know that $\alpha(Z, X_j) \perp \alpha(Y, X_1)$ the last equation implies that in order to prove $\langle \alpha([X_3, X_j], X_1), \alpha(Z, L)\rangle = 0$ it suffices to show $\langle \alpha(Z, X_j), \alpha(Z, X_1)\rangle = 0$. However, this follows from Lemma 2, (ii) and (iv):

$$\begin{aligned}\langle \alpha(Z, X_j), \alpha(Z, X_1)\rangle \;&=\; \langle \alpha(Z, X_j), \alpha([X_2, X_4], X_1)\rangle \;=\; -\langle \alpha(Z, X_j), \alpha([X_1, X_2], X_4)\rangle \\ &=\; -\langle \alpha(Z, X_j), \alpha(Y, X_4)\rangle \;=\; -\langle \alpha(Y, X_j), \alpha(Z, X_4)\rangle \;=\; 0\,.\end{aligned}$$

□

Lemma 8 *If* $\mathfrak{l}$ *is admissible, then* $[X_3, W'] = 0$ *for* $W' := \mathrm{span}\{X_5, \ldots, X_l\}$.

Proof. Suppose $[X_3, W'] \neq 0$. Then we may assume that besides (17), (18), and (19) with $\lambda = 1$ our basis satisfies also

$$[X_3, X_5] = uY + vZ \neq 0\,. \tag{27}$$

Take $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_\sharp$ such that $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. From Lemma 7 we know that

$$\alpha(uY + vZ, \cdot) = 0\,, \tag{28}$$

which implies

$$\begin{aligned}
0 &= d\gamma(uY + vZ, X_1, X_2, X_3) &&= -\gamma(Y, uY + vZ, X_3) + \gamma(Z, uY + vZ, X_2) && (29)\\
0 &= d\gamma(uY + vZ, X_1, X_2, X_4) &&= -\gamma(Y, uY + vZ, X_4) - \gamma(Z, uY + vZ, X_1) && (30)\\
0 &= d\gamma(uY + vZ, X_1, X_2, X_k) &&= -\gamma(Y, uY + vZ, X_k),\ k \geq 5\,. && (31)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
d\alpha(X_1, X_2, X_3) &= -\alpha(Y, X_3) + \alpha(Z, X_2) &= 0\\
d\alpha(X_1, X_2, X_4) &= -\alpha(Y, X_4) - \alpha(Z, X_1) &= 0\,.
\end{aligned}$$

Let us first consider the case $v \neq 0$ in (27). Replacing X_3, X_4, X_5 and Z by

$$X_3' := vX_3 + uX_2,\ X_4' := vX_4 - uX_1,\ X_5' := (1/v) \cdot X_5,\ Z' := uY + vZ$$

we see that we may assume $u = 0$ and $v = 1$ in (27), i.e. $[X_3, X_5] = Z$. Then (28) says $\alpha(Z, \cdot) = 0$, hence Lemma 2 (*ii*) and the above equations for α imply $\alpha(Y, X_j) = 0$ for $j \geq 3$. Equations (29), (30), and (31) imply $\gamma(Y, Z, X_j) = 0$ for $j \geq 3$. Using all this we obtain

$$\gamma(Y, Z, X_i) = d\gamma(Y, X_i, X_3, X_5) = \langle \alpha(Y, X_i), \alpha(X_3, X_5) \rangle, \quad i = 1, 2.$$

In particular, the data $L_0 = Z$, $A_0 = -\alpha(X_3, X_5)$, and $Z_0 = 0$ satisfy the conditions (*i*) and (*ii*) of (A_1). Hence $Z = 0$ by admissibility, which is a contradiction.

If $v = 0$, then we may assume $u = 1$, i.e. $[X_3, X_5] = Y$. Then (28) implies $\alpha(Y, \cdot) = 0$. The above equations for α now give $\alpha(Z, X_j) = 0$ for $j = 1, 2$. Hence $\alpha(X_1, \mathfrak{l}) = \alpha(X_2, \mathfrak{l}) = 0$ and therefore

$$\begin{aligned}
0 &= d\alpha(X_2, X_4, X_3) = -\alpha(Z, X_3)\\
0 &= d\alpha(X_1, X_3, X_k) = -\alpha(Z, X_k), \quad k \geq 4.
\end{aligned}$$

This implies $\alpha(\mathfrak{l}', \mathfrak{l}) = 0$. From (29) and (30) we obtain $\gamma(Y, Z, X_j) = 0$ for $j = 1, 2$. Using this we get

$$2\gamma(Y, Z, X_j) = 2d\gamma(Z, X_1, X_2, X_j) = \langle \alpha \wedge \alpha \rangle (Z, X_1, X_2, X_j) = 0$$

for all $j \geq 3$. Hence $\gamma(\mathfrak{l}', \mathfrak{l}', \mathfrak{l}) = 0$. Again we obtain a contradiction to the admissibility condition (A_1). $\square$

Lemma 9 *If $\mathfrak{l}$ is admissible, then $[X_4, W'] = 0$.*

Proof. Recall that $X_1, \ldots, X_l, Y, Z$ is a basis of $\mathfrak{l}$ which satisfies (17), (18), (19) with $\lambda = 1$. Therefore the basis $X_1', \ldots, X_l', Y', Z'$ of $\mathfrak{l}$ defined by

$$X_1' := X_2,\ X_2' := X_1,\ X_3' := X_4,\ X_4' := X_3,\ X_j' := X_j,\ j \geq 5,\ Y' := -Y,\ Z' := Z$$

also satisfies (17), (18), (19) with $\lambda = 1$. Now Lemma 8 says that

$$[X_4, X_j] = [X_3', X_j'] = 0,\ j \geq 5.$$

□

Proof of the Claim. We know from Equations (17), (18), (19), Lemma 8 and Lemma 9 that $\mathfrak{l}$ is isomorphic to $\mathfrak{l}_1 \oplus \mathbb{R}^k$, where

$$\mathfrak{l}_1 = \{[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_2, X_4] = Z,\ [X_3, X_4] = aY + bZ\}$$

for suitable $a, b \in \mathbb{R}$.

Assume $a = 0$. Then the basis

$$X_1,\ X_2' := X_3 - bX_2,\ X_3' := X_2,\ X_4,\ Y' := Z - bY,\ Z' := Y$$

of $\mathfrak{l}_1$ together with a basis of $\mathbb{R}^k$ satisfies the conditions of case II.1 which contradicts our assumption on $\mathfrak{l}$. Hence $a \neq 0$.

Replacing X_2, X_4, Y by

$$X_2' := X_2 + (b/2a) \cdot X_3,\ X_4' := X_4 + (b/2) \cdot X_1,\ Y' := Y + (b/2a) \cdot Z$$

we obtain a basis $X_1, X_2', X_3, X_4', Y, Z$ of $\mathfrak{l}_1$ satisfying

$$[X_1, X_2'] = Y',\ [X_1, X_3] = Z,\ [X_1, X_4'] = [X_2', X_3] = 0,\ [X_2', X_4'] = \lambda' Z,\ [X_3, X_4'] = aY'$$

where $\lambda' = 1 + b^2/4a$. Since by assumption $\mathfrak{l}$ does not have a basis satisfying the conditions of case II.1 we have $\lambda' \neq 0$. Hence, we may obviously assume $\lambda' = 1$. Putting $\mu = \sqrt{|a|}$ and

$$X_1' = \mu X_1,\ X_3' = (1/\mu) \cdot X_3,\ \bar{Y} = \mu Y'$$

we obtain a basis $X_1', \ldots, X_4', \bar{Y}, Z$ of $\mathfrak{l}_1$ which satisfies

$$[X_1', X_2'] = \bar{Y},\ [X_1', X_3'] = Z,\ [X_1', X_4'] = [X_2', X_3'] = 0,\ [X_2', X_4'] = Z,\ [X_3', X_4'] = \epsilon \bar{Y}$$

with $\epsilon = \pm 1$. If $\epsilon = -1$, then $\mathfrak{l}_1 \cong \mathfrak{g}_{6,5}$. If $\epsilon = 1$, then

$$\mathfrak{l}_1 \cong \{[\tilde{X}_1, \tilde{X}_2] = \tilde{Y},\ [\tilde{X}_3, \tilde{X}_4] = \tilde{Z}\} \cong \mathfrak{h}(1) \oplus \mathfrak{h}(1)$$

for

$$\tilde{X}_1 = \tfrac{1}{2}(X_1' - X_4'),\ \tilde{X}_2 = \tfrac{1}{2}(X_2' + X_3'),\ \tilde{X}_3 = \tfrac{1}{2}(X_1' + X_4'),\ \tilde{X}_4 = \tfrac{1}{2}(X_2' - X_3'),$$
$$\tilde{Y} = \tfrac{1}{2}(\bar{Y} + Z),\ \tilde{Z} = \tfrac{1}{2}(\bar{Y} - Z)\,.$$

