Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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clamped elastic membranes leads to relaxation

by

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We consider the nonlinear elastic energy of a thin membrane whose boundary is kept fixed, and assume that the energy per unit volume scales as h^β, with h the film thickness and $\beta \in (0, 4)$. We derive, by means of Γ convergence, a limiting theory for the scaled displacements, which takes a form similar to the one proposed by Föppl in 1907. The difference can be understood as due to the fact that we fully incorporate the possibility of buckling, and hence derive a theory which does not have any resistence to compression. If forces normal to the membrane are included, then our result predicts that the normal displacement scales as the cube root of the force. This scaling depends crucially on the clamped boundary conditions. Indeed, if the boundary is left free then a much softer response is obtained, as was recently shown by Friesecke, James and Müller.

1 Introduction

Reduced theories for thin elastic bodies have been proposed and used since the early days of the theory of elasticity, but only in the last decade it has become possible to derive them rigorously from three-dimensional nonlinear elasticity. The convergence criterion which has been used for these problems is Γ-convergence, and the different physical regimes are reflected by different energy scalings and different topologies on the space of deformations [13, 14, 8, 9, 16, 17, 10] (we refer to [10] for a review of the recent mathematical literature and of the mechanical context).

One key property of the elasticity of thin bodies is that tangential displacements enter the strain to first order, but normal displacements only to second order (see Figure 1). Therefore linear theories are not usable, as they would describe all normal displacements as completely stress-free (soft). The first nonvanishing contribution of normal displacements to strain is quadratic, and correspondingly the leading energy contribution is of fourth order.

A generalization of the linear theory which incorporates the normal displacements to leading order was proposed by Föppl [7]. In a variational language, and for the special case of isotropic elastic moduli and zero Poisson's ratio, his model corresponds
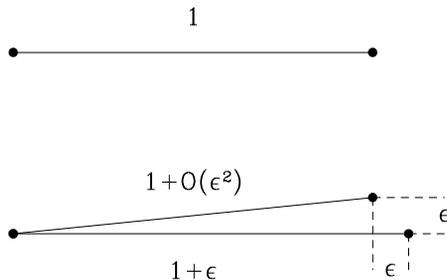
Figure 1: Consider a rod of unit length. If one endpoint is displaced tangentially by ϵ, the length also changes by ϵ. If instead the endpoint is displaced by ϵ in the normal direction, then the length only changes to order ϵ^2.

to minimizing

$$\frac{1}{2} \int_S \left| \nabla u + \nabla u^T + \nabla v \otimes \nabla v \right|^2 dx' \tag{1}$$

subject to appropriate boundary conditions and forces. Here $S \subset \mathbb{R}^2$ represents the cross-section of the membrane, $u : S \to \mathbb{R}^2$ the tangential displacement, and $v : S \to \mathbb{R}$ the normal displacement.

The functional (1) is not lower semicontinuous. Physically, a sheet subject to moderate compression can relax its strain by forming fine-scale folds, which are not penalized by the functional (1) since it does not contain any curvature term. (We note in passing that even if bending energy is included compression is often still relaxed by fine-scale oscillations, see e.g. [3, 6]).

It is therefore to be expected that a variational derivation will not lead to the functional (1), but to its relaxation. Indeed, we show here that under suitable scaling assumptions and with clamped boundary conditions three-dimensional elasticity reduces, in the sense of Γ-convergence, to a functional corresponding to the relaxation of (1), which, for the same special case, takes the form

$$\frac{1}{2} \int_S W_{\text{rel}} \left(\nabla u + (\nabla u)^T + \nabla v \otimes \nabla v \right) dx' \tag{2}$$

where $W_{\text{rel}}(F) = (\lambda_1^+(F))^2 + (\lambda_2^+(F))^2$, $\lambda_1(F)$ and $\lambda_2(F)$ are the eigenvalues of the symmetric matrix F and $\lambda^+ = \max\{\lambda, 0\}$.

Our result, as it will be explained in greater detail in the next section, has important consequences for the scaling behavior of the response of clamped membranes. Consider indeed application of a force $f_h(x') = h^\alpha f(x')$ normal to the membrane.

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If $\alpha \in (0,3)$, then our convergence result applied for $\beta = 4\alpha/3$ implies that the three-dimensional variational problems converge as $h \to 0$ to the relaxed problem $I_0(u,v) + \int_S fv dx'$, for I_0 like in (2). The tangential displacements scale as $h^{\beta/2} = h^{2\alpha/3}$, the normal one as $h^{\beta/4} = h^{\alpha/3}$.

For $\alpha > 3$ one obtains a different limiting theory, which is quadratic and involves only bending energy (see e.g. [10]). The limit functional takes the form $\int |\nabla^2 v|^2 + fv$. In this regime the out-of-plane displacement is linear in the applied force and thus scales like h^α. Understanding the cross-over from the linear to the sublinear scaling, which had also been observed experimentally, was an important motivation for the work of Föppl and von Kármán [20]. Indeed von Kármán points out that his theory interpolates between the linear (pure bending) theory and Föppl's theory [20, p. 350][1].

Notation The vectors e_1, e_2 and e_3 form an orthonormal basis of $\mathbb{R}^3$, and $\mathbb{R}^2$ is the space generated by e_1 and e_2. To every element $x = x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathbb{R}^3$ we associate $x' := x_1 e_1 + x_2 e_2 \in \mathbb{R}^2$. Thus $x = x' + x_3 e_3$.

The space of tensors generated by $\{e_i \otimes e_j\}_{i,j=1,2,3}$ is denoted by $\mathbb{R}^{3 \times 3}$, and $\mathbb{R}^{2 \times 2}$ is the subspace of $\mathbb{R}^{3 \times 3}$ generated by the tensors $\{e_i \otimes e_j\}_{i,j=1,2}$. To every $F = \sum_{i,j=1,2,3} F_{ij} e_i \otimes e_j \in \mathbb{R}^{3 \times 3}$ we associate $F' := \sum_{i,j=1,2} F_{ij} e_i \otimes e_j \in \mathbb{R}^{2 \times 2}$. By $\mathbb{R}^{n \times n}_{\text{sym}}$ we denote the space of symmetric matrices, and by $\mathbb{R}^{n \times n}_+$ the subsets of positive semidefinite symmetric ones (i.e. $\{F \in \mathbb{R}^{n \times n}_{\text{sym}} : F \geq 0\}$). Finally Id_n is the identity matrix in $\mathbb{R}^{n \times n}$.

2 The relaxed Föppl functional

We consider the nonlinear elastic energy of a thin three-dimensional body $\Omega_h := S \times (-h/2, h/2)$, where $S \subset \mathbb{R}^2$ is the cross section and $h > 0$ the (small) thickness. The deformation is a map $w_h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$, and its elastic energy per unit thickness is

$$E(w_h, \Omega_h) := \frac{1}{h} \int_{\Omega_h} W(\nabla w_h(x)) dx.$$

The stored energy function W is assumed to satisfy

(W1) $W : \mathbb{R}^{3 \times 3} \to [0, \infty]$ is a Borel measurable function of class C^2 in an open neighborhood of $SO(3)$;

(W2) $W(RF) = W(F)$ for every $R \in SO(3)$ and every $F \in \mathbb{R}^{3 \times 3}$; furthermore $W(\text{Id}_3) = 0$.

(W3) $W(F) \geq C\text{dist}^2(F, SO(3))$ for every $F \in \mathbb{R}^{3 \times 3}$.

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[1] "In dieser Hinsicht liegt die wirkliche Platte zwischen den beiden Grenzfällen der *vollkommen steifen Platte* nach Gl. (27) und der *vollkommen biegsamen Platte*, deren Gleichungen sich aus dem System (29) mit $D = 0$ ergeben." In this regard the real plate lies in between the two limiting cases of the completely stiff plate according to Eq. (27) and the completely flexible plate, whose equations are obtained from the system (29) [i.e the vK equations] with $D = 0$.

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We study the asymptotic behavior as $h \to 0$ of the minimization problems

$$\inf \left\{ \frac{E(w_h, \Omega_h)}{h^\beta} : w_h \in W^{1,2}(\Omega_h, \mathbb{R}^3), \ w_h(x) = x \text{ on } \partial S \times (-h/2, h/2) \right\}$$

in the range $\beta \in (0,4)$, by means of Γ-convergence theory.

