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by

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MARTIN POINTS ON OPEN MANIFOLDS OF NON-POSITIVE CURVATURE

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Abstract. The Martin boundary of a Cartan-Hadamard manifold describes a fine geometric structure at infinity, which is a sub-space of positive harmonic functions. We describe conditions which ensure that some points of the sphere at infinity belong to the Martin boundary as well. In the case of the universal cover of a compact manifold with Ballmann rank one, we show that Martin points are generic and of full harmonic measure. The result of this paper provides a partial answer to an open problem of S. T. Yau.

1. Introduction

Let $\widetilde{M}$ be a Cartan-Hadamard manifold, a simply connected Riemannian manifold with nonpositive curvature. Then, $\widetilde{M}$ is homeomorphic to an open ball, and there are two natural compactifications of $\widetilde{M}$ associated to the metric.

Fix $x_0 \in \widetilde{M}$. For $z \in \widetilde{M}$, define the continuous function b_z on $\widetilde{M}$ by:

$$b_z(x) \ = \ d(x,z) - d(x_0,z),$$

where d denotes the Riemannian distance on $\widetilde{M}$. The functions $b_z, z \in \widetilde{M}$ are equicontinuous and uniformly bounded on compact subsets of $\widetilde{M}$. They form a relatively compact set of functions for the topology of uniform convergence on compact sets. The closure of $\{z \mapsto b_z\}$ is the geometric compactification of $\widetilde{M}$. Let $\widetilde{M}(\infty)$ be the boundary of $\widetilde{M}$ in its geometric compactification. The set $\widetilde{M}(\infty)$, endowed with the relative topology, is homeomorphic to a sphere. Let $T(\widetilde{M})$ be the tangent bundle of $\widetilde{M}$, and $S_x\widetilde{M} = \{\vec{v} \in T_x(\widetilde{M}) | \quad \|\vec{v}\| = 1\}$ be the unit tangent sphere of $\widetilde{M}$ at x. For any $x \in \widetilde{M}$, the map $P_x : S_x\widetilde{M} \mapsto \widetilde{M}(\infty)$ which associates to $v \in S_x\widetilde{M}$ the point $P_x v = \sigma_v(+\infty)$ realizes this homeomorphism, where σ_v is the geodesic with initial condition v and, for a geodesic σ in $\widetilde{M}$, we denote $\sigma(\pm\infty) = \lim_{t\to+\infty} \sigma(\pm t)$ the corresponding points of $\widetilde{M}(\infty)$.

Assume that $\widetilde{M}$ admits a Green function $G(.,.)$ for the Laplace operator. For $z \in \widetilde{M}$, define the continuous function h_z on $\widetilde{M}$ by:

$$h_z(x) \ = \ \log G(x,z) - \log G(x_0,z).$$

By Harnack inequality, the functions $h_z, z \in \widetilde{M}$ are equicontinuous and uniformly bounded on compact subsets of $\widetilde{M}$ not containing z. They form a relatively compact set of functions for the topology of uniform convergence on compact sets. The closure of $\{z \mapsto h_z\}$ is the Martin compactification of $\widetilde{M}$.

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For Euclidean spaces, the Martin compactification is reduced to the Alexandroff one-point compactification. If the sectional curvatures of $\widetilde{M}$ are pinched between two negative constant, then the Martin compactification coincide with the geometric compactification [AS]. In general, the presence of flats amidst negative curvature is a source of more intricate Martin compactification: for symmetric spaces, the Martin compactification has been described in [GJT] and is a nontrivial continuous extension of the geometric compactification; the general description of the Martin compactification of a product is not known in general, see [MV] and the references therein for the latest results. In these two cases, every geodesic belongs to a flat space. It is believed that, if many geodesics are not within a totally geodesic flat subspace, then the Martin compactification is the geometric compactification. See [Ba4] for a first example.

Following Ancona's programme (see [An]), the same discussion applies to the general uniform elliptic operator $\mathcal{L}$ of second order in a general Cartan-Hadamard manifold $\widetilde{M}$ of dimension $n \geq 2$ with bounded geometry. The elliptic operator $\mathcal{L}$ has the form

$$\mathcal{L}(u) := \operatorname{div}(\mathcal{A}(\nabla u)) + B \cdot \nabla u + \operatorname{div}(uC) + \gamma u.$$

The conditions for the coefficients will be given in the next section. If $\mathcal{L}u = 0$, then u is called a $\mathcal{L}$-harmonic function. We still denote by $G(\cdot, \cdot)$ the Green function of $\mathcal{L}$ and define $h_z(x)$ as before. Again fix $x_0 \in \widetilde{M}$.

Definition 1.1 (Poisson kernel function)**.** A Poisson kernel function $k_\xi(x)$ of $\mathcal{L}$ at $\xi \in \widetilde{M}(\infty)$ is a positive $\mathcal{L}$-harmonic function on $\widetilde{M}$ such that:

$$k_\xi(x_0) = 1, \; k_\xi(y) = O(G_{x_0}(y)) \;\text{ as }\; y \to \xi' \neq \xi, \tag{1.1}$$

Definition 1.2 (Martin point)**.** We say that a point $\xi \in \widetilde{M}(\infty)$ is a Martin point of $\mathcal{L}$ if it satisfies the following properties:

- a) There exists a Poisson kernel function k_ξ of $\mathcal{L}$ at ξ,
- b) the Poisson kernel function is unique, and
- c) if $y_n \to \xi$, then $h_{y_n} \to \log k_\xi$ uniformly on compact sets.

In this paper, we want to describe Martin points of $\mathcal{L}$ for Cartan-Hadamard manifolds. For that purpose, we introduce several local notions of negative curvature along a geodesic in $\widetilde{M}$. For a vector $v \in S\widetilde{M}$, the *rank* of v is the dimension of the space of parallel Jacobi fields along the geodesic σ_v with initial condition v. Clearly, $1 \leq$ rank $v \leq$ dim $\widetilde{M}$. The geodesic rank of the manifold $\widetilde{M}$ is the minimum value of $\{\text{rank } v, v \in S\widetilde{M}\}$. For locally symmetric spaces, the geodesic rank coincide with the real rank of the real algebraic group of isometries of $\widetilde{M}$.

A geodesic σ is called *rank one* if rank of $\sigma'(0)$ is equal to 1. A geodesic in $\widetilde{M}$ is called *regular* if it does not bound a totally geodesic flat half-space. Rank one geodesics are regular. In the next section, we introduce the notion of *hyperbolic* geodesic in $\widetilde{M}$. It is a precise qualitative property which expresses that the geodesic has an infinite number of segments surrounded by enough negative curvature. Geodesics in flats, or even geodesics converging to flats are not hyperbolic. Our main result is:

Theorem 1.3. *Let $\widetilde{M}$ be a Cartan-Hadamard manifold with bounded geometry, $\mathcal{L}$ a uniformly elliptic, weakly coercive and bounded second order operator and σ :*

$\mathbb{R} \mapsto \widetilde{M}$ *a hyperbolic geodesic. Then* $\sigma(+\infty)$ *is a Martin point of* $\mathcal{L}$*. In particular, if the Laplace operator* Δ *is weakly coercive,* $\sigma(+\infty)$ *is a Martin point for* Δ*.*

An *axis* in $\widetilde{M}$ is a geodesic which is invariant by an isometry of $\widetilde{M}$ with two fixed points at infinity. We will see that regular axes are hyperbolic.

Corollary 1.4. *Let* $\widetilde{M}$ *be a Cartan-Hadamard manifold with bounded geometry,* $\mathcal{L}$ *a uniformly elliptic, weakly coercive and bounded second order operator and* $\sigma : \mathbb{R} \mapsto \widetilde{M}$ *an axis such that* σ *is not a boundary of any totally geodesic half-plane. Then* σ *is hyperbolic and* $\sigma(+\infty)$ *is a Martin point of* $\mathcal{L}$*.*

Remark 1.5. If the sectional curvature of $\widetilde{M}$ is pinched, then Ancona([An]) has proved that the Martin boundary $\partial_{\mathcal{L}}\widetilde{M}$ of $\widetilde{M}$ with respect to $\mathcal{L}$ is homeomorphic to the geometrical boundary $\widetilde{M}(\infty)$. Our result extends Ancona's results to non-pinched manifold, at least at extremities of hyperbolic geodesics.

In the rest of the paper, we show that if $\widetilde{M}$ is rank one and admits a cocompact group of isometries, then there are many hyperbolic geodesics. So assume that the manifold $\widetilde{M}$ is the universal cover of a compact manifold M. Then, $\widetilde{M}$ has bounded geometry as soon as the metric is of class C^3, and the Laplace operator admits a Green function as soon as M is not a 2-dimensional torus. Moreover, the geodesic rank rigidity results of Ballmann [Ba2] and Burns-Spatzier [BS] asserts that $\widetilde{M}$ can be written uniquely as a product of Euclidean spaces, symmetric spaces and the universal covers of rank one spaces (see [K1], Appendix, for the existence of a cocompact action on the third factors). We shall therefore concentrate on rank one manifolds. We have:

Corollary 1.6. *Let* $\widetilde{M}$ *be the universal cover of a compact Riemannian manifold of class* C^3*, non-positive curvature and geodesic rank 1,* $\mathcal{L}$ *a uniformly elliptic, weakly coercive and bounded second order operator on* $\widetilde{M}$ *and* $\sigma : \mathbb{R} \mapsto \widetilde{M}$ *a regular axis. Then,* $\sigma(+\infty)$ *is a Martin point of* $\mathcal{L}$*. In particular, Martin points are dense in* $\widetilde{M}(\infty)$*.*

Let $\Gamma = \pi_1(M)$ be the covering group. Recall that the action of Γ by isometries on $\widetilde{M}$ extends to a continuous action on $\widetilde{M}(\infty)$. We set $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$ for the set of pairs of distinct points in $\widetilde{M}(\infty)$. We say that a finite positive measure μ on $\widetilde{M}(\infty)$ is *geodesic ergodic* if

- 1) The support of the measure $\mu \times \mu$ is $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$.
- 2) For $\mu \times \mu$ almost every (η, ξ), there is a unique geodesic $\sigma_{\eta,\xi}$ such that: $\sigma_{\eta,\xi}(-\infty) = \eta, \sigma_{\eta,\xi}(+\infty) = \xi$, and $\sigma_{\eta,\xi}$ is rank one.
- 3) The measure $\mu \times \mu$ is Γ quasi-invariant and ergodic: the diagonal action of Γ preserves the $(\mu \times \mu)$-negligible subsets of $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$; and all Γ-invariant measurable subsets of $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$ are either negligible or co-negligible.

Examples of geodesic ergodic measures are the Patterson-Sullivan measure (see [K1]), other Gibbs measures constructed along the same lines, the harmonic measure for the Laplace operator on $\widetilde{M}$ (see [BL]), or analogously other harmonic measures associated to Markov equivariant symmetric operators on $\widetilde{M}$ or on Γ ([Ka]). It is not known, even for surfaces, whether the visibility measure, obtained by projecting under P_{x_0} the Lebesgue measure of the sphere $S_{x_0}\widetilde{M}$ is geodesic ergodic. We have

Theorem 1.7. *Let $\widetilde{M}$ be the universal cover of a compact Riemannian manifold of class C^3, non-positive curvature and geodesic rank one, and $\mathcal{L}$ a uniformly elliptic, weakly coercive and bounded second order operator on $\widetilde{M}$. Then the set of Martin points is a generic subset of $\widetilde{M}(\infty)$: it contains a countable intersection of open dense subsets. Moreover, the set of Martin points has full measure for any geodesic ergodic measure.*

In the next section, we introduce the necessary definitions and present the general scheme of the proofs. In section 3, we recall the potential theory of weakly coercive operators, and section 4 contains the geometric properties of hyperbolic geodesics we shall use. Theorem 1.3 reduces to Propositions we prove in section 5, and Theorem 1.7 is proven in section 6.

2. Precise statements of results and strategy of the proofs.

Let $\widetilde{M}$ be a complete Riemannian manifold of dimension $n \geq 2$. If $d(y,z)$ is sufficiently small, we let $\mathbb{P}_y^z$ denote the parallel transport from y to z along the unique length-minimizing geodesic segment.

We say that $\widetilde{M}$ has *bounded geometry* if there exists $r_0 > 0$ such that for any ball $B(x,r_0) \subset \widetilde{M}$, there exists a chart $\chi : B(x,r_0) \to \mathbb{R}^n$ satisfying a uniform *first order* quasi-isometry condition:

$$\begin{aligned} &C_0^{-1} d(y,z) \leq ||D\chi_x|_y - D\chi_x|_z||^* \leq C_0 d(y,z), \\ &\forall y,z \in B(x,r_0), \text{with a constant } C_0 \text{ independent of } x, \end{aligned} \tag{2.1}$$

where

$$||D\chi_x|_y - D\chi_x|_z||^* = ||\chi_x(y) - \chi_x(z)|| + \max_{\|\vec{v}\|=1} \{\|(D\chi_x)|_y \vec{v} - (D\chi_x)|_z(\mathbb{P}_y^z \vec{v})\|\}.$$

If the derivative of the curvature of $\widetilde{M}$ is bounded, and if the injectivity radius of $\widetilde{M} > 0$, then (2.1) holds.