□

4 Nilpotent metric Lie algebras of dimension ≤ 10

Recall that a metric Lie algebra is called indecomposable if it is not the direct sum of two non-trivial metric Lie algebras (see also [KO 1]). In this section we will determine all indecomposable nilpotent metric Lie algebras of dimension ≤ 10 (up to isomorphisms).

Let us first consider the following construction. Let $\mathfrak{l}$ be a nilpotent Lie algebra and let $(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ be a pseudo-Euclidean vector space which we consider as a trivial orthogonal $\mathfrak{l}$-module. Let $\mathfrak{d}$ be the vector space $\mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$. Take $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \mathfrak{a})$ and define a bilinear map $[\cdot\,,\cdot] : \mathfrak{d} \times \mathfrak{d} \to \mathfrak{d}$ by

$$
\begin{aligned}
&[\mathfrak{l}^* \oplus \mathfrak{a}, \mathfrak{l}^* \oplus \mathfrak{a}] = 0, \;\; [\mathfrak{l}, \mathfrak{a}] = 0 \\
&[L, Z] = \mathrm{ad}^*(L)(Z) \\
&[A, L] = \langle A, \alpha(L, \cdot)\rangle \\
&[L_1, L_2] = \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1 L_2]_{\mathfrak{l}}
\end{aligned}
$$

for all $L, L_1, L_2 \in \mathfrak{l}$, $A \in \mathfrak{a}$, and $Z \in \mathfrak{l}^*$. Moreover we define an inner product $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{d}$ by

$$
\langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2\rangle \;\; := \;\; \langle A_1, A_2\rangle_{\mathfrak{a}} + Z_1(L_2) + Z_2(L_1)
$$

for $Z_1,\, Z_2 \in \mathfrak{l}^*$, $A_1,\, A_2 \in \mathfrak{a}$ and $L_1,\, L_2 \in \mathfrak{l}$. Then it is not hard to prove that $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}) := (\mathfrak{d}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$ is a nilpotent metric Lie algebra (see also [KO 2] for the case of a general metric Lie algebra).

Let $\mathfrak{l}_i$, $i = 1, 2$ be Lie algebras and let $\mathfrak{a}_i$, $i = 1, 2$ be pseudo-Euclidean vector spaces which we consider as trivial orthogonal $\mathfrak{l}_i$-modules. Consider a pair (S, U) consisting of a homomorphism $S : \mathfrak{l}_1 \to \mathfrak{l}_2$ and an isometry $U : \mathfrak{a}_2 \to \mathfrak{a}_1$. Then $(S, U)^* : C^p(\mathfrak{l}_2, \mathfrak{a}_2) \to C^p(\mathfrak{l}_1, \mathfrak{a}_1)$ induces a map $(S, U)^* : \mathcal{H}^p_Q(\mathfrak{l}_2, \mathfrak{a}_2) \to \mathcal{H}^p_Q(\mathfrak{l}_1, \mathfrak{a}_1)$.

In particular, $G_{(\mathfrak{l},\mathfrak{a})} := \mathrm{Aut}(\mathfrak{l}) \times O(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ acts on $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$.

Definition 2 *Let $\mathfrak{l}$ be a nilpotent Lie algebra and let $(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ be a pseudo-Euclidean vector space considered as a trivial $\mathfrak{l}$-module. A cohomology class $\varphi \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ is called decomposable if there are decompositions $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ and $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$, at least one of them being non-trivial and cohomology classes $\varphi_i \in \mathcal{H}^2_Q(\mathfrak{l}_i, \mathfrak{a}_i)$, $i = 1, 2$ such that $\varphi = (q_1, j_1)^*\varphi_1 + (q_2, j_2)^*\varphi_2$, where $q_i : \mathfrak{l} \to \mathfrak{l}_i$ are the projections and $j_i : \mathfrak{a}_i \to \mathfrak{a}$ are the inclusions. Here we consider $\mathfrak{a}_i$ as trivial $\mathfrak{l}_i$-modules. We denote the subset of indecomposable admissible cohomology classes in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ by $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$.*

One can check easily that $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$ is invariant with respect to the action of $G_{(\mathfrak{l},\mathfrak{a})}$ on $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$. The classification scheme (1) now gives

Proposition 3 *The set of isomorphism classes of nilpotent metric Lie algebras of dimension at most 10 is in bijective correspondence with*

$$
\bigcup_{\mathfrak{l} \in \mathfrak{L}} \bigcup_{\mathfrak{a} \in \mathfrak{A}_{\mathfrak{l}}} \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0 / G_{(\mathfrak{l},\mathfrak{a})},
$$

where $\mathfrak{L}$ is the set of isomorphism classes of nilpotent Lie algebras of dimension at most 5 and for a fixed $\mathfrak{l} \in \mathfrak{L}$ the set $\mathfrak{A}_{\mathfrak{l}}$ consists of all isometry classes of pseudo-Euclidean vector spaces of dimension at most $10 - 2\dim\mathfrak{l}$ which we consider as equivalence classes of trivial orthogonal $\mathfrak{l}$-modules.

In the following we will often abbreviate $G_{(\mathfrak{l},\mathfrak{a})}$ to G. Furthermore, we will use the following conventions. An orthonormal basis of a pseudo-Euclidean vector space $(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ is a basis $A_1, \ldots, A_{p+q}$ consisting of pairwise orthogonal vectors satisfying $\langle A_i, A_i\rangle_{\mathfrak{a}} = -1$ for $1 \le i \le p$ and $\langle A_i, A_i\rangle_{\mathfrak{a}} = 1$ for $p+1 \le i \le p+q$. The pair (p, q) is called signature of $\mathfrak{a}$. We denote the standard pseudo-Euclidean vector space of signature (p, q) by $\mathbb{R}^{p,q}$. A Witt basis of $\mathbb{R}^{1,1}$ is a basis A_1, A_2, where A_1, A_2 are isotropic and $\langle A_1, A_2\rangle = 1$.

Proposition 4 *If $\mathfrak{l}$ is nilpotent and if $\dim\mathfrak{l} = 5$ and $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 \neq \emptyset$ implies $\mathfrak{l} = \mathbb{R}^5$ or $\mathfrak{l} = \mathfrak{g}_{5,2}$.*

Proof. Let $[0,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ be such that $\gamma \neq 0$. Then we know from (A_k) that $\dim\mathfrak{l}^{k+1} \neq 1$ holds for all $k \ge 0$. Since $\mathfrak{l}$ is nilpotent the codimension of $\mathfrak{l}^2$ in $\mathfrak{l}$ cannot be 1, since otherwise $\mathfrak{l}^3 = [\mathfrak{l},\mathfrak{l}^2] = [\mathfrak{l},\mathfrak{l}] = \mathfrak{l}^2$ yields a contradiction. Hence we have only the following possibilities:

(i) $\dim\mathfrak{l}^2 = 0$,

(ii) $\dim\mathfrak{l}^2 = 2, \dim\mathfrak{l}^3 = 0$,

(iii) $\dim\mathfrak{l}^2 = 3, \dim\mathfrak{l}^3 = 0$, or

(iv) $\dim\mathfrak{l}^2 = 3, \dim\mathfrak{l}^3 = 2, \dim\mathfrak{l}^4 = 0$.