In order to define an appropriate convergence criterion for a sequence of deformations w_h, which are all defined on different domains, we rescale (following standard practice) to a unique domain. Precisely, for each $w_h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$ we define $y_h \in W^{1,2}(\Omega_1, \mathbb{R}^3)$ by $y_h(x) = w_h(x' + hx_3e_3)$. Then

$$E_h(w_h, \Omega_h) = \int_{\Omega_1} W(\nabla_h y_h(x))dx,$$

where ∇_h is the operator $\nabla_h := \nabla' + (1/h)\partial_3 \otimes e_3$, i.e.,

$$\nabla_h y(x) = \partial_1 y(x) \otimes e_1 + \partial_2 y(x) \otimes e_2 + \frac{1}{h}\partial_3 y(x) \otimes e_3 \,.$$

In terms of the rescaled deformations, and including the constraint given by the boundary conditions, our problem corresponds to minimizing the functional $I_h : W^{1,2}(\Omega_1, \mathbb{R}^3) \to [0, \infty]$ given by

$$I_h(y) := \begin{cases} \int_{\Omega_1} W(\nabla_h y(x))dx & \text{if } y(x) = x' + hx_3e_3 \text{ for } x \in \partial S \times (-\frac{1}{2}, \frac{1}{2}), \\ +\infty, & \text{else.} \end{cases}$$

Due to the boundary conditions and to the energy regime under consideration, the behavior of a low energy sequence y_h will be understood by considering the scaled displacements

$$u_h(x') \quad := \quad \frac{1}{h^{\beta/2}} \int_0^1 (y_h(x) - x)' \, dx_3, \tag{3}$$

$$v_h(x) \quad := \quad \frac{1}{h^{\beta/4}} \int_0^1 (y_h(x) - hx) \cdot e_3 \, dx_3. \tag{4}$$

Note that for every h we have $u_h \in W_0^{1,2}(S, \mathbb{R}^2)$ and $v_h \in W_0^{1,2}(S)$. However, for a sequence y_h such that $h^{-\beta}I_h(y_h)$ stays bounded, we shall prove that, up to extracting subsequences, (u_h, v_h) is only weakly-$*$ convergent in the larger space $BD(S) \times W_0^{1,2}(S)$ (compare with Part I of Theorem 1 below). We recall that $BD(S)$ denotes the space of the deformations $u \in L^1(S, \mathbb{R}^2)$ such that the symmetric part of the *distributional gradient* $D'u$ is a Radon measure on S, namely

$$\text{sym } D'u \in \mathcal{M}(S, \mathbb{R}^{2\times2}_{\text{sym}})$$

(the symbol $\mathcal{M}$ is used for spaces of Radon measures). The limit of the in-plane displacements u_h will take values in the smaller space

$$X(S) := \{u \in BD(S) : \exists M \in \mathcal{M}(\mathbb{R}^2, \mathbb{R}^{2\times2}_+) \text{ s.t. sym } D'\overline{u} + M \in L^1(\mathbb{R}^2, \mathbb{R}^{2\times2})\}, \tag{5}$$

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where $\overline{u} := u$ in S and $\overline{u} := 0$ in $\mathbb{R}^2 \setminus S$. This corresponds to requiring that the symmetrized distributional derivative is the sum of an L^1 term and a negative definite measure, singular with respect to Lebesgue measure. This sign condition does not bring any additional regularity, as $X(S)$ still contains elements that are not in $BV(S, \mathbb{R}^2)$. The formulation of (5) in terms of the extension $\bar{u}$ corresponds to a sign condition on the boundary values of u (in the sense of inner traces). Precisely, functions $u \in X(S)$ obey $\operatorname{tr}(u) = \lambda \nu_S$, where $\lambda \geq 0$ and ν_S is the outer normal. The structure of $X(S)$ is discussed in more detail in the Appendix.

The main result of this paper is that for all $\beta \in (0, 4)$, as $h \to 0$ the functionals $h^{-\beta} I_h$ converge (in the sense of Γ-convergence) to the limit functional $I_0 : X(S) \times W_0^{1,2}(S) \to [0, \infty]$, defined as

$$I_0(u, v) := \inf \left\{ \frac{1}{2} \int_S Q_2 \left((\operatorname{sym} D'u + M)(x') + \frac{\nabla'v(x') \otimes \nabla'v(x')}{2} \right) dx' \right.$$
$$\left. : M \in \mathcal{M}(\mathbb{R}^2, \mathbb{R}_+^{2 \times 2}), \operatorname{sym} D'\overline{u} + M \in L^1(\mathbb{R}^2, \mathbb{R}_+^{2 \times 2}) \right\}.$$

Here $Q_2 : \mathbb{R}^{2 \times 2} \to [0, \infty)$ is the quadratic form

$$Q_2(A) := \min \left\{ Q_3(\operatorname{sym} A + \operatorname{sym}(a \otimes e_3)) : a \in \mathbb{R}^3 \right\},$$

and $Q_3 : \mathbb{R}^{3 \times 3} \to [0, \infty)$ is the Hessian of the energy at the identity, i.e.

$$Q_3(F) := \nabla^2 W(\operatorname{Id}_3)[F, F].$$

By (W3) the quadratic forms Q_2 and Q_3 are positive definite on symmetric matrices. If $u \in W^{1,1}(S, \mathbb{R}^2)$ and $I_0(u, v) < \infty$, as one can see, the above expression for I_0 reduces to

$$I_0(u, v) = \frac{1}{2} \int_S W_{\text{Fö}}(\nabla'u(x'), \nabla'v(x')) dx' \tag{6}$$

where $W_{\text{Fö}} : \mathbb{R}^{2 \times 2} \times \mathbb{R}^2 \to [0, \infty)$ is defined by

$$W_{\text{Fö}}(A, b) := \min \left\{ Q_2 \left(\operatorname{sym} A + \frac{b \otimes b}{2} + M \right) : M = M^T, M \geq 0 \right\}.$$

We notice that $W_{\text{Fö}}$ is a convex function, see Lemma 3 in the Appendix. In the special case mentioned in the Introduction, which corresponds to $Q_3(F) = |F|^2$, we get $Q_2(A) = |A|^2$ and $W_{\text{Fö}}(A, b)$ coincides, up to a normalization factor, with $W_{\text{rel}}(A + b \otimes b)$ as given after (2).

The minimization over positive-definite matrices entering the definition of $W_{\text{Fö}}$ corresponds to the relaxation of compression by means of oscillations, and implies that $W_{\text{Fö}}$ vanishes on all compressive strains. This minimization was not present in the original theory by Föppl (i.e. he used $\tilde{W}_{\text{Fö}} = Q_2(\operatorname{sym} A + b \otimes b/2)$). This difference is the geometrically linear analogue of the one between the membrane theory rigorously derived by Le Dret and Raoult [13, 14] and the ones that had been heuristically proposed before.

We now give a precise statement of our convergence result.

Theorem 1. *Let $S \in \mathbb{R}^2$ be a bounded, strictly star-shaped, Lipschitz domain and let W satisfy (W1), (W2), (W3). Then for every $\beta \in (0,4)$ the functionals $h^{-\beta} I_h$ Γ-converge (as $h \to 0$) to the relaxed Föppl functional I_0. More precisely we have:*

I. Compactness. *For every sequence $h \to 0$ and every y_h such that*

$$\limsup_{h \to 0} h^{-\beta} I_h(y_h) < \infty$$

the sequences (u_h, v_h) defined by (3-4) have a subsequence such that

$$
\begin{aligned}
u_h &\rightharpoonup u & &\text{weakly in } L^2(S, \mathbb{R}^2), \\
\operatorname{sym} \nabla' u_h &\stackrel{*}{\rightharpoonup} \operatorname{sym} D'u & &\text{weakly-* in } \mathcal{M}(S, \mathbb{R}^{2 \times 2}), \\
v_h &\rightharpoonup v & &\text{weakly in } W_0^{1,2}(S, \mathbb{R}^2)
\end{aligned}
$$

for some $u \in X(S)$ and $v \in W_0^{1,2}(S)$.

II. Lower bound. *Under the same assumptions, and along the same subsequence,*

$$\liminf_{h \to 0} \frac{I_h(y_h)}{h^\beta} \geq I_0(u,v).$$

III. Upper bound. *For every pair of functions $u \in X(S)$ and $v \in W_0^{1,2}(S)$ and every sequence $h \to 0$ there exists a sequence of functions $y_h \in C^\infty(\Omega_1, \mathbb{R}^3)$ with $y_h(x) = x' + h x_3 e_3$ for $x \in \partial S \times (-1/2, 1/2)$ and such that the pair $(u_h, v_h) \in C_0^\infty(S, \mathbb{R}^2) \times C_0^\infty(S)$ defined via (3-4) converges to (u, v) as above, and*

$$\lim_{h \to 0} \frac{I_h(y_h)}{h^\beta} = I_0(u,v).$$

By strictly star-shaped we mean that there is a point $x \in S$ such that for each $y \in \partial S$ the open segment (x, y) is contained in S. Parts I and II of the Theorem hold for generic bounded Lipschitz domains.

We recall that such a Γ-convergence result implies convergence of minimizers, in the sense that Theorem 1 implies that the set of minima of I_0 coincides with the set of accumulation points of asymptotically minimizing sequences for $h^{-\beta} I_h$. Explicitly, (u, v) is a minimizer of I_0 if and only if there is a sequence y_h, converging to (u, v) as above, such that $h^{-\beta}[I_h(y_h) - \inf I_h] \to 0$.

Further, the same holds if a continuous perturbation, such as external forces, is included. In the relevant case of normal forces, this means that the sequence of functionals

$$h^{-\beta} \left[I_h(y_h) + \int_{\Omega_1} f_h(x') \, (y_h(x) - hx) \cdot e_3 dx \right]$$

Γ-converges to

$$I_0(u,v) + \int_S f(x')v(x')dx',$$

provided that $h^{3\beta/4} f_h(x')$ converges to f in $L^2(S)$.

We remark that the range of scalings covered by the present result ($\beta \in (0,4)$) is much broader than the one covered by the corresponding Γ-convergence results obtained without clamped boundary conditions. Indeed, without boundary conditions,

different Γ-limits for $h^{-\beta}I_h$ have been determined for $\beta \in (0, 5/3)$, $\beta = 2$, $\beta \in (2, 4)$ (no result is yet known for $\beta \in [5/3, 2)$). The two extreme cases $\beta = 0$ and $\beta = 4$ are special both in the presence or in absence of clamped boundary conditions. We refer to [10] for a more complete presentation of these different regimes.