Consider the following elliptic operator $\mathcal{L}$:

$$\mathcal{L}(u) := \operatorname{div}(\mathcal{A}(\nabla u)) + B \cdot \nabla u + \operatorname{div}(uC) + \gamma u, \tag{2.2}$$

where $\mathcal{A}$ is a section of $End(T\widetilde{M})$, B and C are vector fields on $\widetilde{M}$ and γ is a function.

Definition 2.1. The operator $\mathcal{L}$ is called uniformly elliptic if there is $\lambda > 1$ such that:

$$\forall (x,u) \in T\widetilde{M}, \lambda^{-1}||u||^2 \leq \langle \mathcal{A}_x(u), u\rangle \leq \lambda ||u||^2. \tag{2.3}$$

Definition 2.2. The operator $\mathcal{L}$ is said to be bounded if there is $\lambda > 0$ such that:

$$\forall x \in \widetilde{M}, ||B||_{L^\infty(B(x,r_0))}, ||C||_{L^\infty(B(x,r_0))}, ||\gamma||_{L^\infty(B(x,r_0))} \leq \lambda. \tag{2.4}$$

Definition 2.3. The function $G : \widetilde{M} \times \widetilde{M} \mapsto (0, +\infty)$ is called a Green function of $\mathcal{L}$, if G is continuous, and for any $x \in \widetilde{M}$, $G_x(y) := G(y, x)$ is a $\mathcal{L}$-potential on $\widetilde{M}$ and is $\mathcal{L}$-harmonic on $\widetilde{M} \setminus \{x\}$ such that

$$\mathcal{L}(G_x) = -\delta_x.$$

Definition 2.4. The operator $\mathcal{L}$ is called weakly coercive, if there exists $\epsilon > 0$ and a positive superharmonic function on $\widetilde{M}$ with respect to the operator $\mathcal{L} + \epsilon I$.

So if $\mathcal{L}$ is weakly coercive for some $\epsilon > 0$, then for any $0 \le t < \epsilon$, the operator $\mathcal{L} + tI$ has a Green function G^t.

Let now $\widetilde{M}$ be a Cartan-Hadamard manifold, and for $\sigma : \mathbb{R} \mapsto \widetilde{M}$ a geodesic line of unit speed, set

$$\mathcal{U}_h(\sigma(\mathbb{R})) = \{y | d(y, \sigma(\mathbb{R})) = h\}.$$

Since $\sigma(\mathbb{R})$ is a closed convex subset of $\widetilde{M}$, there is the nearest-point projection: $\mathcal{P}_\sigma : \widetilde{M} \mapsto \sigma(\mathbb{R})$. We define

$$S_h^\perp(\sigma(t)) = \mathcal{P}_\sigma^{-1}(\sigma(t)) \cap \mathcal{U}_h(\sigma(\mathbb{R})),$$

and

$$\eta_{\sigma([t_1,t_2])}(h) = d_{\mathcal{U}_h}(S_h^\perp(\sigma(t_1)), S_h^\perp(\sigma(t_2))),$$

where $d_{\mathcal{U}_h}(\cdot, \cdot)$ is the distance function of the Riemannian hypersurface $(\mathcal{U}_h, g|_{\mathcal{U}_h})$.

The following notion is a way of expressing at a finite distance that the geodesic σ does not bound a flat half space:

Definition 2.5. A geodesic $\sigma : \mathbb{R} \mapsto \widetilde{M}$ is said to be (h, T, δ)-non flat at t if we have:

$$\eta_{\sigma([t,t+T])}(h) \ > \ T + \delta h.$$

Properties of (h, T, δ)-non flat geodesics are recalled in Section 4. In particular, by Proposition 4.2 there exists a number $\varepsilon^* = \varepsilon^*(\widetilde{M}, h, T)$ such that if the geodesic σ is $(h, T, \pi/2)$-non flat at 0, and τ is another geodesic satisfying

$$\tau(0) \ = \ \sigma(0) \text{ and } \angle_{\sigma(0)}(\tau'(0), \sigma'(0)) \ < \ \varepsilon^*,$$

then the geodesic τ is $(h, T, \pi/4)$-non flat at 0.

Let us now choose $\varepsilon^* < \pi/4$ and set

$$T_1 \ = \ T_1(\widetilde{M}, h, T) \ = \ T + \frac{h}{\tan \varepsilon^*}.$$

Definition 2.6. We say that the geodesic σ admits a (h, T, R) barrier if there exist $t_i, i = 1, 2, \ldots, 6$ with $T_1 < t_{i+1} - t_i < T_1 + R$ and $t_3 + T < 0 < t_4$ such that the geodesic σ is $(h, T, \pi/2)$-non flat at t_i, for $i = 1, 2, \ldots, 6$.

Remark 2.7. Observe that if a geodesic σ is $(h, T, \pi/2)$-non flat at 0, the geodesic $-\sigma$ obtained by reversing time is $(h, T, \pi/2)$-non flat at $-T$. Consequently, if the geodesic σ admits a (h, T, R) barrier, the geodesic $-\sigma$ admits a (h, T, R) barrier as well, with $t'_i = -t_{7-i} - T$.

Definition 2.8. We say that the geodesic σ is hyperbolic if there are h, T, R and a sequence $t_i^* \to +\infty$ such that $\sigma(\cdot - t_i^*)$ admits a (h, T, R) barrier.

We have defined all elements of Theorem 1.3 that we recall:

Theorem 1.3 *Let $\widetilde{M}$ be a Cartan-Hadamard manifold with bounded geometry, $\mathcal{L}$ a uniformly elliptic, weakly coercive and bounded second order operator and $\sigma : \mathbb{R} \mapsto \widetilde{M}$ a hyperbolic geodesic. Then $\sigma(+\infty)$ is a Martin point of $\mathcal{L}$.*

In order to prove Theorem 1.3, we define the families of cones

$$\Gamma_{\sigma,t,\theta} = \{x \in \widetilde{M} | \angle_{\sigma(t)}(\sigma'(t), x) < \theta\}.$$

Theorem 2.9. *Suppose the geodesic τ admits a (h, T, R) barrier. Set $T_2 = 3(T_1 + R) + T$. Then there is a constant $C = C(\widetilde{M}, h, T, R)$ such that the Green function $G(x, y)$ satisfies*

$$G(x, y) \leq CG(x, \tau(0))G(\tau(0), y), \tag{2.5}$$
$$\forall x \in \widetilde{M} \setminus \Gamma_{\tau,-T_2,\pi/2},\ \forall y \in \Gamma_{\tau,T_2,\pi/2}$$

and the Green function $g(x, y)$ in $\widetilde{M} \setminus \Gamma_{\tau,2T_2,\pi/2}$ satisfies

$$g(x, y) \leq Cg(x, \tau(0))g(\tau(0), y), \tag{2.6}$$
$$\forall x \in \widetilde{M} \setminus \in \Gamma_{\tau,-T_2,\pi/2},\ \forall y \in \Gamma_{\tau,T_2,\pi/2} \setminus \Gamma_{\tau,2T_2,\pi/2}.$$

Recall Definition 1.1 of a Poisson kernel function, and call C_ξ the cone of functions positively proportional to a Poisson kernel function at $\xi \in \widetilde{M}(\infty)$. Then,

Proposition 2.10. *Assume τ is a hyperbolic geodesic with $\xi = \tau(+\infty)$. Then, dim $C_\xi \leq 1$.*

Proposition 2.11. *Assume τ is a hyperbolic geodesic with $\xi = \tau(+\infty)$, and consider the functions $k_z(x) = \frac{G(x,z)}{G(x_0,z)}$. Then, if k_ξ is a limit point of k_z as $z \to \xi$, $k_\xi \in C_\xi$.*

Theorem 1.3 follows directly from Propositions 2.10 and 2.11. In Section 5, we prove Theorem 2.9 and explain how Propositions 2.10 and 2.11 follow from Theorem 2.9. In [An], (2.5) is called the Boundary Harnack Inequality and is a key step in the proof. For establishing (2.5), our task is to use as little negative curvature as we find it necessary. The proof follows the ideas from [An], but given the delicate arguments involved, we prefer writing it in whole detail. Then, following [An]'s scheme, Propositions 2.10 and 2.11 follow from Theorem 2.9. Our observation is that it is sufficient to have an infinite number of disjoint barriers converging to ξ, not necessarily a uniform estimate everywhere. Again we write the detailed proof for the sake of completeness.

Assume now that $\tilde{\sigma} : \mathbb{R} \mapsto \widetilde{M}$ is an axis and suppose that $\tilde{\sigma}$ is not the boundary of any totally geodesic half plane. Then, there exist h_0 and δ_0 such that for any $k \in \mathbb{N}$, there is an integer n such that $\tilde{\sigma}$ is $(h_0, nL, k\delta_0)$-non flat at 0, where L is the period of axis $\tilde{\sigma}$.

Indeed, since $\tilde{\sigma}$ is invariant by an isometry, $\tilde{\sigma}$ is not the boundary of any totally geodesic flat two-dimensional quarter. Thus, by corollary 4.4 there exist T_0, h_0 and δ_0 such that

$$\eta_{\tilde{\sigma}([0,T_0])}(h_0) - T_0 \geq \delta_0 > 0.$$

Choose $n_0 > T_0/L$ to be an integer. Thus, since the function $T \mapsto \eta_{\tau[0,T]}(h_0) - T$ is nondecreasing (see Proposition 4.1 and Proposition 4.3(5)):

$$\eta_{\tilde{\sigma}([0,n_0L])}(h_0) - n_0 L \geq \delta_0.$$

For any integer k, we get, using semiaddivity (4.1) and the periodicity of $\tilde{\sigma}$

$$\eta_{\tilde{\sigma}([0,kn_0L])}(h_0) - kn_0 L \geq k\delta_0,$$

which is the desired property by setting $n = n_0 k$.

By invariance under isometries, the axis $\tilde{\sigma}$ is also $(h_0, nL, k\delta_0)$-non flat at KL, for all $K \in \mathbb{N}$. By choosing k such that $k\delta_0 > \frac{\pi}{2} h_0$, and $t_i, i = 1, 2, \ldots, 6$ also multiples of L, we find a number R such that the axis $\tilde{\sigma}$ admits a (h_0, nL, R) barrier. By invariance by isometries again, the axis $\tilde{\sigma}$ is a hyperbolic geodesic. Corollary 1.4 is therefore a particular case of Theorem 1.3.

Consider the case when the Cartan-Hadamard manifold $\widetilde{M}$ is the universal cover of a compact Riemannian manifold M of geodesic rank one. Set SM for the unit tangent bundle of M. A unit tangent vector $v \in SM$ is said to be regular, (h, T, δ)-non flat, admitting a (h, T, R) barrier or hyperbolic if any geodesic $\sigma_{\tilde{v}}$ defined by a lift $\tilde{v}$ of v to $S\widetilde{M}$ has the same property. Ballmann ([Ba1]) showed that unit tangent vectors to regular closed geodesics are dense in SM. Therefore Corollary 1.6 directly follows from Theorem 1.4. The geodesic flow is a one parameter group $\varphi_t, t \in \mathbb{R}$ of diffeomorphisms of SM. There is a unique φ-invariant probability measure $\bar{\nu}$ on SM which realizes the topological entropy. The measure $\bar{\nu}$ has full support on SM and the geodesic flow is ergodic for $\bar{\nu}$ ([K2]). Therefore:

Proposition 2.12. *Let M be a compact Riemannian manifold of nonpositive sectional curvature and geodesic rank 1. Then the set of hyperbolic unit tangent vectors contains a countable intersection of open dense sets in SM. Moreover, it has full measure for $\bar{\nu}$.*

Proof. We know that a unit tangent vector to a regular closed geodesic admits a (h, T, R) barrier for some h, T and R. By proposition 4.2, there is an open neighborhood $\mathcal{O}$ of such a unit vector v such that all $v' \in \mathcal{O}$ also admit a (h, T, R) barrier. Since the measure $\bar{\nu}$ is ergodic and has full support, for all positive K the set $\mathcal{O}_K$ of $v \in SM$ such that the geodesic ray $\sigma_v([K, \infty))$ intersects $\mathcal{O}$ is open dense in SM and has full $\bar{\nu}$ measure. The set $\cap_K \mathcal{O}_K$ is a countable intersection of open dense sets of full $\bar{\nu}$ measure. By definition, any unit vector in $\cap_K \mathcal{O}_K$ is hyperbolic. □

To prove Theorem 1.7, we still have to verify that the large set of unit vectors of Proposition 2.12 lifts and projects to a large subset of $\widetilde{M}(\infty)$. This relies on the properties of the measure $\bar{\nu}$ which have been established in [K2], see Section 6.