If (i) holds, then $\mathfrak{l} \cong \mathbb{R}^5$. If (ii) holds, then $\mathfrak{l} \cong \mathfrak{g}_{5,2}$ by Proposition 2. The conditions in (iii) cannot be satisfied for a 5-dimensional Lie algebra $\mathfrak{l}$, since in this case $\mathfrak{l}' \subset \mathfrak{z}(\mathfrak{l})$, thus $\dim\mathfrak{z} \ge 3$ and therefore $\dim\mathfrak{l}' \le 1$, which contradicts $\dim\mathfrak{l}' = 3$.

Now assume that (iv) holds. Choose linear independent vectors X_1, X_2 in $\mathfrak{l} \setminus \mathfrak{l}'$. Then $X_3 := [X_1, X_2] \notin \mathfrak{l}^3$. For $X_4 := [X_1, X_3]$ and $X_5 := [X_2, X_3]$ we now have $\mathfrak{l}^3 = [\mathfrak{l}',\mathfrak{l}] = [X_3 + \mathfrak{l}^3, \mathfrak{l}] = [X_3, \mathfrak{l}] = \mathrm{span}\{X_4, X_5\}$. Hence, $X_1, \ldots, X_5$ is a basis of $\mathfrak{l}$. Since $X_4, X_5 \in \mathfrak{l}^3$ are central we obtain

$$\begin{array}{rclcrcl} 0 &=& d\gamma(X_1, X_2, X_3, X_4) &=& \gamma(X_1, [X_2, X_3], X_4) &=& \gamma(X_1, X_5, X_4) \\ 0 &=& d\gamma(X_1, X_2, X_3, X_5) &=& \gamma([X_1, X_3], X_2, X_5) &=& \gamma(X_4, X_2, X_5) \\ 0 &=& d\gamma(X_1, X_2, X_4, X_5) &=& -\gamma([X_1, X_2], X_4, X_5) &=& -\gamma(X_3, X_4, X_5)\ , \end{array}$$

thus $\gamma(X_4, X_5, \cdot) = 0$, which contradicts Condition (A_2). $\square$

Proposition 5 *1. If $\mathfrak{l} = \mathbb{R}^5$ and $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$ consists of one element. This element is represented by $[0,\gamma_0] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$, where $\gamma_0 = (\sigma^1\wedge\sigma^2 + \sigma^3\wedge\sigma^4)\wedge\sigma^5$ for a fixed basis $\sigma^1, \ldots, \sigma^5$ of $\mathfrak{l}^*$.*

2. *If $\mathfrak{l} = \mathfrak{g}_{5,2} = \{[X_1, X_2] = Y,\ [X_1, X_3] = Z\}$ and $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ consists of two elements. These elements are represented by $[0, \gamma_1], [0, \gamma_2] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$, where $\gamma_1 = \sigma^1 \wedge \sigma^Y \wedge \sigma^Z$ and $\gamma_2 = \sigma^1 \wedge \sigma^Y \wedge \sigma^Z + \sigma^2 \wedge \sigma^3 \wedge \sigma^Z$ for the basis $\sigma^1, \sigma^2, \sigma^3, \sigma^Y, \sigma^Z$ of $\mathfrak{l}^*$ which is dual to X_1, X_2, X_3, Y, Z.*

Proof. The statement for $\mathfrak{l} = \mathbb{R}^5$ is easy to prove. Take $\mathfrak{l} = \mathfrak{g}_{5,2}$. For $c \in \mathbb{R} \setminus 0$, $A \in GL(2, \mathbb{R})$, $y = (y_1, y_2)$, $y_1, y_2 \in \mathbb{R}^2$, $x \in \mathfrak{gl}(2, \mathbb{R})$ we define a linear map $S(c, A, x, y) : \mathfrak{g}_{5,2} \to \mathfrak{g}_{5,2}$ by

$$S(c, A, x, y) = \begin{pmatrix} c & 0 & 0 \\ y_1 & A & 0 \\ y_2 & x & cA \end{pmatrix}$$

with respect to the basis X_1, X_2, X_3, Y, Z of $\mathfrak{g}_{5,2}$. Using that $\mathfrak{q} := \mathrm{span}\{X_2, X_3, Y, Z\}$ is the unique 4-dimensional abelian ideal of $\mathfrak{g}_{5,2}$ and that $\mathrm{span}\{Y, Z\}$ is the centre of $\mathfrak{g}_{5,2}$ it is not hard to show that the automorphism group of $\mathfrak{g}_{5,2}$ equals

$$\mathrm{Aut}(\mathfrak{g}_{5,2}) = \{S(c, A, x, y) \mid c \in \mathbb{R} \setminus 0, A \in GL(2, \mathbb{R}), y_1, y_2 \in \mathbb{R}^2, x \in \mathfrak{gl}(2, \mathbb{R})\}\,.$$

Obviously, we have $\mathcal{H}^2_Q(\mathfrak{l}, 0) = H^3(\mathfrak{g}_{5,2})$. Using the Hochschild-Serre spectral sequence we see that $H^3(\mathfrak{g}_{5,2})$ is determined by the exact sequence

$$0 \longrightarrow H^1(\mathbb{R} \cdot X_1, H^2(\mathfrak{q})) \longrightarrow H^3(\mathfrak{g}_{5,2}) \longrightarrow H^0(\mathbb{R} \cdot X_1, H^3(\mathfrak{q})) \longrightarrow 0\,.$$

We have

$$H^1(\mathbb{R} \cdot X_1, H^2(\mathfrak{q})) = C^1(\mathbb{R} \cdot X_1, C^2(\mathfrak{q}))/B^1(\mathbb{R} \cdot X_1, C^2(\mathfrak{q})),$$

where

$$B^1(\mathbb{R} \cdot X_1, C^2(\mathfrak{q})) = \left\{ \sigma \in C^1(\mathbb{R} \cdot X_1, C^2(\mathfrak{q})) \;\middle|\; \begin{array}{l} \sigma(X_1)(X_2, Z) + \sigma(X_1)(X_3, Y) = 0 \\ \sigma(X_1)(X_2, Y) = \sigma(X_1)(X_3, Z) = 0 \\ \sigma(X_1)(Y, Z) = 0 \end{array} \right\},$$

and

$$H^0(\mathbb{R} \cdot X_1, H^3(\mathfrak{q})) = C^3(\mathfrak{q})^{X_1} = \{\sigma \in C^3(\mathfrak{q}) \mid \sigma(X_2, Y, Z) = \sigma(X_3, Y, Z) = 0\}\,. \quad (32)$$

Observe that $\mathcal{H}^2_Q(\mathfrak{l}, 0)_0 = \mathcal{H}^2_Q(\mathfrak{l}, 0)_\sharp$ since $\mathfrak{l}$ is not the direct sum of two non-trivial Lie algebras. In particular, Condition (A_1) and Equation (32) imply

$$\mathcal{H}^2_Q(\mathfrak{l}, 0)_0 = \{[\gamma] \in H^3(\mathfrak{g}_{5,2}) \mid \gamma(X_1, Y, Z) \neq 0\}.$$

Using the description of $H^1(\mathbb{R} \cdot X_1, H^2(\mathfrak{q}))$ given above we see that

$$\{S(c, \mathrm{Id}, x, 0) \mid c \in \mathbb{R} \setminus 0, x \in \mathfrak{gl}(2, \mathbb{R})\} \subset \mathrm{Aut}(\mathfrak{g}_{5,2})$$

acts transitively on $\{[\sigma] \in H^1(\mathbb{R} \cdot X_1, H^2(\mathfrak{q})) \mid \sigma(X_1)(Y, Z) \neq 0\}$. Furthermore, using the description of $H^0(\mathbb{R} \cdot X_1, H^3(\mathfrak{q}))$ we see that the action of

$$\{S(1, A, 0, 0) \mid \det A = 1\} \subset \mathrm{Aut}(\mathfrak{g}_{5,2})$$

on $H^0(\mathbb{R} \cdot X_1, H^3(\mathfrak{q}))$ has two orbits represented by $\sigma_1 = 0$ and $\sigma_2 = \sigma^2 \wedge \sigma^3 \wedge \sigma^Z$. Moreover, this group leaves $\sigma^1 \wedge \sigma^Y \wedge \sigma^Z$ invariant.

