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Affine curvature homogeneous 3 dimensional Lorentz manifolds

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AFFINE CURVATURE HOMOGENEOUS 3-DIMENSIONAL LORENTZ MANIFOLDS

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Abstract. We study a family of 3-dimensional Lorentz manifolds. Some members of the family are 0-curvature homogeneous, 1-affine curvature homogeneous, but not 1-curvature homogeneous. Some are 1-curvature homogeneous but not 2-curvature homogeneous. All are 0-modeled on indecomposible local symmetric spaces. Some of the members of the family are geodesically complete, others are not. All have vanishing scalar invariants.

1. Introduction

1.1. **Affine manifolds.** We say that $\mathcal{A} := (M, \nabla)$ is an *affine manifold* if ∇ is a torsion free connection on the tangent bundle TM of a smooth m-dimensional manifold M. Let

$$\mathcal{R}(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

be the associated curvature operator. We say that $\mathcal{A}$ is *locally affine homogeneous* if given any points $P, Q \in M$, there is a diffeomorphism $\Phi_{P,Q}$ from a neighborhood of P to a neighborhood of Q with $\Phi_{P,Q}(P) = Q$ so that $\Phi^*_{P,Q}\nabla = \nabla$. We say that $\mathcal{A}$ is *locally k-affine curvature homogeneous* if given any points $P, Q \in M$, there is a linear isomorphism $\phi_{P,Q}$ from T_PM to T_QM so that $\phi^*_{P,Q}\nabla^i\mathcal{R}_Q = \nabla^i\mathcal{R}_P$ for $0 \le i \le k$. By taking $\phi_{P,Q} = (\Phi_{P,Q})_*$, it is clear that any locally affine homogeneous manifold is locally k-affine curvature homogeneous for all k. What is perhaps somewhat surprising is that given k, there exists a k-affine curvature homogeneous manifold $\mathcal{A}_k$ of dimension $a(k)$ which is not locally affine homogeneous, see, for example, the discussion in [6, 7, 9, 10]; one has that $a(k) \to \infty$ as $k \to \infty$.

1.2. **Pseudo-Riemannian manifolds.** There are similar notions in the metric context. Let $\mathcal{M} := (M, g)$ be a pseudo-Riemannian manifold of signature (p, q). We take ∇ to be the Levi-Civita connection and let $R \in \otimes^4 T^*M$ be the associated curvature tensor:

$$R(X,Y,Z,W) := g(\mathcal{R}(X,Y)Z, W)\,.$$

We say that $\mathcal{M}$ is *locally homogeneous* if given any points $P, Q \in M$, there is an isometry $\Phi_{P,Q}$ from a neighborhood of P to a neighborhood of Q with $\Phi_{P,Q}(P) = Q$. We say that $\mathcal{M}$ is k-curvature homogeneous if given any two points $P, Q \in M$, there is an isometry $\phi_{P,Q}$ from T_PM to T_QM so that $\phi^*_{P,Q}\nabla^iR_Q = \nabla^iR_P$ for $0 \le i \le k$. As (g, R) determines $\mathcal{R}$, locally homogeneous (resp. k-curvature homogeneous) manifolds are locally affine homogeneous (resp. k-affine curvature homogeneous). We refer to the discussion in [1] for a review of some of the literature in this subject.

Given k, there is a pseudo-Riemannian manifold $\mathcal{M}_k$ of dimension $m(k)$ which is k-curvature homogeneous (and hence k-affine curvature homogeneous) but not locally affine homogeneous (and hence not locally homogeneous) [6]; one has that $m(k) \to \infty$ as $k \to \infty$.

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If, however, m is bounded, one has the following result due to Singer [13] in the Riemannian ($p = 0$) setting and to Podesta and Spiro [11] in the general setting:

Theorem 1.1. *There exists an integer $k_{p,q}$ so that if $\mathcal{M}$ is a geodesically complete simply connected pseudo-Riemannian manifold of signature (p, q) which is $k_{p,q}$-curvature homogeneous, then $\mathcal{M}$ is homogeneous.*

We refer to Opozoda [10] for a similar result in the affine setting; there is an additional technical hypothesis which must be imposed.

1.3. **Vanishing scalar invariants.** Adopt the Einstein convention and sum over repeated indices. We can construct scalar invariants by contracting indices. For example, the scalar curvature τ, the norm $|\rho|^2$ of the Ricci tensor, and the norm $|R|^2$ of the full curvature tensor are scalar invariants defined by:

$$\begin{aligned} &\tau := g^{i_1j_1}g^{i_2j_2}R_{i_1i_2j_2j_1},\\ &|\rho|^2 := g^{i_1j_1}g^{i_2j_2}g^{i_3j_3}g^{i_4j_4}R_{i_1i_2j_2i_3}R_{j_1i_4j_4j_3},\\ &|R|^2 := g^{i_1j_1}g^{i_2j_2}g^{i_3j_3}g^{i_4j_4}R_{i_1i_2i_3i_4}R_{j_1j_2j_3j_4}\,. \end{aligned}$$

By Weyl's theorem [14], all universal polynomial scalar invariants of the covariant derivatives of the curvature tensor arise in this way; thus such invariants are called *Weyl scalar invariants.* We say that a pseudo-Riemannian manifold is *VSI* if all the scalar Weyl invariants vanish. This is not possible for non-flat manifolds in the Riemannian setting but is possible in the higher signature setting, see, for example, the discussion in [8, 12].