Remark 2.13. In the case when $\widetilde{M}$ is the universal cover of a compact rank 1 manifold, the Laplace operator Δ is weakly coercive (see below section 3) and clearly uniformly elliptic and bounded. The conclusions of Corollary 1.6 and Theorem 1.7 hold for $\mathcal{L} = \Delta$.

3. Preliminaries (Elliptic operators, Green functions and their estimates)

Let $\widetilde{M}$ be a Cartan-Hadamard manifold with bounded geometry and $\mathcal{L}$ a uniformly elliptic, weakly coercive and bounded second order operator. Let μ be a positive measure on $\widetilde{M}$. Define $G\mu(x) := \int_{\widetilde{M}} G(x,y)d\mu(y)$. If $G\mu$ is not identically $+\infty$, $G\mu$ is the only potential satisfying $\mathcal{L}(G\mu) = -\mu$.

There are two important estimates (see [An]):

- For each $\omega = B(x, r_0) \subset \widetilde{M}$, and every $t, 0 \le t \le 1$, the Green function g^t related to $\mathcal{L} + tI$ over ω satisfies

$$g^t(y,z) \ge C, \forall y, z \in B(x, r_0/2), \text{ and } g^t(y,z) \le C^{-1}, \text{ if } d(y,z) \ge \frac{r_0}{4}, \tag{3.1}$$

  where $C = C(\mathcal{L})$ is independent of x and t.
- (Harnack inequality) If $u > 0$ is a $\mathcal{L} + tI$-harmonic function on $B(x, r_0)$, then

$$C^{-1}u(x) \le u(y) \le Cu(x), \tag{3.2}$$

  where $C = C(\mathcal{L}) > 0$.

The adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is given by the formula:

$$\mathcal{L}^*(u) = \operatorname{div}(\mathcal{A}^*(\nabla u)) - \operatorname{div}(B \cdot u) - C \cdot \nabla u + \gamma u.$$

Note that the Green function $\overset{*}{G}(x,y)$ of $\mathcal{L}^*$ satisfies $\overset{*}{G}(x,y) = G(y,x)$.

Lemma 3.1 ([An], Lemma 1)**.** *For each positive measure μ on $\widetilde{M}$ and each $t, 0 \le t < \epsilon$, we have*

$$G^t(\mu) = G(\mu) + G(G^t(\mu)).$$

Let $\overset{*}{\mu}_x$ be the $\mathcal{L}^*$-harmonic measure of a point $x \in \Omega$, where Ω is a bounded region in $\widetilde{M}$. We have

Lemma 3.2 ([An], Lemma 3)**.** *Let g be the $\mathcal{L}$-Green function of Ω, and let $g_x(y) = 0$ for $y \notin \Omega$, then*

$$\mathcal{L}(g_x) = -\delta_x + \overset{*}{\mu}_x.$$

Proof. We have the representation formula of $g(x,y)$ in terms of $G(x,y)$ and the harmonic measure $\overset{*}{\mu}_x$:

$$g_x(y) = \overset{*}{G}_y(x) - \int_{\partial\Omega} \overset{*}{G}_y(z) d\overset{*}{\mu}_x(z),$$

for $x \in \Omega, y \in \widetilde{M}$. Then we have

$$g_x = G_x - G(\overset{*}{\mu}_x),$$

and so

$$\mathcal{L}(g_x) = -\delta_x + \overset{*}{\mu}_x.$$

□

Denote by g^t the $\mathcal{L} + tI$-Green function of Ω, and by $\overset{*}{\mu}{}^t_x$ the $\mathcal{L}^* + tI$-harmonic measure of x in Ω, we have

Lemma 3.3 ([An], Lemma 4)**.** *If $0 \le t < \epsilon, x \in \Omega$ and $g_x \le k g_x^t$ for some $k > 0$ and outside some compact subset of Ω, then we have*

$$\overset{*}{\mu}_x \le k \overset{*}{\mu}{}_x^t.$$

Definition 3.4. Let Ω be a not necessarily bounded region in $\widetilde{M}$. Let $x \in \Omega$, the "reduit" of G_x on $\overline{\Omega}^c$ is defined as

$$R_{G_x}^{\overline{\Omega}^c} := \inf\{s | s > 0 \text{ is } \mathcal{L} - \text{superharmonic on } \widetilde{M}, \text{and } s \ge G_x \text{ on } \overline{\Omega}^c\}.$$

This reduit is an $\mathcal{L}$-potential, and if we put $\nu_x = -\mathcal{L}(R_{G_x}^{\overline{\Omega}^c})$, then $\forall z \in \widetilde{M} \setminus \overline{\Omega}$, we have the formula:

$$\overset{*}{G}_z(x) = G_x(z) = \int G(z,y) d\nu_x(y),$$

where ν_x is supported by $\partial\Omega$.

Proposition 3.5 ([An], Proposition 7)**.** *There is a constant $C = C(\widetilde{M}, \lambda, \epsilon) > 0$ such that if $x, y \in \widetilde{M}$ and $d(x,y) = 1$, then*

$$\frac{1}{C} \le G^t(x,y) \le C, \text{ for } 0 \le t < \epsilon$$

Lemma 3.6 ([An], Lemma 9)**.** *There exists a constant $\delta = \delta(\widetilde{M}, \lambda, \epsilon), 0 < \delta < 1$, such that for each ball $B(x,1)$ in $\widetilde{M}$, the $\mathcal{L}$-harmonic measure μ_x of x in $B(x,1)$ and the similar $\mathcal{L} + \epsilon I$ harmonic measure μ_x^ϵ satisfy*

$$\mu_x \le (1-\delta)\mu_x^\epsilon.$$

Proposition 3.7 ([An], Proposition 10)**.** *There are positive numbers C and α such that*

$$G(x,y) \le C e^{-\alpha d(x,y)} G^\epsilon(x,y), \forall x, y \in \widetilde{M}, \tag{3.3}$$

where C and α depend only on $\widetilde{M}, \lambda$ and ϵ.

Proof. By induction on $k \in \mathbb{N}, k \ge 1$, we prove that $G(x,y) \le (1-\delta)^{k-1} G^\epsilon(x,y)$, for $d(x,y) = k$ and δ which is given by Lemma 3.6.

When $k = 1$, we have $G(x,y) \le G^\epsilon(x,y)$, since $G^\epsilon(x,y)$ is a $\mathcal{L}$-superharmonic function.

Assume that the inequality holds for $d(x,y) = k$. We want to prove that it holds for $d(x,y) = k+1$. By maximum principle, one has

$$G_x(z) \le (1-\delta)^{k-1} G_x^\epsilon(z), \forall z \in \widetilde{M} \setminus B(x,k).$$

In particular, for $z \in \partial B(y,1)$. Hence

$$G_x(y) = \int_{\partial B(y,1)} G_x(z) d\mu_y(z) \le (1-\delta)^{k-1} \int G_x^\epsilon(z) d\mu_y(z).$$

Now by Lemma 3.6,

$$G_x(y) \le (1-\delta)^k \int G_x^\epsilon(z) d\mu_y^\epsilon(z) = (1-\delta)^k G_x^\epsilon(y).$$

This proves the proposition for $d(x,y)$ being integer. The general case follows by the fact $G_x \le G_x^\epsilon$ and Harnack inequality for G_x^ϵ. ☐

Remark 3.8. Let $\Omega = B(x,r)$. Then proposition 3.7 holds for G_Ω and G^ϵ_Ω, with the constants C, α independent of r. This is because if we proved the estimate for $d(x,y) \leq r-1$. Then if $r-1 \leq d(x,y) < r$, by maximum principle, we have

$$G_\Omega(x,y) \leq Ce^{-\alpha(r-1)} G^\epsilon_\Omega(x,y).$$

Remark 3.9. By Harnack inequality and Proposition 3.5, it is easy to obtain the lower bound estimate of $G(x,y)$:

$$ce^{-\beta d(x,y)} \leq G(x,y),$$

where $c, \beta > 0$ only depend on the bounded geometry of $\widetilde{M}$ and the operator $\mathcal{L}$.

Corollary 3.10 ([An], Corollary 11). *Given $\delta > 0$, there exists $R = R(\widetilde{M}, \lambda, \epsilon, \delta)$ such that $\forall x \in \widetilde{M}$, and $\forall r \geq R$, the $\mathcal{L}$-harmonic measure μ_x of x in $B(x,r)$ and the similar $\mathcal{L} + \epsilon I$ harmonic measure μ^ϵ_x satisfy:*

$$\mu_x \leq \delta \mu^\epsilon_x. \tag{3.4}$$

Proof. For given $\delta > 0$, we can find $R = R(\widetilde{M}, \lambda, \epsilon, \delta)$ such that $Ce^{-\alpha d(x,y)} \leq \delta$ for y near $\partial B(x,r)$ for any $r \geq R$, where C and α are from Remark 3.8. So

$$G_B(x,y) \leq \delta G^\epsilon_B(x,y), \text{ for } y \text{ near } \partial B(x,r),$$

i.e.,

$$\overset{*}{G}_{B,x} \leq \delta \overset{*}{G}{}^\epsilon_{B,x}.$$

By Lemma 3.3, we have

$$\mu_x \leq \delta \mu^\epsilon_x.$$

□

Assume now that the Cartan-Hadamard manifold $\widetilde{M}$ is cocompact, i.e., it is the universal cover of some compact Riemannian manifold M with the lifted metric. Furthermore, we assume M is of geodesic rank 1. It is known that the fundamental group $\pi_1(M)$ of M contains a free group F_2, and hence $\pi_1(M)$ is non-amenable. By Brooks's result, the first eigenvalue of Laplace operator

$$\lambda_1(\widetilde{M}) = \inf_{f \in H^{1,2}(\widetilde{M})} \frac{\int_{\widetilde{M}} |\nabla f|^2}{\int_{\widetilde{M}} |f|^2} > 0.$$

Now let $G(x,y)$ be the Green function of the Laplace operator Δ on $\widetilde{M}$. Since $\widetilde{M}$ is cocompact, the sectional curvature $|K_{\widetilde{M}}|$ and its derivative are bounded and the injectivity radius $inj(\widetilde{M})$ is positive. Thus $\widetilde{M}$ has the "bounded geometry" property (2.1). On the other hand, Laplace operator Δ satisfies (2.3) and (2.4) obviously. If we can prove that Δ is weakly coercive, then all the conclusions in section 2 hold for $\mathcal{L} = \Delta$ and its Green function.

Define the bilinear form

$$a_t(u,\varphi) = \int_{\widetilde{M}} \langle \nabla u, \nabla \varphi \rangle - \int t \langle u, \varphi \rangle$$

from $H^{1,2}(\widetilde{M}) \times H^{1,2}(\widetilde{M})$ to $\mathbb{C}$. The form $a_t(u,\varphi)$ is bounded, since

$$|a_t(u,\varphi)| \leq ||u||_{H^{1,2}} \cdot ||\varphi||_{H^{1,2}},$$

for $0 \le t \le 1$.

The form $a_t(u,\varphi)$ is coercive, since

$$\begin{aligned} a_t(u,u) = \int |\nabla u|^2 - t\int |u|^2 \ge \int |\nabla u|^2 - \frac{t}{\lambda_1-\delta}\int |\nabla u|^2 \\ \ge (1-\frac{t}{\lambda_1-\delta})\frac{1}{\lambda_1-\delta}\int |u|^2, \end{aligned}$$

for $0<\delta<\lambda_1, 0\le t<\lambda_1-\delta$.

Hence

$$a_t(u,u) \ge C_\delta ||u||^2_{H^{1,2}(\widetilde{M})}$$

for $0 \le t < \lambda_1 - \delta$, where

$$C_\delta = \frac{1}{2}(1-\frac{t}{\lambda_1-\delta})\min\{1,\frac{1}{\lambda_1-\delta}\}.$$

If we take $\delta = \frac{\lambda_1}{2}$, then for any $0 \le t < \frac{\lambda_1}{3}$, there is

$$a_t(u,u) \ge C_{\lambda_1}||u||^2_{H^{1,2}},$$

where $C_{\lambda_1} = \frac{1}{6}\min\{1,\frac{2}{\lambda_1}\}$. Now by Lax-Milgram theorem, for any $f \in H^{-1,2}(\widetilde{M})$, there exists a unique $u \in H^{1,2}$ such that

$$a_{\lambda_1/3}(u,v) = \langle f, v\rangle.$$

Take $\varphi \ge 0, \varphi \in C_0^\infty(\widetilde{M})$, then the above equality implies that

$$(\Delta + \frac{\lambda_1}{3})u = -\varphi \le 0.$$

On the other hand,

$$a_{\lambda_1/3}(u^-,u^-) = a_{\lambda_1/3}(-u,u^-) = -\int \varphi u^- \le 0.$$

So by coercivity, there is $u^-=0$ and $u \ge 0$. Therefore if $\varphi \ne 0$, we obtain a positive superharmonic function $u>0$ of the operator $\Delta + \lambda_1/3$.