3 Proof of Theorem 1.

We prove the three parts in sequence. We start from the argument for the compactness part, which is the one more specific to this situation where the energy has very little coercivity and different growth conditions in different variables. The form (1) shows that in this scaling regime one cannot expect to have a local coercivity. Compactness is gained by means of the boundary conditions. Indeed, the boundary values imply that ∇u_h has zero average, hence the integral of $|\nabla v_h|^2$ is controlled by the energy. This gives control of ∇v_h in L^2, but of sym ∇u_h only in L^1.

The lower bound is obtained by a standard argument exploiting the form of W close to the minimum, again with some subtleties arising from the weakness of the topologies.

Finally, in the upper bound an explicit construction is needed, which characterizes the folds which are used to reduce the energy of compressive deformations. In a first step we reduce to smooth displacements (u, v) with compact support, using the star-shapedness of S and the convexity of $W_{\text{Fö}}$. Then we provide a construction which reverses the relaxation. This is based on the explicit definition of oscillatory sequences which reduce the energy of compressive deformations. From the viewpoint of nonlinear elasticity the typical construction can be seen as a laminate between isometric deformations, whose average is, in general, a short deformation - i.e. a deformation whose gradient lies in the convex hull of the set of isometries $O(2, 3)$.

Proof. **Part One: compactness.** We have a family of deformations y_h such that

$$y_h(x) = x' + hx_3 e_3, \quad \forall x \in \partial S \times (-1/2, 1/2); \tag{7}$$

$$\int_{\Omega_1} W(\nabla_h y_h(x)) dx \le Ch^\beta. \tag{8}$$

We now introduce new functions which characterize the deviation of the elastic deformation y_h from the identity $x' + hx_3 e_3$. Since we are dealing with thin sheets it is natural to separate the tangential and the normal displacement. Therefore we consider $U_h \in W^{1,2}(\Omega_1, \mathbb{R}^2)$ and $V_h \in W^{1,2}(\Omega_1)$ defined by

$$y_h(x) = x' + hx_3 e_3 + U_h(x) + V_h(x)e_3 \,.$$

Equivalently,

$$U_h(x) := (y_h(x) - x)', \quad V_h(x) := (y_h(x) - hx) \cdot e_3 \,.$$

The gradients are related by

$$\nabla_h y_h(x) \;=\; \mathrm{Id}_3 + \nabla' U_h(x) + e_3 \otimes \nabla' V_h(x) + \frac{1}{h}(\partial_3 U_h(x) + \partial_3 V_h(x)e_3) \otimes e_3.$$

The tangential nonlinear strain takes the form

$$\left[(\nabla_h y_h)^T \nabla_h y_h - \mathrm{Id}_3\right]' = 2\mathrm{sym}\,\nabla' U_h + (\nabla' U_h)^T(\nabla' U_h) + \nabla' V_h \otimes \nabla' V_h \qquad (9)$$

(recall that F' denotes projection of F onto $\mathbb{R}^{2\times 2}$, and that $(\mathrm{Id}_3 + F)^T(\mathrm{Id}_3 + F) = \mathrm{Id}_3 + 2\mathrm{sym}\,F + F^T F$).

Integrating (9) over $x' \in S$ the first term cancels, since $\int_S \nabla' U(x)dx' = 0$ by (7). Taking the trace and integrating over $x_3 \in (-1/2, 1/2)$ leads to

$$\int_{\Omega_1} |\nabla' U_h(x)|^2 + |\nabla' V_h(x)|^2 dx = \mathrm{Tr}\int_{\Omega_1}\left[(\nabla_h y_h)^T\nabla_h y_h - \mathrm{Id}_3\right]' dx \le Ch^{\beta/2}\,.$$

In the last step we used $|F^T F - \mathrm{Id}| \le C\mathrm{dist}(F, SO(3)) + C\mathrm{dist}^2(F, SO(3))$, (W3) and (8). Plugging this information back into (9) gives an analogous bound for $\mathrm{sym}\,\nabla' U_h$ in $L^1(\Omega_1; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. Summarizing we have

$$\int_{\Omega_1} |\mathrm{sym}\,\nabla' U_h(x)| + |\nabla' U_h(x)|^2 + |\nabla' V_h(x)|^2 dx \le Ch^{\beta/2}. \qquad (10)$$

Therefore it is natural to rescale the tangential displacement U_h by $h^{\beta/2}$, and the normal one V_h by $h^{\beta/4}$.

Taking averages over x_3, we define the rescaled displacements $u_h \in W_0^{1,2}(S, \mathbb{R}^2)$ and $v_h \in W_0^{1,2}(S)$ by

$$u_h(x') := \frac{1}{h^{\beta/2}}\int_{-1/2}^{1/2} U_h(x', x_3)dx_3, \quad v_h(x') := \frac{1}{h^{\beta/4}}\int_{-1/2}^{1/2} V_h(x', x_3)dx_3.$$

This definition is equivalent to (3) and (4) above.

By (10) the sequence $\nabla' v_h$ is bounded in $L^2(S, \mathbb{R}^2)$, hence there is a subsequence such that

$$v_h \rightharpoonup v \quad \text{weakly in } W_0^{1,2}(S). \qquad (11)$$

By (10) the sequence $\mathrm{sym}\,\nabla' u_h$ is bounded in $L^1(S, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, and since $u_h \in W_0^{1,2}$ we can apply the Poincaré-Korn inequality [18] (see also [11, 12] and [19, Sect. II.1]) to find

$$\|u_h\|_{L^2(S, \mathbb{R}^2)} \le C\|\mathrm{sym}\,\nabla' u_h\|_{L^1(S, \mathbb{R}^{2\times 2}_{\mathrm{sym}})} \le C.$$

In particular there is a subsequence and $u \in L^2$ such that

$$u_h \rightharpoonup u \quad \text{weakly in } L^2(S, \mathbb{R}^2)\,. \qquad (12)$$

Further, $\nabla' u_h$ converges to $D'u$ in the sense of distributions, and by (10)

$$\mathrm{sym}\,\nabla' u_h(x')dx' \overset{*}{\rightharpoonup} \mathrm{sym}\,D'u \quad \text{weakly* in } \mathcal{M}(S, \mathbb{R}^{2\times 2}_{\mathrm{sym}})\,. \qquad (13)$$

This is the compactness entailed in the functionals under considerations. We now pass to use these information to obtain a lower bound, that in turn will also allow us to prove that $u \in X(S)$.

Part Two: lower bound. The first part of the argument is along the lines of [9], and in a sense it constitutes the "generic" lower bound argument used in

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the regime $I_h(y_h) \to 0$, i.e. for $\nabla_h y_h$ close to $SO(3)$. In this range it is natural to "normalize" the deformation gradients $\nabla_h y_h$ in order to use the structure of W near $SO(3)$. This amounts in considering a field of rotations $R_h : \Omega_1 \to SO(3)$ such that

$$|\nabla_h y_h(x) - R_h(x)| = \text{dist}(\nabla_h y_h(x), SO(3)).$$

The function R_h can be chosen to be measurable (see Lemma 7 in the Appendix), and hence in $L^\infty(\Omega_1, \mathbb{R}^{3\times3})$. We also note, see Lemma 2 in the Appendix, that

$$R_h(x)^T \nabla_h y_h(x) \in \mathbb{R}^{3\times3}_{\text{sym}}.$$

Consider now

$$G_h := \frac{R_h^T \nabla_h y_h - \text{Id}_3}{h^{\beta/2}}. \tag{14}$$

Since $|G_h| = \text{dist}(\nabla_h y_h, SO(3))/h^{\beta/2}$, from (W3) and (8) we get that G_h is uniformly bounded in L^2, and taking a subsequence

$$G_h \rightharpoonup G \quad \text{weakly in } L^2(\Omega_1, \mathbb{R}^{3\times3}).$$

We now use Taylor's formula to obtain a lower bound in terms of the second derivatives of W at the identity. Precisely, by (W1) and (W2) there is $\rho : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t\to0} \rho(t)/t^2 = 0$ and

$$
\begin{aligned}
W(\nabla_h y_h) &= W(\text{Id}_3 + R_h^T \nabla_h y_h - \text{Id}_3) \\
&\geq \frac{1}{2} Q_3(R_h^T \nabla_h y_h - \text{Id}_3) - \rho\left(|R_h^T \nabla_h y_h - \text{Id}_3|\right).
\end{aligned}
$$

It is convenient to consider separately the part of the domain where $\nabla_h y_h$ is close to a rotation, which is large, and the small exceptional set. To do this, let

$$\omega_h = \{x \in \Omega_1 : \text{dist}(\nabla_h y_h(x), SO(3)) \leq h^{\beta/4}\}.$$

Let χ_h be the characteristic function of ω_h. By (W3) and (8) we get $|\omega_h| \to |\Omega_1|$. Restricting the integration to ω_h we get

$$
\begin{aligned}
\frac{I_h(y_h)}{h^\beta} &\geq \frac{1}{2} \int_{\Omega_1} \chi_h(x) Q_3\left(\frac{R_h(x)^T \nabla_h y_h(x) - \text{Id}_3}{h^{\beta/2}}\right) dx \\
&\quad - \frac{1}{h^\beta} \int_{\Omega_1} \chi_h(x) \rho\left(\text{dist}(\nabla_h y_h(x), SO(3))\right) dx.
\end{aligned} \tag{15}
$$

The second term goes to zero as $h \to 0$, for it is equal to the integral of

$$\frac{\chi_h \, \rho\left(\text{dist}(\nabla_h y_h, SO(3))\right)}{\text{dist}^2(\nabla_h y_h, SO(3))} \cdot \frac{\text{dist}^2(\nabla_h y_h, SO(3))}{h^\beta}.$$

By the definition of ω_h the first fraction converges uniformly to zero as $h \to 0$, at the same time the second one is uniformly bounded in L^1 by (8).