1.4. **Lorentzian manifolds.** In this paper, we shall deal with the 3-dimensional Lorentzian setting – i.e. signature $(1, 2)$. We shall be discussing a number of tensors. For the sake of brevity, we shall only give the non-zero components up to the usual symmetries. Let $\{x, y, \tilde{x}\}$ be coordinates on $\mathbb{R}^3$. Let $f = f(y)$ be a smooth function on $\mathbb{R}$ and let $\mathcal{M}_f := (\mathbb{R}^3, g_f)$ where g_f is the Lorentz metric on $\mathbb{R}^3$ given by:

$$g_f(\partial_x, \partial_x) = -2f(y) \quad \text{and} \quad g_f(\partial_x, \partial_{\tilde{x}}) = g_f(\partial_y, \partial_y) = 1\,.$$

Let $\mathcal{S}_\varepsilon$ be defined by $f_\varepsilon(y) := \frac{1}{2}\varepsilon y^2$ for $\varepsilon = \pm 1$.

Theorem 1.2.

(1) *All scalar Weyl invariants of $\mathcal{M}_f$ vanish.*
(2) *$\mathcal{S}_\varepsilon$ is an indecomposable local symmetric space.*
(3) *If $f''(y) \neq 0$, then $\mathcal{M}_f$ is 0-curvature modeled on $\mathcal{S}_\varepsilon$ for $\varepsilon = \operatorname{sign}(f'')$.*
(4) *Assume that $f''(y) \neq 0$ and that $f'''(y) \neq 0$ for all $y \in \mathbb{R}$.*
    (a) *$\mathcal{M}_f$ is 1-affine curvature homogeneous.*
    (b) *$\mathcal{M}_f$ is 1-curvature homogeneous if and only if $f'' = ae^{by}$.*
    (c) *The following assertions are equivalent:*
        (i) *$\mathcal{M}_f$ is locally homogeneous.*
        (ii) *$\mathcal{M}_f$ is k-curvature homogeneous for all k.*
        (iii) *$\mathcal{M}_f$ is 2-curvature homogeneous.*
        (iv) *$\mathcal{M}_f$ is 2-affine curvature homogeneous*
        (v) *$f' = ae^{by}$.*

1.5. **Completeness.** Let $\exp_P : T_PM \to M$ be the exponential map. We say that an affine manifold $\mathcal{A}$ is *geodesically complete* if all geodesics extend for infinite time.

Theorem 1.3.

(1) *The manifolds $\mathcal{S}_\pm$ are geodesically complete.*
(2) *The map $\exp_P$ for $\mathcal{S}_+$ is not surjective for any point $P \in \mathbb{R}^3$.*
(3) *The map $\exp_P$ for $\mathcal{S}_-$ is a global diffeomorphism from $T_P\mathbb{R}^3$ to $\mathbb{R}^3$ $\forall P \in \mathbb{R}^3$.*

If ∇ is a torsion free connection, the *Jacobi operator* J_∇ and *Ricci form* ρ_∇ are:

$$J_\nabla(x) : \xi \to \mathcal{R}(\xi, x)x \quad \text{and} \quad \rho_\nabla(x, x) := \operatorname{Tr}(J_\nabla(x)) .$$

An affine manifold $\mathcal{A}$ is said to exhibit geodesic Ricci curvature blowup or in brief *Ricci explodes* if there exists a geodesic γ in $\mathcal{A}$ which is defined for $t \in [0, T)$ where $T < \infty$ so $\limsup_{t\to T} |\rho(\dot\gamma, \dot\gamma)(t)| = \infty$. Such a manifold is necessarily geodesically incomplete. Furthermore, such a manifold can not be embedded as an open subset of a geodesically complete affine manifold.

Assume f' never vanishes; by replacing y by $-y$, we may assume $f' > 0$. The growth of f' at $-\infty$ is crucial.

Theorem 1.4. *Assume that $f'(y) > 0$ for all $y \in \mathbb{R}$.*

1. *If $\exists\ C > 0$ so $f'(y) \le C|y|$ for $y \le -1$, $\mathcal{M}_f$ is geodesically complete.*
2. *If $\exists\ \epsilon, \delta > 0$ so $f'(y) \ge \epsilon|y|^{1+\delta}$ for $y \le -1$, $\mathcal{M}_f$ Ricci explodes.*

The remainder of this paper is devoted to the proof of these results. In Section 2, we determine the curvature of the manifolds $\mathcal{M}_f$ and establish Theorem 1.2. In Section 3, we establish Theorem 1.3 by solving the geodesic equations on $\mathcal{S}_\pm$ quite explicitly. In Section 4, we use results from the theory of ordinary differential equations to establish two slightly more general results from which Theorem 1.4 will follow.

The metrics we consider here are a special sub-family of Walker metrics. Various properties of certain of the manifolds in this family have been studied by many authors [2, 3, 4, 5, 8, 12], to give but a few of the many possible references. For example the existence of 1-curvature homogeneous 3-dimensional Lorentzian manifolds which are not locally homogeneous follows from the discussion in [2] and the existence of 3-dimensional VSI Lorentzian manifolds is established in [12]. In this paper, we present a unified treatment of a number of results concerning this family; we discuss some previously known results but also present some new results in affine geometry and deal with questions of geodesic completeness. We feel this family provides a rich family of examples. In particular, one has:

Example 1.5. For $1 \le i \le 6$, let $\mathcal{N}_{i,\pm} := \mathcal{M}_{f_{i,\pm}}$ where

$$\begin{array}{lll} f_{1,-}(y) = -e^{-y}, & f_{2,-}(y) = -e^{-y} + y, & f_{3,-}(y) = -e^{-y} - e^{-2y} \\ f_{1,+}(y) = e^{y}, & f_{2,+}(y) = e^{y} + y, & f_{3,+}(y) = e^{y} + e^{2y} . \end{array}$$

We have $f_i'(y) > 0$, $f_i''(y) \ne 0$, and $f_i''' \ne 0$ for all y. We apply the results of Theorems 1.1, 1.2, 1.3, and 1.4 to see:

1. $\mathcal{S}_-$ is a geodesically complete indecomposible symmetric space.
2. $\mathcal{N}_{1,-}$ is 0-curvature modeled on $\mathcal{S}_-$, locally homogeneous, and Ricci explodes.
3. $\mathcal{N}_{2,-}$ is 0-curvature modeled on $\mathcal{S}_-$, 1-curvature modeled on $\mathcal{N}_{1,-}$, not 2-curvature homogeneous, and Ricci explodes.
4. $\mathcal{N}_{3,-}$ is 0-curvature modeled on $\mathcal{S}_-$, not 1-curvature homogeneous, 1-affine curvature modeled on $\mathcal{N}_1$, and Ricci explodes.
5. $\mathcal{S}_+$ is a geodesically complete indecomposible symmetric space.
6. $\mathcal{N}_{1,+}$ is 0-curvature modeled on $\mathcal{S}_+$, geodesically complete, and homogeneous.
7. $\mathcal{N}_{2,+}$ is 0-curvature modeled on $\mathcal{S}_+$, 1-curvature modeled on $\mathcal{N}_{1,+}$, not 2-curvature homogeneous, and geodesically complete.
8. $\mathcal{N}_{3,+}$ is 0-curvature modeled on $\mathcal{S}_+$, not 1-curvature homogeneous, 1-affine curvature modeled on $\mathcal{N}_{1,+}$, and geodesically complete.

2. Curvature

The following Lemma is immediate from the definition:

Lemma 2.1. *One has for the manifold $\mathcal{M}_f$ that:*

1. *Christoffel symbols:*
   (a) $\nabla_{\partial_x}\partial_x = f'\partial_y$.
   (b) $\nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = -f'\partial_{\tilde{x}}$.
2. *Components of R and of $\mathcal{R}$:*
   (a) $\mathcal{R}(\partial_x,\partial_y)\partial_y = f''\partial_{\tilde{x}}$.
   (b) $\mathcal{R}(\partial_x,\partial_y)\partial_x = -f''\partial_y$.
   (c) $R(\partial_x,\partial_y,\partial_y,\partial_x) = f''$.
3. *Components of the Ricci tensor ρ:*
   (a) $\rho(\partial_x,\partial_x) = f''$.
4. *Components of ∇R and of $\nabla\mathcal{R}$:*
   (a) $\nabla_{\partial_y}\mathcal{R}(\partial_x,\partial_y)\partial_y = f'''\partial_{\tilde{x}}$.
   (b) $\nabla_{\partial_y}\mathcal{R}(\partial_x,\partial_y)\partial_x = -f'''\partial_y$.
   (c) $\nabla R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_y) = f'''$.
5. *Components of $\nabla^2 R$ and of $\nabla^2\mathcal{R}$:*
   (a) $\nabla_{\partial y}\nabla_{\partial y}\mathcal{R}(\partial_x,\partial_y)\partial_y = f''''\partial_{\tilde{x}}$.
   (b) $\nabla_{\partial y}\nabla_{\partial y}\mathcal{R}(\partial_x,\partial_y)\partial_x = -f''''\partial_y$.
   (c) $\nabla_{\partial x}\nabla_{\partial x}\mathcal{R}(\partial_x,\partial_y)\partial_y = f'f'''\partial_{\tilde{x}}$.
   (d) $\nabla_{\partial x}\nabla_{\partial x}\mathcal{R}(\partial_x,\partial_y)\partial_x = -f'f'''\partial_y$.
   (e) $\nabla^2 R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_y,\partial_y) = f''''$.
   (f) $\nabla^2 R(\partial_x,\partial_y,\partial_y,\partial_x;\partial_x,\partial_x) = f'f'''$.

We shall need a technical lemma related to the structure of $\nabla^k R$ when f is a pure exponential. Let $\nabla^k R(\vec{\xi}) := \nabla^k R(\xi_1,\xi_2,\xi_3,\xi_4;\xi_5,...,\xi_{4+k})$ for $\vec{\xi} = (\xi_1,...,\xi_{4+k})$. We suppose $\xi_i = \partial_x$ or $\xi_i = \partial_y$; there is no need to take $\xi_i = \partial_{\tilde{x}}$ since $\nabla^k R(\vec{\xi})$ vanishes if any $\xi_i = \partial_{\tilde{x}}$. Let $\alpha(\vec{\xi})$ denote the number of times that $\xi_i = \partial_x$.

Lemma 2.2. *If $f = ae^{by}$, then $\nabla^k R(\vec{\xi}) = \gamma_{\vec{\xi}}(a,b)e^{\frac{1}{2}\alpha(\vec{\xi})by}$.*

Proof. We proceed by induction on k. Lemma 2.2 follows from Lemma 2.1 when $k = 0,1,2$; $\gamma_{\vec{\xi}}$ is zero if $\alpha(\vec{\xi})$ is odd. We have by definition that:

$$\begin{aligned}
&(2.a)\quad \nabla^k R(\vec{\xi}) = \xi_{k+4}\nabla^{k-1}R(\xi_1,...,\xi_{k+3})\\
&(2.b)\quad \qquad - \textstyle\sum_{1\le i\le k+3}\nabla^{k-1}R(\xi_1,...,\xi_{i-1},\nabla_{\xi_{k+4}}\xi_i,\xi_{i+1},...,\xi_{k+3})\,.
\end{aligned}$$