Theorem 3.11. *There exist two positive numbers C and α depending only on the geometry of M such that $\forall (x,y) \in \widetilde{M}\times\widetilde{M}$ and $d(x,y)\ge 1$, the following holds:*

$$G(x,y) \le Ce^{-\alpha d(x,y)}. \tag{3.5}$$

Proof. This decay estimate was already proved in [SY]. Here we give a different proof. Firstly we prove that for $0<\epsilon<\lambda_1/3$, and for any $x,y\in\widetilde{M}$ satisfying $d(x,y)\ge 1$, we have $G^\epsilon(x,y)\le C$, where C only depends on λ_1.

Let f and g be the characteristic function of the balls $B(x,\rho)$ and $B(y,\rho)$ respectively, where $\rho = \min\{r_0, 1/3\}$. Then $G^\epsilon(fdv)$ is the solution of the equation $\Delta u + \epsilon u = f$. By Schwarz inequality and Lax-Milgram theorem, we have

$$\int G^\epsilon(f)\cdot g \le (\int |G^\epsilon(f)|^2)^{\frac{1}{2}}\cdot ||g||_{L^2} \le C_{\lambda_1/3}||f||_{L^2}||g||_{L^2} = C.$$

Thus we have

$$\int\int_{(\xi,\eta)\in B(x,\rho)\times B(y,\rho)} G^\epsilon(\xi,\eta)d\xi d\eta \le C.$$

Therefore there exists a point pair $(x_1,y_1)\in B(x,\rho)\times B(y,\rho)$ such that

$$G^\epsilon(x_1,y_1)\le C.$$

Using Harnack inequality, we obtain

$$G^{\epsilon}(x,y) \le C,$$

for all (x,y) such that $d(x,y) \ge 1$. Here C only depends on M. By Proposition 3.7, we are done. □

Corollary 3.12. *Given $\delta > 0$, there exists $R = R(M,\delta)$ such that $\forall x \in M$ and $r \ge R$, the Δ-harmonic measure μ_x of x in $B(x,r)$ and the similar $\Delta + \epsilon$ harmonic measure μ_x^{ϵ} satisfy*

$$\mu_x \le \delta \mu_x^{\epsilon}.$$

Proof. It is a direct conclusion from corollary 3.10 and Theorem 3.11. □

By Theorem 3.11, the Green function G of Laplace operator vanishes at infinity. For the Green function of the general elliptic operator $\mathcal{L}$, we need the following definition. Let $\xi \in \widetilde{M}(\infty)$, we say a function u vanishes at ξ in the $\mathcal{L}$-sense, if there exists a positive $\mathcal{L}$-superharmonic function w on $\widetilde{M}$ such that $u = o(w)$ at ξ. If $\mathcal{L}(1) \le 0$, then the vanishing of u at ξ in the $\mathcal{L}$-sense is the same as usual. It is shown in [An], page 509, that for any $x \in \widetilde{M}$, G_x vanishes on $\widetilde{M}$ in the $\mathcal{L}$-sense. Namely, there exists a $\mathcal{L}$-superharmonic function w such that $G_x = o(w)$ at infinity.

Proposition 3.13. *Let $0 < \theta < \pi$, $\Gamma = \Gamma_{\sigma,t_0,\theta}$, and $\Gamma_1 = \Gamma_{\sigma,t_0+T_0,\theta}$ for some t_0 and $T_0 > 0$. If $u(x)$ is a positive $\mathcal{L}$-harmonic function in Γ and vanishes in the $\mathcal{L}$-sense in $\widetilde{M}(\infty) \cap \Gamma$, then the reduit $u_1(x) := R_u^{\Gamma_1}(x)$ is a $\mathcal{L}$-potential on Γ.*

Proof. This is proved in [An], Theorem 2. □

4. Hyperbolicity Estimates

Let $\widetilde{M}$ be a Cartan-Hadamard manifold with bounded geometry, and recall the definition of (h,T,δ)-non flat geodesics. We have the following properties of the distance η:

Proposition 4.1 (Semi-additivity)**.** *For any $h > 0$, we have*

$$\eta_{\sigma([t_1,t_3])}(h) \ge \eta_{\sigma([t_1,t_2])}(h) + \eta_{\sigma([t_2,t_3])}(h) \tag{4.1}$$

for $t_1 \le t_2 \le t_3$.

Proof. Let $\phi : [t_1,t_3] \mapsto \mathcal{U}_h$ be a path from $S_h^{\perp}(\sigma(t_1))$ to $S_h^{\perp}(\sigma(t_3))$. For clear topological reasons, the path ϕ must intersect $S_h^{\perp}(\sigma(t_2))$ at $\phi(t^*)$. Let $L(\phi|_{[s,s+\delta]})$ be the length of $\phi|_{[s,s+\delta]}$. We have

$$L(\phi|_{[t_1,t_3]}) = L(\phi|_{[t_1,t^*]}) + L(\phi|_{[t^*,t_3]}) \ge \eta_{\sigma([t_1,t_2])}(h) + \eta_{\sigma([t_2,t_3])}(h).$$

□

Proposition 4.2 (Continuity)**.** *For fixed t_1, t_2 and h, the function $\eta_{\sigma([t_1,t_2])}(h)$ depends continuously on $\sigma'(0)$. Namely, for fixed δ_0, there exists $\varepsilon = \varepsilon(\widetilde{M}, t_1, t_2, h, \delta_0)$ such that if $d_{S\widetilde{M}}(\sigma'(0), \tau'(0)) < \varepsilon$, then*

$$|\eta_{\sigma([t_1,t_2])}(h) - \eta_{\tau([t_1,t_2])}(h)| \ < \ \delta_0.$$

Proof. Indeed if $\sigma'(0)$ and $\tau'(0)$ are close enough, then the closed sets

$$\mathcal{U}_h(\sigma([t_1,t_2])),\ S_h^\perp(\sigma(t_1)) \text{ and } S_h^\perp(\sigma(t_2))$$

are sufficiently close to respectively the closed sets

$$\mathcal{U}_h(\tau([t_1,t_2])),\ S_h^\perp(\tau(t_1)) \text{ and } S_h^\perp(\tau(t_2))$$

that the respective distances

$$d_{\mathcal{U}_h}(S_h^\perp(\sigma(t_1)), S_h^\perp(\sigma(t_2))) \text{ and } d_{\mathcal{U}_h}(S_h^\perp(\tau(t_1)), S_h^\perp(\tau(t_2)))$$

are close. Moreover, by bounded geometry, if t_1, t_2 and h are bounded, the explicit ε of the above argument can be uniformly chosen, depending only on δ_0. $\square$

The other properties of η we use need some explicitation: Let $F = \exp : \mathcal{N}(\sigma(\mathbb{R})) \mapsto \widetilde{M}$ be the exponential map (Fermi-map) along σ, where $\mathcal{N}(\sigma(\mathbb{R}))$ is the normal bundle along σ. If $\overrightarrow{Y} : \mathbb{R} \mapsto T_\sigma\widetilde{M}$ is a C^2-smooth vector field along σ with $\overrightarrow{Y} \perp \sigma'$ and $|\overrightarrow{Y}| \equiv 1$, we consider the map

$$F = F_{\overrightarrow{Y}} : \mathbb{R}_+ \times [t_1,t_2] \mapsto \widetilde{M}$$
$$(s,t) \mapsto \exp_{\sigma(t)}(s\overrightarrow{Y}(t)).$$

For fixed h, the map $F(s,t) = \exp_{\sigma(t)}[s\overrightarrow{Y}(t)], \forall (s,t) \in [0,h]\times[t_1,t_2]$, gives a two-dimensional embedding surface with image $\square_{t_1,t_2,h}$. Proposition 4.3 below implies that $F : \mathbb{R}^2 \mapsto \widetilde{M}$ is a distance-increasing map, so the intrinsic curvature $K_{\square_{t_1,t_2,h}}$ is well-defined. There is an intrinsic curvature function

$$K_{\square_{t_1,t_2,h}}(s,t) = K(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}) - \frac{|\nabla_{\frac{\partial F}{\partial s}}\frac{\partial F}{\partial t}|^2}{|\frac{\partial F}{\partial s}\wedge\frac{\partial F}{\partial t}|^2}. \tag{4.2}$$

This curvature function is related to the following length function:

$$l(h) = L(F(h,\cdot)|_{[t_1,t_2]}).$$

by the following proposition:

Proposition 4.3. *Let $\sigma, \overrightarrow{Y}$ and $F = F_{\overrightarrow{Y}}$ be as above. Then*

1. *$l(h)$ is a convex function of h;*
2. *If $r(x) = d(x,\sigma(\mathbb{R}))$, then Hess$(r)(X,X) = \langle\nabla_X\nabla r, X\rangle \geq 0$ and*
$$\frac{dl}{dh} = \int_{t_1}^{t_2}\langle\nabla_{\frac{\partial F}{\partial t}}\nabla r, \frac{\partial F}{\partial t}\rangle\frac{1}{|\frac{\partial F}{\partial t}|}dt = \int_{F(h,\cdot)} k_g(\cdot,h)dl \geq 0,$$
where k_g is the geodesic curvature of the curve $t \mapsto F(h,t)$ with respect to ∇r, $k_g = -\langle\nabla_{\frac{\partial F}{\partial t}}(\frac{\partial F}{\partial t}), \nabla r\rangle = $ Hess$(r)(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial t})$.
3. $$-\int_{\square_{t_1,t_2,h}} K_{\square_{t_1,t_2,h}} dA = \frac{\partial l}{\partial h}.$$
4. $$\frac{\partial l}{\partial h}(h) \geq \frac{l(h)-l(0)}{h}$$
5. $$l(h) \ \geq \ t_2 - t_1.$$

Proof. Recall that $h \mapsto F(h,t)$ is a geodesic. Therefore

$$J^t(h) = \frac{\partial F}{\partial t}(h,t)$$

is a Jacobi field along the geodesic ray $\Psi_t : h \mapsto \Psi_t(h) = F(h,t)$.

It is easy to see that if $K_{\widetilde{M}} \le 0$, then the function $h \mapsto ||J^t(h)||$ is a convex function in h, i.e.,

$$\frac{\partial^2 ||J^t(h)||}{\partial h^2} \ge 0.$$

Therefore

$$\frac{\partial^2 l}{\partial h^2} = \int_{t_1}^{t_2} \frac{\partial^2 ||\frac{\partial F}{\partial t}||}{\partial h^2} dt \ge 0.$$

For (2), it is a direct consequence of the first variational formula, where $\nabla r|_{F(h,t)} = \frac{\partial F}{\partial h}(h,t)$. In addition, it is proved in [BGS] that if $\sigma(\mathbb{R})$ is a convex subset, then $r(x)$ is a convex function in $x \in \widetilde{M}$.

The assertion (3) follows from the Gauss-Bonnet formula on $\Box_{t_1,t_2,h}$. To see this we observe that $||\overrightarrow{Y}(t)|| = 1$. It is clear that $r(y) \equiv h$ for all $y \in \mathcal{U}_h(\sigma(\mathbb{R}))$. It follows that $r^{-1}(h) = \mathcal{U}_h(\sigma(\mathbb{R}))$ and $(\nabla r|_{F(h,t)}) \perp \mathcal{U}_h(\sigma(\mathbb{R}))$. Hence, we have a rectangle of curved top.

The discussion above implies that

$$\frac{\partial F}{\partial h} = \nabla r \perp \frac{\partial F}{\partial t}, \tag{4.3}$$

because $\frac{\partial F}{\partial t} \in T_{F(t,h)}[\mathcal{U}_h(\sigma(\mathbb{R}))]$.

Therefore,we apply the Gauss-Bonnet formula to get

$$2\pi = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \int_{F(h,\cdot)} k_g dl + \int_{\Box_{t_1,t_2,h}} K_{\Box_{t_1,t_2,h}} dA.$$

Thus,

$$-\int_{\Box_{t_1,t_2,h}} K_{\Box_{t_1,t_2,h}} dA = \int_{F(h,\cdot)} k_g dl = \frac{\partial l}{\partial h}.$$

This proves (3).