It is easy to check that the orbits of $[0, \gamma_1]$ and $[0, \gamma_2]$ are different. □

Proposition 6 *Take $\mathfrak{l} = \mathfrak{g}_{4,1} = \{[X_1, Z] = Y,\ [X_1, X_2] = Z\}$ and let $\mathfrak{a}$ be a trivial $\mathfrak{l}$-module. If $\mathfrak{a} = 0$ or $\dim \mathfrak{a} \geq 3$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0 = \emptyset$. If $\dim \mathfrak{a} = 1$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ consists of four elements. They are represented by*

$$[\alpha, \gamma] = [\sigma^1 \wedge \sigma^Y \otimes A, r\sigma^2 \wedge \sigma^Y \wedge \sigma^Z + s\sigma^1 \wedge \sigma^Y \wedge \sigma^Z],$$

where $(r, s) \in \{(0, 0),\ (1, 0),\ (0, 1),\ (0, -1)\}$ and A is a fixed unit vector in $\mathfrak{a}$.

If $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ consists of two one-parameter families. They are represented by

$$[\alpha, \gamma] = [\sigma^1 \wedge \sigma^Y \otimes A_1 + \sigma^2 \wedge \sigma^Z \otimes A_2, s\sigma^1 \wedge \sigma^Y \wedge \sigma^Z], \quad s \in \mathbb{R} \tag{33}$$

and

$$[\alpha, \gamma] = [\sigma^1 \wedge \sigma^Y \otimes A_1 + \sigma^2 \wedge \sigma^Z \otimes A_2, r\sigma^2 \wedge \sigma^Y \wedge \sigma^Z], \quad r \in \mathbb{R}_+, \tag{34}$$

where A_1, A_2 is a fixed orthonormal basis in $\mathfrak{a}$.

If $\mathfrak{a} = \mathbb{R}^{1,1}$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ consists of four one-parameter families. They are represented also by (33) and (34), but where now either A_1, A_2 or A_2, A_1 is an orthonormal basis.

Proof. Let us first determine the automorphism group of $\mathfrak{l}$. For $a, b, c \in \mathbb{R}$ and $x = (x_1, \ldots, x_4) \in \mathbb{R}^4$ we define a linear map $S(a, b, c, x) : \mathfrak{l} \to \mathfrak{l}$ by

$$S(a, b, c, x) = \begin{pmatrix} a & 0 & 0 & 0 \\ x_1 & b & 0 & 0 \\ x_2 & c & ab & 0 \\ x_3 & x_4 & ac & a^2 b \end{pmatrix}$$

with respect to the basis X_1, X_2, Z, Y of $\mathfrak{l}$. Using that the unique 3-dimensional abelian ideal $\mathfrak{q} = \mathrm{span}\{X_2, Y, Z\}$ of $\mathfrak{l}$, $\mathfrak{l}' = \mathrm{span}\{Y, Z\}$ and $\mathfrak{l}^3 = \mathbb{R} \cdot Y$ are invariant under each automorphism of $\mathfrak{l}$ it is not hard to check that the automorphism group of $\mathfrak{g}_{4,1}$ equals

$$\mathrm{Aut}(\mathfrak{g}_{4,1}) = \{S(a, b, c, x) \mid a, b \in \mathbb{R} \setminus 0, c \in \mathbb{R}, x \in \mathbb{R}^4\}.$$

The cohomology group $H^2(\mathfrak{g}_{4,1}, \mathfrak{a})$ is determined by the exact sequence

$$0 \longrightarrow H^1(\mathbb{R} \cdot X_1, H^1(\mathfrak{q}, \mathfrak{a})) \longrightarrow H^2(\mathfrak{g}_{4,1}, \mathfrak{a}) \longrightarrow H^0(\mathbb{R} \cdot X_1, H^2(\mathfrak{q}, \mathfrak{a})) \longrightarrow 0.$$

We have

$$H^1(\mathbb{R} \cdot X_1, H^1(\mathfrak{q}, \mathfrak{a})) = C^1(\mathbb{R} \cdot X_1, C^1(\mathfrak{q}, \mathfrak{a}))/B^1(\mathbb{R} \cdot X_1, C^1(\mathfrak{q}, \mathfrak{a})),$$

where

$$B^1(\mathbb{R} \cdot X_1, C^1(\mathfrak{q}, \mathfrak{a})) = \{\sigma \in C^1(\mathbb{R} \cdot X_1, C^1(\mathfrak{q}, \mathfrak{a})) \mid \sigma(X_1)(Y) = 0\},$$

and

$$H^0(\mathbb{R} \cdot X_1, H^2(\mathfrak{q}, \mathfrak{a})) \quad = \quad C^2(\mathfrak{q}, \mathfrak{a})^{X_1} = \{\sigma \in C^2(\mathfrak{q}, \mathfrak{a}) \mid \sigma(X_2, Y) = \sigma(Y, Z) = 0\}.$$

In particular, (A_2) implies $\alpha(Y,X_1)\neq 0$. If $\alpha(Y,X_1)\neq 0$, then also (A_0) and (A_1) are satisfied. Since $d\gamma=0$ for all $\gamma\in C^3(\mathfrak{l})$ the equation $2d\gamma=\langle\alpha\wedge\alpha\rangle$ holds if and only if $\alpha(Y,X_1)\perp\alpha(X_2,Z)$. Hence we obtain

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0=\left\{[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})\;\middle|\;\begin{array}{l}\alpha=(\sigma^1\wedge\sigma^Y)\otimes A_1+(\sigma^2\wedge\sigma^Z)\otimes A_2,\\ \mathfrak{a}=\mathrm{span}\{A_1,A_2\},\ A_1\perp A_2,\ A_1\neq 0\end{array}\right\}.$$

Because of $B^3(\mathfrak{l})=\{\gamma\in C^3(\mathfrak{l})\mid\gamma(X_1,Y,Z)=\gamma(X_2,Y,Z)=0\}$ we may assume that

$$\gamma=r\sigma^2\wedge\sigma^Y\wedge\sigma^Z+s\sigma^1\wedge\sigma^Y\wedge\sigma^Z.$$

Now let us assume that $\dim\mathfrak{a}=1$. Take $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$, $\alpha=(\sigma^1\wedge\sigma^Y)\otimes A_1$, $u:=\langle A_1,A_1\rangle\neq 0$. Suppose $r\neq 0$. Choose $a,b\in\mathbb{R}$ such that $a^3b^3=r^{-1}$ and either $a^3b=|u|^{-1/2}$ or $a^3b=-|u|^{-1/2}$ holds. Applying $S(a,b,0,x)$ for $x=(-s/r,0,0,0)$ to $[\alpha,\gamma]$ we see that we may assume $u=\pm1$, $r=1$ and $s=0$ without changing the G-orbit. Now suppose $r=0$, $s\neq 0$. Choose $a,b\in\mathbb{R}$ such that either $a^4b^2=s^{-1}$ or $a^4b^2=-s^{-1}$ and $a^3b=|u|^{-1/2}$ holds. Applying $S(a,b,0,0)$ to $[\alpha,\gamma]$ we see that here we can achieve $u=\pm1$ and $s=\pm1$ without changing the G-orbit. The four orbits which correspond to $(r,s)\in\{(0,0),(1,0),(0,1),(0,-1)\}$ are pairwise different.