Suppose $\xi_{k+4} = \partial_y$. Let $\vec{\eta} := (\xi_1,...,\xi_{k+3})$; $\alpha(\vec{\xi}) = \alpha(\vec{\eta})$. Since $\nabla_{\partial_y}\xi_i$ is a multiple of $\partial_{\tilde{x}}$, the terms in (2.b) vanish and only the term in (2.a) enters. Thus:

$$\begin{aligned}
\nabla^k R(\vec{\xi}) &= \partial_y\nabla^{k-1}R(\vec{\eta}) = \partial_y\gamma_{\vec{\eta}}(a,b)e^{\frac{1}{2}\alpha(\vec{\eta})by}\\
&= \tfrac{1}{2}\alpha(\vec{\eta})b\gamma_{\vec{\eta}}(a,b)e^{\frac{1}{2}\alpha(\vec{\eta})by} = \gamma_{\vec{\xi}}(a,b)e^{\frac{1}{2}\alpha(\vec{\xi})by}\quad\text{for}\\
\gamma_{\vec{\xi}}(a,b) &:= \tfrac{1}{2}\alpha(\vec{\eta})b\gamma_{\vec{\eta}}(a,b)\,.
\end{aligned}$$

Suppose that $\xi_{k+4} = \partial_x$. Since $\xi_{k+4}\nabla^{k-1}R(\xi_1,...,\xi_{k+3}) = 0$, the term in (2.a) vanishes. Since $\nabla_{\partial_x}\partial_y$ is a multiple of $\partial_{\tilde{x}}$, we can ignore terms where $\xi_i = \partial_y$. Set

$$\vec{\eta}_i := (\xi_1,...,\xi_{i-1},\partial_y,\xi_{i+1},...,\xi_{k+3})\,.$$

We have $\nabla_{\partial_x}\partial_x = f'\partial_y$. If $\xi_i = \partial_x$, then $\alpha(\vec{\eta}_i) = \alpha(\vec{\xi}) - 2$. We compute:

$$\begin{aligned}
\nabla^k R(\vec{\xi}) &= -abe^{by}\textstyle\sum_{i:\xi_i=\partial_x}\nabla^{k-1}R(\vec{\eta}_i) = -abe^{by}\sum_{i:\xi_i=\partial_x}\gamma_{\vec{\eta}_i}(a,b)e^{\frac{1}{2}\alpha(\vec{\eta}_i)by}\\
&= \gamma_{\vec{\xi}}(a,b)e^{\frac{1}{2}\alpha(\vec{\xi})by}\quad\text{for}\\
\gamma_{\vec{\xi}}(a,b) &:= -ab\textstyle\sum_{i:\xi_i=\partial_x}\gamma_{\vec{\eta}_i}(a,b)\,.
\end{aligned}$$

The Lemma now follows from these two special cases. □

Proof of Theorem 1.2. We show that all the scalar Weyl invariants of $\mathcal{M}_f$ vanish as follows. First consider the hyperbolic basis

$$X := \partial_x + f\partial_{\tilde{x}}, \quad \tilde{X} := \partial\tilde{x}, \quad Y := \partial_y\,.$$

We then set

$$e_1^+ := (X + \tilde{X})/\sqrt{2}, \quad e_2^- := (X - \tilde{X})/\sqrt{2}, \quad e_3^+ := \partial_y\,.$$

We form scalar Weyl invariants by contracting indices in pairs and then summing over repeated indices. Since $\nabla^k R(..., e_1^+, ...) = \nabla^k R(..., e_2^-, ...) = \nabla^k R(..., \partial_x, ...)$, since $g^{11} = +1$, and since $g^{22} = -1$, terms where $e_i = e_1^+$ (i.e. $i = 1$) and terms where $e_i = e_2^-$ (i.e. $i = 2$) appear with opposite signs in any Weyl summation and cancel; $\nabla^k R(..., e_3^+, ...) = 0$. Assertion (1) now follows.

If f is quadratic, then $\nabla R = 0$. Thus $\mathcal{S}_+$ and $\mathcal{S}_-$ are local symmetric spaces. The curvature tensor and metric are indecomposible; they are not irreducible as $\operatorname{Span}\{\partial_{\tilde{x}}\}$ is invariant under the isotropy representation. Assertion (2) follows.

Set $\varepsilon_f := \operatorname{sign}(f'')$. We say that a basis $\mathcal{B} = \{X, Y, \tilde{X}\}$ is *normalized* if we have:

$$\begin{aligned} &1) \text{ the non-zero components of } g \text{ are} \quad g(X, \tilde{X}) = g(Y, Y) = 1,\\ &2) \text{ the non-zero components of } R \text{ are} \quad R(X, Y, Y, X) = \varepsilon_f,\\ &3) \text{ we have} \quad \nabla R(\xi_1, \xi_2, \xi_3, \xi_4; X) = 0 \ \forall \xi_1, \xi_2, \xi_3, \xi_4\,. \end{aligned} \tag{2.c}$$

We may define a normalized basis, and thereby establish Assertion (3), by setting

$$X := |f''|^{-1/2}\{\partial_x + f\partial_{\tilde{x}}\}, \quad Y := \partial_y, \quad \tilde{X} := |f''|^{1/2}\partial_{\tilde{x}}\,. \tag{2.d}$$

We say $\mathcal{B}$ is *affine normalized* if the non-zero components of $\mathcal{R}$ and $\nabla\mathcal{R}$ are

$$\begin{aligned} &\mathcal{R}(X, Y)Y = \varepsilon_f\tilde{X}, \quad \mathcal{R}(X, Y)X = -\varepsilon_f Y,\\ &\nabla_Y\mathcal{R}(X, Y)Y = \tilde{X}, \quad \nabla_Y\mathcal{R}(X, Y)X = -Y\,. \end{aligned}$$