For (4), we already proved that $l(h)$ is a convex function. Thus, we have

$$\frac{\partial l}{\partial h} \ge \frac{l(h) - l(0)}{h}.$$

Since, by (4.2), the Left Hand Side of (3) is nonnegative, $\frac{\partial l}{\partial h} \ge 0$ and so $l(h) \ge l(0) = t_2 - t_1$. This proves (5). $\Box$

By definition we have:

$$\eta_{\sigma([t,t+T])}(h) := \inf_{\substack{|\overrightarrow{Y}|=1 \\ \overrightarrow{Y} \perp \sigma'}} \{L(F_{\overrightarrow{Y}}(h,\cdot)|_{[t,t+T]})\}.$$

Therefore, by Proposition 4.3, if a geodesic σ is (h,T,δ)-non flat at t, then it satisfies:

$$\hat{K}_{\sigma,h}(t,t+T) := \inf_{\substack{|\overrightarrow{Y}|=1 \\ \overrightarrow{Y} \perp \sigma'}} \{\int\int_{\Box_{t,t+T,h}} -K_{\Box_{t,t+T,h}}(s,t) \left| \frac{\partial F}{\partial s} \wedge \frac{\partial F}{\partial t} \right| ds\, dt\} \ge \delta. \tag{4.4}$$

We also have:

Corollary 4.4. *If for some t, T and h a geodesic σ satisfies $\eta_{\sigma([t,t+T])}(h) = T$, then there is a field $\overrightarrow{Y}$ along σ such that the rectangle $\Box_{t,t+T,h}$ is totally geodesic and flat.*

Proof. This assertion was indeed implicitly stated in [BGS]. For the convenience of readers, we present a short proof here. Let $\mathcal{P}_\sigma : \widetilde{M} \mapsto \sigma(\mathbb{R})$ be the nearest point projection. Since $\widetilde{M}$ is a Cartan-Hadamard manifold and $\sigma(\mathbb{R})$ is a closed convex subset, it was proved in [BGS] that $\mathcal{P}_\sigma$ is a distance non-increasing map. Thus, we have

$$d_{\widetilde{M}}(x, y) \geq d(\mathcal{P}_\sigma(x), \mathcal{P}_\sigma(y)).$$

Equality holds in above inequality if and only if the four points $\{x, y, \mathcal{P}_\sigma(x), \mathcal{P}_\sigma(y)\}$ are vertices of a totally geodesic flat rectangle $\Box$, see [BGS].

Suppose that $\eta_{\sigma([t,t+T])}(h) = T$. By compactness, there is a point $x \in S_h^\perp(\sigma(t))$, a point $y \in S_h^\perp(\sigma(t+T))$ and a shortest curve on $\mathcal{U}_h(\sigma([t, t+T]))$ realizing $d_{\mathcal{U}_h}(x, y) = \eta_{\sigma([t,t+T])}(h) = T$. Therefore, we have the following equalities and inequalities:

$$T = d_{\mathcal{U}_h}(x, y) \geq d_{\widetilde{M}}(x, y) \geq d(\mathcal{P}_\sigma(x), \mathcal{P}_\sigma(y)) = T.$$

Hence, all inequalities above become equalities. In particular, we have $d_{\widetilde{M}}(x, y) = d(\mathcal{P}_\sigma(x), \mathcal{P}_\sigma(y))$, which implies that the four points $\{x, y, \mathcal{P}_\sigma(x), \mathcal{P}_\sigma(y)\}$ are vertices of a totally geodesic flat rectangle $\Box_{t,t+T,h}$. □

We can describe the geometric consequences of non-flatness we shall use. Let $\widetilde{M}$ be a Cartan-Hadamard manifold and σ be a geodesic line of unit speed. Recall the family of cones $\Gamma_{\sigma,t,\theta} = \{x \in \widetilde{M} | \angle_{\sigma(t)}(\sigma'(t), x) < \theta\}$.

Proposition 4.5. *Suppose that the geodesic σ is $(h, T, \pi/4)$-non flat at 0. Then:*

$$\Gamma_{\sigma,T+h,3\pi/4} \subset \Gamma_{\sigma,0,\pi/2} \text{ and } \Gamma_{\sigma,T,\pi/2} \subset \Gamma_{\sigma,-h,\pi/4}. \tag{4.5}$$

Proof. Let us show the first inclusion, the proof of the other one is similar. It suffices to show that there is no geodesic triangle with one side $\sigma([0, T + h])$, another side τ in $\partial\Gamma_{\sigma,T+h,3\pi/4}$ and the third side in $\partial\Gamma_{\sigma,0,\pi/2}$. Suppose there is such a rectangle geodesic triangle $\triangle_{\sigma(0),\sigma(T+h),\tau(b)}$ with given three vertices $\{\sigma(T + h), \sigma(0), \tau(b)\}$, where $\tau(b) \in [(\partial\Gamma_{\sigma,T+h,3\pi/4}) \cap (\partial\Gamma_{\sigma,0,\pi/2})]$. We derive a contradiction as follows. We choose the vector field $\overrightarrow{Y} : \mathbb{R} \mapsto T_\sigma\widetilde{M}$ along σ with $\overrightarrow{Y} \perp \sigma'$ and $|\overrightarrow{Y}| \equiv 1$ in such a way that $\exp_{\sigma(t)}[S(t)\overrightarrow{Y}]$ lies in τ for some $S(t)$. As before, we let $\Box_{0,T,h} = \{\exp_{\sigma(t)}[s\overrightarrow{Y}] | 0 \leq t \leq T, 0 \leq s \leq h\}$. By comparison with the Euclidean plane, we have $S(t) \geq h$ for $0 \leq t \leq T$. Therefore the triangle $\triangle_{\sigma(0),\sigma(T+h),\tau(b)} = \{\exp_{\sigma(t)}[s\overrightarrow{Y}] | 0 \leq t \leq T + h, 0 \leq s \leq S(t)\}$ contains the subset $\Box_{0,T,h}$. By Proposition 4.3 (3)-(4), we have

$$-\int_{\triangle_{\sigma(0),\sigma(T+h),\tau(b)}} K_\triangle dA \geq -\int_{\Box_{t_1,t_2,h}} K_{\Box_{t_1,t_2,h}} dA = \int_{F(h,\cdot)} k_g dl = \frac{\partial l}{\partial h} > \frac{\pi}{4}.$$

This together with the Gauss-Bonnet formula implies that the sum of inner angles of $\triangle_{\sigma(0),\sigma(T+h),\tau(b)}$ is smaller than $(\pi - \frac{\pi}{4}) = 3\pi/4$, which is impossible. □

The same proof also yields:

Proposition 4.6. *Let $\varepsilon > 0$, and suppose that there is $t_+, t_- > \frac{h}{\tan\varepsilon}$ such that the geodesic σ is $(h, T, \pi/2)$-non flat at t_+ and $-t_-$. Then:*

$$\Gamma_{\sigma,T+t_+,\pi/2} \subset \Gamma_{\sigma,0,\varepsilon} \text{ and } \Gamma_{-\sigma,t_-,\pi/2} \subset \Gamma_{-\sigma,-T,\varepsilon}. \tag{4.6}$$

The main geometric estimate related to the Martin boundary is Ancona's Φ-chain condition. For a cone $\Gamma_{\sigma,0,\theta}$, it says that one can find a time T_0 such that, for $x \in \partial\Gamma_{\sigma,0,\theta}$,

$$d(x, \Gamma_{\sigma,T_0,\theta}) \to \infty \text{ as } d(x, \sigma(0)) \to \infty. \tag{4.7}$$

When $\widetilde{M} = \mathbb{R}^n$ is the Euclidean space, then for $x \in \partial\Gamma_{\sigma,0,\theta}$, $d(x, \Gamma_{\sigma,T_0,\theta}) \leq T_0$. and can NOT be unbounded. For the same reason, if $\sigma(\mathbb{R})$ is a boundary of a totally geodesic flat half plane $\mathbb{R}^2_+$, then (4.7) fails on $\mathbb{R}^2_+ \cap \partial\Gamma_{\sigma,0,\theta}$. However, the cone property (4.5) implies a stronger form of (4.7).

Proposition 4.7. *Let $0 < \theta \leq \frac{\pi}{2}$. If $\Gamma_{\sigma,T_0,\theta+\varepsilon_0'} \subset \Gamma_{\sigma,0,\theta}$ for some $T_0 > 0$ and $\varepsilon_0' > 0$, then*

$$d(x, \Gamma_{\sigma,T_0,\theta}) \geq \varepsilon_0[d(x, \sigma(0)) - \frac{1}{\varepsilon_0}] \tag{4.8}$$

for $x \in \partial\Gamma_{\sigma,0,\theta}$ and some $\varepsilon_0 > 0$ which depends only on ε_0' and T_0.

Proof. By our assumption, if $x \in \partial\Gamma_{\sigma,0,\theta}$ then

$$\angle_{\sigma(T_0)}(x, \Gamma_{\sigma,T_0,\theta}) \geq \varepsilon_0'. \tag{4.9}$$

Recall that $\exp_{\sigma(T_0)} : \mathbb{R}^n \mapsto \widetilde{M}$ is a distance increasing map. If

$$\overset{\circ}{\Gamma}_{\sigma,T_0,\theta} = \{\overrightarrow{\omega} \in T_{\sigma(T_0)}\widetilde{M} | \angle(\overrightarrow{\omega}, \sigma'(T_0)) \leq \theta\}$$

and $\overrightarrow{u}_x = \exp^{-1}_{\sigma(T_0)} x$, then by (4.9) we have

$$d_{\mathbb{R}^n}(\overrightarrow{u}_x, \overset{\circ}{\Gamma}_{\sigma,T_0,\theta}) \geq |\overrightarrow{u}_x| \sin\varepsilon_0'. \tag{4.10}$$

Since $\exp_{\sigma(T_0)} : \mathbb{R}^n \mapsto \widetilde{M}$ is distance increasing, we conclude that

$$d_{\widetilde{M}}(x, \Gamma_{\sigma,T_0,\theta}) \geq d(x, \sigma(T_0)) \sin\varepsilon_0' \geq [d(x, \sigma(0)) - T_0] \sin\varepsilon_0'.$$

Then we choose $\varepsilon_0 = \min\{\sin\varepsilon_0', \frac{1}{T_0 \sin\varepsilon_0'}\}$ and we obtain (4.8). □

5. Boundary Harnack Inequality and Martin boundary

5.1. Boundary Harnack Inequality, proof of Theorem 2.9.

We assume in this section that the geodesic τ admits a (h, T, R) barrier, and we are going to prove (2.5). The proof of (2.6) is the same.

Proposition 5.1. *Assume the geodesic $\tau : \mathbb{R} \mapsto \widetilde{M}$ is $(h, T, \pi/4)$-non flat at 0 and set $T_0 = T + h$. Denote $x_p = \tau(pT_0), p \geq 0$. Then there exists a constant $C = C(\widetilde{M}, h, T)$ such that*

$$G(y, x_p) \leq CG(x_0, x_p)G^\epsilon(y, x_1),$$
$$\forall y \in \widetilde{M} \setminus \Gamma_{\tau,0,\pi/2}, \ \forall p \geq 1. \tag{5.1}$$

Furthermore, for any $x \in \overline{x_p x_{p+1}}$, the line segment between x_p and x_{p+1} ($p \geq 1$), one has

$$G(y, x) \leq CG(x_0, x)G^\epsilon(y, x_1), \forall y \in \widetilde{M} \setminus \Gamma_{\tau,0,\pi/2}. \tag{5.2}$$

Proof. We denote $\Gamma = \Gamma_{\tau,0,\pi/2}, \Gamma_1 = \Gamma_{\tau,T_0,\pi/2}$.

To prove (5.1), we firstly prove the following inequality: there exists $C_p = C(p, \widetilde{M}, h, T)$ such that

$$G(y, x_p) \leq C_p G(x_0, x_p) G^\epsilon(y, x_1), \forall y \in \widetilde{M} \setminus \Gamma. \tag{5.3}$$

By Remark 3.9, we have, with $C_p'' = ce^{-\beta p T_0}$ for some c, β depending only on $(\widetilde{M}, \mathcal{L})$ and

$$G(x_0, x_p) \geq C_p''. \tag{5.4}$$

By construction, $B(x_1, h) \subset \Gamma$.