As $\chi_h(x) \in \{0, 1\}$ we also have $\chi_h Q_3(G_h) = Q_3(\chi_h G_h)$, and since $\chi_h G_h \rightharpoonup G$ weakly in $L^2(\Omega_1, \mathbb{R}^{3\times3})$ we easily conclude from (15) that

$$\liminf_{h\to0} \frac{I_h(y_h)}{h^\beta} \geq \frac{1}{2} \int_{\Omega_1} Q_3(G(x)) dx.$$

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Note that G is symmetric as G_h was.

In order to extract further information on G is useful to express it as a limit of a sequence not involving R_h. Since $\nabla_h y_h = R_h(\mathrm{Id}_3 + h^{\beta/2}G_h)$ we get

$$(\nabla_h y_h)^T(\nabla_h y_h) = \mathrm{Id}_3 + 2h^{\beta/2}G_h + h^\beta G_h^T G_h$$

and thus

$$G_h - \frac{(\nabla_h y_h)^T(\nabla_h y_h) - \mathrm{Id}_3}{2h^{\beta/2}} = -\frac{h^{\beta/2}}{2}G_h^T G_h \to 0 \quad \text{strongly in } L^1(\Omega_1, \mathbb{R}^{3\times3}). \quad (16)$$

In particular

$$\frac{(\nabla_h y_h)^T(\nabla_h y_h) - \mathrm{Id}_3}{2h^{\beta/2}} \rightharpoonup G \quad \text{weakly in } L^1(\Omega_1, \mathbb{R}^{3\times3}). \quad (17)$$

As $G(x)$ is symmetric we have $Q_3(G(x)) \geq Q_2(G(x)')$. Furthermore, as Q_2 is convex, we can apply Jensen's inequality in the x_3 direction and find

$$\liminf_{h\to0} \frac{I_h(y_h)}{h^\beta} \geq \frac{1}{2}\int_S Q_2(A(x'))dx'$$

where

$$A(x') = \int_{-1/2}^{1/2} G(x' + x_3 e_3)' dx_3, \quad \forall x' \in S.$$

It remains to relate A to u and v. To do this, we consider the integral over $x_3 \in (-1/2, 1/2)$ of the nonlinear strain,

$$A_h(x') := \int_{-1/2}^{1/2} \frac{\left[(\nabla_h y_h)^T(\nabla_h y_h) - \mathrm{Id}_3\right]'}{2h^{\beta/2}} dx_3.$$

By (17) we have
$$A_h \rightharpoonup A \quad \text{weakly in } L^1(S, \mathbb{R}^{2\times2}). \quad (18)$$
At the same time, dividing (9) by $2h^{\beta/2}$ and integrating over x_3 gives

$$A_h(x') = \frac{1}{h^{\beta/2}}\int_{-1/2}^{1/2} \mathrm{sym}\,\nabla'U_h(x) + \frac{\nabla'V_h(x) \otimes \nabla'V_h(x)}{2} + \frac{\nabla'U_h(x)^T\nabla'U_h(x)}{2}dx_3.$$

The first term equals $\mathrm{sym}\,\nabla'u_h(x')$, the other two can be bounded via Jensen's inequality leading to

$$A_h(x') \geq \mathrm{sym}\,\nabla'u_h(x') + \frac{\nabla'v_h(x') \otimes \nabla'v_h(x')}{2} + h^{\beta/2}\nabla'u_h(x')^T\nabla'u_h(x').$$

As v_h is bounded in $W^{1,2}(S)$ we have that $\nabla v_h \otimes \nabla v_h$ converges weakly* to a measure $\mu \in \mathcal{M}(S, \mathbb{R}^{2\times2})$, and by a standard lower semicontinuity argument $\mu \geq \nabla v \otimes \nabla v$. Using (13) and the fact that the third term on the right hand side is positive semidefinite we conclude that

$$A(x')\,dx' \geq \mathrm{sym}\,D'u + \frac{\nabla'v(x') \otimes \nabla'v(x')}{2}\,dx'. \quad (19)$$

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The difference of the two sides of this inequality defines a Radon measure on S with values in $\mathbb{R}^{2\times 2}_+$ that we denote by M. In particular $\operatorname{sym} D'u + M$ is absolutely continuous with respect to the Lebesgue measure as

$$\operatorname{sym} D'u + M = \left\{ A(x') - \frac{\nabla'v(x') \otimes \nabla'v(x')}{2} \right\} dx'.$$

Finally,

$$\liminf_{h\to 0} \frac{I_h(y_h)}{h^\beta} \geq \frac{1}{2} \inf \left\{ \int_S Q_2 \left((\operatorname{sym} D'u + M)(x') + \frac{\nabla'v(x') \otimes \nabla'v(x')}{2} \right) dx' \right\}$$

where the infimum runs over all $M \in \mathcal{M}(S, \mathbb{R}^{2\times 2}_+)$ such that $\operatorname{sym} D'u + M \in L^1(S, \mathbb{R}^{2\times 2}_{\text{sym}})$.

Finally, we repeat the argument for $\overline{y}_h(x) := y_h(x)$ if $x \in S \times (-h/2, h/2)$, $\overline{y}_h(x) := x' + hx_3e_3$ if $x \in (\mathbb{R}^2 \setminus S) \times (-h/2, h/2)$. As $W(\operatorname{Id}_3) = 0$ and $Q_3(0) = Q_2(0) = 0$ the above argument can be repeated without any change and we find that there exists a measure $M \in \mathcal{M}(\mathbb{R}^2, \mathbb{R}^{2\times 2}_+)$ such that $\operatorname{sym} D'\overline{u} + M \in L^1(\mathbb{R}^2, \mathbb{R}^{2\times 2}_{\text{sym}})$. Thus $u \in X(S)$.

Part Three: upper bound. We are given $u \in X(S)$ and $v \in W^{1,2}_0(S)$ with $I_0(u, v) < \infty$ (otherwise there is nothing to prove), and we have to construct a recovery sequence. We shall now first use star-shapedness of S to show that it suffices to consider u and v with compact support in S, then use convexity of $W_{\text{Fö}}$ to show that it suffices to consider smooth u and v, and finally provide an explicit construction.

After a translation we can assume that S is star-shaped with respect to the origin. Fix $\varepsilon > 0$ and consider the functions

$$u_\varepsilon(x') = \frac{1}{1+\varepsilon} \overline{u}((1+\varepsilon)x'), \qquad v_\varepsilon(x') = \frac{1}{1+\varepsilon} \overline{v}((1+\varepsilon)x').$$

As above, we denote by a bar extension by zero outside S, so that e.g. $\overline{u} = u$ on S and $\overline{u} = 0$ in $\mathbb{R}^2 \setminus S$. It is clear that u_ε and v_ε are supported on $S/(1+\varepsilon) \subset\subset S$. At the same time $u_\varepsilon \in X(S)$ (as $u \in X(S)$), $v_\varepsilon \in W^{1,2}_0(S)$, and, as $\varepsilon \to 0$,

$$(u_\varepsilon, v_\varepsilon) \rightharpoonup (u, v) \qquad \text{weakly-}* \text{ in } X(S) \times W^{1,2}(S),$$

(i.e., in the convergence stated in Part I). Now we remark that

$$I_0(u_\varepsilon, v_\varepsilon) \leq (1+\varepsilon)^{-2} I_0(u, v). \tag{20}$$

This follows from a change of variables, once one has proven that $\nabla'v_\varepsilon(x') = \nabla'v((1+\varepsilon)x'))$, and that for any $M \in \mathcal{M}(\mathbb{R}^2, \mathbb{R}^{2\times 2}_+)$ such that $\operatorname{sym} D'\overline{u} + M \in L^1(\mathbb{R}^2, \mathbb{R}^{2\times 2}_+)$ we can find $M_\varepsilon \in \mathcal{M}(\mathbb{R}^2, \mathbb{R}^{2\times 2}_+)$ such that

$$\operatorname{sym} D'\overline{u}_\varepsilon + M_\varepsilon = (\operatorname{sym} D'\overline{u} + M)((1+\varepsilon)x')dx'.$$

We now show how to construct M_ε. Since

$$\operatorname{sym} D'\overline{u}_\varepsilon = \frac{1}{(1+\varepsilon)^2} \left[\frac{1}{1+\varepsilon} \operatorname{Id}_2 \# \operatorname{sym} D'\overline{u} \right],$$

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(where # stands for push-forward of measures, that is $f\#\mu(E) := \mu(f^{-1}(E)))$, it suffices to choose

$$M_\varepsilon := \frac{1}{(1+\varepsilon)^2}\left[\frac{1}{1+\varepsilon}\mathrm{Id}_2\#M\right].$$

This concludes the proof of (20). From now on we assume that (u, v) is supported on $S_0 \subset\subset S$.