We construct an affine normalized basis by rescaling the coordinate frame. Let a_1, a_2, and a_3 be constants to be determined. By Lemma 2.1,

$$\begin{aligned} &\mathcal{R}(a_1\partial_x, a_2\partial_y)a_2\partial_y = a_1a_2^2a_3^{-1}f''a_3\partial_{\tilde{x}},\\ &\mathcal{R}(a_1\partial_x, a_2\partial_y)a_1\partial_x = -a_1^2f''a_2\partial_y,\\ &\nabla_{a_2\partial_y}\mathcal{R}(a_1\partial_x, a_2\partial_y)a_2\partial_y = a_1a_2^3a_3^{-1}f'''a_3\partial_{\tilde{x}},\\ &\nabla_{a_2\partial_y}\mathcal{R}(a_1\partial_x, a_2\partial_y)a_1\partial_x = -a_1^2a_2f'''a_2\partial_y\,. \end{aligned}$$

Assume that $f''(y)$ and $f'''(y)$ never vanish. We define an affine normalized basis and prove Assertion (4a) by setting

$$\begin{aligned} &X := a_1\partial_x, && Y := a_2\partial_y, && \tilde{X} := a_3\partial_{\tilde{x}} && \text{where}\\ &a_1 := \{|f''|\}^{-1/2}, && a_2 := |f''|\{f'''\}^{-1}, && a_3 := a_1a_2^2|f''|\,. \end{aligned} \tag{2.e}$$

We note for future reference that

$$\begin{aligned} \nabla_X\nabla_X\mathcal{R}(X, Y)Y &= a_1^3a_2^2\nabla_{\partial_x}\nabla_{\partial_x}\mathcal{R}(\partial_x, \partial_y)\partial_y = a_1^3a_2^2f'f'''\partial_{\tilde{x}}\\ &= f'f'''\{f''\}^{-2}\tilde{X}\,. \end{aligned} \tag{2.f}$$

We study the relevant symmetry group to construct additional invariants of the 1-model. Let $\mathcal{B} = \{X, Y, \tilde{X}\}$ be the normalized basis defined in Equation (2.d). Suppose that $\mathcal{B}_1 = \{X_1, Y_1, \tilde{X}_1\}$ is another normalized basis. Expand:

$$\begin{aligned} X_1 &= a_{11}X + a_{12}Y + a_{13}\tilde{X},\\ Y_1 &= a_{21}X + a_{22}Y + a_{23}\tilde{X},\\ \tilde{X}_1 &= a_{31}X + a_{32}Y + a_{33}\tilde{X}\,. \end{aligned}$$

Since $R(\xi_1, \xi_2, \xi_3, \tilde{X}_1) = 0$ for any ξ_1, ξ_2, ξ_3, we have $a_{31} = a_{32} = 0$. Since one has $\nabla R(\xi_1, \xi_2, \xi_3, \xi_4; X_1) = 0$ for any $\xi_1, \xi_2, \xi_3, \xi_4$, $a_{12} = 0$. Thus

$$X_1 = a_{11}X + a_{13}\tilde{X}, \quad Y_1 = a_{21}X + a_{22}Y + a_{23}\tilde{X}, \quad \tilde{X}_1 = a_{33}\tilde{X}\,.$$

As $g(X_1,\tilde X_1)=1$, $a_{33}a_{11}=1$. As $g(Y_1,\tilde X_1)=0$, $a_{21}=0$. As $g(X_1,X_1)=0$, $a_{13}=0$. Consequently,

$$X_1=a_{11}X,\quad Y_1=a_{22}Y+a_{23}\tilde X,\quad \tilde X_1=a_{11}^{-1}\tilde X\,.$$

As $g(X_1,Y_1)=0$, $a_{23}=0$. As $g(Y_1,Y_1)=1$, $a_{22}^2=1$. As $R(X_1,Y_1,Y_1,X_1)=\varepsilon_f$, $a_{11}^2a_{22}^2=1$. Thus

$$X_1=a_{11}X,\quad Y_1=a_{22}Y,\quad \tilde X_1=a_{11}^{-1}\tilde X\quad\text{where}\quad a_{11}^2=a_{22}^2=1\,.$$

In particular we may use Equation (2.d) and Lemma 2.1 to see:

$$|\nabla R(X_1,Y_1,Y_1,X_1;Y_1)|=|\nabla R(X,Y,Y,X;Y)|=|f'''\{f''\}^{-1}|$$

is an invariant of the 1-model. This is constant if and only if $f'''=cf''$, i.e. $f''=ae^{by}$. Assertion (4b) now follows.

We now establish Assertion (4c). The following implications are immediate:

$$\text{(4c-i)}\Rightarrow\text{(4c-ii)}\Rightarrow\text{(4c-iii)}\Rightarrow\text{(4c-iv)}\,.$$

Suppose that $\mathcal{M}_f$ is 2-affine curvature homogeneous. Let $\mathcal{B}:=\{X,Y,\tilde X\}$ be the affine normalized basis defined in Equation (2.e). Suppose that $\mathcal{B}_1:=\{X_1,Y_1,\tilde X_1\}$ is another affine normalized basis. Let

$$\begin{aligned}
&\mathcal{I}_0:=\operatorname{Span}_{\xi_1,\xi_2,\xi_3}\{\mathcal{R}(\xi_1,\xi_2)\xi_3\}=\operatorname{Span}\{Y,\tilde X\},\\
&\mathcal{K}_0:=\{\eta:\mathcal{R}(\eta,\xi_1)\xi_2=0\text{ for all }\xi_1,\xi_2\}=\operatorname{Span}\{\tilde X\},\\
&\mathcal{K}_1:=\{\eta:\nabla_\eta\mathcal{R}(\xi_1,\xi_2)\xi_3=0\text{ for all }\xi_1,\xi_2,\xi_3\}=\operatorname{Span}\{X,\tilde X\}\,.
\end{aligned}$$