Take $y_0 \in \partial B(x_1, h)$, by Harnack inequality, then we obtain

$$G(y_0, x_p) \leq C_p' G(y_0, x_1).$$

Applying the Harnack inequality to the variable y, we have

$$G(y, x_p) \leq C_{p,1}' G(y, x_1), \forall y \in \partial B(x_1, h), \tag{5.5}$$

with $C_{p,1}' = C_{p,1}'(\widetilde{M}, d(x_1, x_p))$. Similarly we can prove that

$$G(y, x_p) \leq C_{p,2}' G(y, x_1), \forall y \in \partial B(x_p, h), \tag{5.6}$$

with $C_{p,2}' = C_{p,2}'(\widetilde{M}, d(x_1, x_p))$. Let $C_p = \max\{C_{p,1}', C_{p,2}'\}$. Combining (5.5) and (5.6), there is

$$G(y, x_p) \leq C_p G(y, x_1), \forall y \in \partial B(x_1, h) \cup \partial B(x_p, h).$$

Now using maximum principle, we have

$$G(y, x_p) \leq C_p G(y, x_1), \forall y \in \widetilde{M} \setminus B(x_1, h) \cup B(x_p, h).$$

In particular, we have

$$G(y, x_p) \leq C_p G(y, x_1) \leq C_p G^\epsilon(y, x_1), \forall y \in \widetilde{M} \setminus \Gamma. \tag{5.7}$$

By (5.4) and (5.7), we obtain (5.3).

The proof of Proposition 5.1 will consist in showing that one can take the constant in (5.3) independent of p. Observe indeed that to obtain (5.3), we only used the relative distances of x_0, x_1 and x_p and that $B(x_1, h) \subset \Gamma$. Therefore (5.3) can be applied to the cone Γ_1 to get

$$G(y, x_{p+1}) \leq C_p G(x_1, x_{p+1}) G^\epsilon(y, x_2), \forall y \in \widetilde{M} \setminus \Gamma_1. \tag{5.8}$$

Applying the Harnack inequality of $\mathcal{L}^* + \epsilon I$ to its Green function $G^\epsilon(y, x)$, one has

$$G^\epsilon(y, x_2) \leq C' G^\epsilon(y, x_1), \forall y \in \partial B(x_2, h).$$

Then maximum principle and $B(x_2, h) \subset \Gamma_1$ implies that

$$G^\epsilon(y, x_2) \leq C' G^\epsilon(y, x_1), \forall y \in \widetilde{M} \setminus \Gamma_1, \tag{5.9}$$

where $C' = C'(\widetilde{M}, T + h)$. By Harnack inequality, we have

$$G(x_1, x_{p+1}) \leq C'' G(x_0, x_{p+1}), \tag{5.10}$$

where $C'' = C''(\widetilde{M}, T + h)$ is independent of p for $p \geq 1$.

Combining (5.8), (5.9) and (5.10), one has

$$\begin{aligned} G(y, x_{p+1}) \leq & C_p C'' G(x_0, x_{p+1}) C' G^\epsilon(y, x_1) \\ = & C_p C G(x_0, x_{p+1}) G^\epsilon(y, x_1), \forall y \in \widetilde{M} \setminus \Gamma_1, \end{aligned} \tag{5.11}$$

where $C = C(\widetilde{M}, h, T)$.

Now fixing $\delta = \dfrac{1}{C}$, by Corollary 3.10, $\forall \epsilon > 0$, there exists $R_0 = R_0(\widetilde{M}, h, T, \epsilon)$ such that $\forall x \in \widetilde{M}$,and $r \geq R_0$, the harmonic measures μ_x and μ_x^ϵ on balls of radius r about x satisfy

$$\mu_x \leq \delta \mu_x^\epsilon. \tag{5.12}$$

Since the geodesic τ is $(h, T, \pi/4)$-non flat at 0, by Propositions 4.5 and 4.7 there is a ε_0 depending only on T, h such that

$$d(y, \partial\Gamma_1) \geq \varepsilon_0 (d(y, x_0) - \frac{1}{\varepsilon_0}). \tag{5.13}$$

Now we can take $\rho_1 = \rho_1(\varepsilon_0, R_0) = \rho_1(\widetilde{M}, h, T, \epsilon)$ such that for any $y \in \widetilde{M} \setminus \Gamma$ and $d(y, x_0) \geq \rho_1$ the following holds:

$$d(y, \partial\Gamma_1) \geq R_0. \tag{5.14}$$

For such y, the ball $B(y, R_0) \subset \widetilde{M} \setminus \Gamma_1$. By (5.11), for any $z \in \partial B(y, R_0)$, we have

$$G(z, x_{p+1}) \leq C_p C G(x_0, x_{p+1}) G^\epsilon(z, x_1).$$

So

$$\begin{aligned}
&\int_{\partial B(y,R_0)} G(z, x_{p+1}) d\mu_y(z) \\
&\leq C_p C \int_{\partial B(y,R_0)} G^\epsilon(z, x_1) d\mu_y(z) G(x_0, x_{p+1}) \\
&\leq C_p C \delta \int_{\partial B(y,R_0)} G^\epsilon(z, x_1) d\mu_y^\epsilon(z) G(x_0, x_{p+1}) \\
&\leq C_p G(x_0, x_{p+1}) G^\epsilon(y, x_1),
\end{aligned}$$

i.e.,

$$G(y, x_{p+1}) \leq C_p G(x_0, x_{p+1}) G^\epsilon(y, x_1), \tag{5.15}$$

for any $y \in \widetilde{M} \setminus \Gamma$ and $d(y, x_0) \geq \rho_1$.

There exists $C = C(\widetilde{M}, T_0)$ such that $G(x_0, x_1)C \geq 1$. Thus

$$G(y, x_{p+1}) \leq C G(x_0, x_{p+1}) G(y, x_1) \tag{5.16}$$

at $y = x_0$. Using Harnack inequality in the compact set $(\widetilde{M} \setminus \Gamma) \cap B_{\rho_1}(x_0)$, one has

$$G(y, x_{p+1}) \leq C' G(x_0, x_{p+1}) G(y, x_1), \tag{5.17}$$

for any $y \in (\widetilde{M} \setminus \Gamma) \cap B_{\rho_1}(x_0)$, where C' depends on ρ_1 and $C = C(\widetilde{M}, h, T)$ in (5.16), and hence depends only on $\widetilde{M}, h, T$, but not on the the integer p.

Combining (5.17) and (5.15), one obtains

$$G(y, x_{p+1}) \leq \max\{C_p, C'\} G(x_0, x_{p+1}) G^\epsilon(y, x_1). \tag{5.18}$$

So we can improve C_{p+1} such that $C_{p+1} = \max\{C_p, C'\}$.

Hence we can take a uniform constant $C = \max\{C_1, C'\}$ such that for any $p \geq 0$ the following inequality holds:

$$G(y, x_{p+1}) \leq C G(x_0, x_{p+1}) G^\epsilon(y, x_1), \forall y \in \widetilde{M} \setminus \Gamma \text{ and } \forall p \geq 0,$$

where C depends only on $\widetilde{M}, h, T$.

For the general point $x \in \overline{x_p x_{p+1}}$, one can use Harnack inequality because of $d(x_p, x_{p+1}) = T_0$. $\square$

Recall from Section 2 the definition of $\varepsilon^* = \varepsilon^*(\widetilde{M}, h, T)$.

Corollary 5.2. *Assume the geodesic σ is $(h, T, \pi/2)$-non flat at 0. Set $\Gamma = \Gamma_{\sigma,0,\pi/2+\varepsilon^*}$ and $\Gamma_1 = \Gamma_{\sigma,0,+\varepsilon^*} \cap \Gamma_{\sigma,h+T,\pi/2}$. Then, for any $y \in \widetilde{M} \setminus \Gamma$, for any $x \in \Gamma_1$, there is*

$$G(y, x) \leq CG(x_0, x)G^\epsilon(y, x_1),$$

where $C = C(\widetilde{M}, h, T), x_0 = \sigma(0)$, and $x_1 = \sigma(h + T)$.

Proof. By our choice of ε^* and Proposition 5.1, any geodesic τ which satisfies

$$\tau(0) \ = \ \sigma(0) \text{ and } \angle_{\sigma(0)}(\tau'(0), \sigma'(0)) \ < \ \varepsilon^*,$$

is $(h, T, \pi/4)$-non flat at 0. By Proposition 5.1 for any $x \in \tau([h + T, +\infty))$ and any $y \in \widetilde{M} \setminus \Gamma_{\tau,0,\pi/2}$, there is

$$G(y, x) \leq CG(x_0, x)G^\epsilon(y, \tau(h + T)), \tag{5.19}$$

where $C = C(\widetilde{M}, h, T)$. On the other hand, by comparison $d(x_1, \tau(h + T)) \leq \varepsilon^* \sinh(K(h + T))$, where $-K$ is a lower bound for the sectional curvature on $\widetilde{M}$, so that by Harnack inequality, there is a $C = C(\widetilde{M}, h, T)$ such that for any $y \in \widetilde{M} \setminus \Gamma_{\tau,0,\pi/2}$ the following holds

$$G^\epsilon(y, \tau(h + T)) \leq CG^\epsilon(y, x_1). \tag{5.20}$$

Combining (5.19) and (5.20), we have the conclusion for any $x \in \Gamma_{\sigma,0,\varepsilon^*}$ at distance at least $h + T$ from $\sigma(0)$ and any $y \in \widetilde{M} \setminus \cup_\tau \Gamma_{\tau,0,\pi/2}$, in particular for points $x \in \Gamma_1$ and $y \in \widetilde{M} \setminus \Gamma$. □

By Remark 2.7, we can apply Corollary 5.2 to $-\sigma$ and get:

Corollary 5.3. *Assume the geodesic σ is $(h, T, \pi/2)$-non flat at 0. Set $\Gamma' = \Gamma_{-\sigma,-T,\pi/2+\varepsilon^*}$ and $\Gamma'_1 = \Gamma_{-\sigma,-T,+\varepsilon^*} \cap \Gamma_{-\sigma,h,\pi/2}$. Then, for any $y \in \widetilde{M} \setminus \Gamma'$, for any $x \in \Gamma'_1$, there is*

$$G(y, x) \leq CG(x_0, x)G^\epsilon(y, x_1),$$

where $C = C(\widetilde{M}, h, T), x_0 = \sigma(T)$, and $x_1 = \sigma(-h)$.

We can now prove Theorem 2.9:

Proof. Since the geodesic σ admits a (h, T, R) barrier, there is t_4, t_5, with $h/\tan\varepsilon^* \leq t_5 - t_4 \leq T_1 + R$ such that σ is $(h, T, \pi/2)$-non flat at t_4. By Proposition 4.6, then

$$\Gamma_{-\sigma,-t_4,\pi/2} \ \subset \ \Gamma_{-\sigma,-t_5-T,\varepsilon^*}.$$

Moreover, there is t_6, with $T + \frac{h}{\tan\epsilon^*} \leq t_6 - t_5 \leq R$ such that σ is $(h, T, \pi/2)$-non flat at t_6 and by Proposition 4.6,

$$\Gamma_{\sigma,t_6+T,\pi/2} \ \subset \ \Gamma_{\sigma,t_5+T,\varepsilon^*}.$$

Applying Corollary 5.3 we get for any $y \in \Gamma_{\sigma,t_6+T,\pi/2}$ and for any $x \in \widetilde{M} \setminus \Gamma_{\sigma,t_4,\pi/2}$, there is

$$G(y, x) \leq C_1 G(\sigma(t_5 + T), x)G^\epsilon(y, \sigma(t_5 - h)), \tag{5.21}$$

where $C_1 = C_1(\widetilde{M}, h, T)$. Using Harnack inequality, we have with a different $C_1 = C_1(\widetilde{M}, h, T)$:

$$G(y, x) \leq C_1 G(\sigma(t_6 + T), x)G^\epsilon(y, \sigma(0)), \tag{5.22}$$

In the same way, using that the geodesic σ is $(h, T, \pi/2)$-non flat at t_1, t_2 and t_3, and Corollary 5.2, we can obtain that for any $x \in \widetilde{M} \setminus \Gamma_{\sigma,t_1,\pi/2}$ and for any $y \in \Gamma_{\sigma,t_3+T,\pi/2}$, there is

$$G(x,y) \leq C_2 G^{\epsilon}(x, \sigma(t_2+T+h)) G(\sigma(t_2), y), \tag{5.23}$$

and

$$G(x,y) \leq C_2 G^{\epsilon}(x, \sigma(0)) G(\sigma(t_1), y), \tag{5.24}$$

where $C_2 = C_2(\widetilde{M}, h, T)$.

Set $x_0 = \sigma(t_1), x' = \sigma(0)$ and $x_1 = \sigma(t_6+T)$, and let $\Gamma = \Gamma_{\sigma,t_1\pi/2}$, $\Gamma' = \Gamma_{\sigma,0,\pi/2}$ and $\Gamma_1 = \Gamma_{\sigma,t_6+T,\pi/2}$. We claim that the cone pair $\{\Gamma, \Gamma_1\}$ satisfies the conclusion of Theorem 2.9. Since $-T_2 \leq t_1$ and $t_6 + T \leq T_2$, Theorem 2.9 will follow.