To show that (u, v) can be assumed to be smooth, fix $\delta < \mathrm{dist}(S_0, \partial S)$, and set

$$u_\delta(x') = \int_{S_0} \rho_\delta(x' - y')u(y')dy', \qquad v_\delta(x') = \int_{S_0} \rho_\delta(x' - y')v(y')dy'.$$

where ρ_δ is a standard mollification kernel on the scale δ, i.e. $\rho_\delta(x') = \delta^{-2}\rho(x'/\delta)$ for $\rho \in C_c^\infty(B^2)$, $\int_{\mathbb{R}^2}\rho = 1$. Then automatically $(u_\delta, v_\delta) \in C_c^\infty(S, \mathbb{R}^2) \times C_c^\infty(S)$, and as $\delta \to 0$ we have $(u_\delta, v_\delta) \to (u, v)$ weakly in $X(S) \times W^{1,2}(S)$. It remains to show that $\limsup_{\delta\to 0} I_0(u_\delta, v_\delta) \leq I_0(u, v)$. To see this let $M \in \mathcal{M}(\mathbb{R}^2, \mathbb{R}_+^{2\times 2})$ be such that $f = \mathrm{sym}\, D'\overline{u} + M \in L^1(\mathbb{R}^2, \mathbb{R}_{\mathrm{sym}}^{2\times 2})$, and

$$I_0(u, v) \leq \frac{1}{2}\int_S Q_2\left((\mathrm{sym}\, D'u + M)(x') + \frac{\nabla v(x') \otimes \nabla v(x')}{2}\right)dx' + \delta$$

(M and f will depend on δ). Then

$$\nabla' u_\delta(x') = \int_S \rho_\delta(x' - y')f(y')dy' - \int_S \rho_\delta(x' - y')dM(y')$$

where the second integral takes values in the (convex) set $\mathbb{R}_+^{2\times 2}$. We now use that $W_{\mathrm{F\ddot{o}}}$ is nondecreasing in its (matrix-valued) first argument, and that it is convex, to obtain

$$\int_S W_{\mathrm{F\ddot{o}}}(\nabla' u_\delta, \nabla' v_\delta)dx' \quad \leq \quad \int_S W_{\mathrm{F\ddot{o}}}(\rho_\delta * f, \rho_\delta * \nabla' v)dx'$$
$$\leq \quad \int_S W_{\mathrm{F\ddot{o}}}(f, \nabla' v)dx'$$

On the smooth functions (u_δ, v_δ) we can use (6), and since $W_{\mathrm{F\ddot{o}}} \leq Q_2$ we get

$$I_0(u_\delta, v_\delta) \leq I_0(u, v) + \delta.$$

It remains to prove the thesis for the case $u \in C_c^\infty(S, \mathbb{R}^2)$, $v \in C_c^\infty(S)$. We first show that for every $j \in \mathbb{N}$ we can find $M_j \in L^\infty(S, \mathbb{R}_+^{2\times 2})$ and $a_j \in C_c^\infty(S, \mathbb{R}^3)$ such that

$$\frac{1}{2}\int_S Q_3\left(\mathrm{sym}\,(\nabla' u + a_j \otimes e_3) + \frac{\nabla' v \otimes \nabla' v}{2} + M_j\right)dx' \leq I_0(u, v) + \frac{C}{j}, \qquad (21)$$

with M_j taking only a finite number of values, each of them on a Lipschitz subset of S. To see this, consider a subdivision of S into small squares, say of side l_j. The oscillation of the smooth fields ∇u and ∇v on each square is uniformly small, hence

– provided l_j is small enough – on each square we can pick one value of a and one value of M so that

$$Q_3 \left(\text{sym } (\nabla' u + a_j \otimes e_3) + \frac{\nabla' v \otimes \nabla' v}{2} + M_j \right) \leq W_{\text{Fö}}(\nabla' u, \nabla' v) + \frac{1}{j} \, .$$

Further, on the squares intersecting ∂S we can choose $a = 0$, since u and v have zero boundary values. This defines piecewise constant fields a_j and M_j with the required property. Smoothing a_j concludes the proof of (21).

Claim. Given $u \in C_0^\infty(S, \mathbb{R}^2)$, $v \in C_0^\infty(S)$, $a \in C_0^\infty(S, \mathbb{R}^3)$ and $M \in L^\infty(S, \mathbb{R}_+^{2 \times 2})$ taking finitely many values on Lipschitz subsets of S, *there exists* a sequence $y_h \in C^\infty(\Omega_1, \mathbb{R}^3)$ such that $y_h(x) = x' + hx_3 e_3$ for $x \in \partial S \times (-1/2, 1/2)$, the functions u_h and v_h defined as in (3) and (4) satisfy (11), (12), and (13), the scaled nonlinear strain

$$F_h := \frac{(\nabla_h y_h)^T (\nabla_h y_h) - \text{Id}_3}{2h^{\beta/2}}$$

converges to

$$F_h \to \text{sym } (\nabla' u + a \otimes e_3) + \frac{\nabla' v \otimes \nabla' v}{2} + M, \quad \text{strongly in } L^2(\Omega_1, \mathbb{R}^{3 \times 3}), \quad (22)$$

and such that there is a field of rotations $R_h \in L^\infty(\Omega_1, SO(3))$ such that

$$\|R_h^T \nabla_h y_h - \text{Id}_3\|_{L^\infty(S, \mathbb{R}^{3 \times 3})} \leq Ch^{\beta/2}, \quad (23)$$

for some constant C which does not depend on h.

Assume for the moment that this can be done. By (W1) and (W2) we get

$$W(\nabla_h y_h) = W(R_h^T \nabla_h y_h) = \frac{1}{2} Q_3 \left(R_h^T \nabla_h y_h - \text{Id}_3 \right) + o(|R_h^T \nabla_h y_h - \text{Id}_3|^2),$$

so that by (23) it follows

$$\lim_{h \to 0} \frac{1}{h^\beta} \int_{\Omega_1} W(\nabla_h y_h) dx = \lim_{h \to 0} \frac{1}{2} \int_{\Omega_1} Q_3 (G_h) \, dx < \infty,$$

where $G_h := h^{-\beta/2}(R_h^T \nabla_h y_h - \text{Id}_3)$. By (23) G_h is bounded in L^∞. Then $F_h - G_h = 2^{-1} h^{\beta/2} G_h^T G_h$ (compare with (16)) converges strongly to zero in L^∞, while by (22) F_h itself has a strong limit in L^2. Therefore G_h converges strongly in L^2 to the same limit as F_h, and this limit is

$$G(x) := \text{sym } (\nabla' u(x') + a(x') \otimes e_3) + \frac{\nabla' v(x') \otimes \nabla' v(x')}{2} + M(x') \, .$$

This expression does not depend on x_3, and recalling (21) we get

$$\lim_{h \to 0} \frac{1}{h^\beta} \int_{\Omega_1} W(\nabla_h y_h) dx = \int_{\Omega_1} Q_3 (G(x)) \, dx = \int_S Q_3 (G(x')) \, dx' = I_0(u, v) + \frac{C}{j}$$

which is the thesis.

Now we prove the claim. Let us define

$$
\begin{aligned}
y_h(x) \;:=\; & x' + hx_3 e_3 + h^{\beta/2}(u(x') + \xi_h(x')) + h^{\beta/4}(v(x') + \varphi_h(x'))e_3 \\
& + h\, x_3 \left(h^{\beta/4} b_h(x') + h^{\beta/2} s_h(x')e_3 + h^{\beta/2} a(x') \right)
\end{aligned}
$$

where $b_h \in C_0^\infty(S, \mathbb{R}^2)$, $s_h \in C_0^\infty(S)$, $\xi_h \in C_0^\infty(S, \mathbb{R}^2)$ and $\varphi_h \in C_0^\infty(S)$ have to be chosen properly. The choice of these spaces ensures that the boundary condition $y_h(x) = x' + hx_3 e_3$ for $x \in \partial S \times (-1/2, 1/2)$ is satisfied. Further, we shall choose all those functions to be uniformly Lipschitz (i.e. their gradients are bounded by a constant which can depend on M, u and v, but not on h).

The linear term in x_3 cancels under integration over $x_3 \in (-1/2, 1/2)$; the sequences u_h and v_h defined via (3) and (4) satisfy

$$
u_h = u + \xi_h, \quad v_h = v + \varphi_h.
$$

We shall choose $\xi_h \in C_0^\infty(S, \mathbb{R}^2)$ and $\varphi_h \in C_0^\infty(S)$ in such a way that

$$
\begin{aligned}
\xi_h &\rightharpoonup 0 && \text{weakly in } W^{1,2}(S, \mathbb{R}^2) && (24) \\
\varphi_h &\rightharpoonup 0 && \text{weakly in } W^{1,4}(S) && (25) \\
\|(\nabla')^2 \varphi_h\|_{L^\infty(S, \mathbb{R}^{2\times2})} &\leq \frac{C}{\varepsilon_h}, &&&& (26)
\end{aligned}
$$

for a suitable sequence $\varepsilon_h \to 0$ as $h \to 0$. Note that (24) and (25) ensure the convergence properties (11), (12) and (13).