Since these spaces are invariantly defined, we may expand

$$X_1=a_{11}X+a_{13}\tilde X,\quad Y_1=a_{22}Y+a_{23}\tilde X,\quad \tilde X_1=a_{33}\tilde X\,.$$

We have $\mathcal{R}(X_1,Y_1)X_1=a_{11}^2\mathcal{R}(X,Y_1)X$ is a multiple of Y. Since the basis is affine normalized, it is also a multiple of Y_1. Thus $a_{23}=0$. We now compute

$$\begin{aligned}
&\mathcal{R}(X_1,Y_1)Y_1=a_{11}a_{22}^2a_{33}^{-1}\varepsilon_f\tilde X_1,\qquad \mathcal{R}(X_1,Y_1)X_1=-a_{11}^2\varepsilon_fY_1,\\
&\nabla_{Y_1}\mathcal{R}(X_1,Y_1)Y_1=a_{11}a_{22}^3a_{33}^{-1}\tilde X_1,\quad \nabla_{Y_1}\mathcal{R}(X_1,Y_1)X_1=-a_{11}^2a_{22}Y_1\,.
\end{aligned}$$

As the basis is affine normalized, $a_{11}^2=1$, $a_{22}=1$, $a_{33}=a_{11}$. Thus by Equation (2.f),

$$\begin{aligned}
&\nabla_{X_1}\nabla_{X_1}\mathcal{R}(X_1,Y_1)Y_1=a_{11}^3a_{22}^2\nabla_X\nabla_X\mathcal{R}(X,Y)Y\\
=\;&a_{11}^3a_{22}^2f'f'''\{f''\}^{-2}\tilde X=a_{11}^3a_{22}^2a_{33}^{-1}f'f'''\{f''\}^{-2}\tilde X_1\\
=\;&f'f'''\{f''\}^{-2}\tilde X_1\,.
\end{aligned}$$

This shows that $f'f'''\{f''\}^{-2}$ is an invariant of the affine 2-model. Consequently if $\mathcal{M}_f$ is 2-affine curvature homogeneous, then

$$f'f'''\{f''\}^{-2}=c\,.$$

If $c=1$, then $f'=ae^{by}$. If $c\neq1$, then $f'=\alpha(y+\beta)^\gamma$ for some $\{\alpha,\beta,\gamma\}$. This later choice is ruled out as f''' and f'' are assumed to be globally defined and non-zero. Thus we may conclude $f'=ae^{by}$; this establishes the implication:

$$\text{(4c-iv)}\Rightarrow\text{(4c-v)}\,.$$

Finally, we suppose that $f'=ae^{by}$; we take $f=\frac{a}{b}e^{by}$. Consider the normalized basis $\{X,Y,\tilde X\}$ defined in Equation (2.d):

$$X=|ab|^{-1/2}e^{-by/2}\{\partial_x+\tfrac{a}{b}e^{by}\partial_{\tilde x}\},\quad Y=\partial_y,\quad \tilde X=|ab|^{1/2}e^{by/2}\partial_{\tilde x}\,.$$

Let $\nabla^kR(\vec\eta)=\nabla^kR(\eta_1,\eta_2,\eta_3,\eta_4;\eta_5,...,\eta_{4+k})$ for $\vec\eta=(\eta_1,...,\eta_{4+k})$ where $\eta_i=X$ or $\eta_i=Y$ for $1\le i\le 4+k$. Let $\vec\xi$ be the corresponding string where X and Y

are replaced by ∂_x and ∂_y. Let $\alpha(\vec{\eta}) = \alpha(\vec{\xi})$ be the number of times that X or equivalently ∂_x appear. We apply Lemma 2.2 to see:

$$\begin{aligned}\nabla^k R(\vec{\eta}) &= |ab|^{-\alpha(\eta)/2} e^{-\alpha(\vec{\eta})by/2} \nabla^k R(\vec{\xi})\\ &= |ab|^{-\alpha(\eta)/2} e^{-\alpha(\vec{\eta})by/2} \gamma_{\vec{\xi}}(a,b) e^{\alpha(\vec{\xi})by/2}\\ &= |ab|^{-\alpha(\eta)/2} \gamma_{\vec{\xi}}(a,b)\,.\end{aligned}$$

This shows that $\mathcal{M}_f$ is k-curvature homogeneous for all k; a local version of Theorem 1.1 now shows that $\mathcal{M}_f$ is locally homogeneous as desired. Consequently, (4c-v)⇒(4c-i). □

3. Complete manifolds

Let $\gamma(t) = (x(t), y(t), \tilde{x}(t))$ be a path in $\mathcal{M}_f$. The geodesic equation becomes

$$x''(t) = 0, \quad y''(t) = -f'(y(t))x'(t)x'(t), \quad \tilde{x}''(t) = 2f'(y(t))y'(t)x'(t)\,.$$

The first equation yields $x(t) = x_0 + x_1 t$. The remaining equations then become

$$y''(t) = -x_1^2 f'(y(t)), \quad \tilde{x}''(t) = 2x_1 f'(y(t))y'(t)\,.$$

The equation for y is the crucial one; once y is determined, one can express

$$\tilde{x}(t) = \tilde{x}_0 + t\tilde{x}_0' + 2x_1 \int_{s=0}^{t} \int_{u=0}^{s} f'(y(u))y'(u)du ds\,. \tag{3.a}$$

Proof of Theorem 1.3. First set $f_+(y) = \frac{1}{2}y^2$. We then have to solve

$$y''(t) = -x_1^2 y(t)\,.$$

We show $\mathcal{S}_+$ is geodesically complete by solving this equation:

$$y(t) = \begin{cases} y_0 + y_0' t & \text{if} \quad x_1 = 0,\\ y_0\cos(x_1 t) + \frac{1}{x_1} y_0' \sin(x_1 t) & \text{if} \quad x_1 \neq 0\,. \end{cases}$$

A geodesic with $x(0) = x_0$ and $x(1) = x_0 + 2\pi$ has the form:

$$\gamma(t) = (x_0 + 2\pi t, y_0\cos(2\pi t) + \tfrac{1}{2\pi} y_0' \sin(2\pi t), \tilde{x}(t))\,.$$

Thus $y(1) = y(0)$ and the exponential map is not surjective. This establishes the Assertions of the Lemma concerning $\mathcal{S}_+$.