We follow [An]. Let $y \in \widetilde{M} \setminus \Gamma$ and $x \in \Gamma_1$. We have the representation

$$\overset{*}{G}_y(x) = G(y,x) = \int_{\partial\Gamma'} G(y,z) d\mu_x(z),$$

where μ_x is a positive measure supported on $\partial\Gamma'$ and such that

$$G(\mu_x) = R^{\partial\Gamma'}_{G_x} = R^{\omega}_{G_x},\ \omega = \widetilde{M} \setminus \bar{\Gamma}'.$$

Now applying inequality (5.22) to $\overset{*}{G}$, we have

$$\begin{aligned} G(y,x) \leq & C \int_{\partial\Gamma'} G(y,x') G^{\epsilon}(x_0,z) d\mu_x(z) \\ \leq & C G(y,x_1) \int_{\partial\Gamma'} \overset{*}{G}{}^{\epsilon}_{x_0}(z) d\mu_x(z). \end{aligned} \tag{5.25}$$

Since $G^{\epsilon}(x_0,z) = \overset{*}{G}{}^{\epsilon}_{x_0}(z)$ is an $\mathcal{L}^*$-potential, so $\overset{*}{G}{}^{\epsilon}_{x_0}(z) = \overset{*}{G}(\lambda)$ for some positive measure λ on $\widetilde{M}$. We have

$$\int_{\partial\Gamma'} \overset{*}{G}{}^{\epsilon}_{x_0}(z) d\mu_x(z) = \int_{\widetilde{M}} \int_{\partial\Gamma'} G(y,z) d\mu_x(z) d\lambda(y) = \int_{\widetilde{M}} G(\mu_x) d\lambda(y) \tag{5.26}$$

By inequality (5.24), for any $z \in \partial\Gamma'$, there is

$$G_x(z) \leq C G_x(x') G^{\epsilon}_{x_1}(z) \leq C G_x(x_0) G^{\epsilon}_{x_1}(z). \tag{5.27}$$

By the definition of reduit, we have by (5.27),

$$G(\mu_x) = R^{\partial\Gamma'}_{G_x} \leq G_x \leq C G(x_0,x) G^{\epsilon}_{x_1}. \tag{5.28}$$

Substitute (5.28) into (5.26) and then plug in (5.25), then we can obtain

$$G(y,x) \leq C G(y,x_1) G(x_0,x) \int_{\widetilde{M}} G^{\epsilon}_{x_1} d\lambda. \tag{5.29}$$

Now we only need to show that $\int_{\widetilde{M}} G^{\epsilon}_{x_1} d\lambda$ is bounded. By Lemma 3.1, there is

$$\overset{*}{G}{}^{\epsilon}_{x_0} = \overset{*}{G}_{x_0} + \epsilon \overset{*}{G}(\overset{*}{G}{}^{\epsilon}_{x_0}).$$

Since $\overset{*}{G}{}^{\epsilon}_{x_0} = \overset{*}{G}(\lambda)$, so

$$\lambda = -\mathcal{L}^*(\overset{*}{G}{}^{\epsilon}_{x_0}) = \delta_{x_0} + \epsilon \overset{*}{G}{}^{\epsilon}_{x_0} dv_{\widetilde{M}}.$$

Thus

$$\begin{aligned}\int \overset{*}{G}{}^{\epsilon}_{x_1} d\lambda(z) =& G^{\epsilon}(x_1,x_0) + \epsilon \int G^{\epsilon}(z,x_1)G^{\epsilon}(x_0,z)dv_{\widetilde{M}}(z)\\ =& G^{\epsilon}(x_1,x_0) + \epsilon \overset{*}{G}{}^{\epsilon}(\overset{*}{G}{}^{\epsilon}_{x_1})(x_0)\\ =& \overset{*}{G}{}^{\epsilon}_{x_1}(x_0) + \epsilon \overset{*}{G}{}^{\epsilon}(\overset{*}{G}{}^{\epsilon}_{x_1})(x_0) \end{aligned} \tag{5.30}$$

Using Lemma 3.1 again to the operator $\mathcal{L}^* + \epsilon I$ and $\frac{\epsilon}{2}$ instead of $\mathcal{L}$ and ϵ, we have

$$\overset{*}{G}{}^{\epsilon}_{x_1}(x_0) + \epsilon \overset{*}{G}{}^{\epsilon}(\overset{*}{G}{}^{\epsilon}_{x_1})(x_0) \le \widehat{G}(x_0,x_1) \le C, \tag{5.31}$$

where $\widehat{G}(x,y)$ is the Green function of $\mathcal{L}^* + \frac{\epsilon}{2}$. By Proposition 3.5, the constant C here only depends on $M, \mathcal{L}$ and the distance between x_0 and x_1 and hence depend only on $M, \mathcal{L}, \theta, \tau$.

Combining (5.29), (5.30) and (5.31), the whole proof is finished. □

5.2. **Proof of Proposition 2.10.**

Proof. Let $u(x), v(x) \in C_{\xi}$. Since $v(x) = O(G_{x_0}) = o(w)$, for some $\mathcal{L}$-superharmonic function, for $x \in \bar{\Gamma}_{-\tau,-t_k-2T_2,\pi/2} \cap \widetilde{M}(\infty)$, the reduit of $v_1(x)$ of $v(x)$ on $\Gamma_{-\tau,-t_k-T_2,\pi/2}$ with respect to $\Gamma_{-\tau,-t_k-2T_2,\pi/2}$ is a potential by Proposition 3.13. So it has the representation formula:

$$v_1(x) = \int_{\partial\Gamma_{-\tau,-t_k-T_2,\pi/2}} g^k(x,y)d\nu^k(y), \forall x \in \Gamma_{-\tau,-t_k+T_2,\pi/2},$$

where $\nu_k(y)$ is positive measure on $\partial\Gamma_{-\tau,-t_k-T_2,\pi/2}$. According to the definition of reduit, we have

$$v(x) = v_1(x) \le C \int_{\partial\Gamma_{-\tau,t_k,\pi/2}} g^k(x,\tau(t_k))g^k(\tau(t_k - T_2),y)d\nu^k(y), \forall x \in \Gamma_{-\tau,-t_k+T_2,\pi/2}, \tag{5.32}$$

where we used (2.6) and the constant C here is independent of k.

By (5.32), there is

$$v(x) \le Cg^k(x,\tau(t_k + T_2))v(\tau(t_k - T_2)). \tag{5.33}$$

On the other hand, by Harnack inequality and the maximum principle, we have

$$u(x) \ge C'u(\tau(t_k - T_2))g^k(x,\tau(t_k + T_2)), \forall x \in \Gamma_{-\tau,-t_k-2T_2,\pi/2} \setminus B(\tau(t_k + T_2), \frac{T_2}{2}).$$

In particular, one has

$$u(x) \ge C'u(\tau(t_k - T_2))g^k(x,\tau(t_k + T_2)), \forall x \in \Gamma_{-\tau,-t_k+T_2,\pi/2}. \tag{5.34}$$

Combining (5.33) and (5.34), there is

$$\frac{v(x)}{u(x)} \le C\frac{v(\tau(t_k - T_2))}{u(\tau(t_k - T_2))}, \tag{5.35}$$

where C is independent of k. Similarly, one has

$$C^{-1}\frac{v(\tau(t_k - T_2))}{u(\tau(t_k - T_2))} \le \frac{v(x)}{u(x)} \le C\frac{v(\tau(t_k - T_2))}{u(\tau(t_k - T_2))}, \forall x \in \Gamma_{-\tau,-t_k+T_2,\pi/2}. \tag{5.36}$$

Let $x = \tau(t_0)$, then

$$\frac{v(x)}{u(x)} \le C^2\frac{v(\tau(t_0))}{u((\tau(t_0)))} := \lambda, \forall x \in \Gamma_{-\tau,-t_k+T_2,\pi/2}.$$

Since λ is independent of k, we can let $k \to \infty$ and obtain

$$\frac{v(x)}{u(x)} \leq \lambda, \forall x \in \widetilde{M},$$

which implies $\dim C_\xi \leq 1$. $\square$

5.3. **Proof of Proposition 2.11.**

Proof. Let $x_k = \tau(t_k)$, where $t_k \to \infty$ are the barrier times. For $z_i \to \xi$ as $i \to \infty$, define $k^i_{x_q} := \frac{G(x,z_i)}{G(x_q,z_i)}$. Then $k^i_{x_q}(x)$ is a sequence of positive harmonic functions on $\widetilde{M}$ and that satisfy the normalization condition at x_q : $k^i_{x_q}(x_q) = 1$. Therefore there exists a subsequence such that $\lim_{i\to\infty} k^i_{x_q}(x) = k_{x_q}(x)$ exists (where we still use the same index i).

For i large enough, $z_i \in \Gamma_{\tau,t_q+T_2,\pi/2}$ and, setting $\Gamma_q = \Gamma_{\tau,t_q-T_2,\pi/2}$, Theorem 2.9 gives that

$$k_{x_q}(x) \leq CG(x, x_q), \forall x \in \widetilde{M} \setminus \Gamma_q. \tag{5.37}$$

Equation (5.37) means that

$$k_{x_q}(\xi') = 0 \text{ in the } \mathcal{L}\text{-sense}, \ \forall \xi' \in \widetilde{M}(\infty) \setminus \overline{\Gamma_q}. \tag{5.38}$$

Similarly we can consider the positive harmonic function normalized at x_0, i.e., $k_{x_0}(x)$.

By Harnack inequality, there exists a constant C_q such that for sufficiently large i the following holds:

$$C_q^{-1} G(x_q, z_i) \leq G(x_0, z_i) \leq C_q G(x_q, z_i).$$

Therefore we have

$$\frac{G(x, z_i)}{G(x_0, z_i)} \leq C_q \frac{G(x, z_i)}{G(x_q, z_i)},$$

for large i.

So we have

$$k_{x_0}(x) \leq C_q k_{x_q}(x) \leq C_q G(x, x_q), \forall x \in \widetilde{M} \setminus \Gamma_q. \tag{5.39}$$

On the other hand, by Proposition 4.6, for any $\varepsilon > 0$, there is a $q \in \mathbb{N}$ such that $\Gamma_q \subset \Gamma_{\tau,0,\varepsilon}$. Combining this fact and (5.39), we obtain

$$k_{x_0}(\xi') = 0 \text{ in the } \mathcal{L}\text{-sense}, \ \forall \xi' \in \widetilde{M}(\infty) \setminus \{\xi\}.$$

Since $k_{x_0}(x_0) = 1$, k_{x_0} is a nontrivial positive harmonic function on $\widetilde{M}$, and hence is a Poisson kernel function at ξ. $\square$

6. Abundance of Martin points

We assume in this section that $\widetilde{M}$ is the universal cover of a compact Riemannian manifold M of class C^3, nonpositive curvature and geodesic rank one. For each $v \in T\widetilde{M}$, we can write $v = (x, \vec{\theta})$ with $\vec{\theta} \in T_x\widetilde{M}$. If $Exp : T\widetilde{M} \mapsto \widetilde{M}$ is the exponential map given by $Exp(v) = Exp_x(\vec{\theta})$. We always write

$$\sigma_v(t) = Exp_x(t\vec{\theta})$$

and

$$\varphi_s(v) = (Exp_x(s\vec{\theta}), \frac{d[Exp_x(s\vec{\theta})]}{ds}) = (\sigma_v(s), \sigma_v'(s)).$$

The 1-parameter family of diffeomorphisms $\{\varphi_s\}$ is called the geodesic flow on $S\widetilde{M}$.

Let $\pi : S\widetilde{M} \mapsto \widetilde{M}$ be the foot-point projection. For any given $v \in S\widetilde{M}$, we consider the Busemann function

$$\hat{b}_v(x) = \lim_{t\to+\infty}[d(\sigma_v(0), \sigma_v(t)) - d(x, \sigma_v(t))] = \lim_{t\to+\infty}[t - d(x, \sigma_v(t))],$$

for $x \in \widetilde{M}$.

The the level set $\Sigma_v = \hat{b}_v^{-1}(0)$ is called a horosphere with the inner normal vector v. We also let

$$\mathcal{H}_v = \{(y, \nabla\hat{b}_v|_y)| \quad \hat{b}_v(y) = 0\}$$

be the corresponding stable leave. Clearly $\Sigma_v = \pi(\mathcal{H}_v)$ and since M is of class C^3, $\mathcal{H}_v$ is a C^2-smooth embedded disc in $S\widetilde{M}$. As v varies, the sets $\mathcal{H}_v$ form a continuous lamination of $S\widetilde{M}$ (cf [HI], Proposition 3.1).