Now assume that $\dim\mathfrak{a}=2$. Take $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$,

$$\alpha=(\sigma^1\wedge\sigma^Y)\otimes A_1+(\sigma^2\wedge\sigma^Z)\otimes A_2,\quad \gamma=r\sigma^2\wedge\sigma^Y\wedge\sigma^Z+s\sigma^1\wedge\sigma^Y\wedge\sigma^Z.$$

As above we see that we may assume that A_1,A_2 are orthogonal, $\langle A_i,A_i\rangle=\pm1$, $i=1,2$, and that $r=0$ or $s=0$. Moreover, if $s=0$, then we may assume $r\geq 0$. All orbits for $(r,s)\in(\{0\}\times\mathbb{R})\cup(\mathbb{R}_+\times\{0\})$ are different. $\square$

Now let $\mathfrak{l}$ be one of the Lie algebras $\mathfrak{h}(1)\oplus\mathbb{R}=\{[X_1,X_2]=X_3\}\oplus\mathbb{R}\cdot X_4$ or $\mathbb{R}^4=\mathrm{span}\{X_1,\dots,X_4\}$. Let $\sigma^1,\dots,\sigma^4$ be a basis of $\mathfrak{l}^*$ which is dual to $X_1,\dots,X_4$. Let $A_1,A_2,\dots$ be a basis of a vector space $\mathfrak{a}$. We define the following 2-forms

$$\begin{array}{rcl}
\alpha_1&=&(\sigma^1\wedge\sigma^3+\sigma^2\wedge\sigma^4)\otimes A_1+(\sigma^2\wedge\sigma^3+\sigma^1\wedge\sigma^4)\otimes A_2\\
\alpha_2&=&(\sigma^1\wedge\sigma^3-\sigma^2\wedge\sigma^4)\otimes A_1+(\sigma^2\wedge\sigma^3+\sigma^1\wedge\sigma^4)\otimes A_2\\
\alpha_3&=&(\sigma^1\wedge\sigma^3)\otimes A_1+(\sigma^2\wedge\sigma^3+\sigma^1\wedge\sigma^4)\otimes A_2\\
\alpha_4&=&(\sigma^1\wedge\sigma^3)\otimes A_1+(\sigma^2\wedge\sigma^3)\otimes A_2\\
\alpha_5&=&(\sigma^1\wedge\sigma^3)\otimes A_1+(\sigma^1\wedge\sigma^4)\otimes A_2,\quad \alpha_5'=(\sigma^1\wedge\sigma^4)\otimes A_1+(\sigma^1\wedge\sigma^3)\otimes A_2\\
\alpha_6&=&(\sigma^1\wedge\sigma^3)\otimes A_1+(\sigma^2\wedge\sigma^4)\otimes A_2,\quad \alpha_6'=(\sigma^2\wedge\sigma^4)\otimes A_1+(\sigma^1\wedge\sigma^3)\otimes A_2\\
\alpha_7&=&(\sigma^1\wedge\sigma^3)\otimes A_1.
\end{array}$$

Moreover, we define the 3-form γ_0 on $\mathfrak{l}$ by $\gamma_0=\sigma^2\wedge\sigma^3\wedge\sigma^4$.

Proposition 7 *Take $\mathfrak{l}=\mathfrak{h}(1)\oplus\mathbb{R}=\{[X_1,X_2]=X_3\}\oplus\mathbb{R}\cdot X_4$. Let $\mathfrak{a}$ be a trivial orthogonal $\mathfrak{l}$-module. If $\mathfrak{a}=\mathbb{R}^2$ or $\mathfrak{a}=\mathbb{R}^{2,0}$, then the elements in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$ are represented by $[\alpha_1,0]$, $[\alpha_5,0]$, $[\alpha_5,\gamma_0]$, $[\alpha_6,0]$, $[\alpha_6,\gamma_0]$, where A_1,A_2 is a fixed orthonormal basis of $\mathfrak{a}$.*

If $\mathfrak{a} = \mathbb{R}^{1,1}$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$ has eleven elements, three of them are represented by $[\alpha_1, 0]$, $[\alpha_2, 0]$, $[\alpha_3, 0]$, where A_1, A_2 is a fixed Witt basis of $\mathfrak{a}$, eight further elements are represented by $[\alpha_5, 0]$, $[\alpha_5, \gamma_0]$, $[\alpha_6, 0]$, $[\alpha_6, \gamma_0]$, $[\alpha'_5, 0]$, $[\alpha'_5, \gamma_0]$, $[\alpha'_6, 0]$, $[\alpha'_6, \gamma_0]$, where A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$.

If $\mathfrak{a} = \mathbb{R}^1$ or $\mathfrak{a} = \mathbb{R}^{1,0}$, then there is only one element in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$. It is represented by $[\alpha_7, \gamma_0]$, where A_1 is a fixed unit vector in $\mathfrak{a}$.

If $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 = \emptyset$.

Proof. For $A, X \in \mathfrak{gl}(2,\mathbb{R})$ and $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ we define

$$S(A, X, u) = \begin{pmatrix} A & 0 \\ X & U \end{pmatrix} \in \mathfrak{gl}(4,\mathbb{R}), \text{ where } U = \begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{R}).$$

Then the automorphism group of $\mathfrak{l} = \mathfrak{h}(1) \oplus \mathbb{R}$ equals

$$\mathrm{Aut}(\mathfrak{l}) = \Big\{ S(A, X, u) \mid u \in \mathbb{R}^3,\ X \in \mathfrak{gl}(2,\mathbb{R}),\ A \in GL(2,\mathbb{R}),\ \det A = u_1,\ u_3 \neq 0 \Big\},$$

where we consider all automorphisms with respect to the basis $X_1, \ldots, X_4$ of $\mathfrak{l}$.

By direct computations or using the Künneth formula and the explicit description of $H^2(\mathfrak{h}(1), \mathfrak{a})$ in [KO 2], we see that

$$\begin{array}{rcl} Z_{\mathfrak{l}} := \{\alpha \in C^2(\mathfrak{l},\mathfrak{a}) \mid \alpha(X_1, X_2) = \alpha(X_3, X_4) = 0\} & \longrightarrow & H^2(\mathfrak{l},\mathfrak{a}) \\ \alpha & \longmapsto & [\alpha] \end{array}$$

is a bijection.

Now take $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$, $\alpha \in Z_{\mathfrak{l}}$. Since, obviously, $d\gamma = 0$ we have $\langle \alpha \wedge \alpha \rangle = 0$. Condition (A_1) gives $\alpha(X_3, \mathfrak{l}) \neq 0$ and Condition (B_1) says that $\alpha(X_3, \mathfrak{l})$ is non-degenerate. By indecomposability we have $\alpha(\mathfrak{l},\mathfrak{l}) = \mathfrak{a}$. Hence α is an element of the G-invariant subset $C \subset Z_{\mathfrak{l}}$ defined by

$$C := \{\alpha \in Z_{\mathfrak{l}} \mid \langle \alpha \wedge \alpha \rangle = 0,\ \alpha(\mathfrak{l},\mathfrak{l}) = \mathfrak{a},\ 0 \neq \alpha(X_3, \mathfrak{l}) \subset \mathfrak{a} \text{ is non-degenerate}\}.$$

A cocycle $\alpha \in Z_{\mathfrak{l}}$ satisfies $\langle \alpha \wedge \alpha \rangle = 0$ if and only if

$$\langle \alpha(X_1, X_3), \alpha(X_2, X_4) \rangle = \langle \alpha(X_2, X_3), \alpha(X_1, X_4) \rangle. \tag{35}$$

Let us determine the G-orbits in C in the case that $\dim \mathfrak{a} \leq 2$. Take $\alpha \in C$. In particular we have $\dim \alpha(X_3, \mathfrak{l}) = 1$ or $\dim \alpha(X_3, \mathfrak{l}) = 2$.