Let us now note that we have

$$
\nabla_h y_h = \mathrm{Id}_3 + h^{\beta/4} H_1 + h^{\beta/2} H_2 + h^{1+\beta/4} H_3 + h^{1+\beta/2} H_4, \tag{27}
$$

where

$$
\begin{aligned}
H_1 &:= e_3 \otimes \nabla' v_h + b_h \otimes e_3, \\
H_2 &:= \nabla' u_h + a \otimes e_3 + s_h e_3 \otimes e_3, \\
H_3 &:= x_3 \nabla' b_h, \\
H_4 &:= x_3 (\nabla' s_h + \nabla' a).
\end{aligned}
$$

Expanding the nonlinear strain $(\nabla_h y_h)^T (\nabla_h y_h)$ via the rule $(\mathrm{Id}_3 + F)^T (\mathrm{Id}_3 + F) = \mathrm{Id}_3 + 2\mathrm{sym}\, F + F^T F$ we get

$$
(\nabla_h y_h)^T (\nabla_h y_h) - \mathrm{Id}_3 = 2 h^{\beta/4} \mathrm{sym}\, H_1 + h^{\beta/2}(2\mathrm{sym}\, H_2 + H_1^T H_1) + o(h^{\beta/2}) J_h
$$

for a suitable tensor field J_h we shall consider again later on. In order to obtain a strain of order $h^{\beta/2}$ we need to render H_1 antisymmetric, and this can be done by choosing

$$
b_h := -\nabla' v_h. \tag{28}
$$

In this way we find

$$
\begin{aligned}
F_h \;=\; & \mathrm{sym}\, H_2 + \frac{H_1^T H_1}{2} + \frac{o(h^{\beta/2})}{2h^{\beta/2}} J_h \\
\;=\; & \mathrm{sym}\, (\nabla' u_h + a \otimes e_3) + \left(s_h - \frac{|\nabla' v_h|^2}{2} \right) e_3 \otimes e_3 + \frac{\nabla' v_h \otimes \nabla' v_h}{2} + \frac{o(h^{\beta/2})}{2h^{\beta/2}} J_h.
\end{aligned}
$$

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As we are looking for (22) we choose

$$s_h := -\frac{|\nabla' v_h|^2}{2}, \tag{29}$$

and then, in order to have (22), it remains to show (i) that ξ_h and φ_h can be chosen in such a way that (24), (25) hold and

$$\mathrm{sym}\,(\nabla'\xi_h + \nabla'v \otimes \nabla'\varphi_h) + \frac{\nabla'\varphi_h \otimes \nabla'\varphi_h}{2} \to M \quad \text{strongly in } L^2(S,\mathbb{R}^{2\times2}); \tag{30}$$

and that (ii) the resulting tensor field J_h satisfies

$$\frac{o(h^{\beta/2})}{h^{\beta/2}}\,J_h \to 0 \quad \text{strongly in } L^2(S,\mathbb{R}^{2\times2}). \tag{31}$$

This can be done as follows. Let us define

$$\xi_h = \psi_h - \varphi_h \nabla'v,$$

for some $\psi_h \in W_0^{1,\infty}(S,\mathbb{R}^2)$ to be chosen later. Then we find

$$\mathrm{sym}\,(\nabla'\xi_h + \nabla'v \otimes \nabla'\varphi_h) + \frac{\nabla'\varphi_h \otimes \nabla'\varphi_h}{2}$$
$$= \mathrm{sym}\,\nabla'\psi_h + \frac{\nabla'\varphi_h \otimes \nabla'\varphi_h}{2} - \varphi_h(\nabla')^2 v.$$

Accordingly to Lemma 5 below we can find $\psi_h \in C_0^\infty(S,\mathbb{R}^2)$ and $\varphi_h \in C_0^\infty(S)$ uniformly Lipschitz and such that (25) and (26) hold (with an ε_h that we can choose arbitrarily, provided it goes to zero), with

$$\mathrm{sym}\,\nabla'\psi_h + \frac{\nabla'\varphi_h \otimes \nabla'\varphi_h}{2} \to M,$$

strongly in $L^2(S,\mathbb{R}^{2\times2})$ and $\psi_h \rightharpoonup 0$ weakly in $W_0^{1,2}(S,\mathbb{R}^2)$. As a consequence the resulting sequence ξ_h will satisfy (24) and also (30) will hold true. We now prove that (31) is also true and (22) will be established. To this end let us notice that, with the above choices of b_h, s_h, ξ_h and φ_h, we have that, for every h,

$$\|H_1\|_{L^\infty(S,\mathbb{R}^{2\times2})} + \|H_2\|_{L^\infty(S,\mathbb{R}^{2\times2})} \leq C,$$
$$\|H_3\|_{L^\infty(S,\mathbb{R}^{2\times2})} + \|H_4\|_{L^\infty(S,\mathbb{R}^{2\times2})} \leq C(1 + \|(\nabla')^2\varphi_h\|_{L^\infty(S,\mathbb{R}^{2\times2})}).$$

Then

$$\frac{o(h^{\beta/2})}{h^{\beta/2}}|J_h| \leq C\left(h^{\beta/4} + h^{1-\beta/4}|(\nabla')^2\varphi_h|\right) \leq C\left(h^{\beta/4} + \frac{h^{1-\beta/4}}{\varepsilon_h}\right).$$

Since we are working in the regime $0 < \beta < 4$, it suffices to choose $\varepsilon_h = h^{(1-\beta/4)/2}$.

In the end we prove (23). First of all let us notice that for every $F \in \mathbb{R}^{3\times3}$ we have

$$\mathrm{dist}(F, SO(3)) \leq |\mathrm{sym}\,F - \mathrm{Id}_3| + C|F - \mathrm{Id}|^2,$$

an inequality that reflects the fact that the tangent space of $SO(3)$ at Id_3 is the space of antisymmetric matrices. Next we consider a measurable field $\mathbb{R}_h : S \to SO(3)$ such that $\mathrm{dist}(\nabla_h y_h, SO(3)) = |R_h - \nabla_h y_h|$. From (27) we deduce that $\|\nabla_h y_h - \mathrm{Id}_3\|_{L^\infty(S,\mathbb{R}^{3\times3})} \leq Ch^{\beta/4}$ (in particular R_h is uniquely defined) and that $\|\mathrm{sym}\,\nabla_h y_h - \mathrm{Id}_3\|_{L^\infty(S,\mathbb{R}^{3\times3})} \leq Ch^{\beta/2}$, as $\mathrm{sym}\,H_1 = 0$. Thus from the inequality we pointed put above we have $|R_h - \nabla_h y_h| \leq Ch^{\beta/2}$, from which (23) immediately follows. $\qquad\square$

Acknowledgements

Appendix

We start by briefly analyzing the properties of the space $u \in X(S)$, the convex cone in $BD(S)$ that was introduced in (5) and that arises naturally in the determination of the domain of the Γ-limit I_0. General references for the space of functions of bounded deformation $BD(S)$ are, for example, the monograph by Temam [19] and the paper by Ambrosio, Coscia and Dal Maso [2].

Let us recall that if $u \in BD(S)$ then

$$\mathrm{sym}\,D'u = f_u(x')dx' + \mu_u$$

where $\mu_u \in \mathcal{M}(S, \mathbb{R}^{2\times2}_{\mathrm{sym}})$ is singular with respect to the Lebesgue measure on S, and $f_u \in L^1(S, \mathbb{R}^{2\times2})$ is the density of $\mathrm{sym}\,D'u$ with respect to the Lebesgue measure. Then $u \in X(S)$ if and only if $\mu_{\overline{u}} \leq 0$, where $\overline{u} = u$ in S and $\overline{u} = 0$ in $\mathbb{R}^2 \setminus S$.

The structure of the singular part of the strain $\mu_{\overline{u}}$ can be further analyzed: indeed, it turns out that there is a rectifiable set J_u in S and that, once we have fixed an orientation of it $\nu_u \in L^\infty(\mathcal{H}^1\lfloor J_u, S^1)$, there are functions $u^+, u^- \in L^1(\mathcal{H}^1\lfloor J_u, \mathbb{R}^2)$, and a measure $(\mathrm{sym}\,D'u)^c$ singular with respect to both dx' and $\mathcal{H}^1$, such that

$$\mu_{\overline{u}} = (\mathrm{sym}\,D'u)^c + \mathrm{sym}\,((u^+ - u^-) \otimes \nu_u)d\mathcal{H}^1\lfloor J_u + \mathrm{sym}\,(-\,\mathrm{tr}\,(u) \otimes \nu_S)d\mathcal{H}^1\lfloor \partial S,$$

where $\mathrm{tr}\,(u) \in L^1(\mathcal{H}^1\lfloor \partial S, \mathbb{R}^2)$ is the trace of u on ∂S and ν_S is the outer normal to S. In particular the condition $\mu_{\overline{u}} \leq 0$ implies the compatibility condition

$$u^+(x') - u^-(x') = -\lambda(x')\nu_u(x') \quad \text{for } \mathcal{H}^1\text{-a.e. } x' \in J_u,$$

for a suitable $\lambda \in L^1(\mathcal{H}^1\lfloor J_u, [0,\infty))$ (we recall that $\mathrm{sym}\,a \otimes b \leq 0$, with $b \neq 0$, iff $a = -\lambda b$). The sign condition on the boundary term gives analogously

$$\mathrm{tr}\,(u)(x') = \lambda(x')\nu_S(x') \quad \text{for } \mathcal{H}^1 \text{ a.e. } x' \in \partial S$$

for a function $\lambda \in L^1(\partial S, [0, \infty))$. The geometric meaning of the condition $(\operatorname{sym} D'u)^c \le 0$ for the Cantor part of $\operatorname{sym} D'u$ is instead less clear as the validity of the "rank-one property" (established in the space BV by Alberti [1]) in BD is at present unknown.