Furthermore, since all geodesic balls are convex, the sup-level set $\hat{b}_v^{-1}([c,\infty))$ is convex for all $c \in \mathbb{R}$, see [BGS].

Suppose that sectional curvatures K of $\widetilde{M}$ satisfy $-1 \le K \le 0$. The standard Hessian comparison theorem [Pe] asserts

$$\|X\|^2 \ge \mathrm{Hess}(-\hat{b}_v)(X,X) \ge 0.$$

Proposition 6.1. *Suppose that sectional curvatures K of $\widetilde{M}$ satisfy $-1 \le K \le 0$. Then for any given $\varepsilon > 0$, there is $\eta > 0$ such that if $v' \in \mathcal{H}_v$ satisfies $d_{\mathcal{H}_v}(v, v') < \eta$, then $d_{S\widetilde{M}}(\varphi_t(v), \varphi_t(v')) < \varepsilon$ for all $t \ge 0$.*

Proof. If Ω is a convex subset of $\widetilde{M}$, the nearest point projection $\mathcal{P}_\Omega : \widetilde{M} \mapsto \Omega$ is a distance non-increasing map (see [BGS]). Consider $\Omega_s = \hat{b}_v^{-1}([s,\infty))$. Since Ω_s is convex, $\mathcal{P}_{\Omega_s}$ is distance non-increasing map, and

$$d_{\Sigma_{\varphi_t(v)}}(\sigma_v(t), \sigma_{v'}(t)) \le d_{\Sigma_v}(v, v') < \eta$$

for all $t \ge 0$.

Recall that $\mathrm{Hess}(\hat{b}_v)(X,Y) = \langle \nabla_X(\nabla\hat{b}_v), Y\rangle$. Let $\Psi_t : [0,\eta] \mapsto \Sigma_{\varphi_t(v)}$ be a length-minimizing geodesic from $\sigma_v(t)$ to $\sigma_{v'}(t)$ with respect to the induced metric on the horosphere $\Sigma_{\varphi_t(v)} = b_v^{-1}(t)$ of height t. By the fact that $\|X\|^2 \ge \mathrm{Hess}(-\hat{b}_v)(X,X) \ge 0$ and since $d_{\Sigma_{\varphi_t(v)}}(\sigma_v(t), \sigma_{v'}(t)) < \eta$, we obtain, by integrating along the curve Ψ_t,

$$\|\sigma_{v'}'(t) - \mathbb{P}_{\Psi_t}[\sigma_v'(t)]\| = \|\nabla\hat{b}_v|_{\sigma_{v'}(t)} - \mathbb{P}_{\Psi_t}[\nabla\hat{b}_v|_{\sigma_v(t)}]\| \le \int_{\Psi_t} \|\nabla(\nabla\hat{b}_v)\| < \eta,$$

where $\mathbb{P}_\Psi$ is the parallel translation along the curve Ψ_t. It follows that

$$d_{S\widetilde{M}}(\varphi_t(v), \varphi_t(v')) < 2\eta.$$

This completes the proof. □

Fix $x \in \widetilde{M}$ and let ν be the Patterson-Sullivan measure on $\widetilde{M}(\infty)$ associated to x (see [K1]). Recall that $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$ is the set of distinct pairs of points of the geometric boundary $\widetilde{M}(\infty)$ and that the action of the covering group Γ extends to $\widetilde{M}(\infty)$ by continuity and on $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$ by $\gamma(\eta, \xi) = (\gamma\eta, \gamma\xi)$. In [K1], [K2] the following properties of ν are shown:

- 1)([K1], Lemma 4.1) The support of the measure $\nu \times \nu$ is $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$.
- 2)([K2], Corollary 4.4) For $(\nu \times \nu)$ almost every (η, ξ), there is a unique regular geodesic $\sigma_{\eta,\xi}$ such that: $\sigma_{\eta,\xi}(-\infty) = \eta, \sigma_{\eta,\xi}(+\infty) = \xi$.
- 3)([K2], Lemma 2.4) There is a positive continuous function F on $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$ such that the measure $\tilde{\nu} = F(\nu \times \nu)$ is Γ invariant.

To a vector $v \in SM$, one associates $Q(v) \in (\widetilde{M}(\infty) \times \widetilde{M}(\infty))^* \times \mathbb{R}$ by:

$$Q(v) \;=\; (\sigma_v(-\infty), \sigma_v(+\infty), b(v))$$

where $b(v) = \lim_{t\to\infty}(d(x, \sigma_v(t)) - t)$. The map Q is a bijection from $S\widetilde{M}$ on its image. By 2), its image has full $\tilde{\nu} \times dt$ measure. The measure $(Q^{-1})_*(\tilde{\nu} \times dt)$ is therefore a measure on $S\widetilde{M}$. By 3) it is a Γ invariant measure. By definition, it is also invariant under the geodesic flow. It corresponds to a measure $\bar{\nu}$ on SM, which is invariant under the geodesic flow. By 1), the support of $\bar{\nu}$ is SM. By [K2], Theorem 4.3, the measure $\bar{\nu}$ is ergodic under the geodesic flow.

The unit sphere $S_x\widetilde{M}$ is transversal to the foliation $\mathcal{H}$ and to the orbits of the geodesic flow, so that a tubular neighborhood of $S_x\widetilde{M}$ will contain a neighborhood of the form $\cup_{v\in S_x\widetilde{M}} \left(\cup_{s,|s|\le\rho}\varphi_s\mathcal{U}_v\right)$ where $\mathcal{U}_v$ is a neighborhood of v in $\mathcal{H}_v$. Consider the measure ν_x on $S_x\widetilde{M}$ defined by $\nu_x = (P_x^{-1})_*\nu$. On a neighborhood of $S_x\widetilde{M}$ of the above form, the measure $\tilde{\nu}\times dt$ has a positive density with respect to the integral over ν_x of positive measures with full support on $\cup_{s,|s|\ge\rho}\varphi_s\mathcal{U}_v$ (see [L], section 3, for the completely analogous case of negative curvature; another description of this product structure is in [Gu]). This shows the following:

Proposition 6.2. *Let A be a Borel subset of $S_x\widetilde{M}$ with $\nu_x(A) > 0$. Then, for all $\eta > 0$, $(\tilde{\nu} \times dt)(\widetilde{A}_\eta) > 0$, where*

$$\widetilde{A}_\eta \;=\; \cup_{v\in A}\left(\cup_{s,|s|\le\eta}\varphi_s B^{\mathcal{H}_v}(v, \eta)\right)$$

and $B^{\mathcal{H}_v}(v, \eta)$ is the ball of radius η in $\mathcal{H}_v$ centered at v.

We now are able to show that non-hyperbolic directions are ν negligible. More precisely, there is

Proposition 6.3. *There exist h, T and R such that, if $\mathcal{F}_K$ is the set of directions $v \in S_x\widetilde{M}$ such that $\sigma_v(\cdot - t)$ never admits a (h, T, R) barrier for any $t \ge K$, then $\mathcal{F}_K$ has no interior in $S_x\widetilde{M}$ and $\nu_x(\mathcal{F}_K) = 0$.*

Proof. Recall the set $\mathcal{O}_K$ from section 2. For the sake of the proof, we introduce a slightly smaller set $\mathcal{O}'_K$ which is also generic and full measure, but is disjoint from $(\widetilde{\mathcal{F}_K})_\eta$. The conclusion follows then from Proposition 6.2. Fix $\delta > 0$ small.

Definition 6.4. We say that the geodesic σ admits a (h, T, R, δ) barrier if there exist $t_i, i = 1, 2, \ldots, 6$ with $T_1 + i\delta < t_{i+1} - t_i < T_1 + R - i\delta$ and $t_3 + T < 0 < t_4$ such that the geodesic σ is $(h, T, \pi/2 + \delta)$-non flat at t_i, for $i = 1, 2, \ldots, 6$.

As before, one can find h, T and R such that there is an axis that admits a (h, T, R, δ) barrier. By Proposition 4.2 the set $\mathcal{O}'$ of v such that σ_v admits a (h, T, R, δ) barrier is open. Since the measure $\bar{\nu}$ is ergodic and has full support, for all positive K the set $\mathcal{O}'_K$ of $v \in SM$ such that the geodesic ray $\sigma_v([K, \infty))$

intersects $\mathcal{O}'$ is open dense in SM and has full $\bar{\nu}$ measure. By Proposition 4.2 again, we can find a number $\varepsilon > 0$ such that whenever $w \in S\widetilde{M}$ is such that σ_w admits a (h, T, R, δ) barrier and $d_{S\widetilde{M}}(w, w') < \varepsilon$, then $\sigma_{w'}$ admits a (h, T, R) barrier. Choose η associated to ε by Proposition 6.1. Now, if $v \in \mathcal{F}_K$ and $v' \in \cup_{s,|s| \leq \eta} \varphi_s B^{\mathcal{H}_v}(v, \eta)$, then v' cannot belong to $\mathcal{O}'_K$ since it would mean that there is a $t > K$ such that $\sigma_{v'}(t)$ admits a (h, T, R, δ) barrier. Since $d_{S\widetilde{M}}(\sigma_v(t), \sigma_{v'}(t)) < \varepsilon$, $\sigma_v(t)$ would admit a (h, T, R) barrier, contrarily to the definition of $\mathcal{F}_K$. We have shown that $(\widetilde{\mathcal{F}_K})_\eta$ is disjoint from $\mathcal{O}'_K$, an open set of full $\bar{\nu}$ measure. By Proposition 6.2, $\nu_x(\mathcal{F}_K) = 0$. It is also easy to see that for the same reason, $\mathcal{F}_K$ has no interior. $\square$

This proves the first part of Theorem 1.7, since the set of non-hyperbolic geodesics starting from x is exactly the union over $K \in \mathbb{N}$ of the $\mathcal{F}_K$s. For the second part, recall from the introduction the definition of a geodesic ergodic measure on $\widetilde{M}(\infty)$:

- 1) The support of the measure $\mu \times \mu$ is $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$.
- 2) For $\mu \times \mu$ almost every (η, ξ), there is a unique geodesic $\sigma_{\eta,\xi}$ such that: $\sigma_{\eta,\xi}(-\infty) = \eta, \sigma_{\eta,\xi}(+\infty) = \xi$, and $\sigma_{\eta,\xi}$ is rank 1.
- 3) The measure $\mu \times \mu$ is Γ quasi-invariant and ergodic: the diagonal action of Γ preserves the $(\mu \times \mu)$negligible subsets of $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$ and measurable subsets of $(\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$ which are Γ invariant are either negligible or conegligible.

For $(\eta, \xi) \in (\widetilde{M}(\infty) \times \widetilde{M}(\infty))^*$, define $N(\eta, \xi)$ as the number of times, separated by at least $4T_2$, that the geodesic $\sigma_{\eta,\xi}$, if it is unique, admits a (h, T, R) barrier. By property 2) above the function $N(\eta, \xi)$ is $(\mu \times \mu)$ almost everywhere well defined. Moreover, the function $N(\eta, \xi)$ clearly is Γ invariant and therefore $(\mu \times \mu)$ almost everywhere constant. We claim that this constant cannot be a finite K. Indeed, we just proved that there is an open set $\mathcal{O}''$, such that for $\xi \in \mathcal{O}''$, there is a regular geodesic σ_ξ with $\sigma(0) = x, \sigma(+\infty) = \xi$ and at least $K+1$ instants $t_1, \ldots, t_{K+1}$ with $t_j - t_{j+1} > 4T_2$, when $\sigma_\xi(\cdot - t_j)$ admits a (h, T, R) barrier . For such a ξ, we can find, by ([Ba1]) a small neighborhood O_ξ of $\sigma_\xi(-\infty)$ such that for $\eta \in O_\xi$, there is a unique $\sigma_{\eta,\xi}$, and it is close enough to σ_ξ that we still have $N(\eta, \xi) \geq K + 1$. Since $\mu \times \mu \left(\cup_{\xi \in \mathcal{O}''}(O_\xi \times \{\xi\})\right) > 0$, this is a contradiction.

So, for $(\mu \times \mu)$ almost every (η, ξ), $N(\eta, \xi)$ is infinite. Let N_+, N_-, N be the subsets of $\{(\eta, \xi) : N(\eta, \xi) = \infty\}$ where there are an infinite number of barrier times respectively only on the positive side of $\mathbb{R}$, only on the negative side or on both sides. These three sets are disjoint and Γ invariant. Only one of them is of full measure. By Remark 2.7, the sets N_+ and N_- have the same measure, which has to be 0. Therefore, the set N has full measure. In other words, $(\mu \times \mu)$ almost every geodesic is hyperbolic. It follows that for μ almost every $\xi \in \widetilde{M}(\infty)$, there is at least one geodesic which is asymptotic to ξ and hyperbolic. Theorem 1.7 follows from Theorem 1.3.

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