Next, we study $f_-(y) = -\frac{1}{2}y^2$. We then have to solve

$$y''(t) = x_1^2 y(t)\,.$$

We show $\mathcal{S}_-$ is geodesically complete by solving this equation:

$$y(t) = \begin{cases} y_0 + y_0' t & \text{if} \quad x_1 = 0,\\ \frac{1}{2} y_0 \{e^{x_1 t} + e^{-x_1 t}\} + \frac{1}{2x_1} y_0' \{e^{x_1 t} - e^{-x_1 t}\} & \text{if} \quad x_1 \neq 0\,. \end{cases}$$

We take $P := (x_0, y_0, \tilde{x}_0)$ as the initial point. Suppose $Q := (x_1, y_1, \tilde{x}_1)$ is given. The exponential map is given by setting $t = 1$. Thus $x(t) = x_0 + t(x_1 - x_0)$. If $x_1 - x_0 = 0$, then set $y(t) = y_0 + t(y_1 - y_0)$. If $x_1 - x_0 \neq 0$, we determine y_0' uniquely by solving the equation:

$$y_1 = \tfrac{1}{2} y_0 \{e^{x_1 - x_0} + e^{x_0 - x_1}\} + \tfrac{1}{2(x_1 - x_0)} y_0' \{e^{x_1 - x_0} - e^{x_0 - x_1}\}\,.$$

Once x and y have been determined, we then use Equation (3.a) to solve for $\tilde{x}_0'$. This shows that $\mathcal{S}_-$ is geodesically complete and that the exponential map is a diffeomorphism. □

4. The proof of Theorem 1.4

Following the discussion in Section 3, to construct geodesics in the manifold $\mathcal{M}_f$, we must solve the ODE

$$y'' = -x_1^2 f'(y)\,.$$

We shall suppose $x_1 \neq 0$ and set $h = -x_1^2 f'$. We begin with:

Lemma 4.1. *Let $h : \mathbb{R} \to (-\infty, 0)$ be smooth. Let $[0, T)$ be the maximal domain of the solution y to the ODE $y'' = h(y)$ where $y(0) = y_0$ and $y'(0) = y_0'$. If $T < \infty$,*

$$\lim_{t\to T} y(t) = \lim_{t\to T} y'(t) = -\infty \quad \text{and} \quad \limsup_{y\to T} \frac{h(y(t))}{y(t)} = \infty\,.$$

Proof. Since $y'' < 0$, y' is monotonically decreasing and y is bounded from above on $[0, T)$. Suppose first that y is bounded from below on $[0, T)$. This implies that y'' is bounded and hence y' is bounded as well on $[0, T)$. Let

$$y_1 = \lim\inf_{t\to T} y(t) \quad \text{and} \quad y_1' = \lim_{t\to T} y'(t)\,.$$

The fundamental theorem of ODE's shows there exists $\kappa > 0$ so that if

$$|z_1 - y_1| < \kappa, \quad |z_1' - y_1'| < \kappa, \quad \text{and} \quad s \in (T - \kappa, T)$$

then there exists a solution z to the equation $z'' = h(z)$ with initial conditions $z(s) = z_1$ and $z'(s) = z_1'$ which is valid on the interval $[s, s + \kappa)$. We choose

$$s \in (T - \tfrac{1}{2}\kappa, T) \quad \text{so that} \quad |y(s) - y_1| < \kappa \quad \text{and} \quad |y'(s) - y_1'| < \kappa\,.$$

Let $z'' = h(z)$ be defined on $[s, s + \kappa)$ with $z(s) = y(s)$ and $z'(s) = y'(s)$. Then z extends y to the region $[0, T + \frac{1}{2}\kappa)$ which contradicts the assumption that $[0, T)$ was a maximal domain.

Thus y is not bounded from below on $[0, T)$ so $\lim_{t\to T} y'(t) = -\infty$. Consequently, y is monotonically decreasing for t close to T so $\lim_{t\to T} y(t) = -\infty$ as well. Suppose

$$\limsup_{t\to T} \frac{h(y(t))}{y(t)} < \infty$$

i.e. that there exists $C < \infty$ so $|h(y(t))| \le C|y(t)|$ on $[t_0, T)$. We then have

$$\{\ln |y(t)|\}'' = \left\{\frac{y'(t)}{y(t)}\right\}' = \frac{y''(t)}{y(t)} - \left\{\frac{y'(t)}{y(t)}\right\}^2 = \frac{h(y(t))}{y(t)} - \left\{\frac{y'(t)}{y(t)}\right\}^2 \le C\,.$$

This implies $\ln |y(t)|$ is bounded from above and hence $|y(t)|$ is bounded from above on $[t_0, T)$ which is false. This contradiction shows $\limsup_{t\to T} \frac{h(y(t))}{y(t)} = \infty$. □

Proof of Theorem 1.4 (1). We suppose that $f' > 0$ and that $f'(y) \le C|y|$ for $y \le -1$. We set $h = -x_1^2 f'$. Choose a maximal domain $[0, T)$ for the solution to the ODE $y'' = h(y)$ with initial condition $y(0) = y_0$ and $y'(0) = y_0'$. If $T < \infty$, then

$$\limsup_{t\to T} \frac{h(y(t))}{y(t)} = \infty$$

which is false. Thus $T = \infty$ and $\mathcal{M}_f$ is geodesically complete. □

Before proving Theorem 1.4 (2), we must establish:

Lemma 4.2.