One could ask if the sign condition $\mu_{\overline{u}} \le 0$ is sufficient to gain more regularity for the distributional gradient $D'u$. It turns out that this is not the case, in the sense that there are functions in $X(S)$ that are not in $BV(S, \mathbb{R}^2)$. For example, let $S = (-1, 1)^2$, and for $i > 2$ let $Q_i = (2^{-i}, 2^{-i+1})^2$. In each Q_i by [15, Theorem 1] (see also [5, Theorem 1]) there is $u_i \in C_0^\infty(Q_i, \mathbb{R}^2)$ such that

$$\int_{Q_i} |\operatorname{sym} \nabla' u_i| dx' \le 2|Q_i|, \quad \int_{Q_i} |\nabla' u_i| dx' \ge 2^i .$$

We set $u = u_i$ in Q_i, $u = 0$ on $S \setminus \cup Q_i$. It is clear that u is in $BD(S)$ but not in $BV(S; \mathbb{R}^2)$, and that it has zero trace on ∂S. To show that it is in X, it suffices to check that the symmetric part of the distributional gradient is absolutely continuous with respect to the Lebesgue measure. Since $u \in C^1(S \setminus \{0\}, \mathbb{R}^2)$, it suffices to check that the n-dimensional density of $\operatorname{sym} D'u$ at zero is finite. To this end let ρB^2 be the ball of radius ρ and center in the origin, then

$$|\operatorname{sym} D'u|(\rho B^2) \le \sum_{\{i: Q_i \cap \rho B^2 \ne \emptyset\}} |\operatorname{sym} D'u|(Q_i) \le 4|\rho B^2| .$$

This concludes the proof.

It is not clear if for the u constructed above we can find a $v \in W_0^{1,2}(S)$ such that $I_0(u, v) < \infty$. In other words, the question of whether the space $\{u \in X(S) : I_0(u, v) < \infty$ for some $v \in W_0^{1,2}(S)\}$ is contained in $BV(S, \mathbb{R}^2)$ remains open. It is however clear that this space is not more regular than BV. Indeed, let $f : (0, 1) \to (0, 1)$ be a generic monotonic BV function, and extend it to $\mathbb{R}$ by $f(t) = t$. Then set $u(x) = -(f(x_1) - x_1, 0)$, $v = 0$, $S = (-2, 2)^2$. Then $I_0(u, v) < \infty$. This construction provides an example where the jump and Cantor part of Du are nonzero.

The rest of the appendix is devoted to the statement and proof of some lemmas that were used in the proof of the upper bound. Of particular relevance in the description of the relaxation process of compressive deformations are Lemma 4 and Lemma 5.

Lemma 2. *Let $F \in \mathbb{R}^{n \times n}$. Then there is $R \in SO(n)$ such that $\operatorname{dist}(F, SO(n)) = |R^T F - \operatorname{Id}_n|$. For all such R, the product $R^T F$ is symmetric.*

Proof. This is well-known. We recall the argument for the convenience of the reader. Existence is clear. To show symmetry, observe that replacing F by $\tilde{F} = R^T F$ one can reduce to the case $R = \operatorname{Id}_n$, i.e. it suffices to show that $\operatorname{dist}(F, SO(n)) = |F - \operatorname{Id}_n|$ implies that F is symmetric. Consider the function

$$f(Q) = |F - Q|^2 = |F|^2 - 2F : Q + |Q|^2 .$$

(we write $F : G = \operatorname{Tr} F^T G = \sum F_{ij} G_{ij}$). The first and the last term are constant (for $Q \in SO(n)$), hence can be ignored. That $Q = \operatorname{Id}$ is a local minimum among all $Q \in SO(n)$ implies that the gradient of the linear term $-2F : Q$, i.e. $-2F$, is normal to the constraint $SO(n)$ at the identity. The tangent space to $SO(n)$ at the identity is the space of skew-symmetric matrices, hence this requirement corresponds to $-2F$ being symmetric. $\square$

Lemma 3. $W_{F\ddot{o}}$ *is convex.*

Proof. Choose $\lambda \in (0, 1)$, A, $A' \in \mathbb{R}^{2\times 2}_{\text{sym}}$ and b, $b' \in \mathbb{R}^2$, and set

$$A_\lambda = \lambda A + (1-\lambda)A', \qquad b_\lambda = \lambda b + (1-\lambda)b'.$$

We have to show that

$$W_{F\ddot{o}}(A_\lambda, b_\lambda) \le \lambda W_{F\ddot{o}}(A, b) + (1-\lambda)W_{F\ddot{o}}(A', b').$$

The key observation is that

$$b_\lambda \otimes b_\lambda = \lambda b \otimes b + (1-\lambda)b \otimes b - \lambda(1-\lambda)(b-b') \otimes (b-b').$$

Therefore for any $M_\lambda \in \mathbb{R}^{2\times 2}_+$ we have

$$
\begin{aligned}
W_{F\ddot{o}}(A_\lambda, b_\lambda) &\le Q_2\left(\text{sym}\, A_\lambda + \frac{1}{2}b_\lambda \otimes b_\lambda + M_\lambda\right) \\
&= Q_2\left(\lambda\left[\text{sym}\, A + \frac{b \otimes b}{2}\right] + (1-\lambda)\left[\text{sym}\, A' + \frac{b' \otimes b'}{2}\right] - M_b + M_\lambda\right)
\end{aligned}
$$

where $M_b = \lambda(1-\lambda)(b-b') \otimes (b-b') \in \mathbb{R}^{2\times 2}_+$.

Choose now $M, M' \in \mathbb{R}^{2\times 2}_+$ so that

$$W_{F\ddot{o}}(A, b) = Q_2\left(\text{sym}\, A + \frac{1}{2}b \otimes b + M\right),$$

and the same for A', b' and M', and set $M_\lambda = \lambda M + (1-\lambda)M' + M_b \in \mathbb{R}^{2\times 2}_+$. Then the previous expression takes the form

$$Q_2\left(\lambda\left[\text{sym}\, A + \frac{b \otimes b}{2} + M\right] + (1-\lambda)\left[\text{sym}\, A' + \frac{b' \otimes b'}{2} + M'\right]\right)$$

and the convexity of Q_2 concludes the proof. $\square$

Lemma 4. *For each $M \in \mathbb{R}^{2\times 2}_+$ there are $\psi_\delta \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ and $\varphi_\delta \in W^{1,\infty}(\mathbb{R}^2)$ such that*

$$
\begin{aligned}
\psi_\delta &\overset{*}{\rightharpoonup} 0 && \text{weakly* in } W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2), \\
\varphi_\delta &\overset{*}{\rightharpoonup} 0 && \text{weakly* in } W^{1,\infty}(\mathbb{R}^2),
\end{aligned}
$$

as $\delta \to 0$,

$$\text{sym}\, \nabla' \psi_\delta(x') + \frac{\nabla' \varphi_\delta(x') \otimes \nabla' \varphi_\delta(x')}{2} = M,$$

for a.e. $x' \in \mathbb{R}^2$, and $\|\psi_\delta\|_{W^{1,\infty}} + \|\varphi_\delta\|_{W^{1,\infty}} \le C(|M| + 1)$.

Proof. Let $\zeta(t)$ be defined as t if $0 < t < 1/2$, as $(1-t)$ if $1/2 < t < 1$ and extended periodically on the rest of $\mathbb{R}$. Let $\zeta_\delta(t) := \delta\zeta(t/\delta)$ for every $\delta > 0$ so that $\zeta_\delta \rightharpoonup^* 0$ weakly* in $W^{1,\infty}(\mathbb{R})$ as $\delta \to 0$.

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We can write $M = \lambda_1 a_1 \otimes a_1 + \lambda_2 a_2 \otimes a_2$ for $a_1, a_2 \in S^1$ and $\lambda_1, \lambda_2 \geq 0$. We define

$$\psi_\delta(x') := (\sqrt{\lambda_1} a_1 - \sqrt{\lambda_2} a_2) \zeta_\delta((\sqrt{\lambda_1} a_1 + \sqrt{\lambda_2} a_2) \cdot x'),$$

so that

$$\nabla' \psi_\delta(x') := \zeta_\delta'((\sqrt{\lambda_1} a_1 + \sqrt{\lambda_2} a_2) \cdot x')(\sqrt{\lambda_1} a_1 - \sqrt{\lambda_2} a_2) \otimes (\sqrt{\lambda_1} a_1 + \sqrt{\lambda_2} a_2).$$

In particular

$$\operatorname{sym} \nabla' \psi_\delta(x') = \begin{cases} \lambda_1 a_1 \otimes a_1 - \lambda_2 a_2 \otimes a_2 & \text{on } S_\delta^+ \\ \lambda_2 a_2 \otimes a_2 - \lambda_1 a_1 \otimes a_1 & \text{on } S_\delta^- \end{cases}$$

where we have put $S_\delta^- = \mathbb{R}^2 \setminus S_\delta^+$ and

$$S_\delta^+ := \left\{ x' \in \mathbb{R}^2 : \text{for a } k \in \mathbb{N} \text{ we have } (\sqrt{\lambda_1} a_1 + \sqrt{\lambda_2} a_2) \cdot x' \in \left(k\delta, k\delta + \tfrac{1}{2}\delta\right) \right\}.$$