Let us first consider the case $\dim \alpha(X_3, \mathfrak{l}) = 1$. Replacing α by an element in the G-orbit of α we may assume $\alpha(X_1, X_3) = A_1$ and $\alpha(X_2, X_3) = 0$, where $\langle A_1, A_1 \rangle = \pm 1$. From (35) we obtain $\langle \alpha(X_1, X_3), \alpha(X_2, X_4) \rangle = 0$. Hence either $\alpha(X_2, X_4) = 0$ or $\alpha(X_2, X_4) = A_2 \neq 0$ and $A_1 \perp A_2$. In the latter case A_2 cannot be isotropic since $\mathfrak{a}$ is at most two-dimensional. Hence, replacing α by an element in the same G-orbit we may assume that $\langle A_2, A_2 \rangle = \pm 1$ and that $\alpha(X_1, X_4) = 0$, hence α is in the same G-orbit as α_6 or as α'_6 for an orthonormal basis A_1, A_2. In the first case, where $\alpha(X_2, X_4) = 0$ we may assume that $\alpha(X_1, X_4) = 0$ or $\alpha(X_1, X_4) = A_2 \neq 0$ and $A_1 \perp A_2$, $\langle A_2, A_2\rangle = \pm 1$, hence α is in the same orbit as α_5 or α_5' for an orthonormal basis A_1, A_2 or as α_7 for a unit vector A_1. Obviously, the orbit of α_7 contains neither α_5, α_5', α_6, nor α_6'. Also the orbits of α_5 and α_6 are different. Indeed, $\alpha_5(X_2, \mathfrak{l}) = 0$ and $\alpha_6(L, \mathfrak{l}) \neq 0$ for all $L \in \mathfrak{l}, L \neq 0$. Analogously, the orbits of α_5' and α_6' are different. Moreover, α_i and α_i', $i = 5, 6$, are not on the same orbit, since X_3 plays a distinguished role in $\mathfrak{l}$.

Now we consider the case $\dim \alpha(X_3, \mathfrak{l}) = 2$. Then $\alpha(X_1, X_3) =: A_1$ and $\alpha(X_2, X_3) =: A_2$ are linearly independent. First we show that we may assume that A_1, A_2 is an orthonormal basis of $\mathfrak{a}$ if $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$ and that A_1, A_2 is a Witt basis if $\mathfrak{a} = \mathbb{R}^{1,1}$ (replacing α by an element in the same G-orbit). Clearly, we can choose $X_1', X_2' \in \mathrm{span}\{X_1, X_2\}$ such that $\alpha(X_1', X_3)$, $\alpha(X_2', X_3)$ is an orthonormal basis or a Witt basis, respectively. We have $[X_1', X_2'] = rX_3$, $r \in \mathbb{R}$, $r \neq 0$. We choose $s \in \mathbb{R}$ such that $s^3 = r$ and define $\bar{X}_1 = (1/s)X_1'$, $\bar{X}_2 = (1/s)X_2'$, and $\bar{X}_3 = sX_3$. Take $\bar{X}_1, \bar{X}_2, \bar{X}_3, X_4$ as a new basis for $\mathfrak{l}$. Hence, we may assume that A_1, A_2 are as claimed. By (35) we have

$$\langle A_1, \alpha(X_2, X_4)\rangle = \langle A_2, \alpha(X_1, X_4)\rangle. \tag{36}$$

Replacing X_4 by a suitable linear combination of X_4 and X_3 we may assume that $\alpha(X_2, X_4)$ is a multiple of A_1.

Assume that $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$. If $\alpha(X_2, X_4) = 0$, then (36) implies that $\alpha(X_1, X_4)$ is a multiple of A_1. Hence we may assume that either $\alpha(X_1, X_4) = 0$ or that $\alpha(X_1, X_4) = A_1$. Consequently, α is in the same orbit as α_4 or as the 2-form

$$\alpha_1' := (\sigma^1 \wedge \sigma^3 + \sigma^1 \wedge \sigma^4) \otimes A_1 + (\sigma^2 \wedge \sigma^3) \otimes A_2,$$

which is in the same orbit as α_1. Indeed, we have $U^{-1} \circ S^*\alpha_1 = \alpha_1'$ for U, $S = S(A, 0, u)$ with

$$U = A = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad u = (1,\ 1/2,\ 1/2),$$

where we take U with respect to the basis A_1, A_2 of $\mathfrak{a}$.

If $\alpha(X_2, X_4) = rA_1$, $r \neq 0$, then rescaling X_4 we may assume that $\alpha(X_2, X_4) = A_1$. Now (36) yields $\alpha(X_1, X_4) = A_2 + sA_1$. We will show that we may assume $s = 0$. For $s \in \mathbb{R}$ we choose $t \in \mathbb{R}$ such that $s = 2\tan 2t$ and we define

$$a = \sin t,\ b = \cos t,\ u_2 = \sin 2t,\ u_3 = -\cos 2t$$

and

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},\ X = 0 \in \mathfrak{gl}(2, \mathbb{R}),\ u = (1, u_2, u_3) \in \mathbb{R}^3.$$

For this choice of A, X, and u we consider $S = S(A, X, u) \in \mathrm{Aut}(\mathfrak{l})$ and we define $X_i' = SX_i$, $i = 1, \ldots, 4$. Then we have

$$\begin{aligned}
\alpha(X_1', X_3') &= aA_1 + bA_2 =: A_1' \\
\alpha(X_2', X_3') &= -bA_1 + aA_2 =: A_2' \\
\alpha(X_1', X_4') &= (u_2a + au_3s + bu_3)A_1 + (u_2b + au_3)A_2 = A_2' \\
\alpha(X_2', X_4') &= (-u_2b - sbu_3 + au_3)A_1 + (u_2a - bu_3)A_2 = A_1',
\end{aligned}$$

where A_1', A_2' is again an orthonormal basis. Hence, α is in the same orbit as α_1. The 2-forms α_1 and α_4 are on different orbits, since $\alpha_4(X_4, \mathfrak{l}) = 0$ and $\alpha_1(L, \mathfrak{l}) \neq 0$ for all $L \in \mathfrak{l}, L \neq 0$.

Take now $\mathfrak{a} = \mathbb{R}^{1,1}$. Recall that we may assume $\alpha(X_1, X_3) = A_1$ and $\alpha(X_2, X_3) = A_2$ such that A_1, A_2 is a Witt basis and that $\alpha(X_2, X_4) = rA_1$, $r \in \{0, 1\}$. From (36) we get $\alpha(X_1, X_4) = sA_2$ for a real number s. If $r = s = 0$, then α is in the same orbit as α_4. If $r = 0$, $s = 1$ or $r = 1$, $s = 0$, then α is in the same orbit as α_3. If $r = 1$, $s \neq 0$, then we put $x = |s|^{-1/4}$ and $v = (\operatorname{sgn} s) \cdot |s|^{-1/2}$. We define $S = \operatorname{diag}(x, x^{-1}, 1, v) \in \operatorname{Aut}(\mathfrak{l})$ and $U = \operatorname{diag}(x^{-1}, x)$. Then $(S, U)^*\alpha$ equals α_1 or α_2. The 2-forms $\alpha_1, \ldots, \alpha_4$ are on different orbits, since the elements of its orbits differ in the properties of their projections to the isotropic lines in $\mathfrak{a} = \mathbb{R}^{1,1}$.