1. *Let $\alpha > 0$. Let $\{t_n\}_{n\ge1}$ be a sequence of real numbers with $t_1 = 1$ and with $t_{n+1} - t_n \ge n^\alpha$ for $n \ge 1$. Then $t_n \ge \frac{n^{1+\alpha}}{(1+\alpha)2^{1+\alpha}}$.*
2. *Let $\epsilon > 0$ and $\delta > 0$. Suppose that $h(y) < -\epsilon|y|^{1+\delta}$ for $y \le -1$. Let $[0, T)$ be the maximal domain of the solution y to the ODE $y'' = h(y)$ with $y(0) = -1$ and $y'(0) = -1$. Then $T < \infty$ and $\lim_{t\to T} y(t) = -\infty$.*

Proof. We prove Assertion (1) by induction on n; it holds trivially for $n = 1$. We take $n \geq 2$ and use the comparison test to compute:

$$\begin{aligned} t_n &> t_n - t_1 = \sum_{k=2}^{n} \left\{ t_k - t_{k-1} \right\} \geq \int_1^n (x-1)^\alpha dx \\ &= \frac{(n-1)^{1+\alpha}}{1+\alpha} = \frac{n^{1+\alpha}}{(1+\alpha)(1+\frac{1}{n-1})^{1+\alpha}} \geq \frac{n^{1+\alpha}}{(1+\alpha)2^{1+\alpha}} \,. \end{aligned}$$

To prove Assertion (2), we suppose first $T = \infty$ and argue for a contradiction. Choose τ so that

$$\tau\varepsilon \geq 2^{1+\delta/2}(1+\delta/2) \quad \text{and} \quad \tau \geq 1 \,.$$

With our initial conditions, $y'' < 0$ so y' is monotonically decreasing and $y' \leq -1$. This implies y decreases monotonically. Let $\Delta_n = \tau \cdot n^{-1-\delta/2}$. Let $s_1 = 0$ and let $s_{n+1} = s_n + \Delta_n$ for $n \geq 2$. As $\delta > 0$,

$$S := \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} \tau n^{-1-\delta/2} < \infty \,.$$

We wish to show inductively that

(1) $y'(s_n) \leq -n^{1+\delta/2}$.
(2) $y(s_n) \leq -n$.
(3) $y'(s_{n+1}) - y'(s_n) \leq -2^{1+\delta/2}(1+\frac{1}{2}\delta)n^{\delta/2}$.

The first two statements hold $n = 1$ by the choice of our initial conditions. Since y and y' decrease monontonically, we may estimate

$$\begin{aligned} &y''(s) \leq -\epsilon|y(s)|^{1+\delta} \leq -\epsilon|y(s_n)|^{1+\delta} \leq -\epsilon n^{1+\delta} \text{ for } s \in [s_n, s_{n+1}], \\ &y'(s_{n+1}) - y'(s_n) \leq -\Delta_n \epsilon n^{1+\delta} = -\tau n^{-1-\delta/2}\epsilon n^{1+\delta} \leq -2^{1+\delta/2}(1+\delta/2)n^{\delta/2} \,. \end{aligned}$$

Thus statements $(1)_n$ and $(2)_n$ imply assertion $(3)_n$.

Statements $(3)_k$ for $1 \leq k \leq n$ together with Assertion (1) imply Statement $(1)_{n+1}$. Finally, we use Statement $(1)_n$ together with statement $(2)_n$ to establish Statement $(2)_{n+1}$ by computing:

$$\begin{aligned} &y'(s) \leq y'(s_n) \leq -n^{1+\delta/2} \quad \text{for} \quad s \geq s_n, \\ &y(s_{n+1}) \leq y(s_n) + \Delta_n y'(s_n) \leq -n - \tau n^{-1-\delta/2} n^{1+\delta/2} \leq -n-1 \,. \end{aligned}$$

This establishes the truth of all the 3 statements. Thus, $\lim_{s\to S} y(s) = -\infty$. This contradicts the assumption that $T = \infty$.

This shows that y must be defined on a maximal domain $[0,T)$ for $T < \infty$; the fact that $\lim_{t\to T} y(t) = -\infty$ now follows from Lemma 4.1. □

Proof of Theorem 1.4 (2). Suppose $f'(y) > 0$ for all y and that $f'(y) \geq \varepsilon|y|^{1+\delta}$ for $y \leq -1$. Choose a geodesic with $x(0) = 0$, $x'(0) = 1$, $y(0) = -1$, and $y'(0) = -1$. We then have the differential equation

$$y'' = -f'(y) \,.$$

Thus by Lemma 4.2 for some finite time T, we have $\lim_{t\to T} y(t) = -\infty$. Thus $\mathcal{M}_f$ is geodesically incomplete. We have $\rho(\dot\gamma, \dot\gamma) = f''(y(t))$.

If $|f''(y)| \leq K$ on $(-\infty, 0]$, then $f'(y) \leq K|y| + f'(y(0))$ on $(-\infty, 0]$ which is false. Thus $|f''(y)|$ is not bounded on $(-\infty, 0]$. Since $y(t) \to -\infty$ as $t \to T$, $f''(y(t))$ is not bounded on $[0,T)$. This shows, as desired, that $\mathcal{M}_f$ Ricci explodes. □

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