Correspondingly we define

$$\varphi_\delta(x') := \begin{cases} \zeta_\delta(2\sqrt{\lambda_2}\, a_2 \cdot x'), & \text{if } x' \in S_\delta^+, \\ \zeta_\delta(-2\sqrt{\lambda_1}\, a_1 \cdot x'), & \text{if } x' \in S_\delta^-. \end{cases}$$

Note that $\varphi_\delta \in W^{1,\infty}(\mathbb{R}^2)$. Indeed if $x' \in \overline{S_\delta^+} \cap \overline{S_\delta^-}$ we have that for some $j \in \mathbb{N}$

$$j\delta = 2(\sqrt{\lambda_1} a_1 + \sqrt{\lambda_2} a_2) \cdot x',$$

and since ζ_δ is δ-periodic we deduce that φ_δ is continuous on the interfaces, and thus Lipschitz on $\mathbb{R}^2$. On the other hand we have that

$$\nabla' \varphi_\delta(x') = \begin{cases} 2\sqrt{\lambda_2} \zeta_\delta'(2\sqrt{\lambda_2}\, a_2 \cdot x') a_2, & \text{if } x' \in S_\delta^+, \\ -2\sqrt{\lambda_1} \zeta_\delta'(-2\sqrt{\lambda_1}\, a_1 \cdot x') a_1, & \text{if } x' \in S_\delta^-. \end{cases}$$

and since $\zeta_\delta' = \pm 1$ a.e. we get

$$\nabla' \varphi_\delta(x') \otimes \nabla' \varphi_\delta(x') = \begin{cases} 4\lambda_2 a_2 \otimes a_2, & \text{if } x' \in S_\delta^+, \\ 4\lambda_1 a_1 \otimes a_1, & \text{if } x' \in S_\delta^-. \end{cases}$$

The thesis follows. $\qquad\square$

Lemma 5. *Let $M \in L^\infty(S, \mathbb{R}_+^{2\times 2})$ be constant on each of finitely many Lipschitz subsets S_j covering S, and let $\varepsilon_h \to 0$, $\varepsilon_h > 0$. Then there are $\psi_h \in C_0^\infty(S, \mathbb{R}^2)$ and $\varphi_h \in C_0^\infty(S)$ such that*

$$\begin{aligned} \psi_h &\rightharpoonup 0 & \text{weakly in } W^{1,2}(S, \mathbb{R}^2), \\ \varphi_h &\rightharpoonup 0 & \text{weakly in } W^{1,4}(S), \\ \operatorname{sym} \nabla' \psi_h(x') + \frac{\nabla' \varphi_h(x') \otimes \nabla' \varphi_h(x')}{2} &\to M & \text{strongly in } L^2(S, \mathbb{R}^{2\times 2}), \end{aligned}$$

and

$$\begin{aligned} \varepsilon_h \| (\nabla')^2 \varphi_h \|_{L^\infty(S, \mathbb{R}^{2\times 2})} &\leq 1, \\ \| \psi_h \|_{W^{1,\infty}(S, \mathbb{R}^2)} + \| \varphi_h \|_{W^{1,\infty}(S)} &\leq C(\| M \|_{L^\infty(S, \mathbb{R}^{2\times 2})} + 1). \end{aligned}$$

Proof. We can without loss of generality assume that M is constant on the entire S (if not, we perform the construction independently on each S_j).

Let $\tilde{\psi}_\delta$, $\tilde{\varphi}_\delta$ be the functions given by Lemma 4, $S_\rho = \{x' \in S : \text{dist}(x', \partial S) > \rho\}$, and $\eta_\rho \in C_0^\infty(B_\rho, \mathbb{R})$ be a mollification kernel on the scale ρ, i.e. be such that

$$\int_{\mathbb{R}^2} \eta_\rho(x')dx' = 1 \,, \qquad \int_{\mathbb{R}^2} \rho|\nabla'\eta_\rho(x')| + \rho^2|(\nabla')^2\eta_\rho(x')|dx' \leq C \,.$$

We set

$$\psi_{\delta,\rho}(x') = \int_{S_\rho} \tilde{\psi}_\delta(y')\eta_\rho(x' - y')dy'$$

and analogously $\varphi_{\delta,\rho}$. Clearly $\psi_{\delta,\rho} \in C_0^\infty(S, \mathbb{R}^2)$, $\varphi_{\delta,\rho} \in C_c^\infty(S)$, and as $\rho \to 0$

$$\psi_{\delta,\rho} \to \tilde{\psi}_\delta \,, \quad \varphi_{\delta,\rho} \to \tilde{\varphi}_\delta \,, \quad \text{strongly in } W^{1,2}(S, \mathbb{R}^2), \text{ resp. } W^{1,4}(S).$$

It remains to take a suitable diagonal subsequence. Indeed, for each δ we can chose $\rho(\delta)$ such that

$$\|\psi_{\delta,\rho(\delta)} - \tilde{\psi}_\delta\|_{W^{1,2}(S,\mathbb{R}^2)} + \|\varphi_{\delta,\rho(\delta)} - \tilde{\varphi}_\delta\|_{W^{1,4}(S)} \leq \delta \,.$$

This ensures all desired convergence properties as $\delta \to 0$. To include the bound on the second gradient it suffices to choose $\delta(h)$ as the smallest δ for which $\varepsilon_h\|(\nabla')^2\varphi_{\delta,\rho(\delta)}\|_{L^\infty(S,\mathbb{R}^{2\times2})} \leq 1$. This is possible since $\varepsilon_h \to 0$, and for the same reason $\delta(h) \to 0$. Finally, we set $\psi_h = \psi_{\delta(h),\rho(\delta(h))}$ and define φ_h likewise. $\square$

In the proof of Theorem 1 we have stated the existence of certain measurable functions. This can be proved by a rather standard application of the measurable selections principles, which is however typically disregarded in the literature. We therefore chose to provide here the simple details for the case of interest here.

The basic tool is the following slight simplification of Theorem III.6 in [4].

Lemma 6. *Let X be a set with a σ-algebra $\mathcal{F}$, let Y be a complete, separable metric space and for every $x \in X$ let a nonempty subset $F(x)$ of Y be given in such a way that*

$$\{x \in X : F(x) \cap U \neq \emptyset\} \in \mathcal{F} \tag{32}$$

for every open set U in Y.

Then a measurable map $f : X \to Y$ can be defined in such a way that $f(x) \in F(x)$ for every $x \in X$.

For the convenience of the reader we recall the brief proof.

Proof. Let $\{y_k\}_k$ be a countable and dense subset of Y and let $f_0 : X \to Y$ be defined by

$$\begin{aligned} f_0(x) &:= y_{k_0(x)}, \\ k_0(x) &:= \min\{k \in \mathbb{N} : F(x) \cap B(y_k, 2^0) \neq \emptyset\}. \end{aligned}$$

Note that f_0 is measurable as it takes values in $\{y_k\}_k$ and as $(f_0)^{-1}(y_k)$ is measurable for every k, by (32). Assume that a measurable $f_j : X \to Y$ has been defined in

such a way that: $f_j(x) = y_{k_j(x)}$, for $k_j(x)$ such that $F(x) \cap B(y_{k_j(x)}, 2^{-j}) \neq \emptyset$. Then we define $f_{j+1}(x)$ as

$$
\begin{aligned}
f_{j+1}(x) &:= y_{k_{j+1}(x)}, \\
k_{j+1}(x) &:= \min\{k \in \mathbb{N} : F(x) \cap B(y_{k_j(x)}, 2^{-j}) \cap B(y_k, 2^{-j-1}) \neq \emptyset\}.
\end{aligned}
$$

Once again f_{j+1} is measurable by (32). Furthermore we have easily that

$$
\text{dist}(f_j(x), F(x)) \leq 2^{-j}, \quad \text{dist}(f_j(x), f_{j+1}(x)) \leq 2^{-j+1},
$$

so that $\text{dist}(f_j(x), f_{j+h}(x)) \to 0$ as $j \to \infty$ for every h. Since Y is complete for every $x \in X$ we find $f(x) \in F(x)$ such that $f_j(x) \to f(x)$, and in particular the map $f : X \to Y$ is measurable. This completes the proof of the lemma. $\square$

We then state and prove some consequences of this Lemma that we have used in the proof of Theorem 1.

Lemma 7. *Let $M : \Omega \to \mathbb{R}^{n \times n}$ be measurable. Then there is a measurable $R : \Omega \to SO(n)$ such that*

$$
|M(x) - R(x)| = \text{dist}(M(x), SO(n)) \quad \forall x \in \Omega.
$$

Proof. We apply Lemma 6 with $X = \Omega$, $\mathcal{F}$ the σ-algebra of the Lebesgue measurable sets of Ω, $Y = SO(n)$ and $F(x) = \{Q \in SO(n) : |Q - M(x)| = \text{dist}(M(x), SO(n))\}$. Let U be an open set of $SO(3)$ and let U_k be an increasing sequence of compact sets exhausting U. Then

$$
\begin{aligned}
&\{x \in X : F(x) \cap U \neq \emptyset\} \\
&= \{x \in \Omega : \exists Q \in U, |Q - M(x)| = \text{dist}(M(x), SO(n))\} \\
&= \bigcup_{k \in \mathbb{N}} \{x \in \Omega : \text{dist}(M(x), U_k) = \text{dist}(M(x), SO(n))\}
\end{aligned}
$$

and each set in this countable union is measurable as it is the coincidence set of two measurable functions. $\square$

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