We can summarize this as follows. If $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$, then there are four G-orbits in C represented by $\alpha_1, \alpha_4, \alpha_5, \alpha_6$, where A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$. If $\mathfrak{a} = \mathbb{R}^{1,1}$, then there are eight G-orbits in C, four of them are represented by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where A_1, A_2 is a fixed Witt basis of $\mathfrak{a}$, four further orbits are represented by $\alpha_5, \alpha_5', \alpha_6, \alpha_6'$, where now A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$. If $\mathfrak{a} = \mathbb{R}^1$ or $\mathfrak{a} = \mathbb{R}^{1,0}$, then $\alpha_7 \in C$ and G acts transitively on C.

Since $Z^1(\mathfrak{l}, \mathfrak{a}) = \{\tau \in C^1(\mathfrak{l}, \mathfrak{a}) \mid d\tau = 0\} = \{\tau \in C^1(\mathfrak{l}, \mathfrak{a}) \mid \tau(X_3) = 0\}$ we have $\langle \alpha_i \wedge Z^1(\mathfrak{l}, \mathfrak{a})\rangle = C^3(\mathfrak{l})$ for $i = 1, \ldots, 3$. For α_4 we have $B^3(\mathfrak{l}) + \langle \alpha_4 \wedge Z^1(\mathfrak{l}, \mathfrak{a})\rangle = C^3(\mathfrak{l})$, where $B^3(\mathfrak{l}) = \{d\sigma \mid \sigma \in C^2(\mathfrak{l})\} = \mathbb{R} \cdot \sigma^1 \wedge \sigma^2 \wedge \sigma^4$. Hence $[\alpha_i, \gamma] = [\alpha_i, 0] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ for $i = 1, \ldots, 4$ and all $[\alpha_i, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$. Note that $[\alpha_4, 0]$ is decomposable.

If $\alpha \in \{\alpha_5, \alpha_5', \alpha_6, \alpha_6', \alpha_7\}$, then γ_0 spans a complement of $B^3(\mathfrak{l}) + \langle \alpha \wedge Z^1(\mathfrak{l}, \mathfrak{a})\rangle$ in $C^3(\mathfrak{l})$. Hence, for all these α and for all $\gamma \in C^3(\mathfrak{l})$ there exists a real number c such that $[\alpha, \gamma] = [\alpha, c\gamma_0] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$. Let us first determine the G-orbit of $[\alpha_i, c\gamma_0]$ for $i = 6, 7$. If $c \neq 0$, then $S := \operatorname{diag}(c, 1/c^2, 1/c, c^2) \in \operatorname{Aut}(\mathfrak{l})$ and we have $(S, \mathrm{Id})^*[\alpha_i, c\gamma_0] = [\alpha_i, \gamma_0]$. Obviously the orbits of $[\alpha_i, \gamma_0]$ and $[\alpha_i, 0]$ are different. Now consider the G-orbit of $[\alpha_5, c\gamma_0]$. If $c \neq 0$ we put $s = |c|^{1/4}$. Then $S_5 := \operatorname{diag}(s, 1/s^2, 1/s, (\operatorname{sgn} c) \cdot (1/s)) \in \operatorname{Aut}(\mathfrak{l})$, $U := \operatorname{diag}(1, \operatorname{sgn} c) \in O(\mathfrak{a})$ and we get $(S_5, U)^*[\alpha_5, c\gamma_0] = [\alpha_5, \gamma_0]$. The orbits of $[\alpha_5, \gamma_0]$ and $[\alpha_5, 0]$ are different. Analogously, one determines the orbits of $[\alpha_i', \gamma]$ for $i = 5, 6$.

It remains to check admissibility and indecomposability. All cohomology classes $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ with $\alpha \in C$ satisfy (B_0), (A_1), and (B_1). Moreover, it is not hard to see that all cohomology classes listed in the proposition satisfy also (A_0) and are indecomposable. □

Proposition 8 *Let $\mathfrak{l}$ be the abelian Lie algebra $\mathbb{R}^4 = \operatorname{span}\{X_1, \ldots, X_4\}$ and let $\mathfrak{a}$ be a trivial orthogonal $\mathfrak{l}$-module.*

If $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$, then the elements in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ are represented by $[\alpha_1, 0]$ and $[\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4]$, where A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$.

If $\mathfrak{a} = \mathbb{R}^{1,1}$, then the elements in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ are represented by $[\alpha_1, 0]$, $[\alpha_2, 0]$, $[\alpha_3, 0]$, $[\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4]$ where A_1, A_2 is a fixed Witt basis of $\mathfrak{a}$.

If $\mathfrak{a} = \mathbb{R}^1$ or $\mathfrak{a} = \mathbb{R}^{1,0}$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ contains exactly one element. This is repre-

sented by $[\alpha_7, \gamma_0]$*, where* A_1 *is a fixed unit vector in* $\mathfrak{a}$*.*

If $\mathfrak{a} = 0$*, then* $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0 = \emptyset$*.*

Proof. First notice that $[0, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ is in the same G-orbit as either $[0, 0]$ or $[0, \gamma_0]$. Since both of these cohomology classes are decomposable $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$ implies $\alpha \neq 0$. Hence, under the assumptions of the proposition we have $\dim \alpha(\mathfrak{l}, \mathfrak{l}) = 1$ or $\dim \alpha(\mathfrak{l}, \mathfrak{l}) = 2$.

If $\dim \alpha(\mathfrak{l}, \mathfrak{l}) = 2$, then we may assume that $\alpha(X_1, X_3)$ and $\alpha(X_2, X_3)$ are linearly independent and that $\alpha(X_1, X_2) = 0$. This can easily be verified using the same idea as in the proof of (10). Hence, Equation (35) also holds in this case and we can argue as in the proof of Proposition 7 to show that α is in the same G-orbit as α_1 or α_4 if $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$ and in the same orbit as one of the 2-forms $\alpha_1, \ldots, \alpha_4$ if $\mathfrak{a} = \mathbb{R}^{1,1}$ and that all these orbits are different.

If $\dim \alpha(\mathfrak{l}, \mathfrak{l}) = 1$, then by classification of ordinary 2-forms there exists a map $S \in \mathrm{Aut}(\mathfrak{l}) = GL(4, \mathbb{R})$ such that $S^*\alpha = \alpha_7$ or $S^*\alpha = \alpha' := (\sigma^1 \wedge \sigma^3 + \sigma^2 \wedge \sigma^4) \otimes A_1$. Since $\langle \alpha' \wedge \alpha' \rangle \neq 0$ we can exclude the latter case.

Again we have $\langle \alpha_i \wedge Z^1(\mathfrak{l}, \mathfrak{a}) \rangle = C^3(\mathfrak{l})$ for $i = 1, 2, 3$. Furthermore, $\mathbb{R} \cdot \sigma^1 \wedge \sigma^2 \wedge \sigma^4$ is a complement of $\langle \alpha_4 \wedge Z^1(\mathfrak{l}, \mathfrak{a}) \rangle$ in $C^3(\mathfrak{l})$ and $\mathrm{span}\{\sigma^1 \wedge \sigma^2 \wedge \sigma^4, \gamma_0\}$ is a complement of $\langle \alpha_7 \wedge Z^1(\mathfrak{l}, \mathfrak{a}) \rangle$ in $C^3(\mathfrak{l})$. Note that $[\alpha_4, 0]$ and $[\alpha_7, 0]$ are decomposable. Hence, if $[\alpha_4, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$, then $[\alpha_4, \gamma]$ is on the same G-orbit as $[\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4]$. Moreover, if $[\alpha_7, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$, then $[\alpha_7, \gamma] = [\alpha_7, \gamma']$, where $\gamma' = \sigma^2 \wedge (s\sigma^1 + t\sigma^3) \wedge \sigma^4$ for suitable $s, t \in \mathbb{R}$ with $s^2 + t^2 \neq 0$. Eventually, $[\alpha_7, \gamma']$ is on the same G-orbit as $[\alpha_7, \gamma_0]$. $\square$

Combining the description of the moduli space given in Proposition 3 with Propositions 4 – 8 and the computations of $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$ for $\dim \mathfrak{l} \leq 3$ in [KO 1] and [KO 2] we obtain the following result. We use the 2-forms $\alpha_1, \ldots, \alpha_7, \alpha_5', \alpha_6'$ and the 3-form γ_0 introduced before Proposition 7.

Theorem 1 *If* $(\mathfrak{g}, \langle \cdot\,, \cdot \rangle)$ *is an indecomposable non-abelian nilpotent metric Lie algebra of dimension at most 10, then it is isomorphic to* $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a})$ *for exactly one of the data in the following list:*

1. $\mathfrak{l} = \mathbb{R}^5$

   $\mathfrak{a} = 0,\ \alpha = 0,\ \gamma = (\sigma^1 \wedge \sigma^2 + \sigma^3 \wedge \sigma^4) \wedge \sigma^5;$

2. $\mathfrak{l} = \mathfrak{g}_{5,2} = \{[X_1, X_2] = Y,\ [X_1, X_3] = Z\}$

   $\mathfrak{a} = 0,\ \alpha = 0,\ \gamma \in \{\sigma^1 \wedge \sigma^Y \wedge \sigma^Z,\ \sigma^1 \wedge \sigma^Y \wedge \sigma^Z + \sigma^2 \wedge \sigma^3 \wedge \sigma^Z\};$

3. $\mathfrak{l} = \mathfrak{g}_{4,1} = \{[X_1, Z] = Y,\ [X_1, X_2] = Z\}$

   (a) $\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$, $A \in \mathfrak{a}$ *fixed unit vector,*
   $\alpha = \sigma^1 \wedge \sigma^Y \otimes A,$
   $\gamma \in \{0,\ \sigma^2 \wedge \sigma^Y \wedge \sigma^Z,\ \sigma^1 \wedge \sigma^Y \wedge \sigma^Z,\ -\sigma^1 \wedge \sigma^Y \wedge \sigma^Z\};$

(b) $\mathfrak{a} \in \{\mathbb{R}^2,\ \mathbb{R}^{2,0}\}$ *with fixed orthonormal basis* A_1, A_2,
$\alpha = \sigma^1 \wedge \sigma^Y \otimes A_1 + \sigma^2 \wedge \sigma^Z \otimes A_2$,
$\gamma \in \{s\sigma^1 \wedge \sigma^Y \wedge \sigma^Z \mid s \in \mathbb{R}\} \ \cup\ \{r\sigma^2 \wedge \sigma^Y \wedge \sigma^Z \mid r \in \mathbb{R}_+\}$;

(c) $\mathfrak{a} = \mathbb{R}^{1,1}$ *with fixed orthonormal basis* A_1, A_2,
$\alpha \in \{\sigma^1 \wedge \sigma^Y \otimes A_1 + \sigma^2 \wedge \sigma^Z \otimes A_2,\ \sigma^2 \wedge \sigma^Z \otimes A_1 + \sigma^1 \wedge \sigma^Y \otimes A_2\}$,
$\gamma \in \{s\sigma^1 \wedge \sigma^Y \wedge \sigma^Z \mid s \in \mathbb{R}\} \ \cup\ \{r\sigma^2 \wedge \sigma^Y \wedge \sigma^Z \mid r \in \mathbb{R}_+\}$;

4. $\mathfrak{l} = \mathfrak{h}(1) \oplus \mathbb{R}^1$

(a) $\mathfrak{a} \in \{\mathbb{R}^2,\ \mathbb{R}^{2,0}\}$ *with fixed orthonormal basis* A_1, A_2
$(\alpha, \gamma) \in \{(\alpha_1, 0),\ (\alpha_5, 0),\ (\alpha_5, \gamma_0),\ (\alpha_6, 0),\ (\alpha_6, \gamma_0)\}$;

(b) $\mathfrak{a} = \mathbb{R}^{1,1}$,
$(\alpha, \gamma) \in \{(\alpha_1, 0),\ (\alpha_2, 0),\ (\alpha_3, 0)\}$, *where* A_1, A_2 *is a fixed Witt basis, or*
$(\alpha, \gamma) \in \{(\alpha_5, 0),\ (\alpha_5, \gamma_0),\ (\alpha_6, 0),\ (\alpha_6, \gamma_0),\ (\alpha_5', 0),\ (\alpha_5', \gamma_0),\ (\alpha_6', 0),\ (\alpha_6', \gamma_0)\}$,
where A_2, A_1 *is a fixed orthonormal basis of* $\mathfrak{a}$;

(c) $\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$,
$(\alpha, \gamma) = (\alpha_7, \gamma_0)$, *where* A_1 *is a fixed unit vector in* $\mathfrak{a}$;

5. $\mathfrak{l} = \mathbb{R}^4$

(a) $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$ *with fixed orthonormal basis* A_1, A_2,
$(\alpha, \gamma) \in \{(\alpha_1, 0),\ (\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4)\}$;

(b) $\mathfrak{a} = \mathbb{R}^{1,1}$ *with fixed Witt basis* A_1, A_2,
$(\alpha, \gamma) \in \{(\alpha_1, 0),\ (\alpha_2, 0),\ (\alpha_3, 0),\ (\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4)\}$;

(c) $\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$,
$(\alpha, \gamma) = (\alpha_7, \gamma_0)$, *where* A_1 *is a fixed unit vector in* $\mathfrak{a}$;

6. $\mathfrak{l} = \mathfrak{h}(1) = \{[X_1, X_2] = Y\}$

(a) $\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$,
$\alpha = \sigma^1 \wedge \sigma^Y \otimes A$, *where* A *is a fixed unit vector in* $\mathfrak{a}$,
$\gamma = 0$;

(b) $\mathfrak{a} \in \{\mathbb{R}^2,\ \mathbb{R}^{2,0},\ \mathbb{R}^{1,1}\}$ *with fixed orthonormal basis* A_1, A_2,
$\alpha = \sigma^1 \wedge \sigma^Y \otimes A_1 + \sigma^2 \wedge \sigma^Y \otimes A_2$,
$\gamma = 0$;

7. $\mathfrak{l} = \mathbb{R}^3$

(a) $\mathfrak{a} = 0$, $\alpha = 0$, $\gamma = \sigma^1 \wedge \sigma^2 \wedge \sigma^3$;

(b) $\mathfrak{a} \in \{\mathbb{R}^2,\ \mathbb{R}^{2,0},\ \mathbb{R}^{1,1}\}$ *with fixed orthonormal basis* A_1, A_2,
$\alpha = \sigma^1 \wedge \sigma^2 \otimes A_1 + \sigma^1 \wedge \sigma^3 \otimes A_2$,
$\gamma = 0$;

(c) $\mathfrak{a} \in \{\mathbb{R}^3,\ \mathbb{R}^{2,1},\ \mathbb{R}^{1,2},\ \mathbb{R}^{3,0}\}$ *with fixed orthonormal basis* A_1, A_2, A_3,
$\alpha = \sigma^1 \wedge \sigma^2 \otimes A_1 + \sigma^1 \wedge \sigma^3 \otimes A_2 + \sigma^2 \wedge \sigma^3 \otimes A_3$,
$\gamma = 0$;

8. $\mathfrak{l} = \mathbb{R}^2$

$\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$,
$\alpha = \sigma^1 \wedge \sigma^2 \otimes A$, *where A is a fixed unit vector in* $\mathfrak{a}$,
$\gamma = 0$.

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References

[D] J. Dixmier, *Cohomologie des algèbres de Lie nilpotentes.* Acta Sci. Math. Szeged 16 (1955), 246–250.

[KO 1] I. Kath, M. Olbrich, *Metric Lie algebras with maximal isotropic centre.* Math. Z. 246 (2004), 23–53, see also arXiv:math.DG/0209366, 2002.

[KO 2] I. Kath, M. Olbrich, *Metric Lie algebras and quadratic extensions.* to appear in Transf. Groups, see also arXiv:math.DG/0312243, 2004.

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